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Abstract

ABSTRACT: We consider the short time heat content asymptotics for oblique boundary conditions. The first few coefficients in the asymptotic expansion are calculated.

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Let M be a compact Riemannian manifold of dimension m with smooth boundary ∂M and let D be an operator of Laplace type on a vector bundle V over M. The operator D defines a natural connection ∇ - see equation (3) below. Let ∇_m be covariant differentiation with respect to the inward unit normal on the boundary. Let $\mathcal{B}_T$ be a tangential differential operator and let *oblique boundary conditions* [8, 11, 14] be defined by the operator:

$$\mathcal{B}\psi := (\nabla_m + \mathcal{B}_T)\psi|_{\partial M}.$$

Given an initial temperature distribution ϕ, the subsequent temperature distribution $u := e^{-tD_{\mathcal{B}}}\phi$ is defined by the equations:

$$(\partial_t + D)u = 0, \ u|_{t=0} = \phi, \text{ and } \mathcal{B}u = 0. \tag{1}$$

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The specific heat ρ is a section to the dual bundle $\tilde{V}$ and the total heat energy content of the manifold is given by:

$$\beta(\phi,\rho,D,\mathcal{B})(t) := \textstyle\int_M u(t;x)\rho(x)d\nu_M$$

where we integrate with respect to the Riemannian volume element on M. As $t \downarrow 0$, there is a complete asymptotic expansion:

$$\beta(\phi,\rho,D,\mathcal{B})(t) \sim \textstyle\sum_{n\geq 0}\beta_n(\phi,\rho,D,\mathcal{B})t^{n/2}.$$

The *heat content asymptotics* β_n are locally computable and have been studied extensively for Dirichlet, Robin and mixed boundary conditions [3, 4, 5, 12, 13] - see [10] for a recent survey article on the field. In some detail, there exist local invariants β_n^{int} and β_n^{bd} which are bilinear in the jets of ϕ,ρ, with coefficients which are smooth local invariants of the jets of the total symbol of $\{D,\mathcal{B}\}$ so that

$$\beta_n(\phi,\rho,D,\mathcal{B}) = \textstyle\int_M \beta_n^{int}(\phi,\rho,D)d\nu_M + \int_{\partial M}\beta_n^{bd}(\phi,\rho,D,\mathcal{B})d\nu_{\partial M}.$$

The interior terms do not depend on the boundary condition and one may choose [5]

$$\begin{aligned}\beta_{2n+1}^{int}(\phi,\rho,D) &= 0,\\ \beta_{4n}^{int}(\phi,\rho,D) &= \tfrac{1}{(2n)!}(D^n\phi,(\tilde{D})^n\rho),\\ \beta_{4n+2}^{int}(\phi,\rho,D) &= -\tfrac{1}{(2n+1)!}(D^{n+1}\phi,(\tilde{D})^n\rho).\end{aligned}$$

It is the aim of the present letter to find the boundary integrands for the heat content asymptotics for oblique boundary conditions. The heat trace asymptotics has already been extensively discussed in [1, 2, 6, 7].

To express the heat content asymptotics β_n invariantly, we must introduce some additional notation. On the boundary, we let Roman indices a and b index a local orthonormal frame $\{e_1,...,e_{m-1}\}$ for the tangent bundle of the boundary; let e_m be the inward unit normal. We use the connection ∇ and the Levi-Civita connection of the boundary to covariantly differentiate tensors of all types tangentially. We may then express $\mathcal{B}_T = \Gamma_a\nabla_a + \nabla_a\Gamma_a + S$ with auxiliary endomorphisms Γ and S. Let $\tilde{D}$ and $\tilde{\mathcal{B}}$ be the dual operators on the dual bundle $\tilde{V}$. If $\tilde{\nabla}$ is the dual connection and if $\tilde{\Gamma}$ and $\tilde{S}$ are the dual endomorphisms, then $\tilde{\mathcal{B}} = \tilde{\nabla}_m + \tilde{\mathcal{B}}_T$ where $\tilde{\mathcal{B}}_T = \tilde{\Gamma}_a\tilde{\nabla}_a + \tilde{\nabla}_a\tilde{\Gamma}_a + \tilde{S}$. Note that $\tilde{\nabla}$ is also the connection defined by the dual operator $\tilde{D}$. Let L be the second fundamental form; the contraction L_{aa} is the geodesic curvature of the boundary. The following is the main result of this letter.

Theorem 1 *Adopt the notation established above*

1. $\beta_0(\phi,\rho,D,\mathcal{B})=\int_M\phi\rho d\nu_M$.

