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Studying nonlinear pde by geometry in
matrix space

by

Bernd Kirchheim, Stefan Müller, and Vladimír Šverák

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Bernd Kirchheim, Stefan Müller
Max Planck Institute for Mathematics in the Sciences
Inselstr. 22–26
D–04103 Leipzig, Germany
{bk,sm}@mis.mpg.de

Vladimír Šverák
Department of Mathematics
University of Minnesota
Minneapolis, MN 55455, USA
sverak@math.umn.edu

Abstract

We outline an approach to study the properties of nonlinear partial differential equations through the geometric properties of a set in the space of $m \times n$ matrices which is naturally associated to the equation. In particular, different notions of convex hulls play a crucial role. This work draws heavily on Tartar’s work on oscillations in nonlinear pde and compensated compactness and on Gromov’s work on partial differential relations and convex integration. We point out some recent successes of this approach and outline a number of open problems, most of which seem to require a better geometric understanding of the different convexity notions.

Contents

1	Introduction	3
2	Convex integration and rank-one convex hulls	7
2.1	Convex integration for open sets	7
2.2	Closed sets and in-approximations	8

3	Elliptic systems with nowhere smooth solutions	9
3.1	Reduction to first order systems	9
3.2	T_k -configurations	10
3.3	Embedding a T_4 -configuration	11
3.4	Families of T_4 -configurations and dimension counting	14
3.5	Polyconvex examples and obstructions to T_4	17
3.6	Beyond T_k	21
4	Tools to study rank-one convex hulls	21
4.1	Rank-one convex, quasiconvex and polyconvex hulls	21
4.2	The dual objects: laminates and gradient Young measures	22
4.3	Localization and extension	23
5	The simplest polyconvex integrand	25
5.1	A rank-one foliation of Σ	26
5.2	Separating functions on Σ	27
5.3	Extension	28
5.4	More general polyconvex integrands	30
6	Separate convexity	31
6.1	Separate convexity in \mathbb{R}^2	31
6.2	Separate convexity in $\mathbb{R}^2 \oplus \mathbb{R}$	32
6.3	Separate convexity in $\mathbb{R}^2 \oplus \mathbb{R}^2$	43
6.4	A laminate without discrete approximations	46
7	Local hulls, degenerate sets and hyperbolic conservation laws	49
8	Outlook	51

1 Introduction

The purpose of this paper is to outline a connection between nonlinear partial differential equations and simply stated but largely unexplored questions about certain convex hulls, such as the rank-one convex hull or the separately convex hull, in the space of $m \times n$ matrices. While many of the underlying ideas are old and go back to the pioneering work of Tartar [Ta 79, Ta 83], Gromov [Gr 73, Gr 86, Sp 98] and DiPerna [DP 85] there have been a number of recent new successes of this approach including the construction of elliptic and parabolic 2×2 systems with nowhere C^1 solutions [MS 98, MS 99, MRS 02], an analysis of Lipschitz maps with finitely many gradients [Ki 01a], the existence of solutions in mathematical models of martensitic phase transitions [BJ 87, CK 88, MS 96] and a large number of other applications of Gromov's method of convex integration and its variants and extensions, see e.g. [DM 97, DM 99, Ki 01b, Sy 01, MSy 01] for further discussion and references. At the same time a theory of the relevant convexity notions in matrix space is beginning to emerge [MP 98, Ki 01b, Ko 01] even though many basic questions remain open.

In a nutshell, the situation can be described as follows. Many nonlinear systems of pdes for a map $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^d$ can be naturally expressed as a combination of a linear systems of pdes

$$A(Dv) := \sum_{i=1}^n A_i \partial_i v = 0 \tag{1}$$

and a pointwise nonlinear constraint

$$v(x) \in K \subset \mathbb{R}^d \quad \text{a.e.} \tag{2}$$

Then one considers the cone Λ related to one dimensional solutions $v(x) = h(\langle x, \xi \rangle)$ which is defined by

$$\Lambda := \left\{ \xi \in \mathbb{R}^n : \exists a \in \mathbb{R}^d \sum_{i=1}^n \xi_i A_i a = 0 \right\}. \tag{3}$$

Equivalently Λ characterizes the directions of one dimensional high frequency oscillations compatible with (1). Given a cone Λ we say that K is lamination convex (with respect to Λ) if for any two points $A, B \in K$ with $B - A \in \Lambda$ the whole segment $[A, B]$ belongs to K . The lamination convex hull $K^{lc, \Lambda}$ is the smallest lamination convex set containing K (Gromov [Gr 86], who works in the more general setting of jet bundles, calls this the P -convex hull).

The key point in Gromov's method of convex integration (which is a far reaching generalization of the work of Nash [Na 54] and Kuiper [Ku 55] on isometric immersions) is that (1) and (2) admit many interesting solutions provided that $K^{lc,\Lambda}$ is sufficiently large. In applications to elliptic and parabolic systems we always have $K^{lc,\Lambda} = K$ so that Gromov's approach does not directly apply. It turns out, however, that for the construction of Lipschitz (rather than C^1) solutions one can work with the Λ -convex hull K^Λ , defined by duality. More precisely for a compact set K a point does not belong to K^Λ if and only if there exists a Λ -convex function which separates it from K . A crucial fact is that K^Λ can be much larger than the hull $K^{lc,\Lambda}$. This difference already arises for a set consisting of four matrices which form a so-called T_4 -configuration (see Section 3.2 below). This surprising fact was observed independently in different contexts [Sch 74, AH 86, NM 91, CT 93, Ta 93], we learned it from Tartar. In connection with suitable approximations and general position arguments it leads to surprising consequences. We illustrate this by three examples.

Theorem 1 ([MS 99]) (*elliptic systems with nowhere C^1 solutions*) *Let Ω be the unit ball in \mathbb{R}^2 . There exists a smooth function $\phi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ which is strongly quasiconvex and satisfies $|D^2\phi| \leq C$, and a Lipschitz map $w : \Omega \rightarrow \mathbb{R}^2$, which is a weak solution of the elliptic system*

$$-\operatorname{div} D\phi(\nabla w) = 0, \quad (4)$$

such that w is not C^1 in any open subset of Ω . Moreover the system (4) admits solutions with compact support.

We remark in passing that this counterexample is quite different from the classical counterexamples [BDG 69, DG 68, GM 68] and their more recent extensions [HLN 96, SY 00] which are all based on singularities at a point or more generally a set of lower dimension. Scheffer [Sch 74] used T_4 -configurations as a basis of counterexamples to regularity. He proved a weaker version of Theorem 1 with w in the Sobolev space $W^{1,1}$ and ϕ satisfying the Legendre-Hadamard condition (7). Unfortunately, the work [Sch 74] has not appeared in a journal and the original ideas there remained largely unknown and had to be re-discovered by various authors.

Note that (4) is the Euler-Lagrange equation of the functional

$$I(w) = \int_{\Omega} \phi(\nabla w) dx. \quad (5)$$

By a classical result of Evans [Ev 86] *minimizers* of I are smooth outside a closed set of measure zero if ϕ satisfies the assumptions stated in Theorem 1

(the same is true for local minimizers, see [KT 01]). Thus general stationary points of I can behave much worse than minimizers.

We recall that a (continuous) integrand $\phi : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is called *strongly quasiconvex* if

$$\int_{T^n} \phi(X + \nabla \eta) - \phi(X) dx \geq c \int_{T^n} |\nabla \eta|^2 dx \quad (6)$$

for some $c > 0$, all $X \in \mathbb{R}^{m \times n}$ and all periodic Lipschitz maps $\eta : T^n \rightarrow \mathbb{R}^m$ (equivalently one can consider test functions η on bounded domains with zero boundary conditions). If (6) holds with $c = 0$ we say that ϕ is quasiconvex. Using small amplitude test functions $\eta(x)$, Taylor expansion and Fourier transform one easily sees that (6) implies that ϕ is uniformly rank-one convex and one obtains the Legendre-Hadamard (or strong ellipticity) condition

$$D^2\phi(X)(a \otimes b, a \otimes b) \geq c|a|^2|b|^2, \quad (7)$$

so that (4) is indeed an elliptic system. Recently L. Székelyhidi has shown that the conclusion of Theorem 1 also holds for a suitable strictly polyconvex integrand ϕ , i.e. a strictly convex function of F and $\det F$.

The failure of regularity can be extended to parabolic systems with smooth initial data and a small and Hölder continuous right hand side.

Theorem 2 ([MRS 02]) (*parabolic systems with nowhere C^1 solutions*)
Let Ω be the unit ball in \mathbb{R}^2 . Let $\eta > 0, T > 0, \alpha \in (0, 1)$. Then there exists a function $\phi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ such that ϕ is strongly quasiconvex, smooth and $|D^2\phi| \leq C$, a function $f \in C^\alpha(\Omega \times [0, T]; \mathbb{R}^2)$ with $\|f\|_{C^\alpha} < \eta$ and a Lipschitz solution $w : \Omega \times [0, T] \rightarrow \mathbb{R}$ of the parabolic system

$$\partial_t w - \operatorname{div} D\phi(\nabla w) = f \quad \text{in } \Omega \times (0, T) \quad (8)$$

and

$$w(\cdot, 0) \equiv 0, \quad w(t, x) = 0 \text{ for } x \in \partial\Omega \quad (9)$$

such that w is nowhere C^1 in $\Omega \times (0, T)$.

This system exhibits some other unusual features, such as failure of uniqueness and of the energy inequality.

Our last example concerns Lipschitz maps whose gradient takes only finitely many values (except on a set of measure zero).

Theorem 3 ([Ki 01a]) (*maps with finitely many gradients*). Let Ω be the unit ball in \mathbb{R}^2 . There exist five matrices $A_1, \dots, A_5 \in \mathbb{R}^{2 \times 2}$ with

$$\text{rk}(A_i - A_j) \neq 1 \tag{10}$$

and a Lipschitz map $u : \Omega \rightarrow \mathbb{R}^2$ which satisfies

$$\nabla u \in \{A_1, \dots, A_5\} \quad \text{a.e.} \tag{11}$$

and $\nabla u \neq A_i$.

Interestingly, the corresponding statement for four matrices turns out to be false [CK 00]. The condition (10) rules out trivial maps which depend only on one direction and whose gradient takes two values. It also implies that the sets $\Omega_i = \{x : \nabla(x) = A_i\}$ must be very complicated. Indeed if Ω_i and Ω_j meet at a smooth (or rectifiable) boundary then a straightforward blow-up argument shows that $A_i - A_j$ must have rank one and the common boundary of Ω_i and Ω_j is flat with normal b , where $A_i - A_j = a \otimes b$. In this context (10) can be seen as an ellipticity condition for the partial differential relation (11). Nonetheless, as in Theorem 1, ellipticity is not strong enough to rule out large scale oscillations of ∇u . These are, in a certain sense, encoded in the T_4 -configurations alluded to above (see Section 3.2 below).

The rest of this paper is organized as follows. We specialize to the situation that $v \in \mathbb{R}^{m \times n}$ and that the differential constraint (1) is simply $\text{curl } v = 0$ (where the curl is taken along rows). Then the combination of (1) and (2) leads to the first order partial differential relation $\nabla u \in K$ and Λ is the cone of rank-one matrices. In Section 2 we review the general results on convex integration and reduce the existence of (highly oscillatory) solutions to the computation of the rank-one convex hull of K . To illustrate this idea we outline in Section 3 the main ideas of the proof of Theorem 1. One first finds one T_4 -configuration in the relevant set K and then uses a dimension counting argument to show that the abstract conditions reviewed in Section 2 are satisfied.

The constructions related to Theorems 1–3 all use the simplest set K which has the property that $K^{lc, \Lambda} = K$, but K^Λ is much bigger, the T_4 -configuration. We hope that a better understanding of the geometry of rank-one convexity will lead to new applications and to insights how to formulate structure conditions on ϕ which guarantee regularity results for elliptic systems (so far nothing is known beyond monotonicity or quasimonotonicity [Fu 87, Zh 88, Ha 95] of $D\phi$).

To this end we first recall in Section 4 some general tools to study rank-one convex hulls and then consider in Sections 5–7 a number of case studies.

In Section 5 we study the set K related to the simplest polyconvex integrand $\phi(F) = (\det F)^2$ and we show that its rank-one convex hull is trivial. Interestingly, it is not known whether the same holds for the set related to the integrand $\varepsilon|F|^2 + (\det F)^2$ for small positive ε . Already the restriction to diagonal matrices leads to interesting questions about separately convex functions which we discuss in Section 6. In Section 7 we discuss an example related to compactness for hyperbolic conservation laws. It leads to a set K which is degenerate in the sense that while K contains no rank-one connections its tangent spaces do. The study of such sets was initiated in a pioneering paper by DiPerna [DP 85], following the program outlined by Tartar [Ta 79, Ta 83]. Finally in Section 8 we give a brief outlook.

2 Convex integration and rank-one convex hulls

For simplicity we will mostly consider the situation $v \in \mathbb{R}^{m \times n}$ and the simplest linear differential constraint $\operatorname{curl} v = 0$ (where the curl is taken by rows). If we restrict attention to simply connected domains Ω in \mathbb{R}^n then the combination of (1) and (2) reduces to the first order partial differential relation

$$\nabla u \in K \quad \text{a.e. in } \Omega, \quad (12)$$

where $K \subset \mathbb{R}^{m \times n}$ is given and where we seek a map $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$.

In this case the cone Λ defined by (3) is simply the cone of rank-one matrices and for brevity we use the notation $K^{lc} := K^{lc, \Lambda}$ and $K^{rc} := K^\Lambda$ for the lamination convex hull and the rank-one convex hull. A function $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is rank-one convex if it is convex along each rank-one line $A + ta \otimes b$. For a compact set K the rank-one convex hull K^{rc} consists of all points which cannot be separated by rank-one convex functions, i.e.

$$K^{rc} := \{A \in \mathbb{R}^{m \times n} : f(A) \leq \sup_K f, \text{ for all rank-one convex } f\}. \quad (13)$$

For a general set E we set

$$E^{rc} := \bigcup_{K \subset E \text{ compact}} K^{rc}. \quad (14)$$

If $m = 1$ or $n = 1$ then K^{rc} and K^{lc} agree with the convex hull.

2.1 Convex integration for open sets

The key result in the theory of convex integration is that the partial differential relation (12) admits many solutions if K^{rc} is large. Here we just

recall the relevant results and refer to [MS 99] for the proofs and further discussion. We first consider the case that K is open. In the following we say that a map $u : \Omega \rightarrow \mathbb{R}^m$ is *piecewise affine* if it is Lipschitz and there exists finite or countably many open sets Ω_i such that u is affine on Ω_i and the union of the Ω_i has full measure.

Theorem 4 ([MS 99], Thm. 3.1) *Let $K \subset \mathbb{R}^{m \times n}$ be open and let $L \subset K^{rc}$ be compact. Let $u_0 : \Omega \rightarrow \mathbb{R}^m$ be a piecewise affine map with $\nabla u_0 \in L$ a.e. Then there exists a piecewise affine map $u : \Omega \rightarrow \mathbb{R}^m$ such that*

$$\nabla u \in K \quad \text{a.e. in } \Omega, \tag{15}$$

$$u = u_0 \quad \text{on } \partial\Omega. \tag{16}$$

Remark. In fact there exist many such solutions u . One can show that u_0 admits a fine approximation by solutions of (15), i.e. for each continuous function η with $\eta > 0$ in Ω there exists a solution u with $|u - u_0|(x) < \eta(x)$.

2.2 Closed sets and in-approximations

One crucial step in convex integration is the passage from open sets K to general sets (which may have high codimension). It is now understood that at least for Lipschitz solutions this can be done in different ways, e.g. by the Baire category theorem [DM 97, DM 99] (for earlier applications to odes see e.g. [Ce 80, DP 82, DP 91]), a refinement of it using Baire-1 functions or the Banach-Mazur game [Ki 01b] or by direct construction [Sy 01, MSy 01]. As in [MS 96] we follow here Gromov's original approach based on an in-approximation. Whatever approach one uses the basic theme is the same as in the construction of continuous, nowhere differentiable functions: at each step of the construction one adds a highly oscillatory correction whose frequency is much larger and whose amplitude is much smaller than those of the previous corrections. This leads to strong convergence of the gradient in L^1 but typically to a very irregular limiting Lipschitz map.

For simplicity we consider only compact sets K (for the application to elliptic systems discussed in Section 3 below it suffices to intersect the set in (24) with a large ball).

Definition 5 *We say that a sequence of open sets $\{U_i\}$ is an in-approximation of a compact set K if $U_i \subset U_{i+1}^{rc}$ and $\sup_{X \in U_i} \text{dist}(X, K) \rightarrow 0$ as $i \rightarrow \infty$.*

Theorem 6 ([MS 99], Thm. 3.2.) *Suppose that the compact set K admits an in-approximation by open sets $\{U_i\}$. Let $u_0 : \Omega \rightarrow \mathbb{R}^m$ be a C^1 map which satisfies*

$$\nabla u_0 \in U_1 \quad \text{in } \Omega. \quad (17)$$

Then there exists a Lipschitz map $u : \Omega \rightarrow \mathbb{R}^m$ such that

$$\nabla u \in K \quad \text{a.e. in } \Omega, \quad (18)$$

$$u = u_0 \quad \text{on } \partial\Omega. \quad (19)$$

Remark. Again each such u_0 admits in fact a fine approximation by solutions u of (18).

An illustrative example is given by equidimensional isometric (Lipschitz) immersions. In this case $K = O(n) = \{F \in \mathbb{R}^{n \times n} : F^T F = \text{Id}\}$. We have $K^{rc} = K^{lc} = \{F : F^T F \leq \text{Id}\}$ and an in-approximation of K is given by $U_i = \{F : \lambda_i \text{Id} < F^T F < \text{Id}\}$, where $0 < \lambda_1 < \lambda_2 < \dots < 1$, $\lambda_i \rightarrow 1$. Let $\|F\| = \sup\{|Fx| : |x| \leq 1\}$ denote the operator norm of F with respect to the Euclidean norm in \mathbb{R}^n . Then any C^1 map u_0 with $\sup\|\nabla u_0\| < 1$ can be approximated in C^0 by Lipschitz maps u with $\nabla u \in O(n)$ which satisfy the same boundary conditions. For other examples, including applications to models of solid-solid phase transitions in crystals, see [MS 98, DM 99].

3 Elliptic systems with nowhere smooth solutions

In this section we sketch the proof of Theorem 1, following partly the exposition in [MRS 02].

3.1 Reduction to first order systems

For definiteness let Ω be the unit ball in \mathbb{R}^2 . We seek solutions $w : \Omega \rightarrow \mathbb{R}^2$ of the system

$$-\text{div } D\phi(\nabla w) = 0, \quad (20)$$

where

$$\phi \text{ strongly quasiconvex, smooth, } |D^2\phi| \leq C. \quad (21)$$

In particular ϕ satisfies the Legendre-Hadamard condition (7). Now the condition (20) is equivalent to the existence of a potential W such that $D\phi(\nabla w)J = \nabla W$ where J is the 90° rotation. If we introduce

$$u = \begin{pmatrix} w \\ W \end{pmatrix}, \quad u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^4 \quad (22)$$

then (20) is equivalent to

$$\nabla u \in K \subset \mathbb{R}^{4 \times 2}, \quad (23)$$

where

$$K = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} : Y = D\phi(X)J, X \in \mathbb{R}^{2 \times 2} \right\}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (24)$$

3.2 T_k -configurations

To construct ‘wild’ solutions of (23) and hence (20) we have to show that K^{rc} is sufficiently large so that in-approximation of K can be constructed. A first attempt might be to show that K^{lc} is large. This, however, is doomed since the Legendre-Hadamard condition (7) implies that $K^{lc} = K$. Indeed if $A = \begin{pmatrix} X \\ Y \end{pmatrix}$ and $A + \begin{pmatrix} a \otimes n \\ b \otimes n \end{pmatrix}$ belong to K , then $D\phi(X + ta \otimes n) - D\phi(X) = (b \otimes n)J^T = b \otimes Jn$. Hence $\langle D\phi(X + ta \otimes n) - D\phi(X), a \otimes n \rangle = 0$ and this contradicts the strict convexity of the map $t \rightarrow \phi(X + ta \otimes n)$, see also [Ba 80].

