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A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity

by

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Abstract. The energy functional of nonlinear plate theory is a curvature functional for surfaces first proposed on physical grounds by G. Kirchhoff in 1850. We show that it arises as a Γ-limit of three-dimensional nonlinear elasticity theory as the thickness of a plate goes to zero. A key ingredient in the proof is a sharp rigidity estimate for maps $v \in W^{1,2}(U, I\!R^n)$, $U \subset I\!R^n$. We show that the L^2 distance of ∇v from a single rotation matrix is bounded by a multiple of the L^2 distance from the group SO(n) of all rotations.

1 Introduction

A classical theorem due to Liouville says that if a smooth mapping $v : \Omega \to I\!R^n$, $\Omega \in I\!R^n$, satisfies $\nabla v \in \mathrm{SO(n)}$, then it is affine, $v(x) = Rx + c$. There are numerous generalizations of this fundamental result, the most general being due to Reshetnyak [37]: if a sequence $v^{(k)}$ converging weakly in $W^{1,2}(\Omega, I\!R^n)$ satisfies $\nabla v^{(k)} \to \mathrm{SO(n)}$ in measure, then $\nabla v^{(k)}$ converges strongly in $L^2(\Omega)$ to a single matrix on SO(n).[1] These theorems play a pivotal role in solid mechanics and differential geometry.

However, the latter fall just short of being useful when specific information about the rate of convergence of the sequence is important. This is exactly the case when one tries to rigorously

[1] For a short modern proof using Young measures, see [20]. By an approximation result for Young measures [41], the result also holds with the spaces $W^{1,2}$ and L^2 above replaced by $W^{1,1}$ and L^1.

derive two-dimensional plate or shell theories (in the case that bending is considered) from three dimensional nonlinear elasticity, the small parameter being the thickness h of the plate. Such a derivation begins with an underlying smooth stored energy function W defined on 3×3 matrices which is minimized exactly on SO(3). A three-dimensional deformation $v^{(h)}$ defined, say, on a thin domain $\Omega_h = S \times (-\frac{h}{2}, \frac{h}{2})$, $S \subset I\!R^2$, has elastic energy

$$\int_{\Omega_h} W(\nabla v^{(h)}(x))\, dx,$$

and one seeks to understand the behaviour as $h \to 0$ of minimizers subject to appropriate boundary conditions. For compressive boundary conditions such as, for instance,

$$\Omega_h = (-1,1)^2 \times (-\frac{h}{2}, \frac{h}{2}), \quad v^{(h)}(x)|_{x_1 = \pm 1} = x \mp (a, 0, 0), \tag{1}$$

where $a \in (0,1)$ is fixed, the minimum energy scales like h^3. (The well known heuristic argument is made rigorous in Section 6 below. It is based on the intuition that the plate will accommodate the boundary conditions by bending, while keeping its mid-surface unstretched.) By contrast the volume of the domain scales like h, i.e. tends to zero much slower. This means that $\nabla v^{(h)}$ tends in a certain sense to SO(3). But the Reshetnyak theorem is insufficient to nail down the convergence properties sufficiently to calculate, for example, the limiting energy

$$\frac{1}{h^3} \int_{\Omega_h} W(\nabla v^{(h)}(x))\, dx. \tag{2}$$

Because of the presence of the scales $1/h^3$ in front of the integral, and h in the domain of integration, a quantitative understanding is needed. Because such an understanding has hitherto been lacking, rigorous passage to the thin plate limit has remained an open problem (see Ciarlet [10] for a recent survey of mathematical research in this area).

The main results of this paper are 1) a quantitative rigidity theorem which generalizes the results of Liouville, Reshetnyak [37] and F. John [22, 23] and would appear to be widely applicable and, 2) a rigorous derivation of the thin-plate limit of 3-D nonlinear elasticity theory, under not just the special boundary condition (1) but indeed any boundary condition compatible with keeping the mid-surface unstretched.

The rigidity theorem is discussed following its precise statement in Section 3. The remainder of this Introduction is devoted to passage to the thin plate limit.

The derivation of plate/shell theories is a problem having a long history with major contributions from Euler, D. Bernoulli, Cauchy, Kirchhoff, Love, E. and F. Cosserat, von Karman, and a great many modern authors. The classical lines of research are reviewed by Love [33]. Nearly all are based on ansatzes for (exact or approximate) minimizers of (2), leading to a great variety of plate/shell theories in the literature which are not consistent with each other. In terms of the question of which plate theory, if any, is actually predicted by nonlinear elasticity for a thin plate, the field has hitherto been in a state of confusion. One particular line of research going back to the Cosserats is the ansatz that the energy density of the shell can be expressed as a function of the deformation gradient of the middle surface together with a number of vectors, and possibly their gradients up to some order, that model shear and compression of the plate relative to the middle surface. These models are called Cosserat models (for further discussion and references see Antman [3]).

Recently rigorous results have begun to appear which compare the 3D minimizers to their 2D counterparts ([4], [6], [8], [10], [18] [27], [28], [29], [36]). The natural mathematical setting in which these results are usually formulated is that of variational or $\Gamma-$convergence which was introduced by De Giorgi ([13], [12]). Here we discuss only such derivations that begin with nonlinear elasticity; there is a large body of related research based on linearized elasticity in which SO(3) is replaced by the linear space of skew matrices, but, in view of the fact that thin plates can easily undergo large rotations that invalidate the assumption upon which linear elasticity is based, these have limited applicability (however, this research does shed light on the subject of "moderately thin plates" [10]). It is remarkable and quite unexpected that the rigorous study of the 3D minimizers in the limit $h \to 0$ often leads to Cosserat models. The $\Gamma-$limit of the energy,

$$\frac{1}{h} \int_{\Omega_h} W(\nabla v(x))\, dx, \tag{3}$$

is now reasonably well understood: this yields the so-called *membrane theory* ([8], [27], [28], [29]). It captures the energy that is proportional to the thickness h which includes stretching and shearing of the plate relative to the middle surface, as induced for example by tensile boundary conditions (1) with $a < 0$. It assigns zero energy to typical bent states of the plate, whose energy scales like h^3. Here we determine the $\Gamma-$limit of the energy (2). This is more difficult since the limit functional contains higher derivatives and one is thus dealing with a singular perturbation problem.

We now describe the limiting plate theory we obtain. For simplicity we restrict ourselves in this Introduction to the case when the stored-energy function is isotropic (that is to say, $W(F) = W(QFR)$ for all $F \in M^{3\times 3}$ and all Q, $R \in \mathrm{SO}(3)$). In this case, the second derivative of W at the identity is

$$\frac{\partial^2 W}{\partial F^2}(I)(A, A) = 2\mu\, |e|^2 + \lambda(\mathrm{trace}\, e)^2, \quad e = \frac{A + A^T}{2},$$

for some constants λ, $\mu \in I\!R$. The limiting 2D energy functional to which (2) $\Gamma-$converges is then

$$I^0(v) = \begin{cases} \frac{1}{24} \int_S \Big(2\mu |\mathrm{II}|^2 + \frac{\lambda\mu}{\mu+\lambda/2}(\mathrm{trace}\, \mathrm{II})^2\Big) & \text{on isometries } v : S \to I\!R^3, \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

Here II denotes the second fundamental form of the surface, i.e. $\mathrm{II} = (\nabla v)^T \nabla b$ where $b = \frac{\partial v}{\partial x_1} \wedge \frac{\partial v}{\partial x_2}$ is the surface normal. The limiting energy is thus a quadratic form in the (extrinsic) curvature tensor. See Section 6 for a detailed discussion, including the interesting issue of the limiting boundary conditions and a natural variational explanation for the emergence of the renormalized Lamé constant $\frac{\lambda\mu}{\mu+\lambda/2}$ in place of the naively expected stiffer constant λ.