2. $\beta_1(\phi,\rho,D,\mathcal{B})=0$.

3. $\beta_2(\phi,\rho,D,\mathcal{B})=-\int_M D\phi\cdot\rho d\nu_M+\int_{\partial M}\mathcal{B}\phi\cdot\rho d\nu_{\partial M}$.

4. $\beta_3(\phi,\rho,D,\mathcal{B})=\frac{4}{3\sqrt{\pi}}\int_{\partial M}\mathcal{B}\phi\cdot\tilde{\mathcal{B}}\rho d\nu_{\partial M}$.

5. $$\beta_4(\phi,\rho,D,\mathcal{B})=\frac{1}{2}\int_M D\phi\cdot\tilde{D}\rho d\nu_M+\int_{\partial M}\{-\frac{1}{2}\mathcal{B}\phi\cdot\tilde{D}\rho-\frac{1}{2}D\phi\cdot\tilde{\mathcal{B}}\rho$$ $$+(\frac{1}{2}\mathcal{B}_T+\frac{1}{4}L_{aa})\mathcal{B}\phi\cdot\tilde{\mathcal{B}}\rho\}d\nu_{\partial M}.$$

Assertion (1) follows by setting $t=0$. The remainder of this letter is devoted to the proof of the remaining assertions.

If we set $\Gamma=0$, then we recover Robin boundary conditions and Theorem 1 follows from results given in [3]. Thus the whole interest lies in the Γ dependence - we have encoded this dependence in the operators $\mathcal{B}$ and $\mathcal{B}_T$. To establish Theorem 1, we will need some functorial properties of these invariants. As always, one can use dimensional analysis to see the boundary integrands β_n^{bd} are homogeneous of order $n-1$ in the data. The primary difficulty is that Γ has weight 0 and thus the dependence upon Γ in various coefficients is not controlled by this homogeneity argument.

We begin by noting that Lemma 2.1 of [3] generalizes to this setting as:

Lemma 2 1. $\beta_n(\phi,\rho,D,\mathcal{B})=\beta_n(\rho,\phi,\tilde{D},\tilde{\mathcal{B}})$.

2. *If* $\mathcal{B}\phi=0$, *then* $\beta_n(\phi,\rho,D,\mathcal{B})=-\frac{2}{n}\beta_{n-2}(D\phi,\rho,D,\mathcal{B})$.

There is another useful functorial property. Let the torus $\mathbb{T}^k=S^1\times..\times S^1$ have the usual periodic parameters $\vec{\theta}:=(\theta_1,...,\theta_k)$ and let r be the usual parameter on the interval $[0,1]$. Give $N:=[0,1]$ the usual metric dr^2 and give the manifold $M:=[0,1]\times\mathbb{T}^k$ a metric

$$ds_M^2=dr^2+g_{ab}(r)d\theta^a\circ d\theta^b$$

which only depends on the radial parameter r. Let D_M be an operator of Laplace type on the space of smooth sections to the trivial bundle $M\times\mathbb{C}^\nu$

over M and let $\mathcal{B}_M$ define oblique boundary conditions where the coefficients of D_M and $\mathcal{B}_M$ only depend on the radial parameter. Let

$$\begin{array}{ll} \vec{n}=(n_1,...,n_k)\in\mathbb{Z}^k, & \vec{n}\cdot\vec{\theta}:=n_1\theta_1+...+n_k\theta_k, \\ \phi_{\mathbb{T}}(\vec{\theta}):=e^{\sqrt{-1}\vec{n}\cdot\vec{\theta}}, & D_N:=\phi_{\mathbb{T}}^{-1}D_M\phi_{\mathbb{T}}, \text{ and} \\ \mathcal{B}_N:=\phi_{\mathbb{T}}^{-1}\mathcal{B}_M\phi_{\mathbb{T}}. & \end{array}$$

Then D_N is an operator of Laplace type and $\mathcal{B}_N$ defines Robin boundary conditions on $[0,1]\times\mathbb{C}^\nu$ since there is no θ dependence. Let $\phi_N=\phi_N(r)$ and $\rho_N=\rho_N(r)$. Let $u_N(t;r):=e^{-tD_{N,\mathcal{B}_N}}\phi_N$ be the solution to the equations:

$$(\partial_t+D_N)u_N=0,\ u_N|_{t=0}=\phi_N,\ \text{and}\ \mathcal{B}_Nu_N=0.$$

We set $\phi_M:=\phi_N\phi_{\mathbb{T}}$, $\rho_M:=\rho_N\phi_{\mathbb{T}}^{-1}$, and $u_M:=u_N\phi_{\mathbb{T}}$. Since

$$D_Mu_M=D_Nu_N\cdot\phi_{\mathbb{T}}\text{ and }\mathcal{B}_Mu_M=\mathcal{B}_Nu_N\cdot\phi_{\mathbb{T}}, \tag{2}$$

u_M solves the equations

$$(\partial_t+D_M)u_M=0,\ u_M|_{t=0}=\phi_N\phi_{\mathbb{T}},\ \text{and}\ \mathcal{B}_Mu_M=0.$$