A crucial observation is that there are simple sets which have a nontrivial rank-one convex hull K^{rc} even though K^{lc} is trivial.

Definition 7 *Let $k \geq 4$ and consider k -tuples $\mathbf{M} = (M_1, \dots, M_k)$ of matrices $M_j \in \mathbb{R}^{m \times n}$. We say that \mathbf{M} is a T_k -configuration if there exist rank-one matrices C_1, \dots, C_k with $\sum_{j=1}^k C_j = 0$, scalars $\kappa_1, \dots, \kappa_k$ with $\kappa_j > 0$ and matrices $P_j \in \mathbb{R}^{m \times n}$ such that the relations*

$$P_{j+1} - P_j = C_j, \quad M_j - P_{j+1} = \kappa_j C_j$$

hold, where the index j is counted modulo k (see Fig. 1).

Let \mathbf{M} be a T_k -configuration and let $K = \{M_1, \dots, M_k\}$. One easily sees that every rank-one convex function which vanishes on K must also be nonpositive at all the P_j (see Fig. 1). Hence the P_j belong to K^{rc} . On the other hand there may be no rank-one connection in K . The simplest example arises already for $k = 4$ in diagonal 2×2 matrices. One may take

$$M_1 = -M_3 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, M_2 = -M_4 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

We emphasize that in general a T_4 -configuration need not lie in a plane. To construct ‘wild’ solutions of (23) we will show in the next subsections that K contains sufficiently many T_4 -configurations.

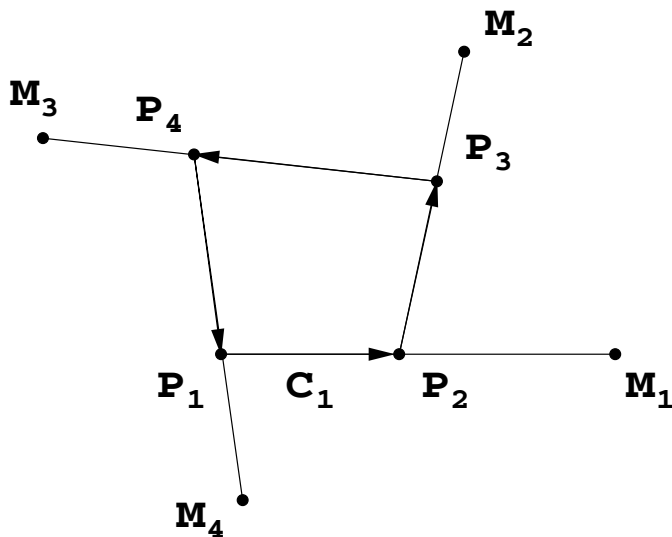


Figure 1: T_4 -configuration with $P_1 = P$, $P_2 = P + C_1$, $P_3 = P + C_1 + C_2$, $P_4 = P + C_1 + C_2 + C_3$. The lines indicate rank-1 connections. Note that the figure need not be planar

3.3 Embedding a T_4 -configuration

The first observation is that there exists a strongly quasiconvex and smooth function $\phi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ (with $|D^2\phi| \leq C$) such that the set

$$K = \left\{ \begin{pmatrix} X \\ D\phi(X)J \end{pmatrix} : X \in \mathbb{R}^{2 \times 2} \right\} \subset \mathbb{R}^{4 \times 2}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

admits a T_4 -configuration \mathbf{M}^0 with $M_i^0 \in K$, see [MS 99], Lemma 4.3. One may take

$$M_1^0 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \\ 0 & -1 \\ 3 & 0 \end{pmatrix}, M_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}, M_3^0 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \\ 0 & 1 \\ -3 & 0 \end{pmatrix}, M_4^0 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \\ 0 & -3 \\ -1 & 0 \end{pmatrix}. \quad (25)$$

To illustrate the geometric ideas behind the construction of such an integrand we first sketch the construction of a ϕ which satisfies at least the Legendre-Hadamard condition (7). In this case it essentially suffices to construct ϕ on diagonal 2×2 matrices. Then one can use general extension

arguments in the spirit of Proposition 13 below to conclude. Let

$$f \begin{pmatrix} s \\ t \end{pmatrix} = \phi \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}, \quad \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \nabla f.$$

We look for T_4 -configurations in the set

$$\begin{pmatrix} s & 0 \\ 0 & t \\ 0 & -\sigma(s, t) \\ \tau(s, t) & 0 \end{pmatrix}.$$

The Legendre-Hadamard condition reduces to

$$\frac{\partial \sigma}{\partial s} \geq c, \quad \frac{\partial \tau}{\partial t} \geq c, \quad c > 0. \quad (26)$$

If we drop the constraint that (σ, τ) is a gradient we can easily embed a T_4 -configuration in the plane $\{\sigma = \tau = 0\}$. It suffices to make the ansatz

$$\sigma(s, t) = s - g(t), \quad \tau(s, t) = t - h(s).$$

Then the Legendre-Hadamard condition imposes no constraint on g and h and hence we can embed in the plane $\{\sigma = \tau = 0\}$ any set which is a graph both over s and over t . For definiteness we chose the T_4 -configuration $(-3, 1), (1, 3), (3, -1), (-1, -3)$.

One can check, however, that the plane $\{\sigma = \tau = 0\}$ cannot contain a T_4 -configuration if (σ, τ) is a gradient (it suffices to integrate around the inner square of the T_4 -configuration and to use (26) to obtain a contradiction). To overcome this difficulty we tilt the T_4 -configuration, i.e. we look for a linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^{4 \times 2}$ which preserves rank-one connections. A natural choice is

$$(s, t) \mapsto \begin{pmatrix} s & 0 \\ 0 & t \\ 0 & \mu t \\ \mu s & 0 \end{pmatrix},$$

where μ is a constant. This leads to the conditions $\sigma = -\mu t, \tau = \mu s$ or equivalently

$$\nabla f \begin{pmatrix} s \\ t \end{pmatrix} = \mu J \begin{pmatrix} s \\ t \end{pmatrix}, \quad \text{for } \begin{pmatrix} s \\ t \end{pmatrix} = J^k \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad k \in \{0, 1, 2, 3\}, \quad (27)$$

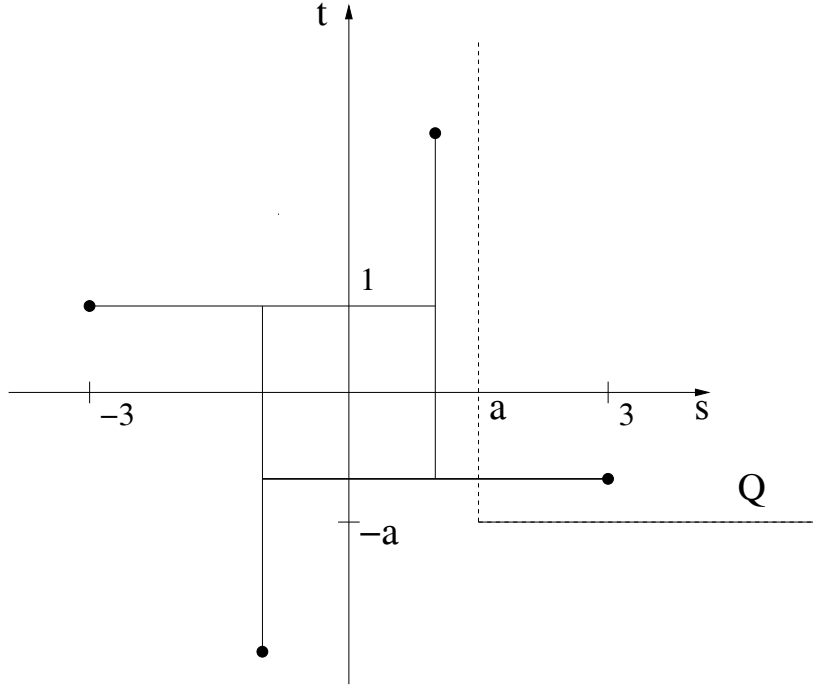


Figure 2: Construction of an integrand for which the set K contains a T_4 -configuration. The function f_0 is bilinear in the quadrant $Q = \{(s, t) : s > a, t > -a\}$ and vanishes outside Q . It vanishes in particular on the rotated quadrants $J^k Q$, where $k = 1, 2, 3$ and where J is the 90° rotation.

where J is the 90° rotation. We make an ansatz for f which reflects the 90° rotation symmetry of the planar T_4 -configuration:

$$f\left(\begin{matrix} s \\ t \end{matrix}\right) = \sum_{k=0}^3 f_0\left(J^k\left(\begin{matrix} s \\ t \end{matrix}\right)\right) + \lambda(s^2 + t^2), \quad \text{where } \lambda > 0.$$

Then it suffices to verify (27) for $k = 0$ and this is equivalent to

$$\langle \nabla f(x), x \rangle = 0, \quad \text{with } x = \begin{pmatrix} 3 \\ -1 \end{pmatrix}. \quad (28)$$

We now make the choice (see Fig. 2)

$$f_0\left(\begin{matrix} s \\ t \end{matrix}\right) = (s - a)_+ (t - a)_+, \quad \text{where } 1 < a < 3$$

and where $s_+ = \max(s, 0)$. In the quadrant $\{s > a, t > -a\}$ we have $f(s, t) = f_0(s, t) + \lambda(s^2 + t^2)$ and one easily checks that (28) is satisfied if, for example, $a = 5/4$, $\lambda = 1/20$. Multiplying of f by a suitable factor we thus see that the T_4 -configuration (25) lies in K .

To construct a quasiconvex ϕ for which K contains (25) we first use that the rank-one convex function s_+t_+ on diagonal 2×2 matrices can be extended to a quasiconvex function on symmetric 2×2 matrices, see [Sv 92b]. Then ϕ can be extended to all 2×2 matrices by adding a high quadratic penalty for the skew-symmetric part, see [MS 99] for the details.

3.4 Families of T_4 -configurations and dimension counting

In order to construct an in-approximation of K , which is needed for the application of Theorem 6, we first show that

$$\mathcal{K} := K \times K \times K \times K \subset (\mathbb{R}^{4 \times 2})^4 \quad (29)$$

contains not only the special T_4 -configuration \mathbf{M}^0 but an eight-dimensional family of T_4 -configurations. Then we will show that the corresponding corner points P_j cover an open set in the eight dimensional space $\mathbb{R}^{4 \times 2}$ and use this fact to construct the in-approximation.

The dimension of the set of T_4 -configurations can be guessed by a simple parameter count. First note that the rank-one cone in $\mathbb{R}^{4 \times 2}$ is a five dimensional manifold (away from its vertex). In view of Definition 7 the set

$$\mathcal{M} = \{\mathbf{M} \in (\mathbb{R}^{4 \times 2})^4 : \mathbf{M} \text{ is a } T_4 \text{ - configuration}\} \subset \mathbb{R}^{32}. \quad (30)$$

involves $4 \times 5 + 4 \times 1 + 8 = 32$ parameters (namely the C_j , the κ_j and P_1) which are subject to 8 constraints (namely $\sum C_j = 0$). Hence we expect \mathcal{M} to be a 24 dimensional manifold in the neighbourhood of \mathbf{M}^0 . Since \mathcal{K} is 16 dimensional, we expect the intersection to be have dimension 8, as desired.

To verify this and to actually construct the in-approximation we consider the maps

$$\begin{aligned} \pi_j : \mathcal{M} \cap \mathcal{K} &\longrightarrow \mathbb{R}^{4 \times 2} \\ (M_1, M_2, M_3, M_4) &\longmapsto P_j \\ \mu_j : \mathcal{M} \cap \mathcal{K} &\longrightarrow \mathbb{R}^{4 \times 2} \\ (M_1, M_2, M_3, M_4) &\longmapsto M_j \end{aligned}$$

Let $T_{M_j^0}K$ be the tangent space of K at M_j^0 , let Q_j^\perp denote the projection onto its orthogonal complement and define the map

$$\begin{aligned} \psi_j : \mathcal{M} \cap \mathcal{K} &\longrightarrow \mathbb{R}^{4 \times 2} \\ (M_1, M_2, M_3, M_4) &\longmapsto (M_j, Q_j^\perp(P_j - P_j^0)) \end{aligned}$$

Proposition 8 ([MS 99]) *There exists a choice of ϕ such that $\mathbf{M}^0 \in \mathcal{K}$ and*

- (i) *in a neighbourhood of \mathbf{M}^0 the sets \mathcal{M} and \mathcal{K} intersect transversely in an eight dimensional manifold,*
- (ii) *π_j and ψ_j are local diffeomorphisms from a neighbourhood of \mathbf{M}^0 in $\mathcal{M} \cap \mathcal{K}$ to open sets in $\mathbb{R}^{4 \times 2}$.*

With this result at hand one can construct an in-approximation as follows (see Fig. 3). Let $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \dots$ be (small) open neighbourhoods of \mathbf{M}^0 in $\mathcal{M} \cap \mathcal{K}$ which are diffeomorphic to eight dimensional balls. For $1/2 < \lambda_2 < \lambda_3 < \dots < 1$ consider the maps $\lambda_i \mu_j + (1 - \lambda_i) \pi_j$ and define for $i \geq 2$ the sets \mathcal{U}_i by

$$\mathcal{U}_i = \cup_{j=1}^4 \mathcal{U}_i^j, \quad \mathcal{U}_i^j = (\lambda_i \mu_j + (1 - \lambda_i) \pi_j)(\mathcal{O}_i).$$

Using the non-degeneracy of ψ_j one can show that the \mathcal{U}_i are open (if λ_2 is chosen sufficiently close to 1), see [MS 99]. Let $\mathcal{U}_1 = \mathcal{U}_2^c$. Then $\{\mathcal{U}_i\}_{i \geq 1}$ is an in-approximation of K . In fact it is an in-approximation of the set K intersected with a small neighbourhood of the set $\{M_1^0, M_2^0, M_3^0, M_4^0\}$. Moreover \mathcal{U}_1 contains the points $P_1^0, P_2^0, P_3^0, P_4^0$. Hence it also contains 0.

This shows that (23) admits a non-trivial solution with zero boundary conditions whose gradient is always close to the set $\{M_1^0, M_2^0, M_3^0, M_4^0\}$. It is easy to see that one can achieve in addition $0 \in K$. Hence extension of the solution by zero yields a solution of (4) with compact support.

To see that this solution is nowhere C^1 one has to trace back the general construction used in the proof of Theorem 6 a bit more carefully, see [MS 99] for the details. The main point is that at each step of the construction a (locally) affine map is replaced by a piecewise affine map whose gradient takes values near all of the four points M_i^0 . This leads to a limit map whose gradient has an oscillation of order 1 in every open set.

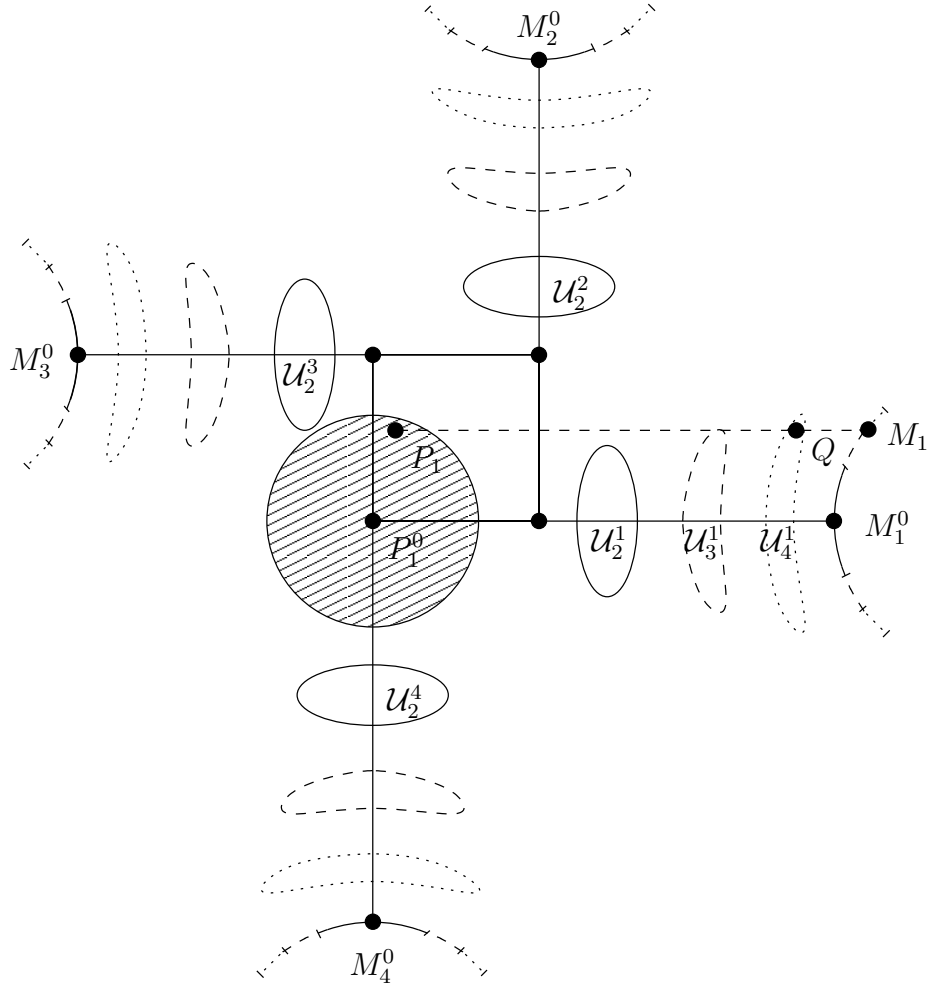


Figure 3: Schematic illustration of the sets $\mathcal{U}_i^j \subset \mathbb{R}^{4 \times 2}$. The solid (resp. dashed, or dotted) lines through the point M_1^0 are the sets $\mu_1(\mathcal{O}_2)$, $\mu_1(\mathcal{O}_3)$, $\mu_1(\mathcal{O}_4)$, respectively, i.e. the projections of the sets $\mathcal{O}_i \subset (\mathbb{R}^{4 \times 2})^4$ to the first component. The sets $\mu_1(\mathcal{O}_i)$ are not open in $\mathbb{R}^{4 \times 2}$ since they are contained in K . The shaded set is $\pi_1(\mathcal{O}_4)$ and is open in $\mathbb{R}^{4 \times 2}$ and the sets $\mathcal{U}_i^1 = [(1 - \lambda_i)\pi_1 - \lambda_i\mu_1](\mathcal{O}_i)$ are also open. A typical point Q in \mathcal{U}_4^1 is given by $(1 - \lambda_4)\pi_1(\mathbf{M}) + \lambda_4 M_1$, where $\mathbf{M} = (M_1, M_2, M_3, M_4) \in \mathcal{O}_4$. In particular Q lies on the rank-one segment $[P_1, M_1]$ and hence in the rank-one convex hull of $\{M_1, M_2, M_3, M_4\}$. It also lies in the rank-one convex hull of four points in \mathcal{U}_5 which are close the M_i .