The limit energy (4) agrees with the expression proposed in the original work of Kirchhoff ([24], equation (9.)), but not with the expression obtained by suppressing the geometric nonlinearity (i.e. by approximating the constraint that v must be isometric by $v_1 = v_2 = 0$ and replacing II by $-\nabla^2 v_3$) which much of the subsequent literature has associated with Kirchhoff's name. Likewise, expression (4) does not agree with the expression obtained via a standard nonlinear Cosserat ansatz (sometimes called nonlinear Kirchhoff-Love ansatz in the literature even though not due to Kirchhoff) for the 3D deformation which assumes that the fibers orthogonal to the mid-surface deform linearly,

$$v^{(h)}(x_1, x_2, x_3) = y(x_1, x_2) + x_3 b(x_1, x_2). \tag{5}$$

This leads to an energy of correct functional form but containing the incorrect constant λ in place of $\frac{\lambda\mu}{\mu+\lambda/2}$. A simple physical explanation for why the amount of stretch of the fibers is in fact nonconstant along the fibers is given in Section 7. This variation along the fibers, missed by (5) and quantified exactly in Section 7, turns out to contribute to the energy at same order as the variation of the fiber direction, captured by (5).

Finally, the following special case of our result may be of some geometric interest: the functional $\frac{1}{h^3}\int_{\Omega_h} \mathrm{dist}(\nabla y(x), \mathrm{SO}(3))^2 dx$ Γ-converges to the Willmore functional arising in differential geometry [40] restricted to isometric surfaces, $I^0(y) = \frac{1}{12}\int_S |\mathrm{II}|^2\, dx$ when y is an isometry and $+\infty$ otherwise. As an immediate corollary of this Γ-convergence we obtain existence of surfaces in $I\!R^3$ which minimize the Willmore functional in the class of isometries for appropriate boundary conditions (see Section 6).

Modern interest in plate theories has blossomed with the ubiquitous presence of thin films in science and technology. Interesting mechanical problems have arisen out of studies of the delamination of films from substrates (e.g., Gioia and Ortiz [19], [35]) and the behavior of so-called "active thin films". The latter have been modeled by energy densities W with multiple energy wells (Bhattacharya and James [8]) of the form $\mathrm{SO}(3)A \cup \mathrm{SO}(3)B \cup ...$, where $A, B, ...$ are constant 3×3 matrices.

We believe our methods will be generally useful, but a great many interesting open problems remain:

i) Shell theory (reference state not flat). Shells can be more rigid than plates, depending on their reference state. For example, bending a corrugated shell (as in a corrugated roof) around an axis perpendicular to the direction of the corrugations immediately activates the membrane energy, and is therefore expected to lead to a different energy scaling than h^3 (see also the next item). On the other hand, the extension to thin rods is relatively straightforward.

ii) The case when the membrane theory is not trivial, so membrane and bending energies are both present, i.e., the boundary conditions are such as to forbid a simple overall scaling. Recent work of Ben Belgacem, Conti, DeSimone and Müller [6] and Jin and Sternberg [21] reveals the subtlety of this issue. They show that for boundary conditions that exert a small uniform compression, the energy scales like h^2, between membrane (h) and bending (h^3). In fact, the practical case in the delamination of thin films under compression studied by Gioia and Ortiz appears to be of this type.

iii) The case of multiple energy wells. Our results are decidedly 'one-well', and quite different and unexpected shell theories may arise in the case appropriate to active films.

iv) The multilayer case. This is important in films and is modelled by an explicit x_3 dependence of W. This case relates to the most important measurement of stress in films (the wafer-curvature measurement) as well as the behavior of the classic bimetallic strip.

v) Predictions of the theory, in particular, the connection with recent studies of singularities of "paper-folding" (see, e.g., Ben Amar and Pomeau [2], Cerda, Chaieb, Melo, Mahadevan [11], DiDonna, Venkataramani, Witten, Kramer [14] and Lobkovsky [32]). These authors argue that certain canonical singularities that arise during the crumpling of paper necessarily involve both membrane and bending energies and they construct an associated deformation that exhibits a scaling of $h^{8/3}$.

2 Notation, bending energy and Euler–Bernoulli theory

We will be concerned with variational integrals of the type

$$\int_\Omega W(\nabla v(z))dz \tag{6}$$

that arise in the theory of nonlinear elasticity. Mathematically, Ω is a bounded open subset of $I\!R^3$, $v:\Omega\to I\!R^3$ is a sufficiently smooth mapping and W is defined on 3×3 matrices, denoted $M^{3\times3}$. (Physically, Ω is the region occupied by an elastic body in a reference configuration, v is the deformation and (6) its elastic energy.) A superimposed T indicates the transpose and I the identity matrix. The set of $n\times n$ rotation matrices (or simply rotations), $\{R\in M^{n\times n}: R^TR=I,\ \det R=1\}$, is denoted SO(n). For $A\in M^{n\times n}$ let $\operatorname{cof}A$ denote the matrix of cofactors of A, i.e.,

$$(\operatorname{cof}A)_{ij}=(-1)^{i+j}\det \widehat{A}_{ij}, \tag{7}$$

where $\widehat{A}_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column. It is well-known that for $v\in W^{1,2}(\Omega)$, $\operatorname{div}\operatorname{cof}\nabla v=0$. In this paper C is a generic absolute constant (Its value can vary from line to line, but each line is valid with C being a pure positive number, independent of all other quantities).

For $A\in M^{3\times3}$ we denote the Euclidean norm by $|A|=\sqrt{\operatorname{tr}AA^T}$. The distance from A to SO(n) is denoted $\operatorname{dist}(A,\mathrm{SO(n)})$. If $\det A>0$ and $A=RU$ is its polar decomposition ($R\in\mathrm{SO(n)}$, and $U=\sqrt{A^TA}$), a short calculation shows that $\operatorname{dist}(A,\mathrm{SO(n)})=|U-I|$. More generally, if the condition $\det A>0$ is dropped, we still have the inequality $\operatorname{dist}(A,\mathrm{SO(n)})\geq|(A^TA)^{1/2}-I|$.

The key assumption of this paper is the usual assumption of geometrically-nonlinear elasticity theory that the stored energy function $W: M^{3\times3}\to I\!R$ has a single energy well at SO(3). Altogether, we assume,

i) $W\in C^0(M^{3\times3})$, $W\in C^2$ in a neighbourhood of SO(3)

ii) W is frame-indifferent: $W(F)=W(RF)$ for all $F\in M^{3\times3}$ and all $R\in\mathrm{SO(3)}$,

iii) $W(F)\geq C\operatorname{dist}^2(F,\mathrm{SO(3)})$, $W(F)=0$ if $F\in\mathrm{SO(3)}$.

We do not impose any growth condition from above; in fact, the condition $W\in C^0(M^{3\times3})$ in (i) can be weakened to include W's which take the value $+\infty$ outside an open neighbourhood of SO(3), such as the following model functional for isotropic materials which goes back to St Venant and Kirchhoff

$$W(F)=\begin{cases}\mu\Big(\sqrt{F^TF}-I\Big)^2+\frac{\lambda}{2}\Big(\operatorname{trace}\Big(\sqrt{F^TF}-I\Big)\Big)^2, & \det F>0\\ +\infty, & \text{otherwise;}\end{cases}$$

see Section 6.

In the application to nonlinear plate theory we shall be concerned with regions of the form $\Omega_h=S\times(-\frac{h}{2},\frac{h}{2})$, where $S\subset I\!R^2$ is strongly Lipschitz and $h>0$ is the small parameter. Consider an orthonormal basis $\{e_1,e_2,e_3\}$ with e_3 pointing in the direction normal to S, and an associated rectangular Cartesian co-ordinate system (z_1,z_2,z_3). In order to deal with sequences of deformations defined on a fixed domain we change variables,

$$x_1=z_1,\quad x_2=z_2,\quad x_3=\frac{1}{h}z_3, \tag{8}$$

and rescale deformations according to the rule $y(x) = v(z(x))$ so that $y : \Omega_1 \to I\!R^3$. We use the notation $\nabla' y = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ for the gradient in the plane, so that,

$$\nabla v = (\nabla' y, \frac{1}{h} y_{,3}). \tag{9}$$

The total free energy of a plate of thickness h and cross-section S is,

$$\int_{\Omega_h} W(\nabla v(z))\, dz = h \int_{\Omega_1} W(\nabla' y, \frac{1}{h} y_{,3})\, dx =: E^{(h)}(y) \tag{10}$$

which is well-defined for $y \in W^{1,2}(\Omega_1, I\!R^3)$ as an element of $[0, \infty) \cup \{\infty\}$.