Let $g=\det(g_{ab})^{1/2}$. Then $d\nu_M=gdrd\theta$. We compute:

$$\begin{array}{rcl} \beta(\phi_M,\rho_M,D_M,\mathcal{B}_M)(t) & = & \int_M u_M(t;r,\theta)\rho_M(r,\theta)g(r)drd\theta \\ & = & \int_M u_N(t;r)\rho_N(r)g(r)drd\theta \\ & = & \operatorname{vol}(\mathbb{T}^k)\beta(\phi_N,g\rho_N,D_N,\mathcal{B}_N). \end{array}$$

We equate powers of t in the associated asymptotic expansions to prove:

Lemma 3 *Adopt the notation established above. Then*

$$\beta_n(\phi_M,\rho_M,D_M,\mathcal{B}_M)=\operatorname{vol}(\mathbb{T}^k)\cdot\beta_n(\phi_N,g\rho_N,D_N,\mathcal{B}_N).$$

We remark that it is not necessary to take $N=[0,1]$; an analogous Lemma holds for more general products with a toroidal factor where the coefficients in D_M and $\mathcal{B}_M$ only depend on the coordinates on N.

We now begin the proof of Theorem 1. We express

$$\begin{array}{l} \beta_1(\phi,\rho,D,\mathcal{B})=\int_{\partial M}\mathcal{E}_1(\phi,\rho,D,\mathcal{B})d\nu_{\partial M} \\ \beta_2(\phi,\rho,D,\mathcal{B})=-\int_M D\phi\cdot\rho d\nu_M+\int_{\partial M}\{\mathcal{B}\phi\cdot\rho+\mathcal{E}_2(\phi,\rho,D,\mathcal{B})\}d\nu_{\partial M} \\ \beta_3(\phi,\rho,D,\mathcal{B})=\frac{4}{3\sqrt{\pi}}\int_{\partial M}\{\mathcal{B}\phi\cdot\tilde{\mathcal{B}}\rho+\mathcal{E}_3(\phi,\rho,D,\mathcal{B})\}d\nu_{\partial M} \\ \beta_4(\phi,\rho,D,\mathcal{B})=\frac{1}{2}\int_M D\phi\cdot\tilde{D}\rho d\nu_M+\int_{\partial M}\{-\frac{1}{2}\mathcal{B}\phi\cdot\tilde{D}\rho-\frac{1}{2}D\phi\cdot\tilde{\mathcal{B}}\rho \\ \qquad +(\frac{1}{2}S+\frac{1}{4}L_{aa})\mathcal{B}\phi\cdot\tilde{\mathcal{B}}\rho+\mathcal{E}_4(\phi,\rho,D,\mathcal{B})\}d\nu_{\partial M}, \end{array}$$

where by construction $\mathcal{E}_i(\phi, \rho, D, \mathcal{B}) = 0$ for $\Gamma = 0$ to ensure the above results agree with the results of Desjardins et al [5] for Robin boundary conditions. We may use Lemma 2 to see that

$$\begin{aligned}
&\mathcal{E}_\nu(\phi, \rho, D, \mathcal{B}) = \mathcal{E}_\nu(\rho, \phi, \tilde{D}, \tilde{\mathcal{B}}) \\
&\mathcal{E}_\nu(\phi, \rho, D, \mathcal{B}) = -\tfrac{2}{\nu}\mathcal{E}_{\nu-2}(D\phi, \rho, D, \mathcal{B}) \text{ if } \mathcal{B}\phi = 0 \\
&\mathcal{E}_\nu(\phi, \rho, D, \mathcal{B}) = -\tfrac{2}{\nu}\mathcal{E}_{\nu-2}(\phi, \tilde{D}\rho, D, \mathcal{B}) \text{ if } \tilde{\mathcal{B}}\rho = 0.
\end{aligned}$$