3.5 Polyconvex examples and obstructions to T_4

Can one carry out the construction outlined above also for an integrand ϕ which is uniformly polyconvex (i.e. $\phi(X) = g(X, \det X)$ where g is uniformly convex) ? It turns out that T_4 -configurations are no longer sufficient.

Proposition 9 *Suppose that $\phi : \mathbb{R}^{2 \times 2}$ is strictly polyconvex. Then the set K given by (24) does not contain any T_4 -configuration.*

Nonetheless Székelyhidi [Sz 02] has shown that there exists a uniformly polyconvex ϕ such that the elliptic system (4) admits a Lipschitz, nowhere C^1 solution. His proof uses the fact that one can embed sufficiently many T_5 -configurations in K .

The statement of the proposition above is in fact the corollary of a slightly more general result. It gives some insight into the consequences of the monotonicity condition on the gradient of the convex function g representing our polyconvex integrand. It states that K_ϕ can not support a discrete laminate whose second $\mathbb{R}^{2 \times 2}$ -coordinate is just a linear image of the first $\mathbb{R}^{2 \times 2}$ -coordinate, see Proposition 10 below. As we will see this in particular rules out the existence of a T_4 configuration in K_ϕ .

We begin with a little algebra. Let

$$\phi(X) = g(X, \det X) \text{ for } X \in \mathbb{R}^{2 \times 2},$$

where $g : \mathbb{R}^5 \rightarrow \mathbb{R}$ is strictly convex. Using that

$$\nabla \phi(X) = \nabla_X G(X, \det X) + G_{,5}(X, \det X) \operatorname{cof} X,$$

and writing $\rho(X) = G_{,5}(X, \det X)$ we obtain from the strict monotonicity of ∇G that for any two different $X, \tilde{X} \in \mathbb{R}^{2 \times 2}$

$$\begin{aligned} 0 &< \langle \nabla \phi(\tilde{X}) - \nabla \phi(X) - \rho(\tilde{X}) \operatorname{cof} \tilde{X} + \rho(X) \operatorname{cof} X, \tilde{X} - X \rangle \\ &\quad + (\rho(\tilde{X}) - \rho(X))(\det \tilde{X} - \det X). \end{aligned}$$

We abbreviate $\rho = \rho(X)$, $Y = \nabla \phi(X)$, $\tilde{\rho} = \rho(\tilde{X})$ and $\tilde{Y} = \nabla \phi(\tilde{X})$. Expanding the difference of the determinants we conclude

$$\begin{aligned} 0 &< \langle \tilde{Y} - Y, \tilde{X} - X \rangle + \langle \rho \operatorname{cof} X - \tilde{\rho} \operatorname{cof} \tilde{X}, \tilde{X} - X \rangle \\ &\quad + (\tilde{\rho} - \rho)(\langle \operatorname{cof} X, \tilde{X} - X \rangle + \det(\tilde{X} - X)) \\ &= \langle \tilde{Y} - Y, \tilde{X} - X \rangle - \langle \tilde{\rho}(\operatorname{cof} \tilde{X} - \operatorname{cof} X), \tilde{X} - X \rangle + (\tilde{\rho} - \rho) \det(\tilde{X} - X) \\ &= \langle \tilde{Y} - Y, \tilde{X} - X \rangle - 2\tilde{\rho} \det(\tilde{X} - X) + (\tilde{\rho} - \rho) \det(\tilde{X} - X). \end{aligned}$$

Therefore, we have for any $X \neq \tilde{X}$

$$0 < \langle \nabla \phi(\tilde{X}) - \nabla \phi(X), \tilde{X} - X \rangle - (\rho(X) + \rho(\tilde{X})) \det(X - \tilde{X}). \quad (31)$$

Proposition 10 Suppose that $\phi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ is strictly polyconvex, and that for

$$K = \left\{ \begin{pmatrix} X_i \\ Y_i \end{pmatrix}; i = 1, \dots, n \right\} \subset K_\phi$$

there is a matrix $A \in \mathbb{R}^{2 \times 2}$ such that

$$(Y_i - Y_j) = A(X_i - X_j) \text{ for all } i, j \leq n.$$

Then $K^{rc} = K$.

Proof. From the definition of K we obtain $\nabla\phi(X_i) = -Y_i J$. Since for all $X \in \mathbb{R}^{2 \times 2}$ the equation $XJ = J \operatorname{cof} X$ holds, we infer from (31) that for any $i \neq j$

$$\begin{aligned} 0 &< \langle -A(X_i - X_j)J, X_i - X_j \rangle - (\rho(X_i) + \rho(X_j)) \det(X_i - X_j) \\ &= \langle -AJ \operatorname{cof}(X_i - X_j), X_i - X_j \rangle - (\rho(X_i) + \rho(X_j)) \det(X_i - X_j) \\ &= \operatorname{tr}((-AJ) \operatorname{cof}(X_i - X_j)(X_i - X_j)^T) - (\rho(X_i) + \rho(X_j)) \det(X_i - X_j) \\ &= \det(X_i - X_j)(\operatorname{tr}(-AJ) - \rho(X_i) - \rho(X_j)). \end{aligned}$$

We denote

$$\sigma(X) = -\left(\frac{1}{2} \operatorname{tr}(AJ) + \rho(X)\right)$$

and have therefore

$$\det(X_i - X_j)(\sigma(X_i) + \sigma(X_j)) > 0 \text{ for } i \neq j.$$

Lemma 11 below ensures now that in case $K^{rc} \neq K$ we find a closed cycle $i_0, i_1, \dots, i_l, i_{l+1} \in \{1, \dots, n\}$ with $i_0 = i_l, i_1 = i_{l+1}$ such that

$$\det(X_{i_k} - X_{i_{k-1}}) \det(X_{i_{k+1}} - X_{i_k}) < 0 \text{ for all } k = 1, \dots, l.$$

Up to a shift of indices we can therefore suppose that

$$(-1)^k (\sigma(X_{i_k}) + \sigma(X_{i_{k+1}})) > 0 \text{ if } k = 1, \dots, l$$

and that l is even. Summing over this cycle we get

$$-\sigma(X_{i_1}) + \sigma(X_{i_2}) - \sigma(X_{i_2}) + \sigma(X_{i_3}) - \dots + \sigma(X_{i_{l+1}}) > 0.$$

In other words, $0 > 0$ - this contradiction finishes our proof.

Lemma 11 *Assume a laminate $\mu \in \mathcal{M}^{rc}(\mathbb{R}^{2 \times 2})$ is supported in a finite set $\{X_1, \dots, X_n\}$ satisfying $\det(X_i - X_j) \neq 0$ if $i \neq j$. (See Section 4.2 for the definition of \mathcal{M}^{rc} .) Then there is a closed cycle*

$$i_0, i_1, \dots, i_l, i_{l+1} \in \{1, \dots, n\} \text{ with } i_0 = i_l, i_1 = i_{l+1}$$

satisfying

$$\det(X_{i_k} - X_{i_{k-1}}) \det(X_{i_{k+1}} - X_{i_k}) < 0 \text{ for } k = 1, \dots, l.$$

Proof. We claim that

$$\text{for each } i \leq n \text{ there are } j, k : \det(X_j - X_i) \det(X_k - X_i) < 0. \quad (32)$$

Indeed, if $\det X_j - X_i$, has a fixed sign for all j different from i then one can show that the laminate μ is either a Dirac mass at X_i or does not charge X_i (see Proposition 15 below). This contradicts the hypothesis that the support of μ is exactly $\{X_1, \dots, X_n\}$.

A little combinatorial argument will now show that (32) implies the existence of the required cycle. Indeed, (32) certainly enforces the existence of

$$i_1, \dots, i_l, i_{l+1} \text{ with } i_1 = i_{l+1}, (-1)^k \det(X_{i_{k+1}} - X_{i_k}) < 0 \text{ if } 1 \leq k \leq l.$$

Consequently, the only problem that might occur is that l is odd and so $\det(X_{i_2} - X_{i_1}), \det(X_{i_l} - X_{i_1}) > 0$. But, we observe that then necessarily

$$\det(X_{i_k} - X_{i_1}) > 0 \text{ also for all } k \in \{3, \dots, l-1\}.$$

Indeed, if this inequality fails then we extend the starting negative connection i_1, i_k into the adjacent positive connection and keep then running in this direction inside the already built cycle back to i_1 . In other words, for even k we can take $i_k, i_1, i_2, \dots, i_k, i_1$ as the desired cycle and if k is odd then $i_l, i_1, i_k, i_{k+1}, \dots, i_l, i_1$ does the job.

However, by (32) also X_{i_1} has to have a negative connection, so we find

$$X_{i_0} \text{ with } \det(X_{i_0} - X_{i_1}) < 0 \text{ and } X_{i_0} \notin \{X_{i_1}, \dots, X_{i_l}\}.$$

The same reasoning as before now shows that

$$\det(X_{i_0} - X_{i_k}) < 0 \text{ for all } k = 2, \dots, l.$$

Next, we find $X_{i_{-1}}$ with $\det(X_{i_{-1}} - X_{i_k}) > 0$ if $k \geq 0$, and so on. Obviously, repeating this argument finally leads to a contradiction, because our set $\{X_1, \dots, X_n\}$ is finite.

Proof of Proposition 9. In light of Proposition 10 we only need to show that for any T_4 -configuration $M_i = (X_i, Y_i) \in \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 2}$ there exists an $A \in \mathbb{R}^{2 \times 2}$ such that $Y_i - Y_j = A(X_i - X_j)$. According to Definition 7 we find $n_i, x_i, y_i \in \mathbb{R}^2$ and $\kappa_i > 0$ for $i = 1, \dots, 4$ such that

$$\sum_{i=1}^4 \begin{pmatrix} x_i \\ y_i \end{pmatrix} \otimes n_i = 0 \text{ and } M_{i+1} - M_i = (\kappa_{i+1} + 1) \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \otimes n_{i+1} - \kappa_i \begin{pmatrix} x_i \\ y_i \end{pmatrix} \otimes n_i.$$

So we are done, if we find $A \in \mathbb{R}^{2 \times 2}$ with $y_i = Ax_i$ for all i . For this purpose we notice that any consecutive n_i, n_{i+1} are linearly independent and that x_1, x_2, x_3, x_4 span the whole \mathbb{R}^2 . Indeed, the first statement is obvious from $\text{rk}(M_i - M_{i+1}) > 1$. So suppose the second fails, then all matrices X_1, \dots, X_4 are contained in a single rank-one plane $S \subset \mathbb{R}^{2 \times 2}$. Of course, ϕ must be strictly convex on S which gives the monotonicity condition

$$0 < \langle \nabla \phi(X_i) - \nabla \phi(X_j), X_i - X_j \rangle = -\langle (Y_i - Y_j)J, X_i - X_j \rangle$$

for all $i \neq j$. Because $\langle YJ, X \rangle = (X_{11}Y_{12} - X_{12}Y_{11}) + (X_{21}Y_{22} - X_{22}Y_{21})$ we see that a certain sum of quadratic minors is negative on all differences in the set $\{M_1, \dots, M_4\}$. By [Sv 93] this even implies that all polyconvex measures on $\{M_1, \dots, M_4\}$ have to be Dirac masses. So it is clear that after reshuffling the indices if necessary, we can assume that $\{n_1, n_2\}$ and $\{x_3, x_4\}$ are both bases for \mathbb{R}^2 .

Next, we observe that

$$u_1 \otimes v_1 + u_2 \otimes v_2 = (u_1 \ u_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

if the u_i 's are column and the v_i 's are row vectors from \mathbb{R}^2 . So we have

$$-(x_1 \ x_2) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = (x_3 \ x_4) \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}.$$

In other words our zero sum assumption implies

$$(x_1 \ x_2) = (x_3 \ x_4)B \text{ and } (y_1 \ y_2) = (y_3 \ y_4)B, \text{ for } B = -\begin{pmatrix} n_3 \\ n_4 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^{-1}.$$

Thus, if we define A by $y_j = Ax_j$ for $j = 3, 4$ then

$$(y_1 \ y_2) = (y_3 \ y_4)B = [A(x_3 \ x_4)]B = A[(x_3 \ x_4)B] = A(x_1 \ x_2)$$

as required. \square

3.6 Beyond T_k

In the above examples it is enough to embed a T_4 -configuration (or a T_5 -configuration) to show that K^{rc} is sufficiently rich. The same strategy essentially works for Theorems 2 and 3. To understand more general examples and to find structure conditions which would exclude ‘wild’ solutions one would like to compute (or at least estimate) K^{rc} rather than just trying to embed T_k -configurations (we will see below examples that K^{rc} can be non-trivial even if K contains no T_k -configuration). Ultimately this will require a deeper understanding of the geometry of rank-one convexity. Due to the high dimensions of the rank-one cone and the surrounding space this seems rather difficult at the moment. To build up some intuition we discuss below some examples where by different means one can reduce the dimensionality of the problem and thus gain some geometric insight.

One interesting example is set K related to the simplest polyconvex integrand

$$\varepsilon|F|^2 + (\det F)^2, \quad \varepsilon \geq 0.$$

For $\varepsilon = 0$ we show in Section 5 that $K^{rc} = K$. Interestingly, it is not known whether the same holds for the seemingly better case $\varepsilon > 0$, see Section 6 for some partial results. In both cases no partial regularity results for weak solutions are known, except for the case of local minimizers.

In Section 7 we raise the question what one can say about the local rank-one convex hull. One issue is whether one can formulate higher order conditions which give triviality of the hull when the tangent space of K contains rank-one lines. We first review some general tools to study rank-one convex hulls.

4 Tools to study rank-one convex hulls

In this section we collect some definitions and tools which are useful for the study of rank-one convex hulls and related hulls. For more detailed accounts see e.g. [Da 89, Pe 97, Mu 99].

4.1 Rank-one convex, quasiconvex and polyconvex hulls

In the following we always assume that $m, n \geq 2$ since otherwise all the convexity notions introduced below agree with ordinary convexity. A function $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is rank-one convex if is convex on each rank-one line, it is

polyconvex if it can be expressed as a convex functions of minors (subdeterminants) and it is quasiconvex if

$$\int_{T^n} f(A + \nabla\eta) - f(A) dx \geq 0 \quad (33)$$

for all $A \in \mathbb{R}^{m \times n}$ and for all periodic Lipschitz maps $\eta : T^n \rightarrow \mathbb{R}^m$. We have the following implications:

$$f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank-one convex.} \quad (34)$$

For a compact set we define the rank-one convex, quasiconvex and polyconvex hull as the set of those points which can not be separated by the corresponding class of functions.

$$K^* := \{A \in \mathbb{R}^{m \times n} : f(A) \leq \sup_K f : f \text{ is } *\}, \quad * = \{\text{rc, qc, pc}\}. \quad (35)$$

For a general set E we set

$$E^* := \bigcup_{K \subset E \text{ compact}} K^*, \quad * = \{\text{rc, qc, pc}\}. \quad (36)$$

In view of (34)

$$K^{\text{pc}} \supset K^{\text{qc}} \supset K^{\text{rc}}. \quad (37)$$

As we have seen in Sections 2 and 3 above a large rank-one convex hull allows one to construct many solutions of $\nabla u \in K$. The quasiconvex convex hull is related to the stability of the partial differential relation under weak convergence. More specifically if

$$\nabla u^{(j)} \xrightarrow{*} \nabla u \quad \text{and} \quad \text{dist}(\nabla u^{(j)}, K) \rightarrow 0 \text{ in } L^1 \quad (38)$$

then

$$\nabla u \in \bar{K}^{\text{qc}}, \quad (39)$$

and this property characterizes K^{qc} for compact K . The polyconvex hull provides an upper bound for both K^{qc} and K^{rc} .

4.2 The dual objects: laminates and gradient Young measures

Let $\mathcal{P}(K)$ denote the set of probability measures supported on K . For $\mu \in \mathcal{P}(K)$ we denote by $\bar{\mu} = \int A d\mu(A)$ its barycentre. We consider the

following subsets of $\mathcal{P}(K)$ which satisfy a Jensen's inequality with respect to the above convexity notions.

$$\begin{aligned} \mathcal{M}^{rc}(K) &= \{\mu \in \mathcal{P}(K) : \int f(A) d\mu(A) \geq f(\bar{\mu}), \forall f \text{ rc}\} \\ \mathcal{M}^{qc}(K) &= \{\mu \in \mathcal{P}(K) : \int f(A) d\mu(A) \geq f(\bar{\mu}), \forall f \text{ qc}\} \\ \mathcal{M}^{pc}(K) &= \{\mu \in \mathcal{P}(K) : \int f(A) d\mu(A) \geq f(\bar{\mu}), \forall f \text{ pc}\} \\ &= \{\mu \in \mathcal{P}(K) : \int M(A) d\mu(A) = M(\bar{\mu}), \text{ for all minors } M\}. \end{aligned}$$

For a compact set K we have

$$K^* = \{\bar{\mu} : \mu \in \mathcal{M}^*(K)\}, \quad * \in \{\text{rc, qc, pc}\}.$$

The elements of $\mathcal{M}^{qc}(\mathbb{R}^{m \times n})$ are called homogeneous gradient Young measures since they arise as distribution functions of the gradients of periodic functions. More precisely let $\mu \in \mathcal{M}^{qc}(\mathbb{R}^{m \times n})$ be a measure with compact support and let $\bar{A} = \bar{\mu}$ be its barycentre. Then there exists periodic maps $u^{(j)} : T^n \rightarrow \mathbb{R}^m$ with uniform Lipschitz constant such that the measures μ^j defined by $\int_{\mathbb{R}^{m \times n}} \eta(A) d\mu^j(A) := \int_{T^n} \eta(\bar{A} + \nabla u^{(j)}) dx$ converge weak* to μ , see [KP 91, Sy 99]. For a compact set K the set $\mathcal{M}^{qc}(K)$ is trivial (i.e. contains only Dirac masses) if and only if all sequences $\nabla u^{(j)}$ satisfying (38) converge strongly in L^1 .

The elements of $\mathcal{M}^{rc}(K)$ are called laminates and they can be obtained as weak* limits of so called laminates of finite order (with uniformly bounded support), see [Pe 93, MS 99]. The class \mathcal{L} of laminates of finite order is defined by inductive splitting along rank-one segments as follows. First, each Dirac mass δ_A belongs to \mathcal{L} . Second, suppose that $\lambda_i \geq 0$, $\sum_{i=1}^m \lambda_i = 1$, that $\nu = \sum \lambda_i \delta_{A_i} \in \mathcal{L}$ and that $A_m = (1-s)B_1 + sB_2$ is a rank-one convex combination of B_1 and B_2 . Then $\mu = \sum_{i=1}^{m-1} \lambda_i \delta_{A_i} + (1-s)\lambda_m \delta_{B_1} + s\lambda_m \delta_{B_2}$ also belongs to \mathcal{L} . Thus laminates are the distribution functions of gradients which arise from essentially one-dimensional constructions.