The Γ–limit of $\frac{1}{h} E^{(h)}$ has been discussed by many authors, as summarized in the introduction. This first Γ–limit is the so-called *membrane theory*; it governs stretching, as well as shear and compression parallel to e_3, of the plate. We shall be concerned with the case, arising from compressive boundary conditions such as (1), when the membrane theory is trivial and the total energy scales as h^3; the latter is also the case of Euler–Bernoulli theory, as explained below. We shall say that a sequence $y^{(h)} \in W^{1,2}(\Omega, I\!R^3)$ has finite bending energy if

$$\limsup_{h \to 0} \frac{1}{h^3} E^{(h)}(y^{(h)}) < \infty. \tag{11}$$

Euler–Bernoulli theory concerns, say, a strip $S = (0, L) \times (0, w)$ bent in the $x_1 - x_3$ plane. The kinematics of Euler–Bernoulli theory is described by an isometric deformation $y : (0, L) \times (0, w) \to I\!R^3$ of this strip,

$$y(x_1, x_2) = x_2 e_2 + \left(\int_0^{x_1} \cos\theta(s)\, ds \right) e_1 + \left(\int_0^{x_1} \sin\theta(s)\, ds \right) e_3, \tag{12}$$

where $\theta \in W^{1,2}(0, L)$. The Euler–Bernoulli energy of the deformed strip is,

$$\int_0^L \frac{1}{2} E I \theta'(s)^2 ds. \tag{13}$$

Here E is a phenomenological elastic modulus (which Euler, in his fundamental 1744 paper [16], did not attempt to derive from the three-dimensional elastic moduli), and the moment of inertia is $I = wh^3/12$, where h is the thickness of the strip[2]. Below in Theorem 4.1 we show that if a sequence $y^{(h)}$ has finite bending energy then $(\nabla' y^{(h)}, \frac{1}{h} y_{,3}^{(h)})$ converges strongly to a particular $(\nabla' y, b)$ with values on SO(3) and with $(\nabla' y, b)$ independent of x_3. It follows that y is an isometric mapping of $S \subset I\!R^2$, which in the case of deformations in the $x_1 - x_3$ plane, agrees with Euler–Bernoulli kinematics. The detailed form of the Euler–Bernoulli energy will be shown in Section 6 to agree with the rigorous thin-plate limit of 3D nonlinear elasticity, with a particular evaluation of the modulus E.

[2]In Euler–Bernoulli theory the only appearance of the thickness h of the strip is in the formula for the moment of inertia.

3 Geometric rigidity

The basic rigidity result relevant to passage to the thin plate limit is the following.

Theorem 3.1 *Let U be a bounded Lipschitz domain in $I\!R^n$, $n \geq 2$. There exists a constant $C(U)$ with the following property. For each $v \in W^{1,2}(U, I\!R^n)$ there is an associated rotation $R \in \mathrm{SO(n)}$ such that,*

$$\|\nabla v - R\|_{L^2(U)} \;\leq\; C(U)\,\|\mathrm{dist}(\nabla v, \mathrm{SO(n)})\|_{L^2(U)}. \tag{14}$$

The result also holds in L^p for $1 < p < \infty$, as will be shown elsewhere. It is sharp in the sense that neither the norm on the right hand side nor the power with which it appears can be improved.

An estimate in terms of $\epsilon + \sqrt{\epsilon}$, where $\epsilon = ||\mathrm{dist}(\nabla v, \mathrm{SO(n)})||_{L^2(U)}$, is much easier to prove, but is insufficient for the application to plate theory, where one needs to sum the estimate over many small cubes of size h.

Corollary 3.1 *(F. John* [22, 23]*) If Q is an n-dimensional cube, and if $v \in C^1$ with*

$$||\mathrm{dist}(\nabla v, \mathrm{SO(n)})||_{L^\infty(\mathrm{Q})} \leq \delta \tag{15}$$

for δ sufficiently small, then (14) holds for $U = \mathrm{Q}$. In particular, for all such v

$$[\nabla v]_{BMO(\mathrm{Q})} := \sup_{\mathrm{Q}'\subset\mathrm{Q}} \tfrac{1}{|\mathrm{Q}'|} \int_{\mathrm{Q}'} \Big|\nabla v - \tfrac{1}{|\mathrm{Q}'|} \textstyle\int_{\mathrm{Q}'} \nabla v\Big| \leq C(n)\delta \tag{16}$$

where the supremum is taken over all cubes $\mathrm{Q}' \subseteq \mathrm{Q}$.

(Theorem 3.1 shows that (16) in fact holds for arbitrary $\delta > 0$ and arbitrary maps $v \in W^{1,1}(\mathrm{Q}, I\!R^n)$ with $||\mathrm{dist}(\nabla v, \mathrm{SO(n)})||_{L^\infty(\mathrm{Q})} \leq \delta$. This is immediate from equivalence of the BMO-seminorm and the BMO^2-seminorm (see e.g. [7], Corollary 7.8) and (14).) For the application to plate theory it is crucial to remove F. John's restrictions on v, since they do not follow from smallness of the elastic energy.

Kohn [25] established optimal L^p estimates for $v - Rx + const$, but not $\nabla v - R$, without these restrictions.

Corollary 3.2 *(Yu. G. Reshetnyak* [37]*) If $v^j \rightharpoonup v$ in $W^{1,2}(U; I\!R^n)$ and $\mathrm{dist}(\nabla v^j, \mathrm{SO(n)}) \to 0$ in measure then $\nabla v^j \to R$ in $L^2(U)$ where R is a constant rotation.*

Reshetnyak established related results for the more general case of nearly conformal maps. An interesting open question raised by our work is whether these results can also be made quantitative.

Before giving the proof of Theorem 3.1 we motivate some of its steps, by considering the special case when the right hand side in (14) is zero. Theorem 3.1 then reduces to the Liouville theorem that a $W^{1,2}(U; I\!R^n)$ map v which satisfies the partial differential relation

$$Dv(x) \in \mathrm{SO(n)} \quad a.e. \tag{17}$$

is a rigid motion, i.e. $Dv(x) \equiv const$. (In the setting of Sobolev maps this was first proved by Reshetnyak [37].) A short modern proof consists of two observations. First, (17) implies that v is harmonic, and in particular smooth. (Proof: $Dv(x) = \operatorname{cof} Dv(x)$ a.e.; take the divergence and use that $\operatorname{div} \operatorname{cof} Dv(x) = 0$ for all $v \in W^{1,2}$.) Second, the second gradient squared of any harmonic map can be expressed pointwise via derivatives of the inner products $v_{,i} \cdot v_{,j}$,

$$\frac{1}{2}\Delta\Big(|\nabla v|^2 - n\Big) = \nabla v \cdot \Delta \nabla v + |\nabla^2 v|^2 = |\nabla^2 v|^2; \tag{18}$$

but $|\nabla v|^2 - n = 0$ when v satisfies (17).

Theorem 3.1 deals with approximate rather than exact solutions of the partial differential relation, but both observations above will continue to play a certain role. We will show that every approximate solution can be decomposed into a harmonic part and a small part (see Step 1), and generalize (18) into a smallness estimate for $\nabla^2 v$ in terms of the L^2 distance of v from SO(n) (see Step 2).

It will be useful in the proof to work with functions whose gradients have a bounded L^∞ norm. For this purpose we need an approximation lemma similar to one that appears in the literature ([31], [42], [17]); this is proved in Appendix A (Proposition A.1).

We begin the proof of Theorem 3.1 by establishing a corresponding interior estimate when U is a cube.