As $\mathcal{E}_{-1} = \mathcal{E}_0 = 0$, we have $\mathcal{E}_\nu = 0$ if $\mathcal{B}\phi = 0$ or $\tilde{\mathcal{B}}\rho = 0$ for $\nu = 1, 2$. Thus $\mathcal{E}_\nu$ is divisible by expressions which are bilinear in $\mathcal{B}\phi$, $\tilde{\mathcal{B}}\rho$, and tangential covariant derivatives of these quantities. This means ϕ appears only in combination with $\mathcal{B}$ as $\mathcal{B}\phi$ or tangential covariant derivatives of $\mathcal{B}\phi$. Similarly, ρ only appears as $\tilde{\mathcal{B}}\rho$ or tangential covariant derivatives of $\tilde{\mathcal{B}}\rho$. Since $\mathcal{E}_\nu$ is homogeneous of degree $\nu - 1$ and $\mathcal{B}\phi$ and $\tilde{\mathcal{B}}\rho$ are homogeneous of degree 1, we conclude $\mathcal{E}_\nu = 0$ for $\nu = 1, 2$ which establishes assertions (2) and (3) of Theorem 1. We may also conclude now similarly that $\mathcal{E}_\nu$ is divisible by expressions which are bilinear in $\mathcal{B}\phi$, $\tilde{\mathcal{B}}\rho$, and tangential covariant derivatives of these quantities for $\nu = 3, 4$. Thus

$$\begin{aligned}
\mathcal{E}_3(\phi, \rho, D, \mathcal{B}) &= \int_{\partial M} \alpha_0(\Gamma)\mathcal{B}\phi \cdot \tilde{\mathcal{B}}\rho d\nu_{\partial M} \\
\mathcal{E}_4(\phi, \rho, D, \mathcal{B}) &= \int_{\partial M} \{\alpha_1(\Gamma, S) + \alpha_2(\Gamma, L) + \alpha_3(\Gamma, \nabla\Gamma) \\
&+ \alpha_{4,a}(\Gamma)\nabla_a + \nabla_a \alpha_{4,a}(\Gamma)\}\mathcal{B}\phi \cdot \tilde{\mathcal{B}}\rho d\nu_{\partial M}.
\end{aligned}$$

For dimensional reasons, $\alpha_1(\Gamma, S)$ is linear in S, $\alpha_2(\Gamma, L)$ is linear in L, and $\alpha_3(\Gamma, \nabla\Gamma)$ is linear in $\nabla\Gamma$. Furthermore, the terms α_ν depend smoothly on Γ. We complete the proof of Theorem 1 by studying these universal multipliers.

Give $M = [0, 1] \times \mathbb{T}^k$ the metric

$$ds^2 = dr^2 + g_{ab}(r)d\theta^a \circ d\theta^b$$

and let $g := \det(g_{ab})^{1/2}$ define the volume element on M. Let

$$D_M = -(\partial_r^2 + g^{-1}\partial_r(g) \cdot \partial_r + g^{ab}\partial_a^\theta \partial_b^\theta)$$

be the associated Laplacian. Let S_M and Γ be arbitrary. Let $\check{\mathcal{B}}_N := \rho_{\mathbb{T}}^{-1}\tilde{\mathcal{B}}_M \rho_{\mathbb{T}}$ and let $\tilde{\mathcal{B}}_N$ be the dual operator determined by $\mathcal{B}_N$. These two boundary operators satisfy the intertwining property:

$$\tilde{\mathcal{B}}_N g = g\check{\mathcal{B}}_N.$$

We compute:

$$\begin{aligned}\beta_3(\phi_M,\rho_M,D_M,\mathcal{B}_M) &= \tfrac{4}{3\sqrt{\pi}}\textstyle\int_{\partial M}(1+\alpha_0(\Gamma))(\mathcal{B}_M\phi_M\cdot\tilde{\mathcal{B}}_M\rho_M)gdrd\theta\\ &= \operatorname{vol}(\mathbb{T}^k)\cdot\tfrac{4}{3\sqrt{\pi}}\textstyle\int_{\partial N}(1+\alpha_0(\Gamma))(\mathcal{B}_N\phi_N\cdot\tilde{\mathcal{B}}_N g\rho_N)dr\\ \beta_3(\phi_N,\rho_N,D_N,\mathcal{B}_M) &= \tfrac{4}{3\sqrt{\pi}}\textstyle\int_{\partial N}(\mathcal{B}_N\phi_N\cdot\tilde{\mathcal{B}}_N g\rho_N)dr.\end{aligned}$$

By Lemma 3,

$$\beta_3(\phi_M,\rho_M,D_M,\mathcal{B}_M)=\operatorname{vol}(\mathbb{T}^k)\cdot\beta_3(\phi_N,\rho_N,D_N,\mathcal{B}_N).$$

Consequently $\alpha_0(\Gamma)\equiv 0$ which proves Theorem 1 (4).