4.3 Localization and extension

Rank-one convexity is a local property (while qc and pc are not, at least for $m \geq 3$, see [Kr 99]) and this greatly simplifies the construction of separating functions. We first recall that it suffices to separate locally.

Proposition 12 ([MS 99], Lemma 2.3) *Let K be compact and let U be an open neighbourhood of K^{rc} . Suppose that $f : U \rightarrow \mathbb{R}$ is rank-one convex in U , i.e. convex on each rank-one segment entirely contained in U . Then there exists a rank-one convex function $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ which agrees with f in a neighbourhood of K^{rc} .*

One can also extend from lower dimensional sets, see [Sv 92a] for the details.

Proposition 13 *Let L be a subspace $\mathbb{R}^{m \times n}$ and suppose that $f : L \rightarrow \mathbb{R}$ is C^2 and is rank-one convex on L . Let $\delta > 0$ and let $E \subset L$ be compact. Then there exists a rank-one convex function $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ which satisfies $\sup_E |F - f| \leq \delta$.*

If K^{rc} has several components then the rank-one convex hull can be computed for each piece separately.

Proposition 14 ([Ki 01b], Thm. 4.7) *Let K be a compact set.*

(i) *Let B be bounded. Then*

$$K^{rc} \cap B = [(K \cap B) \cup (K^{rc} \cap \partial B)]^{rc} \cap B. \quad (40)$$

(ii) *Let C_1, \dots, C_k be disjoint compact sets and suppose $K^{rc} \subset \cup_i C_i$. Then $K^{rc} = \cup_i (K \cap C_i)^{rc}$.*

Part (ii) appears already in [Pe 93, MP 98] (see [Ma 01] for a detailed proof) and it can easily be deduced from part (i) by taking B as a sufficiently small neighbourhood U_i of C_i which does not intersect the other C_j . The following application of the proposition will be useful later.

Proposition 15 *Let K be a compact subset of $\{X \in \mathbb{R}^{2 \times 2} : \det X > 0\}$. Then $(K \cup \{0\})^{rc} = K^{rc} \cup \{0\}$ and every laminate supported on $K \cup \{0\}$ is either supported on K or is a Dirac mass at zero. In particular, if*

$$E = \{X \in \mathbb{R}^{2 \times 2} : \det X = 1\} \cup \{0\}, \quad \text{then } E^{rc} = E.$$

Remark. Astala and Faraco [AF 02] have shown that the same assertion holds for the quasiconvex hull and measures in $\mathcal{M}^{qc}(K \cup \{0\})$, i.e., gradient Young measures. Their proof uses ideas from the theory of quasiregular maps, in particular a careful analysis of the Beltrami equation.

Proof. By compactness there exist $\varepsilon > 0, R > 0$ such that $\det X \geq \varepsilon$ and $|X| \leq R$ for all $X \in K$. The polyconvex function $f(X) = \varepsilon|X| - R \det X$

is ≤ 0 in $K \cup \{0\}$, but is positive for $0 < |X| < 2\varepsilon/R$. Thus we can apply Proposition 14 with $C_1 = \{X \in \mathbb{R}^{2 \times 2} : |X| \leq \varepsilon/R\}$ and $C_2 = \{X \in \mathbb{R}^{2 \times 2} : |X| \geq 2\varepsilon/R\}$ and we obtain $(K \cup \{0\})^{rc} = K^{rc} \cup \{0\}$.

Let μ be a laminate supported on K , with barycentre $\bar{\mu}$. Suppose first $\bar{\mu} \in K^{rc}$. Let U_i be small neighbourhoods of C_i and define $g : U = U_1 \cup U_2 \rightarrow \mathbb{R}$ by $g = -1$ on U_1 and $g = 1$ on U_2 . Since g is constant on each component of U it is trivially rank-one convex. By Proposition 12 there exists a rank-one convex function $\tilde{g} : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ which agrees with g on K . Hence $1 = \tilde{g}(\bar{\mu}) \leq \int \tilde{g} d\mu = \mu(K) - \mu(\{0\})$. Thus μ must be supported on K . If $\bar{\mu} = 0$ one concludes similarly that $\mu = \delta_0$ by starting from the function $-g$. \square

5 The simplest polyconvex integrand

Here we consider the simplest polyconvex integrand

$$\phi(X) = \frac{1}{2}(\det X)^2.$$

The Euler-Lagrange equation $\operatorname{div} D\phi(\nabla w) = 0$ is equivalent to the first order partial differential relation

$$\nabla u \in K := \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^{4 \times 2} : Y = \det X (\operatorname{cof} X) J, X \in \mathbb{R}^{2 \times 2} \right\}, \quad (41)$$

where J is the 90° rotation.

Theorem 16 $K^{rc} = K$.

To prove this result it is convenient to make the change of variables $\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} X \\ -JY \end{pmatrix}$. This bijection maps rank-one lines onto rank-one lines and therefore does not affect the computation of rank-one convex hulls. Since $-J \operatorname{cof} X J = X$ it thus suffices to consider the set

$$\tilde{K} = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^{4 \times 2} : Y = (\det X) X, X \in \mathbb{R}^{2 \times 2} \right\}. \quad (42)$$

The set \tilde{K} is contained in the cone

$$\Sigma = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{pmatrix} : \alpha \wedge \gamma = \beta \wedge \delta = 0, \alpha \wedge \delta = \beta \wedge \gamma \right\}, \quad (43)$$

where $\alpha \wedge \gamma = \alpha_1 \gamma_2 - \alpha_2 \gamma_1$ etc. Since Σ is defined by minors it is polyconvex and thus $\tilde{K}^{rc} \subset \Sigma$. To separate points in $\Sigma \setminus \tilde{K}$ we first use separating rank-one convex functions defined only on Σ . Then we extend these function to $\mathbb{R}^{4 \times 2}$. Since Σ is not smooth this requires some care.

5.1 A rank-one foliation of Σ

The construction of rank-one convex functions in Σ is largely simplified by the fact that most of Σ can be decomposed into simpler sets which contain all the rank-one connections.

Proposition 17 *We have*

$$\Sigma = \bigcup_{\lambda \in \mathbb{R}} L_\lambda \cup L_\infty \cup N, \quad (44)$$

where

$$L_\lambda = \left\{ \begin{pmatrix} X \\ \lambda X \end{pmatrix} : X \in \mathbb{R}^{2 \times 2} \right\}, \quad L_\infty = \left\{ \begin{pmatrix} 0 \\ Y \end{pmatrix} : Y \in \mathbb{R}^{2 \times 2} \right\},$$

$$N = \{a \otimes b : a \in \mathbb{R}^4, b \in \mathbb{R}^2\}.$$

Moreover if $A, B \in \Sigma$ satisfy $\text{rk}(A - B) = 1$ then the rank-one line through A and B lies entirely in N or entirely in one of the spaces L_λ , where $\lambda \in \mathbb{R} \cup \{\infty\}$.

Proof. Clearly N , L_λ and L_∞ are contained in Σ . To see the converse suppose first that $A = (\alpha, \beta, \gamma, \delta)^T$ satisfies $\alpha \wedge \beta \neq 0$ or $\alpha \wedge \delta = \beta \wedge \gamma \neq 0$. Then $\alpha \neq 0$ and $\beta \neq 0$ and the first two conditions in the definition of Σ give $\gamma = \lambda \alpha$, $\delta = \mu \beta$ while the last condition gives $\mu = \lambda$. Hence $A \in L_\lambda$. Now assume $\alpha \wedge \beta = \alpha \wedge \delta = \beta \wedge \gamma = 0$. If $\gamma \wedge \delta \neq 0$ then $\alpha = \lambda \gamma$, $\beta = \mu \delta$ and this yields $\alpha = \beta = 0$, so that $A \in L_\infty$. Finally, if also $\gamma \wedge \delta = 0$ then all 2×2 minors of A are zero. Hence $A \in N$.

Since Σ is given by minors any rank-one line

$$A(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} X + tx \otimes n \\ Y + ty \otimes n \end{pmatrix}$$

lies entirely in Σ if two points lie in Σ . If $X(t) \equiv 0$ then $A(t)$ lies in L_∞ . Next assume that $X(t) \not\equiv 0$ but $\det X(t) \equiv 0$. We claim that $A(t) \in N$. Indeed $X(t) \neq 0$ for all but one value of t . If $A(t)$ was not in N then

$Y(t) = \lambda(t)X(t)$. Since $\det X(t) = 0$ this yields again $A(t) \in N$ for all but one t and hence for all t . Finally assume that $\det X(t) \neq 0$. Then $\det X(t) \neq 0$ for all but one t . In particular $Xn^\perp \neq 0$ where $n^\perp = Jn$ is perpendicular to n . Moreover $Y(t) = \lambda(t)X(t)$. Applying this identity to n^\perp we deduce that $\lambda(t)$ must be constant, so that the rank-one line lies in one L_λ . \square

5.2 Separating functions on Σ

The foliation of Σ almost allows one to compute the rank-one convex hull separately on all the spaces L_λ and on N . More precisely we make the following ansatz for the separating function

$$f(X, Y) = h(X, Y) - \begin{cases} \varphi(\lambda) \det Y & \text{if } \begin{pmatrix} X \\ Y \end{pmatrix} \in L_\lambda, \\ 0 & \text{if } \begin{pmatrix} X \\ Y \end{pmatrix} \in N \cup L_\infty. \end{cases} \quad (45)$$

Here $h : \Sigma \rightarrow \mathbb{R}$ is rank-one convex and $\varphi \in C_0^\infty(\mathbb{R})$. Since each rank-one line stays either in L_λ or in N the function f is automatically rank-one convex. On \tilde{K} the expression for f simplifies since $\tilde{K} \cap (N \cup L_\infty) = \{0\}$ and in $\tilde{K} \cap L_\lambda$ we have $\lambda = \det X$. We thus have

$$f(X, Y) = h(X, X \det X) - \varphi(\det X)(\det X)^3 \quad (46)$$

Now we specify h and ϕ in dependence of the point $\bar{A} = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \in \Sigma \setminus \tilde{K}$ we seek to separate.

Case 1: $\bar{A} \in L_{\bar{\lambda}}$, $\bar{\lambda} \notin \{0, \infty\}$. The point $A = \begin{pmatrix} X \\ Y \end{pmatrix}$ belongs to $\tilde{K} \cap L_{\bar{\lambda}}$ if and only if

$$Y \in E := \{Z \in \mathbb{R}^{2 \times 2} : \det Z = \bar{\lambda}^3\} \cup \{0\}.$$

By Proposition 15 the rank-one convex hull of E is trivial. Hence there exists a rank-one convex function $\bar{h} : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ such that $\bar{h}(\bar{Y}) = 1$ and $\bar{h} \leq 0$ on E . Set $h(X, Y) = \bar{h}(Y)$. In view of (46) it suffices to show that (for each R) there exists φ such that

$$\varphi(\bar{\lambda}) = 0,$$

$$\bar{h}(X \det X) - \varphi(\det X)(\det X)^3 \leq \frac{1}{2}, \quad \forall X \text{ with } |X| \leq R$$

To see that this is possible first note that if $\det X = \bar{\lambda}$ then $\bar{h}(X \det X) \leq 0$ by construction. Hence $\bar{h}(X \det X) \leq 1/2$ for $|\det X - \bar{\lambda}| \leq \delta_0(R)$. Similarly for $\det X = 0$ we have $\bar{h}(X \det X) = 0$. Hence $\bar{h}(X \det X) \leq 1/2$ for

$|\det X| \leq \delta_0(R)$. For the remaining values of $\det X$ the desired inequality can be achieved by a suitable choice of φ (note that $\det X \leq R^2$ so that there is no obstruction to choosing φ with compact support).

Case 2: $\bar{A} = \begin{pmatrix} \bar{X} \\ 0 \end{pmatrix} \in L_0 \setminus N$. We have $\det \bar{X} \neq 0$ and we take

$$h(X, Y) = \det X \operatorname{sgn}(\det \bar{X}).$$

Then we can choose φ such that

$$|\lambda| - \varphi(\lambda)\lambda^3 \leq \frac{|\det \bar{X}|}{2}, \quad \text{for } |\lambda| \leq R^2. \quad (47)$$

Case 3: $\bar{A} \in N \cup L_\infty$. Let

$$h(X, Y) = |Y|^2, \quad \varphi = R^2 \hat{\varphi},$$

where $\lambda^2 - \hat{\varphi}(\lambda)\lambda^3 \leq R^{-2}|\bar{Y}|^2/2$. On the set \tilde{K} the function f is bounded by

$$|X \det X|^2 - \varphi(\det X)(\det X)^3 \quad (48)$$

$$\leq R^2[(\det X)^2 - \hat{\varphi}(\det X)(\det X)^3] \leq \frac{|\bar{A}|^2}{2}. \quad (49)$$

5.3 Extension

We show that the function $f : \Sigma \rightarrow \mathbb{R}$ constructed above can be approximated (uniformly on compact subsets of Σ) by functions which are rank-1 convex in a neighbourhood of Σ . In view of Proposition 12 this will finish the proof of Theorem 16.

The main point is to define an analogue of the parameter λ in (45) for a general matrix $A \in \mathbb{R}^{4 \times 2}$. Consider the set

$$\tilde{\Sigma} = \left\{ \begin{pmatrix} X \cos \alpha \\ X \sin \alpha \end{pmatrix} : X \in \mathbb{R}^{2 \times 2}, \alpha \in (-\pi/2, \pi/2] \right\}.$$

A point $A = \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^{4 \times 2}$ has a unique best approximation $\pi_{\tilde{\Sigma}}(A)$ in $\tilde{\Sigma}$ if and only if

$$P(A) := (|X|^2 - |Y|^2)^2 + 4\langle X, Y \rangle^2 \neq 0.$$

In this case the optimal angle $\bar{\alpha}(A)$ is determined by the condition

$$\begin{pmatrix} \cos 2\bar{\alpha} \\ \sin 2\bar{\alpha} \end{pmatrix} = \frac{1}{P(A)^{1/2}} \begin{pmatrix} |X|^2 - |Y|^2 \\ 2\langle X, Y \rangle \end{pmatrix}.$$

On the set $P \neq 0$ the closest point projection $\pi_{\bar{\Sigma}}$ is smooth. We define $\hat{\lambda}(A) = \tan \bar{\alpha}(A)$ and replace the ansatz (45) by

$$f_{\delta}(A) = h(A) - \varphi(\hat{\lambda}(A))\eta\left(\frac{P(A)}{\delta^4}\right) \det Y, \quad (50)$$

where $\varphi \in C_0^{\infty}(\mathbb{R})$, where $1 - \eta \in C_0^{\infty}(-1, 1)$ with $\eta|_{(-1/2, 1/2)} = 1$ and where $\delta > 0$ is a small parameter. Then f_{δ} is well-defined on $\mathbb{R}^{4 \times 2}$ and smooth. We first claim that on Σ the function f_{δ} is close to f and nearly rank-one convex.

Clearly $f_{\delta} = f$ on N since $\det Y = 0$ on N . On each L_{λ} (including $\lambda = \infty$) we have $P(A) = |A|^4$. Hence $f_{\delta} = f$ on $\Sigma \setminus B_{\delta}$ and thus $\sup_{\Sigma} |f - f_{\delta}| \leq C\delta^2$. Using again that $P(A) = |A|^4$ on L_{λ} and the homogeneity of $\hat{\lambda}$ and \det it is easy to verify that for each rank-one line $A + ta \otimes b$ in Σ we have $D^2 f_{\delta}(A)(a \otimes b, a \otimes b) \geq -C|a|^2|b|^2$. Thus the proof is concluded by the following extension result.

For $A = (\alpha, \beta, \gamma, \delta)^T \in \mathbb{R}^{4 \times 2}$ consider the minors

$$U_1(A) = \alpha \wedge \gamma, \quad U_2(A) = \beta \wedge \delta, \quad U_3(A) = \alpha \wedge \delta - \beta \wedge \gamma$$

and recall that Σ was defined as the set where all three minors vanish.

Proposition 18 *Suppose that $f \in C^2(\mathbb{R}^{4 \times 2})$. Assume that for all rank-one lines $A + ta \otimes b$ contained in Σ one has*

$$\begin{aligned} D^2 f_{\delta}(A)(a \otimes b, a \otimes b) &\geq 0 && \text{if } |A| \geq \delta, \\ D^2 f_{\delta}(A)(a \otimes b, a \otimes b) &\geq -C_0|a|^2|b|^2 && \text{if } |A| < \delta. \end{aligned}$$

Then there exists a smooth convex function $g : \mathbb{R}^{4 \times 2} \rightarrow \mathbb{R}$ with

$$0 \leq g(A) \leq CC_0\delta|A|$$

and for each $\varepsilon > 0$ there exists $\mu \geq 0$ such that the function

$$F := f_{\delta} + g + \mu \sum_{i=1}^3 U_i^2 + \varepsilon|A|^2$$

is rank-one convex in a neighbourhood of Σ .

Proof. First note that there exists a smooth convex function g with the properties stated in the proposition which satisfies $D^2 g(A) \geq C_0 \text{Id}$ for $|A| < \delta$. Fix $\varepsilon > 0$ and suppose the assertion of the proposition was

false. Then there exist points $A_k \rightarrow A$ with $A \in \Sigma$, rank-one directions $B_k = a_k \otimes b_k$ with $|B_k| = 1$ and $B_k \rightarrow B$, and $\mu_k \rightarrow \infty$ such that

$$D^2 f_\delta(A_k)(B_k, B_k) + D^2 g(A_k)(B_k, B_k) + 2\mu_k \sum_{i=1}^3 \langle DU_i(A_k), B_k \rangle^2 \leq -2\varepsilon.$$

Taking the limit $k \rightarrow \infty$ we see that

$$D^2 f_\delta(A)(B, B) + D^2 g(A)(B, B) \leq -2\varepsilon \quad (51)$$

and $\langle DU_i(A), B \rangle = 0$. Hence $A + tB$ is a rank-one line on Σ and thus (51) leads to a contradiction with the hypotheses on $D^2 f$ and the choice of g . \square

Remark. The main idea behind formula (50) can be understood as follows. Let $\Sigma_{reg} = \bigcup_{\lambda \in \mathbb{R}} L_\lambda \setminus \{0\}$. We note that λ can be thought of as a smooth function on Σ_{reg} defined by $A \in L_{\lambda(A)}$. For $\varphi \in C_0^\infty(\mathbb{R})$ the function $A \rightarrow \varphi(\lambda)$ can be smoothly extended from $\Sigma_{reg} \cap \mathbb{S}^{(4 \times 2)-1}$ to all of this unit sphere. Such a function can now be extended to a smooth zero homogeneous function $\tilde{\varphi}$ on $\mathbb{R}^{4 \times 2} \setminus \{0\}$ which still agrees with the original function on all of Σ_{reg} . Moreover, the function

$$A = \begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \tilde{\varphi}(A) \det(Y)$$

is 2-homogeneous on $\mathbb{R}^{4 \times 2}$, smooth away from 0 and rank-one convex on Σ . The role of η and Proposition 18 is to handle the singularity at 0. Since the second derivatives of $\tilde{\varphi}(A) \det(Y)$ are bounded, this does not present a problem.