Proposition 3.1 *Let* Q *be an n-dimensional cube, and let* Q′ *be a concentric cube having half the side length of* Q*. For each $v \in W^{1,2}(\mathrm{Q}, I\!R^n)$ there exists an associated rotation $R \in \mathrm{SO(n)}$ such that*

$$\|\nabla v - R\|_{L^2(\mathrm{Q}')} \;\le\; C(n)\, \|\operatorname{dist}(\nabla v, \mathrm{SO(n)})\|_{L^2(\mathrm{Q})}. \tag{19}$$

Proof of Proposition 3.1 We first observe that it suffices to prove Proposition 3.1 for maps v with $\|\nabla v\|_{L^\infty(\mathrm{Q})} \le M$, for some constant M depending only on the dimension n. Indeed, note that $|A| \le 2\operatorname{dist}(A, \mathrm{SO(n)})$ if $|A| \ge 2\sqrt{n}$. Hence an application of Proposition A.1 with $\lambda = 4\sqrt{n}$ yields a map $V \in W^{1,\infty}(\mathrm{Q}, I\!R^n)$ satisfying

$$\begin{aligned}
\|\nabla V\|_{L^\infty(\mathrm{Q})} &\;\le\; 4\sqrt{n}\, C := M, \\
\|\nabla V - \nabla v\|^2_{L^2(\mathrm{Q})} &\;\le\; C\int_{\{x\in \mathrm{Q}\,:\,|\nabla v(x)|>2\sqrt{n}\}} |\nabla v|^2 dx, \\
&\;\le\; 4\,C\int_{\mathrm{Q}} \operatorname{dist}^2(\nabla v, \mathrm{SO(n)})\, dx.
\end{aligned} \tag{20}$$

Hence, if we prove (19) (or (14)) for V the assertion for v follows by two applications of the triangle inequality, viz.,

$$\begin{aligned}
\|\nabla v - R\|_{L^2(\mathrm{Q}')} &\;\le\; \|\nabla V - R\|_{L^2(\mathrm{Q})} \;+ \|\nabla v - \nabla V\|_{L^2(\mathrm{Q})} \\
&\;\le\; C\,\|\operatorname{dist}(\nabla V, \mathrm{SO(n)})\|_{L^2(\mathrm{Q})} + 2\sqrt{C}\,\|\operatorname{dist}(\nabla v, \mathrm{SO(n)})\|_{L^2(\mathrm{Q})} \\
&\;\le\; C\,\|\operatorname{dist}(\nabla v, \mathrm{SO(n)})\|_{L^2(\mathrm{Q})}.
\end{aligned} \tag{21}$$

Hence we can assume from now on that $\|\nabla v\|_{L^\infty(\mathrm{Q})} \le M$ for some constant M depending only on the dimension n.

Step 1. Let

$$\varepsilon = \|\operatorname{dist}(\nabla v, \mathrm{SO(n)})\|_{L^2(\mathrm{Q})}. \tag{22}$$

We may suppose $\epsilon \leq 1$. Since $\operatorname{div} \operatorname{cof} \nabla v = 0$, we have

$$-\triangle v = \operatorname{div}(\operatorname{cof} \nabla v - \nabla v). \tag{23}$$

The quantity $|A - \operatorname{cof} A|^2$ is smooth and nonnegative, and vanishes on SO(n). Hence, there is an absolute constant C such that

$$|A - \operatorname{cof} A|^2 \leq C \operatorname{dist}^2(A, \mathrm{SO(n)}) \text{ for } |A| \leq M. \tag{24}$$

Now (23), (24) motivate the decomposition $v = w + z$, where $z \in W^{1,2}(\mathrm{Q})$ is the unique solution to

$$-\triangle z = \operatorname{div}(\operatorname{cof} \nabla v - \nabla v) \text{ in Q}, \quad z = 0 \text{ on } \partial \mathrm{Q} \tag{25}$$

and $w := v - z$ satisfies $\triangle w = 0$ in Q. Testing (25) with z and using (24) we get,

$$\int_{\mathrm{Q}} |\nabla z|^2 dx \leq \int_{\mathrm{Q}} |\operatorname{cof} \nabla v - \nabla v|^2 dx \leq C \varepsilon^2. \tag{26}$$

Hence it suffices to show

$$\int_{\mathrm{Q}} \operatorname{dist}(\nabla w, \hat{R})^2 \, dx \leq C\epsilon^2 \tag{27}$$

for some $\hat{R} \in \mathrm{SO(n)}$. In other words we need to show that the harmonic part, which carries information about the boundary values of v, is approximately linear with gradient on SO(n). To estimate the oscillation of its gradient on the subset Q$'$, we proceed in two steps. First we derive a bound in terms of $\varepsilon^{1/2}$. This by itself is not good enough. It allows us, however, to linearize about SO(n) and to derive a bound of order ε for the oscillation of the symmetric part of the gradient. Then Korn's inequality can be used to control the skew part as well.

Step 2. The harmonic part w satisfies the identity (18). Let $\frac{1}{2}\mathrm{Q} = \mathrm{Q}' \subset \mathrm{Q}'' \subset \mathrm{Q}$ be strictly increasing concentric cubes. Choose a cutoff function $\eta \in C_0^\infty(\mathrm{Q})$ with $\eta \geq 0$ and $\eta = 1$ on Q$''$. Then

$$\begin{aligned}
\int_{\mathrm{Q}} |\nabla^2 w|^2 \eta \, dx &\leq \sup_{\mathrm{Q}} (\triangle \eta) \int_{\mathrm{Q}} \left| |\nabla w|^2 - n \right| dx, \\
&\leq C \left(\int_{\mathrm{Q}} \left| |\nabla v|^2 - n \right| \, dx + 2 \int_{\mathrm{Q}} |\nabla v| |\nabla z| \, dx + \int_{\mathrm{Q}} |\nabla z|^2 \, dx \right), \\
&\leq C \left(\int_{\mathrm{Q}} |\operatorname{dist}(\nabla v, \mathrm{SO(n)}| \, dx + \left(\int_{\mathrm{Q}} |\nabla z|^2 \, dx \right)^{\frac{1}{2}} + \int_{\mathrm{Q}} |\nabla z|^2 \, dx \right).
\end{aligned} \tag{28}$$

Hence,

$$\int_{\mathrm{Q}''} |\nabla^2 w|^2 \, dx \leq C\varepsilon. \tag{29}$$

Since w (and hence $\nabla^2 w$) is harmonic on Q, the mean value property with $r = \operatorname{dist}(\mathrm{Q}'', \partial \mathrm{Q}')$ gives

$$\sup_{x \in \mathrm{Q}'} |\nabla^2 w(x)|^2 = \sup_{x \in \mathrm{Q}'} \left| \frac{1}{|B(x,r)|} \int_{B(x,r)} \nabla^2 w(y) \, dy \right|^2 \leq C\varepsilon. \tag{30}$$

Hence, there is an $R \in M^{n \times n}$ such that

$$\sup_{\mathrm{Q}'} |\nabla w - R| \leq C\varepsilon^{1/2}, \tag{31}$$

and, in fact, we can choose R in SO(n), because

$$\int_{\mathrm{Q}} \mathrm{dist}^2(\nabla w, \mathrm{SO(n)})\, dx \;\le\; 2\int_{\mathrm{Q}} \Big(\mathrm{dist}^2(\nabla v, \mathrm{SO(n)}) + |\nabla z|^2\Big) dx \;\le\; C\varepsilon^2, \tag{32}$$

according to (26) and (22). For the rest of the proof we may assume without loss of generality that $R = I$, for otherwise we could apply the following arguments to $R^T v$ and $R^T w$ in place of v and w.

Step 3. Linearizing dist($\cdot$, SO(n)) near the identity we get

$$\mathrm{dist}(G, \mathrm{SO(n)}) = |\tfrac{1}{2}(G + G^T) - I| + \mathcal{O}(|G - I|^2). \tag{33}$$

Let $e = \frac{1}{2}(\nabla w + (\nabla w)^T) - I$. We have on Q′,

$$|e| \le \mathrm{dist}(\nabla w, \mathrm{SO(n)}) + C\varepsilon, \tag{34}$$

so that, using (32),

$$\int_{\mathrm{Q}'} |e|^2\, dx \le C\varepsilon^2. \tag{35}$$

By Korn's inequality for the displacement $u(x) := w(x) - x$ we have (letting $\hat{R} := \frac{1}{|\mathrm{Q}'|}\int_{\mathrm{Q}'} \nabla w\, dx$)

$$\int_{\mathrm{Q}'} |\nabla w - \hat{R}|^2\, dx = \int_{\mathrm{Q}'} \Big|\nabla u - \tfrac{1}{|\mathrm{Q}'|}\textstyle\int_{\mathrm{Q}'} \nabla u\Big|^2 dx \le C\int_{\mathrm{Q}'} |e|^2\, dx \le C\varepsilon^2.$$

But $\mathrm{dist}(\hat{R}, \mathrm{SO(n)}) \le C\varepsilon$ by (32), so $\hat{R}$ can be replaced by a matrix on SO(n), completing the proof of Proposition 3.1.