The connections defined by D_M and D_N differ. In general, if

$$D=-(g^{\mu\nu}\partial_\mu\partial_\nu+A^\mu\partial_\mu+B)$$

is an arbitrary operator of Laplace type, then the associated connection 1 form is given by:

$$\omega_\delta=\tfrac{1}{2}g_{\nu\delta}(A^\nu+g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu) \tag{3}$$

where $\Gamma_{\mu\sigma}{}^\nu$ are the Christoffel symbols of the Levi-Civita connection; see [9] for details. Thus

$${}^M\nabla_r=\partial_r \text{ and } {}^N\nabla_r=\partial_r+\tfrac{1}{2}g^{-1}\partial_r g.$$

Let $\varepsilon(1)=-1$ and $\varepsilon(0)=+1$ so that $\varepsilon\partial r$ is the inward unit normal on ∂N. We suppose $g_{ab}=\delta_{ab}$ on ∂N. Then $L_{ab}=-\frac{\varepsilon}{2}\partial_r g_{ab}$ and $g^{-1}\partial_r g=-\varepsilon L_{aa}$ so

$$\begin{aligned}&{}^N\nabla_m+\tfrac{1}{2}L_{aa}=\varepsilon(\partial_r+\tfrac{1}{2}g^{-1}\partial_r g)+\tfrac{1}{2}L_{aa}=\varepsilon\partial_r={}^M\nabla_m\\ &\phi_{\mathbb{T}}^{-1}\mathcal{B}_M\phi_T=\varepsilon\partial_r+2\sqrt{-1}\Gamma_a n_a+S_M \text{ so}\\ &S_N=S_M+\tfrac{1}{2}L_{aa}+2\sqrt{-1}n_a\Gamma_a.\end{aligned}$$

We then have:

$$\begin{aligned}&\textstyle\int_{\partial N}\{\tfrac{1}{2}S_N\mathcal{B}_N\phi_N\cdot\tilde{\mathcal{B}}_N(g\rho_N)\}d\nu_{\partial N}\\ =&\textstyle\int_{\partial N}\{\{\tfrac{1}{2}S_M+\tfrac{1}{4}L_{aa}+\alpha_1(\Gamma,S_M)+\alpha_2(\Gamma,L)\\ +&2\sqrt{-1}\alpha_{4,a}(\Gamma)n_a\}\mathcal{B}_N\phi_N\cdot\check{\mathcal{B}}_N\rho_N\}gd\nu_{\partial N}.\end{aligned}$$

Since $\tilde{\mathcal{B}}_N(g\rho_N)=g\check{\mathcal{B}}_N(\rho_N)$, we may conclude:

$$\alpha_1\equiv 0,\ \alpha_2\equiv 0,\ \text{and } \alpha_{4,a}=\tfrac{1}{2}\Gamma_a.$$

We complete the proof of Theorem 1 by evaluating α_3. We take the flat metric on M and set

$$D_M := -(\partial_r^2 + \partial_a^\theta \partial_a^\theta + 2\omega_a \partial_a^\theta).$$

We take $S_M = 0$ and $\phi_{\mathbb{T}} = \rho_{\mathbb{T}} = 1$. By equation (3), ${}^M\nabla_a = \partial_a^\theta + \omega_a$. Consequently $S_N = \Gamma_a \omega_a + \omega_a \Gamma_a$. We compute:

$$\begin{aligned}
& \textstyle\int_{\partial N} \{\frac{1}{2} S_N \mathcal{B}_N \phi_N \cdot \tilde{\mathcal{B}}_N \rho_N\} d\nu_{\partial N} \\
= \ & \textstyle\int_{\partial N} \{\{\frac{1}{2}(\omega_a \Gamma_a + \Gamma_a \omega_a) + \alpha_3(\Gamma, \nabla\Gamma)\} \mathcal{B}_N \phi_N \cdot \tilde{\mathcal{B}}_N \rho_N\} d\nu_{\partial N}.
\end{aligned}$$

This shows that $\alpha_3 \equiv 0$ and completes the proof of Theorem 1.

Remark: Whereas in the heat trace asymptotics the breakdown of the classic Lopatinski condition is clearly reflected in the heat kernel coefficients [2, 6], the heat content coefficients do not show any signs of the loss of strong ellipticity and they are defined for arbitrary endomorphisms Γ.

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