5.4 More general polyconvex integrands

It seems natural to expect that Theorem 16 can be extended to all the strictly polyconvex integrands

$$\phi(F) = \frac{1}{2}\varepsilon|F|^2 + \frac{1}{2}(\det F)^2, \quad \varepsilon > 0, \quad (52)$$

but whether this is true is not known. If we restrict X and Y to diagonal matrices then the corresponding set \tilde{K} becomes

$$\tilde{K}_{\text{diag}} = \left\{ \left(\begin{pmatrix} s & 0 \\ 0 & t \\ (st)s + \varepsilon t & 0 \\ 0 & (st)t + \varepsilon s \end{pmatrix} : s, t \in \mathbb{R} \right) \right\}.$$

This set is contained in a four-dimensional subspace and on this subspace rank-one convexity in $\mathbb{R}^{4 \times 2}$ reduces to separate convexity in $\mathbb{R}^2 \oplus \mathbb{R}^2$. Even in this simplified setting it is not known whether $\tilde{K}_{\text{diag}}^{rc} = \tilde{K}_{\text{diag}}$. In the next section we will establish this result at least for finite sets (this in particular implies that no T_k -configuration can be embedded in \tilde{K}_{diag}). To do so we study separate convexity in more detail.

6 Separate convexity

6.1 Separate convexity in \mathbb{R}^2

This corresponds to the cone $\Lambda = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \subset \mathbb{R}^2$ which arises if we restrict the rank-one convex cone in 2×2 matrices to diagonal matrices. This situation is relatively well understood. In particular every nontrivial configuration must contain a T_4 -configuration.

Proposition 19 ([Ta 93], Remark 10; [MP 98], Proposition 5.3) *Let K be a compact set in diagonal 2×2 matrices.*

- (i) *Every point $A \in K^{rc}$ is contained in the rank-one convex hull of a subset of K consisting of at most five points.*
- (ii) *If K contains no rank-one connections but $K^{rc} \neq K$ then K must contain a T_4 -configuration.*

There also exists an efficient algorithm for the computation of the rank-one convex hull [MP 98]. Moreover on diagonal 2×2 matrices rank-one convexity and quasiconvexity agree, in the sense that the spaces \mathcal{M}^{rc} of laminates and \mathcal{M}^{qc} of gradient Young measures agree [Mu 99b].

Separate convexity in $\mathbb{R}^n = \mathbb{R} \oplus \dots \oplus \mathbb{R}$ which arises by restricting rank-one convexity in $\mathbb{R}^{n \times n}$ to diagonal matrices is already more subtle, see [MP 98]. Here we are more interested in separate convexity in $\mathbb{R}^2 \oplus \mathbb{R}^2$ and as an intermediate step we consider $\mathbb{R}^2 \oplus \mathbb{R}$.

One key tool is the following separation argument for sets which are supported in two opposite quadrants [Ta 93]. In order to use the same notions as in Section 4 we formally view separate convexity in $\mathbb{R}^m \oplus \mathbb{R}^n$ as a special case of rank-one convexity (see (54)).

Lemma 20 *For $x_0, a \in \mathbb{R}^m$ and y_0, b in \mathbb{R}^n consider the generalized quadrants*

$$Q_{\pm, \pm} = \{(x, y) \in \mathbb{R}^m \oplus \mathbb{R}^n : \pm \langle x - x_0, a \rangle > 0, \pm \langle y - y_0, b \rangle > 0\}$$

and the set

$$Q_{00} = \{(x, y) \in \mathbb{R}^m \oplus \mathbb{R}^n : \langle x - x_0, a \rangle = \langle y - y_0, b \rangle = 0\}.$$

Let the set $K \subset Q_{++} \cup Q_{--} \cup Q_{00}$ be compact and let μ be laminate supported on K , with barycentre $\bar{\mu}$. Then one of the following three assertions holds

- (i) $\bar{\mu} \in Q_{00}$ and $\text{supp } \mu \subset Q_{00}$,
- (ii) $\bar{\mu} \in Q_{++}$ and $\text{supp } \mu \subset Q_{++} \cup Q_{00}$,
- (iii) $\bar{\mu} \in Q_{--}$ and $\text{supp } \mu \subset Q_{--} \cup Q_{00}$.

If, in addition, $K \cap Q_{++}$ and $K \cap Q_{--}$ are compact (e.g., if K is finite) then in (ii) and (iii) one has $\text{supp } \mu \subset Q_{++}$ and $\text{supp } \mu \subset Q_{--}$, respectively.

Proof. We may suppose $x_0 = y_0 = 0$. Consider the separately affine function

$$f(x, y) = \langle x, a \rangle \langle y, b \rangle.$$

Then $f(\bar{\mu}) = \int_K f d\mu \geq 0$. If $f(\bar{\mu}) = 0$ then $\text{supp } \mu \subset Q_{00}$ and hence $\bar{\mu} \in Q_{00}$. If $f(\bar{\mu}) > 0$ then $\bar{\mu} \in Q_{++} \cup Q_{--}$. Suppose $\bar{\mu} \in Q_{++}$ and consider the separately convex function

$$g(x, y) = \langle x, a \rangle_+ \langle y, b \rangle_+ - \langle x, a \rangle \langle y, b \rangle.$$

where $a_+ = \max(a, 0)$. Then $g \leq 0$ on K and $g < 0$ on $K \cap Q_{--}$. Since $0 = g(\bar{\mu}) \leq \int g d\mu$ it follows that $\text{supp } \mu \subset Q_{++} \cup Q_{00}$. If $K \cap Q_{++}$ is compact we can find $\varepsilon > 0$ such that $\min(\langle x, a \rangle, \langle y, b \rangle) \geq \varepsilon$ for all $(x, y) \in K \cap Q_{++}$. In addition we may assume that $\langle \bar{\mu}_1, a \rangle \geq \varepsilon$ and $\langle \bar{\mu}_2, b \rangle \geq \varepsilon$, where $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2)$. Then we can replace $\langle x, a \rangle$ with $\langle x, a \rangle - \varepsilon$ and $\langle y, b \rangle$ with $\langle y, b \rangle - \varepsilon$ in the definition of g and we conclude easily. The case $\bar{\mu} \in Q_{--}$ is analogous. \square

6.2 Separate convexity in $\mathbb{R}^2 \oplus \mathbb{R}$

Here we consider the cone

$$\Lambda = \mathbb{R}^2 \times \{0\} \cup \{0\} \times \mathbb{R} \subset \mathbb{R}^3. \quad (53)$$

We can view Λ as a subset of the rank-one cone in 3×2 matrices if we identify \mathbb{R}^3 with the space

$$L = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \\ 0 & z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}. \quad (54)$$

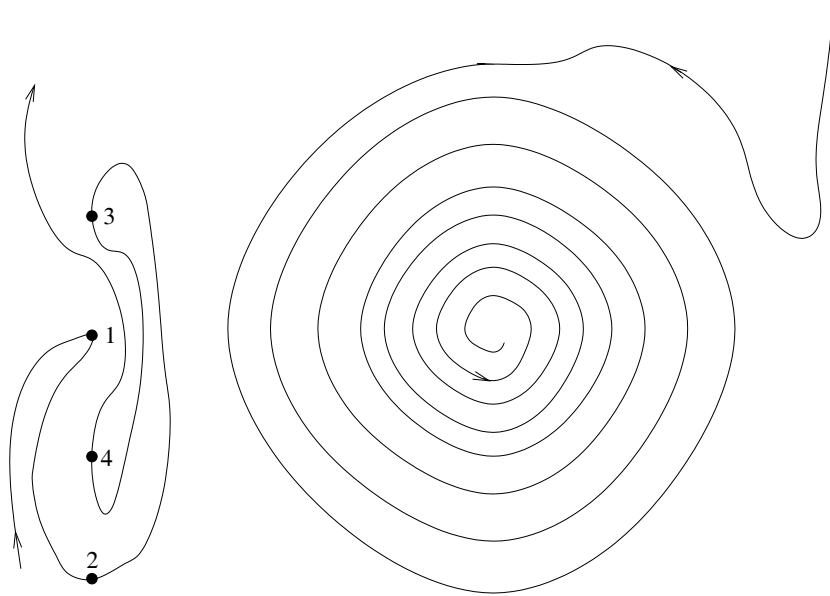


Figure 4: The figure on the left shows the T_4 configuration embedded on a non selfintersecting curve. The highly nonconvex spiral on the right, however, leads to a trivial rank-one convex hull

With this identification in mind we continue to write K^{rc} instead of K^Λ for the rank-one convex hull. We wish to understand how the more complicated geometry of separate convexity in $\mathbb{R}^2 \oplus \mathbb{R}$ can be used to construct examples which do not exist in $\mathbb{R} \oplus \mathbb{R}$. We first summarize the results and then turn to the proofs.

Proposition 21 *Let L be given by (54). Suppose that $K \subset L$ consists of at most five points and contains no rank-one connection. If $K^{rc} \neq K$ then K contains a T_4 -configuration.*

Proposition 22 *Let L be given by (54). There exists a set $K \subset L$ which consists of six points, contains no T_4 -configuration and has a non-trivial rank-one convex hull $K^{rc} \neq K$.*

For the construction see Fig. 6. The short proof that the example contains no T_k configuration is given below. An interesting class of sets without rank-one connections are (monotone) graphs over curves without self-intersections. For curves which are ‘spiral-like’ we can show that K^{rc} is trivial (see Fig. 4).

Proposition 23 *Identify L given by (54) with \mathbb{R}^3 . Let $\gamma : [0, T] \rightarrow \mathbb{R}^2$ have a regular C^1 -image (i.e. if the curve is parametrized by arclength, the derivative exists everywhere and varies continuously) which satisfies for all $t \in [0, T)$ that $\gamma(t)$ is not in the convex hull of $\{\gamma(s) : s > t\}$. Set $K = \{(\gamma(t), t) : t \in [0, T]\}$. Then $K^{rc} = K$.*

Without the hypotheses on the convex hull the result may fail, since one can embed a T_4 configuration (see Fig. 4). Interestingly, even with that hypotheses the result can fail if we allow Lipschitz curves rather than C^1 curves, see Example 24 and Fig. 7 below.

Proof of Proposition 21. The proof shows that the simple geometric separation argument, Lemma 20, becomes quite powerful when combined with the localization formula (40) in Proposition 14 (i). We first eliminate the simpler cases when K contains four or less points.

If K contains three or less points, then it is well-known that the absence of rank-one connections implies $K^{rc} = K$. For two points one can use a suitable 2×2 minor to show this. We give a proof for three points for our example since the same argument can be used for four points. Let $K = \{P_1, P_2, P_3\}$ with $P_i = (p_i, z_i) = (x_i, y_i, z_i)$. Since there are no rank-one connections we can assume (after a permutation of indices if necessary) that $z_1 < z_2 < z_3$. We claim that $p_1 \in \{p_2, p_3\}^{co}$. If this fails then there exists $a \in \mathbb{R}^2$ such that $\langle p_i - p_1, a \rangle > 0$, for $i = 2, 3$. Thus the points P_2 and P_3 lie in the generalized quadrant

$$Q_{++} = \{(p, z) : \langle p - p_1, a \rangle > 0, z - z_1 > 0\},$$

and $K \subset Q_{++} \cup Q_{00}$. It follows from Lemma 20 that a nontrivial laminate supported on K must be supported on $\{P_2, P_3\}$, but this possibility has already been ruled out. Therefore $p_1 \in \{p_2, p_3\}^{co}$. Similarly we conclude $p_3 \in \{p_1, p_2\}^{co}$. Thus $p_1 = p_2 = p_3$ and K lies in a rank-one line, in contradiction with our assumption.

If $K = \{P_1, P_2, P_3, P_4\}$ we can again assume $z_1 < z_2 < z_3 < z_4$ and the separation argument given above implies that $p_1 \in \{p_2, p_3, p_4\}^{co}$ and $p_4 \in \{p_1, p_2, p_3\}^{co}$. Thus the convex set $\{p_1, p_2, p_3, p_4\}^{co}$ has only the two extreme points p_2, p_3 and hence $\{p_1, p_2, p_3, p_4\} \subset \{p_2, p_3\}^{co}$. If $\{p_1, p_2\}^{co} \cap \{p_3, p_4\}^{co} = \emptyset$ we can again use the separation argument to obtain a contradiction. Hence $\{p_1, p_2\}^{co} \cap \{p_3, p_4\}^{co} \neq \emptyset$ and drawing the picture of K in the planar strip $[p_2, p_3] \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R}$ we see the usual T_4 -configuration (see Fig. 5).

Finally, if K contains five points, we suppose again $z_1 < z_2 < \dots < z_5$. As before, we infer $p_1, p_5 \in \{p_2, p_3, p_4\}^{co}$. If p_2, p_3 and p_4 lie on a line l then

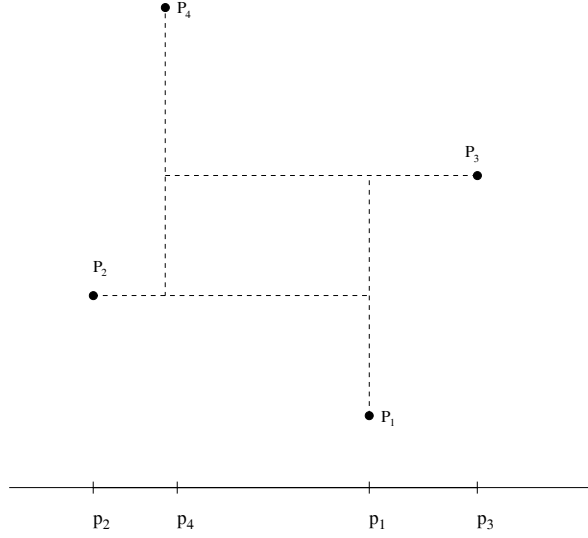


Figure 5: Finding the T_4 -configuration in the plane spanned by $[p_2, p_3]$ and the z -axis

K is contained in $l \times \mathbb{R}$ and hence K must contain a T_4 -configuration by Proposition 19 (ii) for separate convexity in \mathbb{R}^2 (alternatively this can be checked directly arguing as above and distinguishing a few cases).

Using separately convex functions of the type $(p, z) \rightarrow (\langle p, a \rangle - b)_\pm (z - c)_\pm$ that are chosen to vanish on K , we see that $(p, z) \notin K^{rc}$ if

- $z < z_1$ or $z > z_5$, or
- $(z \in [z_1, z_2]$ but $p \neq p_1$) or $(z \in (z_4, z_5]$ but $p \neq p_5$) or ,
- $(z \in [z_2, z_3]$ but $p \notin [p_1, p_2])$ or $(z \in (z_3, z_4]$ but $p \notin [p_4, p_5])$.

} (55)

Now we will use the localization formula for the rank-one convex hull (see Proposition 14(i))

$$K^{rc} \cap U = [(K \cap U) \cup (K^{rc} \cap \partial U)]^{rc} \cap U, \quad (56)$$

which is valid for any compact K and any bounded U . Let B_R be an open ball which contains K^{co} and let

$$U = \{(p, z) : z < z_3, \text{dist}(p, [p_1, p_2]) < z_3 - z\} \cap B_R.$$

Then

$$\partial U \subset ([p_1, p_2] \times \{z_3\}) \cup \{(p, z) : z < z_3, \text{dist}(p, [p_1, p_2]) = z - z_3\} \cup \partial B_R.$$

Thus (56) and (55) yield

$$\begin{aligned} K^{rc} \cap \{(p, z) : z < z_3\} &= K^{rc} \cap U \\ &\subset U \cap (\{P_1, P_2\} \cup ([p_1, p_2] \times \{z_3\} \cap K^{rc}))^{rc}. \end{aligned} \quad (57)$$

Analogously, above the z_3 -level we see

$$K^{rc} \cap \{(p, z) : z > z_3\} \subset (\{P_4, P_5\} \cup ([p_4, p_5] \times \{z_3\} \cap K^{rc}))^{rc}. \quad (58)$$

We claim that

$$C := K^{rc} \cap \{(p, z) : z = z_3\} = \{P_3\}.$$

Once this is shown we are done. Indeed (57) and (58) then imply that $K^{rc} \setminus \{(p, z) : z = z_3\} = \{P_1, P_2, P_4, P_5\}$ since the rank-one convex hull of three points without rank-one connections is trivial.

The compact convex set C is the convex hull of its extreme points. Suppose that $C \neq \{P_3\}$ and let $P_0 = (p_0, z_3) \in C \setminus \{P_3\}$ be an extreme point. We claim that $p_0 \in [p_1, p_2] \cap [p_4, p_5]$. To see this suppose first $p_0 \notin [p_1, p_2]$. Choosing a ball $B_\varepsilon(P_0)$ around P_0 which is so small that $\bar{B}_\varepsilon(P_0) \cap (K \cup ([p_1, p_2] \times \mathbb{R})) = \emptyset$ we see from the localization formula that

$$P_0 \in (K^{rc} \cap \partial B_\varepsilon(P_0))^{rc} \subset [(C \cap \partial B_\varepsilon(P_0)) \cup (K^{rc} \cap \{(p, z) : z > z_3\})]^{co}.$$

Since in the computation of the convex hull points above $\{z = z_3\}$ can not be compensated for by points below $\{z = z_3\}$ we get $P_0 \in (C \cap \partial B_\varepsilon(p_0))^{co}$. This contradicts the extremality of P_0 in C . Similarly one shows $p_0 \in [p_4, p_5]$. For future reference we note that $p_0 \neq p_2$. Otherwise $p_2 \in [p_4, p_5]$ and thus $p_5 = p_2$, since p_5 is in the convex hull of p_2, p_3 and p_4 and these three points do not lie on a line. This contradicts the assumption that P_2 and P_5 are not rank-one connected. Similarly $p_0 \neq p_4$.

We next claim that the segments $[p_1, p_2]$ and $[p_4, p_5]$ intersect transversely in a single point, which we denote again by p_0 . Indeed, otherwise $p_1, p_5 \in [p_4, p_2]$ (since $p_1, p_5 \in \{p_2, p_3, p_4\}^{co}$) and thus p_4, p_1, p_5, p_2 form a T_4 -configuration in the plane generated by the line through p_4 and p_2 and the z -axis. Thus the intersection must be transversal, p_0 and P_0 are uniquely determined and $C \subset [P_0, P_3]$.