Proof of Theorem 3.1. As in the proof of Proposition 3.1 we may assume

$$\|\nabla v\|_{L^\infty(U)} \le M, \tag{36}$$

M being a constant only depending on the domain U. We again write $v = w + z$ as in the proof of Proposition 3.1 (cf., $(24)_{\mathrm{ff}}$). The bound (26), whose proof applies equally to general bounded Lipschitz domains, already holds on all of U so it remains to estimate the harmonic part w. To this end let $\mathrm{Q}(a, r) = a + r\,(-\frac{1}{2}, \frac{1}{2})^n$ be the cube of side length $r > 0$ centered at $a \in I\!R^n$. We exhaust U by cubes $\mathrm{Q}(a_i, r_i)$ with

$$2r_i \le \mathrm{dist}(a_i, \partial\mathrm{Q}) \le Cr_i \tag{37}$$

and such that each $x \in U$ is contained in at most N cubes $\mathrm{Q}(a_i, 4r_i)$. By Proposition 3.1 applied to w, there are rotations R_i such that

$$\int_{\mathrm{Q}(a_i,2r_i)} |\nabla w - R_i|^2\, dx \le C \int_{\mathrm{Q}(a_i,4r_i)} \mathrm{dist}^2(\nabla w, \mathrm{SO(n)})\, dx. \tag{38}$$

Since w is harmonic we deduce that

$$r_i^2 \int_{\mathrm{Q}(a_i,r_i)} |\nabla^2 w|^2\, dx \le C \int_{\mathrm{Q}(a_i,2r_i)} |\nabla w - R_i|^2\, dx. \tag{39}$$

Using the fact that for $x \in \mathrm{Q}(a_i, r_i)$ the distance between x and ∂U is comparable to r_i, we obtain

$$\int_{\mathrm{Q}(a_i,r_i)} \mathrm{dist}^2(x, \partial U)|\nabla^2 w|^2\, dx \le C \int_{\mathrm{Q}(a_i,4r_i)} \mathrm{dist}^2(\nabla w, \mathrm{SO(n)})\, dx. \tag{40}$$

Sum this over i, using the the inequality $\sum_i \chi_{\mathrm{Q}(a_i,4r_i)} \leq N$ to get the following global result:

$$\int_U \mathrm{dist}^2(x,\partial U)|\nabla^2 w|^2\,dx \leq C \int_U \mathrm{dist}^2(\nabla w, \mathrm{SO(n)})\,dx. \tag{41}$$

Now we use a weighted Poincaré inequality of the form,

$$\min_{G\in M^{n\times n}} \int_U |f-G|^2\,dx \leq C\int_U \mathrm{dist}^2(x,\partial U)|\nabla f|^2\,dx, \tag{42}$$

for $f \in W^{1,2}(U, M^{n\times n})$. This is an immediate consequence of Theorem 1.5 of [34] or Theorem 8.8 of [26]:

$$\int_U |g|^2\,dx \leq C_U^1 \int_U (|g|^2 + |\nabla g|^2)\mathrm{dist}^2(x,\partial U)\,dx \tag{43}$$

for $g \in W^{1,2}_{loc}(U) \cap L^2(U)$. To pass from (43) to (42), fix $\delta > 0$ such that $C_U^1\,\delta^2 \leq \frac{1}{2}$ and let $q = \{x \in U : \mathrm{dist}(x,\partial U) > \delta\}$. By the ordinary Poincaré inequality for q there exists $a \in I\!R$ such that,

$$\int_q |f-a|^2\,dx \leq C_q \int_q |\nabla f|^2\,dx \leq \frac{C_q}{\delta^2}\int_q |\nabla f|^2\,\mathrm{dist}^2(x,\partial U)\,dx. \tag{44}$$

Application of (43) with $g = f - a$ and the use of $\mathrm{dist}^2(x,\partial U)C_U^1 \leq \frac{1}{2}$ for $x \in U \setminus q$ yields,

$$\int_U |f-a|^2\,dx \leq \tfrac{1}{2}\int_{U\setminus q} |f-a|^2\,dx + (\frac{C_q}{\delta^2}+1)\,C_U^1 \int_U |\nabla f|^2\,\mathrm{dist}^2(x,\partial U)\,dx, \tag{45}$$

and this implies (42).

Apply the inequality (42) to (41) to yield the existence of R (which, as above, can be chosen on SO(n) using (32)) such that,

$$\|\nabla w - R\|_{L^2(U)} \leq C\,\|\mathrm{dist}(\nabla w, \mathrm{SO(n)})\|_{L^2(U)}. \tag{46}$$

Combining this with the estimate (26) with the domain Q replaced by U yields the assertion of the theorem. □

Remark. Theorem 3.1 is invariant under uniform scaling and translation of the domain, e.g., the same value of C serves for $\lambda U + c$, and the rescaled function $\lambda v((x-c)/\lambda)$ may be associated with the same choice of $R \in \mathrm{SO(n)}$. Finally, we note that, trivially, sequences of linear deformations with gradients approaching SO(n) serve to show that the exponent on the right hand side of (14) cannot be improved.

4 Compactness of sequences having finite bending energy

The quantitative rigidity estimate applies to a fixed domain, whereas the sets of interest in plate theory are of fixed cross-section and very thin, with lateral diameter h. The plate can then be viewed, except for a boundary layer near its edges, as a union of cubes of side length h. On each of these a deformation with finite bending energy is nearly rigid, according to the quantitative rigidity estimate. In this way of thinking, the goal of a compactness argument is to estimate how much this rigid deformation can vary from cube to cube in the lateral direction.

Theorem 4.1 *Suppose a sequence $y^{(h)} \subset W^{1,2}(\Omega; I\!R^3)$ has finite bending energy, that is to say*

$$\limsup_{h\to 0} \frac{1}{h^2} \int_\Omega \operatorname{dist}^2((\nabla' y^{(h)}, \frac{1}{h} y,_3^{(h)}), \mathrm{SO}(3))\, dx < \infty. \tag{47}$$

Then $\nabla_h y^{(h)} = (\nabla' y^{(h)}, \frac{1}{h} y,_3^{(h)})$ is precompact in $L^2(\Omega)$ as $h \to 0$: there exists a subsequence (not relabelled) such that

$$\nabla_h y^{(h)} \to (\nabla' y, b) \quad \text{in } L^2(\Omega) \tag{48}$$

with $(\nabla' y, b) \in \mathrm{SO}(3)$ a.e. Furthermore, $(\nabla' y, b)$ is independent of x_3 and $(\nabla' y, b) \in H^1(\Omega)$.

Remarks.

i) One interesting aspect of this result is that $(\nabla' y, b)$ is much more regular than naively expected.

ii) If the factor $1/h^2$ in hypothesis (47) is replaced by any factor $\eta(h)$ tending slower to infinity with $h \to 0$, then precompactness fails. See Section 5.

Proof. The main technical object to be studied is a piecewise constant approximation of the rescaled deformation gradient, obtained via Theorem 3.1.

Consider a lattice of squares

$$S_{a,h} = a + (-\frac{h}{2}, \frac{h}{2})^2, \quad a \in h\mathbb{Z}^2, \tag{49}$$

and let

$$S'_h = \bigcup_{S_{a,3h} \subset S} S_{a,h}\,. \tag{50}$$

Undo the rescaling and apply Theorem 3.1 to $v^{(h)}(z) = y^{(h)}(z', \frac{1}{h} z_3)$ restricted to the cubes $a + (-\frac{h}{2}, \frac{h}{2})^3$; this yields a piecewise constant map $R^{(h)} : S'_h \to \mathrm{SO}(3)$ such that,

$$\int_{S'_h \times (-\frac{1}{2}, \frac{1}{2})} |(\nabla' y^{(h)}, \frac{1}{h} y,_3^{(h)}) - R^{(h)}|^2\, dx \le C h^2. \tag{51}$$

To simplify the notation, let $\nabla_h y^{(h)}(x) = (\nabla' y^{(h)}(x), \frac{1}{h} y,_3^{(h)}(x)),\ x \in S \times (-\frac{1}{2}, \frac{1}{2})$. To estimate the variation of $R^{(h)}$ from a cube to a neighboring cube, we begin with the following simple estimate. Let $b = a + x_1 e_1 + x_2 e_2$, $x_1, x_2 \in \{-h, 0, h\}$. Then $S_{b,h} \subset S_{a,3h}$ so

$$\begin{aligned} |S_{b,h}||R^{(h)}(b) - R^{(3h)}(a)|^2 &\le 2 \int_{S_{b,h} \times (-\frac{1}{2}, \frac{1}{2})} |R^{(h)}(b) - \nabla_h y^{(h)}(x)|^2\, dx \\ &+ 2 \int_{S_{b,h} \times (-\frac{1}{2}, \frac{1}{2})} |R^{(3h)}(a) - \nabla_h y^{(h)}(x)|^2\, dx. \end{aligned} \tag{52}$$

Enlarge the second integral to the domain $S_{a,3h} \times (-\frac{1}{2}, \frac{1}{2})$ and apply Theorem 3.1 to the flattened cube $S_{a,3h} \times (-\frac{h}{2}, \frac{h}{2})$. Therefore, we have, using (51) and its analog for the flattened cube,

$$|S_{b,h}||R^{(h)}(b) - R^{(3h)}(a)|^2 \le C \int_{S_{a,3h} \times (-\frac{1}{2}, \frac{1}{2})} \operatorname{dist}^2(\nabla_h u^{(h)}, \mathrm{SO}(3))\, dx. \tag{53}$$

Since $|R^{(h)}(a) - R^{(h)}(b)|^2 \leq 2(|R^{(h)}(a) - R^{(3h)}(a)|^2 + |R^{(h)}(b) - R^{(3h)}(a)|^2)$, by (53) and its special case $a = b$

$$|S_{b,h}||R_h(a) - R_h(b)|^2 \leq C \int_{S_{a,3h}\times(-\frac{1}{2},\frac{1}{2})} \text{dist}^2(\nabla_h u^{(h)}, \text{SO}(3))\, dx, \tag{54}$$

which also can be written, using the piecewise constancy of $R^{(h)}$,

$$\int_{S_{a,h}} |R_h(x' + x_1e_1 + x_2e_2) - R_h(x')|^2\, dx' \leq C \int_{S(a,3h)\times(-\frac{1}{2},\frac{1}{2})} \text{dist}^2(\nabla_h u^{(h)}, \text{SO}(3))\, dx. \tag{55}$$

Hence for $\zeta \in I\!R^2$ satisfying $|\zeta|_\infty := \max\{|\zeta \cdot e_1|, |\zeta \cdot e_2|\} \leq h$,

$$\int_{S_{a,h}} |R^{(h)}(x' + \zeta) - R^{(h)}(x')|^2\, dx' \leq C \int_{S_{a,3h}\times(-\frac{1}{2},\frac{1}{2})} \text{dist}^2(\nabla_h y^{(h)}, \text{SO}(3))\, dx. \tag{56}$$

Now let S' be a compact subset of S, and consider a difference quotient with more general translation vector $\zeta \in I\!R^2$, $|\zeta|_\infty \leq c\,\text{dist}(S', \partial S)$. Let $N := \max\{[\frac{\zeta}{h} \cdot e_1], [\frac{\zeta}{h} \cdot e_2]\}$, where [ ] denotes the integer part, and pick $\zeta_0, \ldots, \zeta_{N+1}$ such that $\zeta_0 = 0$, $\zeta_{N+1} = \zeta$, $|\zeta_{k+1} - \zeta_k|_\infty \leq h$. Then $|R^{(h)}(x' + \zeta) - R^{(h)}(x')|^2 \leq (N + 1) \sum_{k=0}^{N} |R^{(h)}(x' + \zeta_{k+1}) - R^{(h)}(x' + \zeta_k)|^2$ and hence

$$\int_{S_{a,h}} |R^{(h)}(x' + \zeta) - R^{(h)}(x')|^2 dx' \leq C(N + 1) \sum_{k=0}^{N} \int_{S_{a+\zeta_k,3h}\times(-\frac{1}{2},\frac{1}{2})} \text{dist}^2(\nabla_h y^{(h)}, \text{SO}(3))\, dx.$$

Summing over all $S_{a,h} \cap S' \neq \emptyset$ and using that each $x \in S \times (-\frac{1}{2}, \frac{1}{2})$ is contained in at most $(N+1)C$ of the sets $S_{a+\zeta_k,3h} \times (-\frac{1}{2}, \frac{1}{2})$,

$$\int_{S'} |R^{(h)}(x' + \zeta) - R^{(h)}(x')|^2\, dx' \leq C\Big(\Big|\frac{\zeta}{h}\Big| + 1\Big)^2 \int_{S\times(-\frac{1}{2},\frac{1}{2})} \text{dist}^2(\nabla_h y^{(h)}, \text{SO}(3))\, dx \leq C(|\zeta| + h)^2. \tag{57}$$

This key estimate readily implies compactness of $R^{(h_j)}$ in $L^2(S')$, for any sequence $h_j \to 0$, as we shall now detail. Compactness is equivalent to validity of the Frechet-Kolmogorov criterion (see e.g. [1])

$$\limsup_{|\zeta|\to 0} \sup_{h_j} ||R^{(h_j)}(\cdot + \zeta) - R^{(h_j)}||_{L^2(S')} = 0. \tag{58}$$

Fix $\epsilon > 0$. Clearly the supremum over the finite set $\{h_j \,|\, h_j \geq \epsilon\}$ tends to zero as $|\zeta| \to 0$, since $||f(\cdot + \zeta) - f||_{L^2(S')}$ tends to zero for any fixed $f \in L^2(S)$. On the other hand the supremum over the remaining set $\{h_j \,|\, h_j < \epsilon\}$ satisfies $\limsup_{|\zeta|\to 0} \sup_{h_j<\epsilon} ||R^{(h_j)}(\cdot + \zeta) - R^{(h_j)}||^2_{L^2(S')} \leq C\epsilon^2$, by (57). Since ϵ was arbitrary, this establishes (58). Hence a subsequence of $R^{(h_j)}$ converges strongly in $L^2(S')$ to some $\bar{R} \in L^2(S')$ with $\bar{R}(x') \in \text{SO}(3)$ for a.e. $x' \in S'$.

We proceed to show strong convergence of the unapproximated sequence $\nabla_{h_j} y^{(h_j)}$, on the whole domain $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$. Since the sequence has bounded bending energy, one immediately has subsequential weak convergence $\nabla_{h_j} y^{(h_j)} \rightharpoonup (\nabla' y, b)$ in $L^2(\Omega)$. By (51), $R^{(h_j)} - \nabla_{h_j} y^{(h_j)} \to 0$ strongly in $L^2(S' \times (-\frac{1}{2}, \frac{1}{2}))$. Consequently $(\nabla' y, b) = \bar{R}$ for a.e. $x \in S' \times (-\frac{1}{2}, \frac{1}{2})$. In particular $(\nabla' y, b)$ is independent of x_3, and lies in SO(3) for a.e. $x \in S' \times (-\frac{1}{2}, \frac{1}{2})$. Since S' was an arbitrary compact subset of S, the above properties hold in all of Ω. Since $\text{dist}(\nabla_h y^{h_j}, \text{SO}(3)) \to 0$ in $L^2(\Omega)$ we have $|\nabla_{h_j} y^{(h_j)}|^2 \to 3 = |\bar{R}|^2$ in $L^1(\Omega)$, so that $||\nabla_{h_j} y^{(h_j)}||_{L^2(\Omega)} \to ||(\nabla' y, b)||_{L^2(\Omega)}$, which together with weak convergence in $L^2(\Omega)$ implies strong convergence in $L^2(\Omega)$.