Moreover, the segment $[p_0, p_3]$ intersects both $[p_1, p_2]$ and $[p_4, p_5]$ in $\{p_0\}$ only, since else we would find a T_4 -configuration over one of the edges $[p_2, p_3]$

or $[p_4, p_3]$ (note that $p_0 \in [p_2, p_3]$ implies $p_5 = p_0$ since $p_4 \notin [p_2, p_3]$). Because $C \subset [P_0, P_3]$ the inclusion (57) yields

$$K^{rc} \cap \{(p, z) : z < z_3\} \subset \{P_1, P_2, P_0\}^{rc} = \{P_1, P_2, P_0\},$$

except when $p_0 = p_1$. Suppose first $p_0 \neq p_1$. Then $K^{rc} \cap \{(p, z) : z \in (z_2, z_3)\} = \emptyset$. As before this contradicts the fact P_0 is not in K (i.e. different from P_3) but extreme in C . If $p_0 = p_1$, we have $p_0 \neq p_5$ and we can use (58) to find

$$K^{rc} \cap \{(p, z) : z > z_3\} \subset \{P_4, P_5, P_0\}^{rc} = \{P_4, P_5, P_0\}$$

and to get the same contradiction. In conclusion we must have $C := K^{rc} \cap \{(p, z) : z = z_3\} = \{P_3\}$ and the proof is finished. \square

Proof of Proposition 22. For the construction of the set K see Fig. 6. Now suppose that the set K contained a T_k configuration (M_1, \dots, M_k) . Then $M_{j+1} - M_j = (1 + \kappa_{j+1})C_{j+1} - \kappa_j C_j$, where the C_j are rank-one matrices. Since there are no rank-one connections in K , the C_j must alternate between vertical and horizontal vectors. Thus if C_j is horizontal, both C_{j-1} and C_{j+1} are vertical and thus the projections of M_{j-1} , M_j and M_{j+1} to the base plane lie on a line. Fig. 6 shows that no three points in K have this property. \square

Proof of Proposition 23. If the conclusion fails then $T > 0$ and, perhaps after cutting away unused parts of the curve, we can certainly find a laminate μ which is not concentrated on a proper closed subinterval of $[0, T]$. In other words

$$(\gamma(0), 0), (\gamma(T), T) \in \text{supp}(\mu) \subset K. \quad (59)$$

The separation argument which is a central theme of this section now tells that for any $t \in (0, T)$ the sets $\gamma([0, t])$ and $\gamma((t, T])$ can not strictly lie on two different sides of some line. Indeed, else there is an $a \in \mathbb{R}^2$ such that

$$\langle \gamma(r), a \rangle < \langle \gamma(t), a \rangle < \langle \gamma(s), a \rangle \text{ for } r < t < s. \quad (60)$$

Then $\{(\gamma(s), s) : s \in [0, T]\} \subset Q_{++} \cup Q_{--} \cup Q_{00}$, where

$$Q_{\pm\pm} = \{\pm \langle \gamma(s) - \gamma(t), a \rangle > 0, \pm(s - t) > 0\}.$$

Thus Lemma 20 implies that $\text{supp} \mu$ is contained in $\mathbb{R}^2 \times [t, T]$, or in $\mathbb{R}^2 \times [0, t]$ or in $\mathbb{R}^2 \times \{t\}$. This contradicts (59).

This shows that (60) can not occur, and we will finish our proof by verifying that the assumption on γ anyhow enforces (60) to hold for some

This figure presents the six point configuration with a nontrivial separately convex hull in $\mathbb{R}^2 \oplus \mathbb{R}$. The set K consists of $\{1, 2, 3, 4, 5, 6\}$ - each point at the corresponding height.

The picture below shows the projection into the base plane - at each point it is indicated for which height about this given point we are in the separately convex hull of the set.

The six smaller pictures on the right present the intersections of the hull with a plane of given height $1, \dots, 6$. Here \bullet denotes a point from the original set, and \times and \circ denote points in the hull, but not in K - the \times -points are extreme on their vertical line, while the \circ -points are not.

To prove that a separately convex function which vanishes on K can not be positive in any of the \times or \circ -points it is enough to check that each \times -point is not convex extreme in its corresponding plane. A special feature is the occurrence of the auxiliary point P - this makes it difficult to find simple grid-based algorithms to compute the hull.

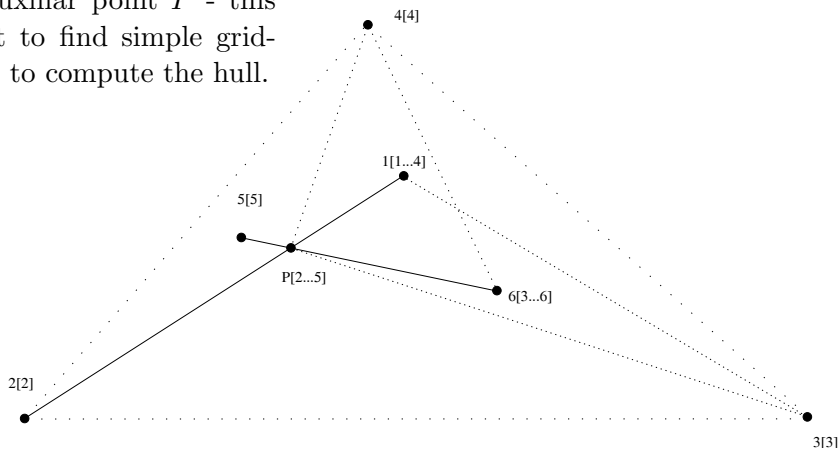
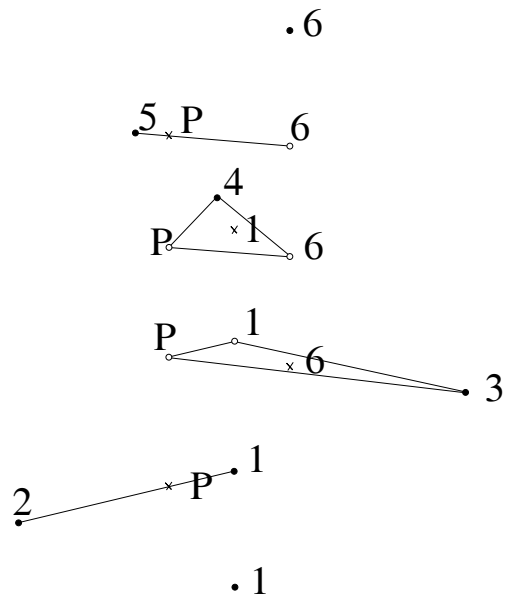


Figure 6: The new C_6 -configuration

t . The assumptions of Proposition 23 and (60) involve γ rather than K and remain unchanged if we (monotonously) reparametrize γ and apply some affine map of the plane to it. Therefore, due to the hypothesis $\gamma(0) \notin (\gamma((0, T]))^{\text{co}}$ we can suppose in the sequel that

- $\gamma' : [0, T] \rightarrow \mathbb{S}^1$ is continuous,
- $\gamma(0) = (0, 0)$
- $\gamma_2(t) \geq 0$ for all t and hence $\gamma'_2(0) \geq 0$.

First, we assume $\gamma_2(t) > 0$ for all $t \in (0, T]$. Then a simple compactness argument gives that for $\delta > 0$ but sufficiently small

$$\langle \gamma(t), (\delta\gamma'_1(0), 1) \rangle > 0 \text{ if } t \in (0, T].$$

Because also

$$\langle \gamma'(t), (\delta\gamma'_1(0), 1) \rangle > 0 \text{ for all } t \text{ positive but small enough,}$$

the mean value theorem says that for k sufficiently large

$$\langle \gamma(t), (\delta\gamma'_1(0), 1) \rangle < \langle \gamma(\frac{1}{k}), (\delta\gamma'_1(0), 1) \rangle < \langle \gamma(s), (\delta\gamma'_1(0), 1) \rangle \text{ if } t < \frac{1}{k} < s.$$

So we arrived at the impossible relation (60) and are done with the first case.

Thus, we can in addition, perhaps after a reparametrisation and a reflection at the x_2 axis, suppose that

$$\gamma(T_0) = (-1, 0) \text{ for some } T_0 \in (0, T].$$

Hence

$$\gamma(t) \notin (0, \infty) \times \{0\} \text{ for } t > 0$$

since otherwise $\gamma(0) \in [\gamma(t), \gamma(T_0)]$. If $\gamma'(0) \neq (1, 0)$ then as before we find $\delta > 0$ such that

$$\langle \gamma'(0), (-\delta, 1) \rangle > 0 \text{ and } \langle \gamma(t), (-\delta, 1) \rangle > 0 \text{ for all } t > 0.$$

Again we conclude that (60) holds for $a = (-\delta, 1)$ and $t > 0$ but sufficiently small. So we are left with the most difficult case when

- $\gamma'(0) = (1, 0)$.

Now we can moreover require that T_0 was chosen such that $\gamma(T_0)$ is the point on $\gamma((0, T]) \cap (\mathbb{R} \times \{0\})$ which is closest to 0. Then

- $\gamma(t) \notin (-1, \infty) \times \{0\}$ if $t \in (0, T]$.

We use again compactness arguments to choose a few more constants. First fix

- $\delta_0 > 0$ with $\gamma'_1 > 1/2$ on $[0, 2\delta_0]$, then
- $\eta > 0$ such that for any $t \in [\delta_0, T]$ we have $\gamma_2(t) > 2\eta$ or $\gamma_1(t) \leq -9/10$.
Finally, pick
- $\delta_1 \in (0, \delta_0)$ with $10(\text{diam}(K) + 10)\gamma_2(t) < \eta$ for all $t \in [0, \delta_1]$.

Because $\frac{d}{dt} \log(\gamma_2)$ has to be unbounded from above in any neighbourhood of the root of γ_2 at 0, we find $t_0 \in (0, \delta_1)$ with

$$\frac{\gamma'_2(t_0)}{\gamma'_1(t_0)} > \frac{\gamma_2(t_0)}{\gamma_1(t_0) + \frac{1}{2}}. \quad (61)$$

Finally, we select t_1 to maximize

$$c_t = \frac{\gamma_2(t) - \gamma_2(t_0)}{\gamma_1(t) - \gamma_1(t_0)} \text{ under the constraint } \gamma_1(t) \leq -\frac{9}{10}.$$

The key observation is now that there is an $\varepsilon > 0$ such that

- (i) $c_t > c_{t_1} + 2\varepsilon$ if $t \in [0, t_0)$ or $\gamma_1(t) > \gamma_1(t_0)$, and
- (ii) $\gamma_2(t) \geq \gamma_2(t_0) + c_{t_1}(\gamma_1(t) - \gamma_1(t_0))$ if $t \in (t_0, T]$ and $\gamma_1(t) \leq \gamma_1(t_0)$.

It is easy to check that these two conditions give for $t = t_0$ and $a = (-c_{t_1} - \varepsilon, 1)$ the forbidden separation (60).

Now for $\gamma_1(t) \leq -9/10$ assertion (ii) is just a reformulation of the maximality property of c_{t_1} . If $\gamma_1(t) \in (-9/10, \gamma_1(t_0)]$ then $t > t_0$ implies $t > \delta_0$. Thus $\gamma_2(t) > \eta > \gamma_2(t_0)$. Since $c_{t_1} \geq c_{T_0} > 0$ this proves assertion (ii).

It remains to verify (i). First we note that $c_t - c_{t_1}$ is larger then some positive constant for t near t_0 – this is just a consequence of (61), which gives

$$\frac{\gamma'_2(t_0)}{\gamma'_1(t_0)} > \frac{\gamma_2(t_0)}{\gamma_1(t_0) + \frac{1}{2}} > \frac{\gamma_2(t_0) - \gamma_2(t_1)}{\gamma_1(t_0) - \gamma_1(t_1)} = c_{t_1}.$$

If $c_t = c_{t_1}$ and $t \in [0, t_0)$ then $\gamma(t) \in \{\gamma(t_1), \gamma(t_0)\}^{co}$. Combining this with the above estimate for t near t_1 we conclude that $\inf_{t \in [0, t_0)} c_t - c_{t_1} > 0$.

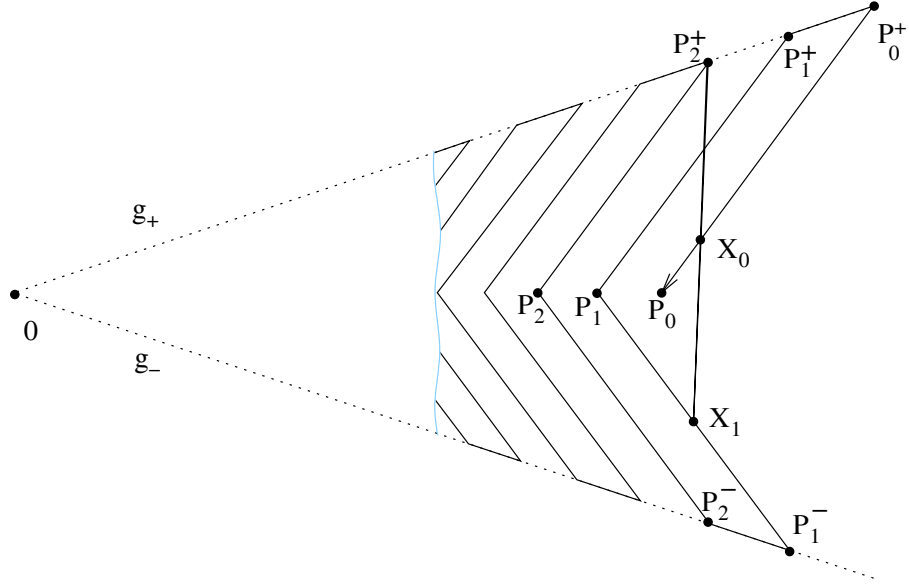


Figure 7: A continuous curve with small future but large hull

Similarly $\inf_{t \in (t_0, \delta_0]} c_t - c_{t_1} > 0$ follows from $\gamma(t_0) \notin \{\gamma(t_1), \gamma(t)\}^{co}$ for $t > t_0$. The last situation to deal with is $\gamma_1(t) > \gamma_1(t_0)$ but $t > \delta_0$. Then the bound is a consequence of

$$\gamma_2(t_0) + c_{t_1}(\gamma_1(t) - \gamma_1(t_0)) \leq \gamma_2(t_0) + \frac{\gamma_2(t_0)}{(\frac{1}{2})} \text{diam}(\mathbf{K}) < \eta < \gamma_2(t) - \eta.$$

□

Example 24

We construct an example which shows that the conclusion of Proposition 23 need not to hold for Lipschitz curves. We fix in the plane the two rays

$$g_+ = \{(x, \frac{1}{2}x) : x \geq 0\} \text{ and } g_- = \{(x, -\frac{1}{2}x) : x \geq 0\},$$

and the point $P_0 = (1, 0)$. For $\lambda \in (0, 1)$ we define $P_1 = \lambda P_0$ and the two segments $s_i^+ = [P_i P_i^+]$, where $P_i^+ \in g_+$ satisfies $(P_i^+ - P_i) \parallel (1, 2)$ for $i = 0, 1$. Together with the connecting segment $c_0 = [P_1^+ P_0^+]$ they form

$M_0 = s_0^+ \cup c_0 \cup s_1^+$. We use the linear map $A_\lambda = \text{diag}(\lambda, -\lambda)$ to build the selfsimilar set

$$M = \bigcup_{i=0}^{\infty} M_i \cup \{0\} \text{ where } M_{i+1} = A_\lambda(M_i),$$

consisting of the segments $s_{i+1}^- = A_\lambda(s_i^+)$, $s_{i+1}^+ = A_\lambda(s_i^-)$ and $c_{i+1} = A_\lambda(c_i)$ and containing the points $P_{i+1} = A_\lambda(P_i)$, $P_{i+1}^- = A_\lambda(P_i^+)$ and $P_{i+1}^+ = A_\lambda(P_i^-)$, see Figure 7.

Let $X_1 = (P_1 + P_1^-)/2$ and denote by $X_0 = X_0(\lambda)$ the intersection of the line through X_1 and $P_2^+(\lambda)$ with the line containing s_0^+ . It is clear that for $\lambda \rightarrow 1_-$ the point $X_0(\lambda)$ tends to P_0^+ , but if λ gets smaller, it will leave s_0^+ through its endpoint P_0 . Hence, we can find $\lambda_0 \in (0, 1)$ such that $X_0(\lambda_0) = (P_0 + P_0^+)/2 = A_\lambda^{-1}(X_1)$.

Now we can consider any injective continuous curve $\gamma : [0, T] \rightarrow \mathbb{R}^2$ with $\gamma([0, T]) = M = M(\lambda_0)$ and $\gamma(0) = 0$. Because $s_{i+1}^+ \cup c_{i+1} \cup s_{i+1}^-$ intersects the convex hull of $(s_i^+ \cup c_i \cup s_i^-)$ only in an extreme point and since

$$\gamma \text{ first runs through } s_{i+1}^+ \cup c_{i+1} \cup s_{i+1}^- \text{ and then through } s_i^+ \cup c_i \cup s_i^-, \quad (62)$$

it is easy to verify that $\gamma(t)$ never belongs to the convex hull of $\gamma([t, T])$. On the other hand, $K = \{(\gamma(t), t) : t \in [0, T]\}$ has a nontrivial rank-one convex hull. To verify this note that $X_0 \in [P_0^+, X_1]$ and thus

$$(X_0, \gamma^{-1}(P_2^+)) \in [(P_2^+, \gamma^{-1}(P_2^+)), (X_1, \gamma^{-1}(P_2^+))].$$

Now

$$(X_1, \gamma^{-1}(P_2^+)) \in [(X_1, \gamma^{-1}(P_3^-)), (X_1, \gamma^{-1}(X_1))]$$

which implies that

$$(X_0, \gamma^{-1}(P_2^+)) \in (K \cup \{(X_1, \gamma^{-1}(P_3^-))\})^{rc}.$$

Using the (A_λ) -selfsimilarity of M to iterate this we have

$$(X_0, \gamma^{-1}(P_2^+)) \in (K \cup \{(X_i, \gamma^{-1}(P_{i+2}^{(-1)^i}))\})^{rc} \text{ for all } i.$$

Since $\lim_{i \rightarrow \infty} (X_i, \gamma^{-1}(P_{i+2}^{(-1)^i})) = (0, 0) \in K$ and any separately convex function is continuous, we get as required

$$(X_0, \gamma^{-1}(P_2^+)) \in K^{rc} \setminus K.$$

□

6.3 Separate convexity in $\mathbb{R}^2 \oplus \mathbb{R}^2$

We consider the integrand

$$\phi(X) = \frac{1}{2}|X|^2 + h(\det X), \quad X \in \mathbb{R}^{2 \times 2}, \quad (63)$$

the corresponding Euler-Lagrange equation $\operatorname{div} D\phi(\nabla w) = 0$ and its reformulation as a first order partial differential relation

$$\nabla u \in K_\phi = \left\{ \begin{pmatrix} X \\ D\phi(X)J \end{pmatrix} : X \in \mathbb{R}^{2 \times 2} \right\}. \quad (64)$$

Theorem 25 *Suppose $h : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and strictly convex. Suppose further that E is a finite subset of K and that the elements of E are of the form $\begin{pmatrix} X \\ Y \end{pmatrix}$, where X is a diagonal 2×2 matrix. Then $K^{rc} = K$ and the laminates $\mathcal{M}^{rc}(K)$ supported on K are Dirac masses.*

Proof. Geometrically the heart of the matter is again a good foliation as in Section 5. After some transformation we will see that E lies in a certain two-dimensional set Γ in some $\mathbb{R}^2 \oplus \mathbb{R}^2$, where the \mathbb{R}^2 's correspond to the rank-one directions. The point is that Γ can be foliated by a one-parameter family of curves whose projection to either \mathbb{R}^2 is a foliation (of one quadrant) by strictly convex curves (see Step 4 below; the different curves are distinguished by $d = st$). Then the separation result, Lemma 20, in connection with the finiteness of E can be used once more to finish the argument. First we need to make some normalizations and to show that E lies in one quadrant.

Step 1: Reduction to $h'(0) = 0$. Adding a linear term to h does not affect the Euler-Lagrange equation since \det is a null Lagrangian, i.e. for all function w we have $\operatorname{div}(D \det) \nabla w = \operatorname{div} \operatorname{cof} \nabla w = 0$. On the level of the partial differential relation we have

$$\begin{aligned} K_{\phi + \lambda \det} &= \left\{ \begin{pmatrix} X \\ D\phi(X)J + \lambda \operatorname{cof} X J \end{pmatrix} : X \in \mathbb{R}^{2 \times 2} \right\} \\ &= \left\{ \begin{pmatrix} X \\ Y + \lambda JX \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix} \in K_\phi \right\}. \end{aligned}$$

Since $\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y + \lambda JX \end{pmatrix}$ is a linear isomorphism which preserves rank-one lines we may suppose $h'(0) = 0$.