Finally, letting $h \to 0$ in (57) yields

$$\int_{S'} \left| \frac{(\nabla' y, b)(x' + \zeta) - (\nabla' y, b)(x')}{|\zeta|} \right|^2 dx' \leq C,$$

which implies $(\nabla' y, b) \in H^1(S')$. Because C is independent of S', in fact we have $(\nabla' y, b) \in H^1(S)$. □

5 Noncompactness by wrinkling

Bending energy occurs at order h^3 while membrane energy occurs at order h (Recall that one power of h was absorbed by the change of variables leading to (10)). It is therefore interesting to ask whether a sequence $y^{(h)}$ is compact if we assume the energy is bounded by a power of h between h and h^3 (resp., between 1 and h^2 in rescaled variables). The answer is no: the simple examples below, which only involve bending in the $x_1 - x_3$ plane as captured by Euler-Bernoulli kinematics generalized to finite thickness, show that (in rescaled variables) there are sequences $y^{(h)}$ that satisfy

$$\limsup_{h\to 0} \frac{1}{h^\alpha} \int_\Omega \mathrm{dist}^2((\nabla' y^{(h)}, \frac{1}{h} y,_3^{(h)}), \mathrm{SO}(3))\, dx < \infty \tag{59}$$

for $\alpha < 2$ but arbitrarily close to 2, such that $(\nabla' y^{(h)}, \frac{1}{h} y,_3^{(h)})$ converges weakly but not strongly to SO(3). In particular, infinite bending energy is compatible with zero membrane energy.

Let $\theta^{(h)} \in W^{1,2}(I\!R)$, $S = (0, L) \times (-b/2, b/2)$ and consider a sequence of deformations of Euler–Bernoulli type (12), modified to account for finite thickness in a way that preserves zero membrane energy:

$$y^{(h)}(x', x_3) = x_2 e_2 + \Big(\int_0^{x_1} \cos\, \theta^{(h)}(s)\, ds\Big) e_1 + \Big(\int_0^{x_1} \sin\, \theta^{(h)}(s)\, ds\Big) e_3 + h\, x_3 b^{(h)}(x_1), \tag{60}$$

where,

$$b^{(h)}(x_1) = -\sin\, \theta^{(h)}(x_1)\, e_1 + \cos\, \theta^{(h)}(x_1)\, e_3. \tag{61}$$

We have,

$$\begin{aligned} (\nabla' y^{(h)}, \frac{1}{h} y,_3^{(h)}) &= (\cos\, \theta^{(h)}\, e_1 + \sin\, \theta^{(h)}\, e_3 + h\, x_3 \frac{db^{(h)}}{dx_1}\, , e_2\, , -\sin\, \theta^{(h)}\, e_1 + \cos\, \theta^{(h)}\, e_3), \\ &= R^{(h)}(I - h\, x_3 \frac{d\theta^{(h)}}{dx_1} e_1 \otimes e_1), \end{aligned} \tag{62}$$

where $R^{(h)} = \cos\, \theta^{(h)} e_1 \otimes e_1 + \sin\, \theta^{(h)} e_3 \otimes e_1 - \sin\, \theta^{(h)} e_1 \otimes e_3 + \cos\, \theta^{(h)} e_3 \otimes e_3 + e_2 \otimes e_2 \in \mathrm{SO}(3)$. For $|\frac{1}{2} h\, (d\theta^{(h)}/dx_1)| < 1$ (62) is the polar decomposition, so in that case

$$\mathrm{dist}^2((\nabla' y^{(h)}, \frac{1}{h} y,_3^{(h)}), \mathrm{SO}(3)) = |h\, x_3 \frac{d\theta^{(h)}}{dx_1} e_1 \otimes e_1)|^2 = h^2\, x_3^2 \Big(\frac{d\theta^{(h)}}{dx_1}\Big)^2 \tag{63}$$

Then (59) becomes

$$\limsup_{h\to 0} h^{(2-\alpha)} \int_0^L \frac{1}{12} (\frac{d\theta^{(h)}}{dx_1})^2\, dx_1 < \infty. \tag{64}$$

As a particular example we may choose $\theta^{(h)}$ to be smooth and periodic with period h^β satisfying

$$\theta^{(h)}(x_1) = \begin{cases} \theta_1 & \text{on} \quad (0, (1/4)h^\beta - \frac{1}{2}\, h^\gamma], \\ \theta_2 & \text{on} \quad ((1/4)h^\beta + \frac{1}{2}\, h^\gamma, (3/4)h^\beta - \frac{1}{2}\, h^\gamma], \\ \theta_1 & \text{on} \quad ((3/4)h^\beta + \frac{1}{2}\, h^\gamma, h^\beta], \end{cases} \tag{65}$$

with $\theta_2 > \theta_1$, $1 > \gamma > \beta > 0$, and $|d\theta^{(h)}/dx_1| < 2(\theta_2 - \theta_1)h^{-\gamma}$. The condition $\gamma < 1$ ensures that $|\frac{1}{2}\, h\, (d\theta^{(h)}/dx_1)| < 1$ for h sufficiently small, validating (63). Thus,

$$h^{(2-\alpha)} \int_0^L \frac{1}{12} (\frac{d\theta^{(h)}}{dx_1})^2\, dx_1 \le L(\theta_2 - \theta_1)^2 h^{(2-\alpha-\beta-\gamma)} \tag{66}$$

So, if $\beta + \gamma$ is chosen sufficiently small then (64) holds, but clearly the sequence $(\nabla' y^{(h)}, \frac{1}{h} y,_3^{(h)})$ is not compact in L^2.

6 The limiting plate theory for minimizing deformations having finite bending energy

Theorem 4.1 says that a sequence $(\nabla' y^{(h)}, \frac{1}{h} y,_3^{(h)})$ with finite bending energy is compact and its limit $(\nabla' y, b))$ lies on SO(3); in particular $b = y_{,1} \wedge y_{,2}$. We now show that if this sequence is (exactly or approximately) minimizing subject to appropriate boundary conditions, then its limiting bending energy can be expressed solely in terms of y, and there is a variational principle for the limit.

In the spirit of Γ-convergence we first study arbitrary sequences with finite bending energy, not required to satisfy boundary conditions.

Theorem 6.1 *For $h \to 0$, the functional $\frac{1}{h^3} E^{(h)}$ (as defined in (10)) converges to the limit functional I^0 given below, in the following sense (amounting to Γ-convergence on $W^{1,2}(\Omega; I\!R^3)$ in the language of [13, 12, 9]):*

(i) (Ansatz-free lower bound) If a sequence $y^{(h)} \subset W^{1,2}(\Omega; I\!R^3)$ converges to y in $W^{1,2}$ then $\liminf_{h\to 0} \frac{1}{h^3} E^{(h)}(y^{(h)}) \ge I^0(y)$,

(ii) (Attainment of lower bound) For all $y \in W^{1,2}(\Omega; I\!R^3)$ there exists a sequence $y^{(h)} \subset W^{1,2}$ converging to y in $W^{1,2}$ such that $\lim_{h\to 0} \frac{1}{h^3} E^{(h)}(y^{(h)}) = I^0(y)$.