Step 2: Special separately convex functions. We now write out the elements $\begin{pmatrix} X \\ Y \end{pmatrix}$ of K_ϕ for which X is a diagonal matrix more explicitly

$$\begin{pmatrix} X \\ D\phi(X)J \end{pmatrix} = \begin{pmatrix} X \\ X + h'(\det X) \operatorname{cof} X J \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & t \\ 0 & -s - h'(st)t \\ t + h'(st)s & 0 \end{pmatrix}.$$

After exchanging rows and multiplying one row by -1 (both of which correspond to making a linear change of the dependent variable u in (64)) we may suppose that

$$E \subset \left\{ \begin{pmatrix} s & 0 \\ \sigma & 0 \\ 0 & t \\ 0 & \tau \end{pmatrix} : s, t \in \mathbb{R}, \sigma = t + h'(st)s, \tau = s + h'(st)t \right\}. \quad (65)$$

On the linear subspace

$$L = \{A \in \mathbb{R}^{4 \times 2} : A_{12} = A_{22} = A_{31} = A_{41} = 0\},$$

which contains E , rank-one convexity agrees with separate convexity in the variables (A_{11}, A_{21}) and (A_{32}, A_{42}) .

In particular, for each $a, b \in \mathbb{R}^2$ we have the separately convex functions

$$(A_{11}, A_{21}, A_{32}, A_{42}) \mapsto \langle (A_{11}, A_{21}), a \rangle_\pm \langle (A_{32}, A_{42}), b \rangle_\pm$$

at our disposal.

Step 3: Restriction to generalized quadrants. We argue by contradiction. If the claim fails then there is a minimal set $E_0 \subset E$ which supports a nontrivial laminate μ_0 . Our goal is to contradict minimality by a separation argument similar to [Sv 92b, Ta 93]. We denote by $P_0 = (s_0, \sigma_0, t_0, \tau_0)$ the centre of mass of μ_0 .

Recall that strict convexity and the normalization condition $h'(0) = 0$ imply that $h'(x)x > 0$ for $x \neq 0$. Thus all $(s, \sigma, t, \tau) \in E$ satisfy

$$|\tau| \geq |s|, \tau s \geq 0 \quad \text{and} \quad s = 0 \Leftrightarrow \tau = 0. \quad (66)$$

Thus we can apply Lemma 20 to the generalized quadrants

$$Q_{\pm\pm} = \{(s, \sigma, t, \tau) : \pm s > 0, \pm\tau > 0\}.$$

Since E_0 is finite this shows that either $s_0 > 0, \tau_0 > 0$ and $s > 0, \tau > 0$ in E_0 , or $s_0 < 0, \tau_0 < 0$ and $s < 0, \tau < 0$ on E_0 , or $s_0 = \tau_0 = 0$ and

$s = \tau = 0$ on E_0 . In the last case we have $\sigma = t$ on E_0 and therefore the quadratic minor $\sigma t \hat{=} A_{21}A_{32} - A_{22}A_{31}$ has a (strictly) positive value at all non vanishing differences of elements in E_0 . This implies that even all polyconvex measures supported on E_0 must be Dirac masses [Sv 93] and thus yields a contradiction. Hence we have shown that there exists a $c_s \in \{-1, 1\}$ such that

$$c_s s_0, c_s \tau_0 > 0, \quad c_s s, c_s \tau > 0 \text{ on } E_0.$$

Since the problem is invariant under the exchange of variables $(s, \tau) \leftrightarrow (t, \sigma)$ we also find $c_t \in \{-1, 1\}$ such that

$$c_t t_0, c_t \sigma_0 > 0 \quad c_t t, c_t \sigma > 0 \text{ on } E_0.$$

Multiplication of all variables by -1 leaves the right hand side of (65) invariant. Hence we may assume $c_s = 1$. If we multiply only t and σ by -1 and replace $h(x)$ by its reflection $h(-x)$ then (65) remains invariant. Hence we may suppose $c_s = c_t = 1$, i.e.

$$s_0, \sigma_0, t_0, \tau_0 > 0, \quad s, \sigma, t, \tau > 0 \text{ on } E_0. \quad (67)$$

Step 4: Separation. Let $d = \min\{st : (s, \sigma, t, \tau) \in E_0\}$. Since E_0 is finite we have $d > 0$. Pick any point $P_1 \in E_0$ with $s_1 t_1 = d$. Projecting E_0 into the (s, σ) and (t, τ) planes we see that the set

$$E_0^s := \{(s, \sigma) : (s, \sigma, t, \tau) \in E_0\}$$

is contained in the (closed) epigraph of the strictly convex function

$$s \mapsto \frac{d}{s} + h'(d)s,$$

because for fixed $s > 0$ the expression on the right hand side is increasing in $d > 0$. Since (s_1, σ_1) lies on the graph of this strictly convex function it is an extreme point of E_0^s . Thus there exists $a_s \in \mathbb{R}^2$ with

$$\langle (s_1, \sigma_1), a_s \rangle < \langle (s, \sigma), a_s \rangle \quad \text{for } (s, \sigma, t, \tau) \in E_0 \setminus \{P_1\}. \quad (68)$$

Here we used that fact that the projection from E_0 to E_0^s is injective since K_ϕ contains no rank-one connections (this in turn follows from the strict convexity of h). In view of the invariance under the exchange of variables $(s, \sigma) \leftrightarrow (t, \tau)$ the same reasoning yields $a_t \in \mathbb{R}^2$ with

$$\langle (t_1, \tau_1), a_t \rangle < \langle (t, \tau), a_t \rangle \quad \text{for } (s, \sigma, t, \tau) \in E_0 \setminus \{P_1\}. \quad (69)$$

Invoking once more Lemma 20 we deduce that μ_0 is either supported on $\{P_1\}$ or on $E_0 \setminus \{P_1\}$. This contradicts the minimality of μ_0 and the proof is finished. \square

6.4 A laminate without discrete approximations

One might hope to extend Theorem 25 to more general sets through an approximation by finite sets. The following example shows, however, that the rank-one convex hull can shrink drastically, when we pass from continua to discrete sets. It was motivated by J.M. Ball's construction of sets of gradients with no rank-one connections ([Ba 90]) and is based on separate convexity in $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, which arises by restricting rank-one convexity in $\mathbb{R}^{3 \times 3}$ to diagonal matrices.

Proposition 26 *We consider separate convexity in $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and define for $t \in [0, 1]$*

$$p(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix}, \quad p_i(t) = p(t) + te_i \text{ for } i = 1, 2, 3.$$

Let $S_i = p_i([0, 1])$. Then $p(t) \in (S_1 \cup S_2 \cup S_3)^{rc}$ for $t \in [0, 1]$ and there is only one laminate supported in $S_1 \cup S_2 \cup S_3$ which generates this point. Moreover, this laminate does not charge points. More precisely,

$$\{\mu \in \mathcal{M}^{rc}(S_1 \cup S_2 \cup S_3) : \bar{\mu} = p(t)\} = \{\mu_t\},$$

where $\mu_0 = \delta_0$ and where for $t > 0$

$$\mu_t(A) = \frac{1}{t^3} \int_0^t \sum_{i=1}^3 \chi_A(p_i(\tau)) \tau^2 d\tau \text{ if } A \subset \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}.$$

In particular, if K is a compact subset of $(S_1 \cup S_2 \cup S_3)$ but $p_i(t_0) \notin K$, then $p(t) \notin K^{rc}$ for $t \in (t_0, 1]$.

Proof of Proposition 26. In the sequel we identify $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ with \mathbb{R}^3 and suppose the measures μ_t are defined as above. First we show that for all $t_0 \in [0, 1]$ and all $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ separately convex the Jensen-type inequality

$$f(p(t_0)) \leq \int f d\mu_{t_0} \tag{70}$$

holds. Choosing for f constant and linear functions we can then conclude that μ_{t_0} is a probability measure, $\bar{\mu}_{t_0} = p(t_0)$ and $\mu_{t_0} \in \mathcal{M}^{rc}(S_1 \cup S_2 \cup S_3)$.

To establish (70) we notice that for any $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ separately convex, for $x \in \mathbb{R}^3$ and $t_1, t_2, t_3 > 0$ we have

$$\limsup_{t \rightarrow 0_+} \frac{f(x + p(t)) - f(x)}{t} \leq \sum_{i=1}^3 s_i, \text{ where } s_i = \frac{f(x + t_i e_i) - f(x)}{t_i}. \tag{71}$$

Indeed, after addition of an affine function and translation we may suppose that $x = 0 = (s_1, s_2, s_3)$. Suppose now that the upper limit of these difference quotients was positive. Since separately convex functions are locally Lipschitz, a suitable subsequence of the rescaled functions $f_\varepsilon(y) = \varepsilon^{-1}(f(\varepsilon y) - f(0))$ converges (uniformly on compact subsets) to a limit f_0 with $f_0(p(1)) > 0$. Moreover f_0 is globally Lipschitz, separately convex and satisfies $f_0(te_i) \leq 0 = f_0(0)$ for $t > 0$ and $i = 1, 2, 3$. This implies that for any $y \in \mathbb{R}^3$ and i the function $t \rightarrow f_0(y + te_i)$ is nonincreasing. Otherwise $t \mapsto f_0(y + te_i)$ is bounded from below by a function with positive slope and $f_0(y + te_i) - f_0(te_i)$ tends to infinity as $t \rightarrow +\infty$, in contradiction with the Lipschitz property of f_0 . Using that $t \mapsto f_0(y + te_i)$ is nonincreasing we obtain

$$0 < f_0(p(1)) \leq f_0(e_1 + e_2) \leq f_0(e_1) \leq f_0(0) = 0,$$

a contradiction proving (71).

Since $t \mapsto f(p(t))$ is Lipschitz it is differentiable for almost every $t \in [0, 1]$ and (71) implies that

$$\frac{d}{dt}f(p(t)) \leq \frac{1}{t}(f(p_1(t)) + f(p_2(t)) + f(p_3(t)) - 3f(p(t))).$$

This is equivalent to

$$3f(p(t)) + t \frac{d}{dt}f(p(t)) \leq f(p_1(t)) + f(p_2(t)) + f(p_3(t)),$$

and therefore

$$t_0^3 f(p(t_0)) \leq \int_0^{t_0} \sum_{i=1}^3 f(p_i(t)) t^2 dt = \int f d(t_0^3 \mu_{t_0})$$

follows and ensures (70).

The uniqueness of μ_{t_0} requires a bit more effort, so assume there is an other $\mu' \in \mathcal{M}^{rc}(S_1 \cup S_2 \cup S_3)$ with $\bar{\mu}' = p(t_0)$. Since these two measures differ, we find some

$$f : S_1 \cup S_2 \cup S_3 \rightarrow \mathbb{R} \text{ such that } \int f d\mu_{t_0} > \int f d\mu'$$

and, after adding a constant and making a suitable approximation if necessary, we also may assume that for $i = 1, 2, 3$

$$t \rightarrow f(p_i(t)) \text{ is } C^\infty(\mathbb{R}) \text{ with support in } (\eta_0, 2] \text{ for some } \eta_0 > 0.$$

Now, for $\varepsilon > 0$ let $g_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be the solution of

$$g'_\varepsilon(t) = \begin{cases} \frac{1}{t}(f(p_1(t)) + f(p_2(t)) + f(p_3(t)) - 3g_\varepsilon(t)) - \varepsilon & \text{if } t > \eta_0, \\ -\frac{\varepsilon}{4}t & \text{if } t \leq \eta_0, \end{cases}$$

with $g_\varepsilon(0) = 0$. It is easily checked that g_ε is also C^∞ . Since $\lim_{\varepsilon \rightarrow 0^+} g_\varepsilon(t) = \int f d\mu_t$ for $t \in [0, 1]$ we can choose $\varepsilon = \varepsilon_0 > 0$ such that

$$g = g_{\varepsilon_0} \text{ satisfies } \int f d\mu' < g(t_0).$$

For convenience we also introduce the smooth functions

$$s_i(t) = \begin{cases} \frac{1}{t}(f(p_i(t)) - g(t)) & \text{if } t > \eta_0, \\ \frac{\varepsilon}{4} & \text{if } t \leq \eta_0 \end{cases}, i = 1, 2, 3,$$

and fix a finite constant c_0 such that

$$10 \sum_{i=1}^3 |s'_i(t)| + |s''_i(t)| < c_0 \text{ if } t \in [-4, 4].$$

Finally, we define the auxiliary function

$$f_1(x) = \prod_{j \in \{1,2,3\} \setminus \{i\}} (x_j - x_i) \text{ if } x \in \mathbb{R}^3 \text{ and } x_i = \min_j x_j,$$

and the function

$$F(x) = g(t) + \langle x - p(t), \begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{pmatrix} \rangle + c_0 f_1(x) \text{ for } x \in \mathbb{R}^3 \text{ and } t = \min_j x_j.$$

Suppose we had already shown that F is separately convex in the open cube $(-4, 4)^3$ of \mathbb{R}^3 . Then Proposition 12 combined with Proposition 13 imply that F satisfies the Jensen inequality for all laminates supported in $(-2, 2)^3$. Because $F(p_i(t)) = g(t) + ts_i(t) = f(p_i(t))$ and $F(p(t)) = g(t)$ for $t \in [0, 1]$, we obtain the contradiction

$$\int F d\mu' = \int f d\mu' < g(t_0) = F(p(t_0)) = F(\bar{\mu}')$$

finishing our proof.

Therefore, it remains to study separate convexity of F near an arbitrary point $x^0 \in (-4, 4)^3$. Essentially only $x^0 \in (0, 4)^3$ is interesting but the general case is not more complicated. So, let $t^0 = \min_j x_j^0$ and we consider

$$h(r) = f(x^0 + r e_i) \text{ for } i \in \{1, 2, 3\} \text{ fixed and } r \text{ near } 0.$$

First, suppose $t^0 < x_i^0$. Hence $\min_j(x^0 + re_j) \equiv t^0$, and

$$h(r) = (x_i^0 + r - t^0)s_i(t^0) + c_0 f_1(x^0 + re_i) + \text{const}$$

is obviously convex in r .

Next, let $t^0 = x_i^0 < \min_{j \neq i} x_j^0$. Then $\min_j(x^0 + re_i) = t^0 + r$ and

$$h(r) = g(t^0 + r) + \sum_{j \neq i} (x_j^0 - t - r)s_j(t^0 + r) + c_0 \prod_{j \neq i} (x_j^0 - t^0 - r).$$

Since $g'(t) = s_1(t) + s_2(t) + s_3(t) - \varepsilon_0$, we have

$$h'(r) = s_i(t^0 + r) - \varepsilon_0 + \sum_{j \neq i} (x_j^0 - t^0 - r)s'_j(t^0 + r) + c_0(2r + 2t^0 - \sum_{j \neq i} x_j^0)$$

and

$$h''(0) = s'_i(t^0) + \sum_{j \neq i} -s'_j(t) + (x_j^0 - t^0)s''_j(t^0) + 2c_0 \geq 2c_0 - c_0 > 0.$$

So we are left with the last case when $t^0 = x_i^0 = x_j^0 \leq x_k^0$ where $\{i, j, k\} = \{1, 2, 3\}$. For $r < 0$ we compute h and h' as before and obtain

$$\lim_{r \rightarrow 0_-} h'(r) = s_i(t^0) - \varepsilon_0 + (x_k^0 - t^0)(s'_i(t^0 + r) - c_0).$$

For $r > 0$ we have

$$h(r) = g(t^0) + \langle (x^0 + re_i) - p(t^0), s(t^0) \rangle + r(x_k^0 - t^0)$$

and thus

$$\lim_{r \rightarrow 0_+} h'(r) = s_i(t^0) + (x_k^0 - t^0) \geq \lim_{r \rightarrow 0_-} h'(r) + \varepsilon_0$$

and again h has a (local) subdifferential at zero. This finishes the proof. \square

7 Local hulls, degenerate sets and hyperbolic conservation laws

In this section we formulate some ‘local’ problems for the various hulls K^* and the classes of measures \mathcal{M}^* associated with a given set $K \subset \mathbb{R}^{m \times n}$. As we sketch below, such problems are relevant for example in connection with compactness properties of L^∞ entropy solutions of $l \times 2$ systems of hyperbolic conservation laws.

Let us consider a smooth submanifold $K \subset \mathbb{R}^{m \times n}$. We introduce the following properties of K .

- (P1) Each point $A \in K$ has a neighbourhood $U \subset \mathbb{R}^{m \times n}$ such that $(K \cap \bar{U})^{rc} = K \cap \bar{U}$.
- (P2) Each point $A \in K$ has a neighbourhood $U \subset \mathbb{R}^{m \times n}$ such that $\mathcal{M}^{rc}(K \cap \bar{U})$ is trivial.
- (P3) Each point $A \in K$ has a neighbourhood $U \subset \mathbb{R}^{m \times n}$ such that $(K \cap \bar{U})^{pc} = K \cap \bar{U}$.
- (P4) Each point $A \in K$ has a neighbourhood $U \subset \mathbb{R}^{m \times n}$ such that $\mathcal{M}^{pc}(K \cap \bar{U})$ is trivial.

A sufficient condition for (P1) and (P2) to be satisfied is that the tangent spaces $T_A K$ do not contain any rank-one connections, see e.g. [Ta 83] (this condition in fact implies the stronger assertion obtained by replacing ‘rc’ with ‘qc’). The same condition is also sufficient for (P3) and (P4) when $n = 2$. This follows from the fact that a rank-one convex quadratic form on $\mathbb{R}^{m \times n}$ is polyconvex, see [Se 83].

If $T_A K$ does contain rank-one connections, the situation is more complicated. For simplicity, assume that for some $A_0 \in K$ the rank-one cone Λ and $T_{A_0} K$ intersect transversely (away from the origin). Then $T_A K \cap \Lambda$ behaves ‘well’ for A close to A_0 , and one might speculate that there is some natural ‘higher order’ condition which would imply (P1)–(P4), or at least some of these properties, in a neighbourhood of A_0 . This situation arises in connection with compactness properties of L^∞ entropy solutions of hyperbolic conservation laws, as was pointed out already in DiPerna’s work [DP 85], where some very interesting examples are considered and shown to have property (P4).

It seems that the problem of determining whether (P1)–(P4) are satisfied remains open even in some simple and very natural situations. Consider for example the following problem taken from the above paper by DiPerna. We look at the one dimensional wave equation

$$\varphi_{tt} - a(\varphi_x)_x = 0, \tag{72}$$

where a is strictly convex, increasing and satisfies $a(0) = 0$. There is a natural energy associated with (72):

$$\eta(\varphi_t, \varphi_x) = \frac{1}{2} \varphi_t^2 + F(\varphi_x), \quad \text{where } F(\xi) = \int_0^\xi a(s) ds.$$

One has $\eta_t - q_x = 0$ for each regular solution of (72), where $q = q(\varphi_t, \varphi_x) = \varphi_t a(\varphi_x)$. Letting $u = \varphi_t$, $v = \varphi_x$ we write (72) as a first order system

$$\begin{aligned} u_t - a(v)_x &= 0, \\ v_t - u_x &= 0. \end{aligned} \tag{73}$$

L^∞ entropy solutions of (73) can be defined as L^∞ functions (u, v) satisfying

$$\begin{aligned} u_t - a(v)_x &= 0, \\ v_t - u_x &= 0, \\ \eta_t - q_x &\leq 0 \end{aligned} \tag{74}$$

in the sense of distributions. One can also add further entropies to (74), to get a more restrictive class of solutions, see [DP 85]. In view of Murat's lemma [Mu 81] it is reasonable - at least in a first approximation - to replace the inequality in (74) by an equality when studying compactness properties. This enables us to introduce stream functions as follows.

$$\begin{aligned} (u, -a(v)) &= (\psi_x^1, -\psi_t^1), \\ (v, -u) &= (\psi_x^2, -\psi_t^2), \\ (\eta, -q) &= (\psi_x^3, -\psi_t^3). \end{aligned}$$

This can be rewritten as

$$\nabla\psi \in K \subset \mathbb{R}^{3 \times 2},$$

with

$$K = \left\{ \begin{pmatrix} u & a(v) \\ v & u \\ \eta(u, v) & q(u, v) \end{pmatrix} : u, v \in \mathbb{R} \right\}.$$

As far as we know it is an open problem to determine for which functions a the set $K \subset \mathbb{R}^{3 \times 2}$ given above satisfies any of the conditions (P1)–(P4). We refer the reader to the paper of DiPerna [DP 85] for a study of a situation with additional entropies, in which one can determine that (P4) holds.

8 Outlook

Here we formulate some further questions and briefly mention a few related results.

Question 1 *Is it possible to characterize (or at least give some non-trivial examples of) smooth uniformly rank-one convex functions $\phi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ for which $K_\phi^{pc} = K_\phi$?*

Perhaps we can expect that weak solutions of the Euler-Lagrange equation $\operatorname{div} D\phi(\nabla u) = 0$ for ϕ with $K_\phi^{pc} = K_\phi$ have some extra regularity properties.

Characterizing ϕ with $K_\phi^{qc} = K_\phi$ would be even more interesting but that looks out of reach.

Question 2 *Is it possible to identify some geometric properties of smooth “elliptic” sets $K \subset \mathbb{R}^{m \times 2}$ which imply a-priori estimates (for example C^α -estimates) for smooth solutions of $\nabla u \in K$?*

By a smooth elliptic set we mean a set K which is a smooth submanifold of $\mathbb{R}^{m \times 2}$ such that the tangent spaces $T_A K$ contain no rank-one connections.

While rank-one connections, T_4 -configurations or, in general, non-trivial measures in \mathcal{M}^{rc} are obstructions to regularity of Lipschitz solutions of $\nabla u \in K$, their rôle in connection with a-priori estimates for smooth solutions is less clear. Of course, they do become relevant even in this setting if they occur “infinitesimally”, but that is not possible for “elliptic” sets.

Question 3 *Is it possible to develop some general methods which would work for the local problems in Section 7?*

The construction of examples by convex integration rests on the subtle combination of one-dimensional constructions (and hence is closely related to rank-one convexity).

Question 4 *Can one extend convex integration using genuinely multi-dimensional constructions as building blocks?*

This would allow one to obtain interesting examples under the weaker (and more natural) condition that the quasicontinuous hull is sufficiently trivial. The main difficulty is that it is much harder to ‘glue’ multidimensional constructions. One interesting test case is an example by Šverák (see, e.g., [Mu 99], Section 4.7, equation (4.25)) of a set $K \subset \mathbb{R}^{6 \times 2}$ for which $\nabla u \in K$ admits periodic solutions. It is not known whether solutions with compact support exist. For possible extensions of convex integration see also [Gr 86], Section 2.4.12.

Question 5 *Is there an effective algorithm to decide whether a given probability measure supported on a finite subset of $\mathbb{R}^{2 \times 2}$ is a laminate?*

Such an algorithm would enable us to effectively test (at least in a probability sense) whether rank-one convexity implies quasi-convexity on 2×2 matrices. For example, one would generate (by some random or pseudo-random process) on a computer piecewise affine maps $u : T^2 \rightarrow \mathbb{R}^2$ and check whether the corresponding distribution μ of their gradients given by

$$\langle f, \mu \rangle = \frac{1}{|T^2|} \int_{T^2} f(\nabla u(x)) dx$$

are laminates. At present no effective algorithm for this is known. For the related question to determine numerically the rank-one convex hull of a function, see [Do 02] and the references therein.

Many simple geometric questions about rank-one convexity are open, even in low dimensions. In view of applications the case of 2×2 matrices is particularly interesting.

Question 6 *Does each T_k -configuration in $\mathbb{R}^{2 \times 2}$ contain some T_4 -configuration? More generally, does every compact set $K \subset \mathbb{R}^{2 \times 2}$ with nontrivial rank-one convex hull contain a T_4 -configuration?*

This is even open for symmetric 2×2 matrices. The answer to both questions is yes for diagonal 2×2 matrices (see Proposition 19). The answer to the second question is no for 3×2 matrices (see Proposition 22). Székelyhidi [Sz 02] constructs a T_5 -configuration in $\mathbb{R}^{4 \times 2}$ which does not contain a T_4 -configuration.

Many examples arise from nonlinear elasticity and thus have at least an $SO(n)$ symmetry.

Question 7 *How can we exploit symmetries efficiently?*

A nice example is the computation of the rank-one convex and polyconvex hull for an energy function which describes nematic elastomers [DD 02, Si 02]. The role of discrete symmetries which arise in crystalline materials is also important [Fr], but largely unexplored.

Question 8 *Is there an efficient algorithm to decide if $K^{rc} = K$? Can one at least efficiently check whether K contains a T_k configuration?*

A typical ingredient in efficient algorithms is the use of extreme points in connection with the Krein-Milman theorem. With a sufficiently abstract (dual) definition of extreme points the Krein-Milman theory holds in our setting [Al 71, Kr 00, Zh 98]. The question is whether one can obtain a more geometric characterization of extreme points. For a nearly optimal result for 2×2 matrices see [Ki 01b], Thm. 4.20.

Question 9 *Can one combine Theorems 1 and 3 ? In other words, is there an elliptic system (with a quasiconvex or polyconvex energy function) which admits a solution whose gradient takes only finitely many values ?*

This might be easier if one goes beyond 2×2 systems. In this case $v = \begin{pmatrix} X \\ Y \end{pmatrix}$, with $X, Y \in \mathbb{R}^{n \times n}$ and $A(Dv) = \begin{pmatrix} \operatorname{curl} X \\ \operatorname{div} Y \end{pmatrix}$.

Question 10 *Is the quasiconvex hull of the set K related to $\phi(X) = (\det X)^2$ trivial ?*

This is related to the compactness and regularity properties of solutions to the corresponding (degenerate) Euler-Lagrange equation

$$\operatorname{div}[(\det \nabla w) \operatorname{cof} \nabla w] = 0.$$

Since $\operatorname{div} \operatorname{cof} \nabla w = 0$ one can argue formally that all solutions must satisfy $\det \nabla w = \operatorname{const}$ and this argument is correct for C^1 solutions. If one applies the formal argument to the ‘linearized’ problem

$$\operatorname{div}[f(x) \operatorname{cof} \nabla w] = 0$$

one would similarly conclude that $f = \operatorname{const}$, at least if $\det \nabla w \neq 0$. There exist, however, a Lipschitz map $w : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\det \nabla w \in \{-1, 1\}$ a.e. and a non constant f with values in $\{-1, 1\}$ which satisfy the above equation.

In this paper we have mainly studied sets, but the study of rank-one convex functions is equally interesting. Many seemingly simple questions are open.

Question 11 *Can one characterize the traces of separately convex functions $f : \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R}$ on the diagonal ?*

Tartar showed that any smooth function can arise as a trace, see [Ta 93], Remark 11 there. Motivated by results due to Šverák and Preiss mentioned in [Ta 93], it was shown that C^1 is a necessary and $C^{1,\alpha}$ a sufficient condition for being a trace. The lack of a precise characterization seems related to our

partial understanding of rank-one convexification procedures. The situation is even less clear if we go into 3 or more dimensions.

Finally, throughout this work we have focused on oscillations effects only and restricted attention to bounded sets K . If one drops this assumptions one also needs to study the possible interaction of oscillation and concentration effects and new tools are required, see e.g. [Ta 90]. More on the technical side one can ask to which differential operators A one can extend the general theory (for questions of compactness, A gradient Young measures, and relaxation see [FM 99, BFL 00]).

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References

- [Al 71] E.M. Alfsen, *Compact convex sets and boundary integrals*, Springer, 1971.
- [AF 02] K. Astala and D. Faraco, Quasiregular mappings and Gradient Young measures, to appear in *Proc. Roy. Soc. Edinburgh*.
- [AH 86] R. Aumann and S. Hart, Bi-convexity and bi-martingales, *Israel J. Math.* **54** (1986), 159–180.
- [Ba 80] J.M. Ball, Strict convexity, strong ellipticity and regularity in the calculus of variations, *Math. Proc. Cambridge Phil. Soc.* **87** (1980), 501–513.
- [Ba 90] J.M. Ball, Sets of gradients with no rank-one connections, *J. Math. Pures Appl.* **69** (1990), 241–259.

- [BJ 87] J.M. Ball and R.D. James, Fine phase mixtures as minimizers of energy, *Arch. Rat. Mech. Anal.* **100** (1987), 13–52.
- [BDG 69] E. Bombieri, E. De Giorgi and E. Giusti, Minimal cones and the Bernstein theorem, *Invent. Math.* **7** (1969), 243–269.
- [BFL 00] A. Braides, I. Fonseca and G. Leoni, A -quasiconvexity: relaxation and homogenization, *ESAIM Control Optim. Calc. Var.* **5** (2000), 539–577.
- [CT 93] E. Casadio-Tarabusi, An algebraic characterization of quasi-convex functions, *Ricerche Mat.* **42** (1993), 11–24.
- [Ce 80] A. Cellina, On the differential inclusion $x \in [-1, 1]$, *Atti Accad. Naz. Lincei Rend. Sci. Fis. Mat. Nat.* **69**, 1–6.
- [CK 88] M. Chipot and D. Kinderlehrer, Equilibrium configurations of crystals, *Arch. Rat. Mech. Anal.* **103** (1988), 237–277.
- [CK 00] M. Chlebik and B. Kirchheim, Rigidity for the four gradient problem, Preprint MPI-MIS 35/2000.
- [Da 89] B. Dacorogna, *Direct methods of the calculus of variations*, Springer, 1989.
- [DM 97] B. Dacorogna and P. Marcellini, General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases, *Acta Math.* **178** (1997), 1–37.
- [DM 99] B. Dacorogna and P. Marcellini, *Implicit partial differential equations*, Birkhäuser, 1999.
- [Do 02] G. Dolzmann, *Variational Methods for Crystalline Microstructure - Analysis and Computation*, to appear as Lecture Notes in Mathematics, Springer.
- [DP 82] F.S. De Blasi and G. Pianigiani, Baire category approach to the existence of solutions of multivalued differential equations in Banach spaces, *Funkcial. Ekvac.* **25** (1982), 153–162.
- [DP 91] F.S. De Blasi and G. Pianigiani, Non convex valued differential inclusions in Banach spaces, *J. Math. Anal. Appl.* **157** (1991), 469–494.

- [DG 68] E. De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. U.M.I.* **4** (1968), 135–137.
- [DD 02] A. DeSimone and G. Dolzmann, Macroscopic response of nematic elastomers via relaxation of a class of $SO(3)$ invariant energies, *Arch. Rat. Mech. Anal.* **161** (2002), 181–204.
- [DP 85] R.J. DiPerna, Compensated compactness and general systems of conservation laws, *Trans. Amer. Math. Soc.* **292** (1985), 383–420.
- [Ev 86] L.C. Evans, Quasiconvexity and partial regularity in the calculus of variations, *Arch. Rat. Mech. Anal.* **95** (1986), 227–252.
- [FM 99] I. Fonseca and S. Müller, A -quasiconvexity, lower semicontinuity, and Young measures, *SIAM J. Math. Anal.* **30** (1999), 1355–1390.
- [Fr] G. Friesecke, personal communication.
- [Fu 87] M. Fuchs, Regularity theorems for nonlinear systems of partial differential equations under natural ellipticity conditions, *Analysis* **7** (1987), 83–93.
- [GM 68] E. Giusti and M. Miranda, Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni, *Boll. U.M.I.* **2** (1968), 1–8.
- [Gr 73] M. Gromov, Convex integration of differential relations, *Izv. Akad. Nauk S.S.S.R* **37** (1973), 329–343.
- [Gr 86] M. Gromov, *Partial differential relations*, Springer, 1986.
- [Ha 95] C. Hamburger, Quasimonotonicity, regularity and duality for nonlinear systems of partial differential equations, *Ann. Mat. Pure Appl. Ser. IV* **169** (1995), 321–354.
- [HLN 96] W. Hao, S. Leonardi, J. Nečas, An example of irregular solution to a nonlinear Euler-Lagrange elliptic system with real analytic coefficients, *Ann. SNS Pisa Ser. IV* **23** (1996), 57–67.
- [KP 91] D. Kinderlehrer and P. Pedregal, Characterizations of Young measures generated by gradients, *Arch. Rat. Mech. Anal.* **115** (1991), 329–365.

- [Ki 01a] B. Kirchheim, Deformations with finitely many gradients and stability of quasiconvex hulls, *C. R. Acad. Sci. Paris Sr. I Math.* **332** (2001), 289–294.
- [Ki 01b] B. Kirchheim, Habilitation thesis, University Leipzig, 2001.
- [Ko 01] J. Kolář, Non-compact lamination convex hulls, Preprint MPI-MIS 17/2001.
- [Kr 99] J. Kristensen, On the non-locality of quasiconvexity, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16** (1999), 1–13.
- [KT 01] J. Kristensen and A. Taheri, Partial regularity of strong local minimizers in the multi-dimensional calculus of variations, Preprint MPI-MIS 59/2001.
- [Kr 00] M. Kružík, Bauer’s maximum principle and hulls of sets, *Calc. Var.* **11** (2000), 321–332.
- [Ku 55] N.H. Kuiper, On C^1 isometric embeddings, I., *Nederl. Akad. Wetensch. Proc. A* **58** (1955), 545–556.
- [Ma 01] J. Matoušek, On directional convexity, *Discrete Comput. Geom.* **25** (2001), 389–403.
- [MP 98] J. Matoušek and P. Plecháč, On functional separately convex hulls, *Discrete Comput. Geom.* **19** (1998), 105–130.
- [MS 96] S. Müller and V. Šverák, Attainment results for the two-well problem by convex integration, *Geometric analysis and the calculus of variations*, (J. Jost, ed.), International Press, 1996, 239–251.
- [Mu 99] S. Müller, *Variational models for microstructure and phase transitions*, in: Calculus of Variations and Geometric Evolution Problems, Cetaro 1996 (F. Bethuel, G. Huisken, S. Müller, K. Steffen, S. Hildebrandt, and M. Struwe, eds.), Springer, Berlin, 1999.
- [Mu 99b] S. Müller, Rank-one convexity implies quasiconvexity on diagonal matrices, *Int. Math. Research Not.* 1999, 1087–1095.
- [MS 98] S. Müller and V. Šverák, Unexpected solutions of first and second order partial differential equations, *Documenta Math.*, Special volume Proc. ICM 1998, Vol. II, 691–702.

- [MS 99] S. Müller and V. Šverák, Convex integration for Lipschitz mappings and counterexamples to regularity, Preprint MPI-MIS 26/1999.
- [MRS 02] S. Müller, M.O. Rieger and V. Šverák, Parabolic systems with nowhere smooth solutions, Preprint.
- [MSy 01] S. Müller and M. Sychev, Optimal existence theorems for non-homogeneous differential inclusions, *J. Funct. Anal.* **181** (2001), 447–475.
- [Mu 81] F. Murat, L'injection du cone positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl.* **60** (1981), 309–322.
- [Na 54] J. Nash, C^1 isometric embeddings, *Ann. Math.* **60** (1954), 383–396.
- [NM 91] V. Nesi and G.W. Milton, Polycrystalline configurations that maximize electrical resistivity, *J. Mech. Phys. Solids* **39** (1991), 525–542.
- [Pe 93] P. Pedregal, Laminates and microstructure, *Europ. J. Appl. Math.* **4** (1993), 121–149.
- [Pe 97] P. Pedregal, *Parametrized measures and variational principles*, Birkhäuser, 1997.
- [Sch 74] V. Scheffer, Regularity and irregularity of solutions to nonlinear second order elliptic systems of partial differential equations and inequalities, Dissertation, Princeton University, 1974.
- [Se 83] D. Serre, Formes quadratiques et calcul des variations, *J. Math. Pures Appl.* **62** (1983), 177–196.
- [Si 02] M. Šilhavý, Relaxation in a class of $SO(n)$ -invariant energies related to nematic elastomers, Preprint.
- [Sp 98] D. Spring, *Convex integration theory*, Birkhäuser, 1998.
- [Sv 90] V. Šverák, Examples of rank-one convex functions, *Proc. Roy. Soc. Edinburgh A* **114** (1990), 237–242.
- [Sv 92a] V. Šverák, Rank-one convexity does not imply quasiconvexity, *Proc. Roy. Soc. Edinburgh* **120** (1992), 185–189.

- [Sv 92b] V. Šverák, New examples of quasiconvex functions, *Arch. Rat. Mech. Anal.* **119** (1992), 293–300.
- [Sv 93] V. Šverák, On Tartar’s conjecture, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **10** (1993), 405–412.
- [Sv 95] V. Šverák, Lower semicontinuity of variational integrals and compensated compactness, in: Proc. ICM 1994 (S.D. Chatterji, ed.), vol. 2, Birkhäuser, 1995, 1153–1158.
- [SY 00] V. Šverák and Xiadong Yan, A singular minimizer of a smooth strongly convex functional in three dimensions, *Calc. Var.* **10** (2000), 213–221.
- [Sy 99] M.A. Sychev, A new approach to Young measure theory, relaxation and convergence in energy, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16** (1999), 773–812.
- [Sy 01] M.A. Sychev, Comparing two methods of resolving homogeneous differential inclusions, *Calc. Var.* **13** (2001), 213–229.
- [Sz 02] L. Székelyhidi, PhD thesis, in preparation.
- [Ta 79] L. Tartar, Compensated compactness and partial differential equations, in: *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium Vol. IV*, (R. Knops, ed.), Pitman, 1979, 136–212.
- [Ta 83] L. Tartar, The compensated compactness method applied to systems of conservation laws, in: *Systems of Nonlinear Partial Differential Equations*, (J.M. Ball, ed.), NATO ASI Series, Vol. C111, Reidel, 1983, 263–285.
- [Ta 90] L. Tartar, H -measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **115** (1990), 193–230.
- [Ta 93] L. Tartar, Some remarks on separately convex functions, in: *Microstructure and phase transitions*, IMA Vol. Math. Appl. **54**, (D. Kinderlehrer, R.D. James, M. Luskin and J.L. Ericksen, eds.), Springer, 1993, 191–204.
- [Zh 88] Kewei Zhang, On the Dirichlet problem for a class of quasilinear elliptic systems of partial differential equations in divergence form,

in: *Partial differential equations* (Tianjun, 1986), Lecture Notes in Mathematics **1306**, Springer, 1988.

[Zh 98] Kewei Zhang, On the structure of quasiconvex hulls, *Ann. Inst. H. Poincaré Analyse Non Linéaire* **15** (1998), 663–686.