The limit functional I^0 is given by

$$I^0(y) := \begin{cases} \frac{1}{24} \int_S Q_2(\mathrm{II}) dx' & \text{if } y(x) \text{ is independent of } x_3 \text{ and } y \in \mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases}$$

Here the class $\mathcal{A}$ of admissible maps consists of isometries from S into $I\!R^3$,

$$\mathcal{A} = \{y \in W^{2,2}(S; I\!R^3) \,:\, |y_{,1}| = |y_{,2}| = 1,\, y_{,1} \cdot y_{,2} = 0\}$$

and II is the second fundamental form (or extrinsic curvature tensor) $\mathrm{II}_{ij} = ((\nabla' y)^T \nabla' b)_{ij} = y_{,i} \cdot b_{,j}$, $\quad b = y_{,1} \wedge y_{,2}$. The quadratic form Q_2 on $M^{2\times 2}$ is defined by

$$Q_2(G) := \min_{c \in I\!R^3} Q_3(\hat{G} + c \otimes e_3) \tag{67}$$

where $\hat{G}$ is the 3×3 matrix $\sum_{i,j=1}^{2} G_{ij} e_i \otimes e_j$, and Q_3 is the quadratic form of linear elasticity theory on $M^{3\times 3}$,

$$Q_3(F) := \frac{\partial^2 W}{\partial F^2}(I)(F,F) = \sum_{i,j,k,l=1}^{3} \frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(I) F_{ij} F_{kl}. \tag{68}$$

Remarks.

i) In particular, as proved earlier (Theorem 4.1), if the sequence has bounded bending energy then the limit y has the higher regularity $y \in W^{2,2}(S; I\!R^3)$.

ii) The result remains valid if strong $W^{1,2}$ convergence in (i) is replaced by weak convergence, as proved below.

iii) An interesting technical aspect of our result is that no growth condition from above was imposed on W. This means that in order to establish (ii), we will have to construct approximating sequences whose gradient stays bounded in L^∞, even when the gradient of the normal of the limit map is not in L^∞. This will be achieved with the help of fine truncation arguments for Sobolev maps, discussed in Appendix A. In fact, for any given $\epsilon > 0$ the approximating sequences constructed can be chosen to satisfy $\mathrm{dist}(\nabla_h y^{(h)}, \mathrm{SO}(3)) \le \epsilon$ for all sufficiently small h. (This follows from (90), noting that the constant C in that estimate is independent of c and that c, introduced below (89), can be chosen as small as we wish.) Consequently, the proof below shows that Theorem 6.1 remains valid when hypothesis (i) on W is replaced by (i') $W \in C^0(U)$ for some open set $U \supset \mathrm{SO}(3)$, $W = +\infty$ outside U, $W \in C^2$ in a neighbourhood of SO(3). This allows one in particular to prove the full Gamma convergence result in the setting considered by Pantz [36] (adapted here to the case without boundary conditions). He works with modified energies $\tilde{E}^{(h)}(y)$ which are $+\infty$ unless $y \in C^1$ and $|(\nabla y)^T \nabla y - I| \le \delta$.

iv) Consider the case when W is isotropic, i.e $W(RFQ) = W(F)$ for all $F \in M^{3\times 3}$ and all R, $Q \in \mathrm{SO}(3)$, so that

$$Q_3(F) = 2\mu\,|e|^2 + \lambda(\mathrm{trace}\, e)^2, \quad e = \frac{F+F^T}{2}, \tag{69}$$

for some λ, $\mu \in I\!R$. Then an elementary calculation shows that the quadratic form on $M^{2\times 2}$ defined by (67) is

$$Q_2(G) = 2\mu \left|\frac{G+G^T}{2}\right|^2 + \frac{\lambda\mu}{\mu + \frac{\lambda}{2}}(\mathrm{trace}\, G)^2.$$

Since $\mathrm{II}(x')$ is automatically symmetric for every x', it follows that

$$I^0(y) = \begin{cases} \frac{1}{24}\int_S \Big(2\mu|\mathrm{II}|^2 + \frac{\lambda\mu}{\mu+\lambda/2}(\mathrm{trace}\,\mathrm{II})^2\Big)\, dx' & \text{if } y \text{ is independent of } x_3 \text{ and } y \in \mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases} \tag{70}$$

This agrees with the expression proposed on the basis of insightful ad hoc assumptions by Kirchhoff in 1850 [24], but not with a well known simplified expression which replaces the isometry constraint $y \in \mathcal{A}$ by the condition $y_1 = y_2 = 0$ and II by $-\nabla^2 y_3$, which much of the subsequent literature has associated with Kirchhoff's name, as noted in the Introduction.

A large literature exists devoted to deriving the bending energy of an isotropic plate under unproven assumptions on the 3D deformations weaker than those of [24]. The furthest results are those of [36] who showed that Kirchhoff's functional (70) is a lower bound for the Γ-limit of a certain constrained elasticity functional, which is set equal to $+\infty$ except when the 3D deformation $v : \Omega_h \to I\!R^3$ is a C^1 diffeomorphism with $\mathrm{dist}(\nabla v(x), \mathrm{SO}(3)) < \delta$ for all $x \in \Omega_h$ and some sufficiently small δ. As emphasized in Section 3 in our discussion of F. John's classical rigidity results (on which the results in [36] are based), such restrictions on v do not follow from smallness of the elastic energy. For our ansatz-free derivation of I^0 the sharp results of Section 3 are essential.

v) Specializing further, if $W(F) = \mathrm{dist}(F, \mathrm{SO}(3))^2$ then $I^0(y) = \frac{1}{12}\int_S |\mathrm{II}|^2 dx'$ on $\mathcal{A}$, which, up to the numerical prefactor, agrees on isometries with the Willmore functional arising in differential geometry.

vi) For W isotropic as in (iv), $S = (0, L) \times (0, w)$, and deformations $y \in \mathcal{A}$ of Euler-Bernoulli form (12), we have $\mathrm{II}_{11}(x') = -\theta'(x_1)$ and the remaining components of II vanish, whence

$$I^0(y) = \frac{1}{2} EI \int_0^L \theta'(x_1)^2 dx_1, \quad E = 2\mu + \frac{\mu\lambda}{\mu + \lambda/2}. \tag{71}$$

Thus the functional form of I^0 agrees with that proposed in Euler's celebrated 1744 paper [16]; in addition, our result yields the plate modulus. To our knowledge, ours is the first rigorous derivation of the functionals (71) and (70) from 3D elasticity.

vii) Pantz ([36], Remark 1) raised the question whether in the description of the admissible set $\mathcal{A}$ it suffices to assume regularity of the normal, i.e., whether in fact

$$\mathcal{A} = \{y \in W^{1,\infty}(S, I\!R^3) : |y_{,1}| = |y_{,2}| = 1,\ y_{,1} \cdot y_{,2} = 0,\ y_{,1} \wedge y_{,2} \in W^{1,2}(S, I\!R^3)\}. \tag{72}$$

This is indeed the case. First, by a density argument any two maps y and z in $W^{1,2}(S, I\!R^3)$ satisfy $(y_{,2} \wedge z)_{,1} - (y_{,1} \wedge z)_{,2} = y_{,2} \wedge z_{,1} - y_{,1} \wedge z_{,2}$ in the sense of distributions. Second, if y is an isometry and $z = y_{,1} \wedge y_{,2}$ then $y_{,2} \wedge z = y_{,1}$ and $-y_{,1} \wedge z = y_{,2}$. Thus $\triangle y \in L^2$ in the sense of distributions. Hence $y \in W^{2,2}_{loc}$ and we can apply the chain rule to differentiate the isometry conditions and obtain $y_{,ij} = (y_{,ij} \cdot z)z = -(y_{,i} \cdot z_{,j})z$. This yields $y \in W^{2,2}(S, I\!R^3)$ as claimed.

Proof of Theorem 6.1 (i) Consider an arbitrary sequence $y^{(h)}$ converging weakly in $W^{1,2}(\Omega; I\!R^3)$, and let y denote its weak limit. To measure the deviation of $\nabla_h y^{(h)} = (\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3})$ from SO(3) we recall the lattice of squares S'_h and the piecewise constant approximation $R^{(h)} : S'_h \to \mathrm{SO}(3)$ introduced in (49), (50), (51), and consider the quantity $G^{(h)} : S'_h \to M^{3\times3}$ defined by

$$G^{(h)}(x', x_3) = \frac{R^{(h)}(x')^T \nabla_h y^{(h)}(x', x_3) - I}{h}. \tag{73}$$

By the basic estimate (51) which followed from Theorem 3.1, $||G^{(h)}||_{L^2(S'_h \times(-1/2,1/2))} \le C$. Hence, extending $G^{(h)}$ by zero to all of $S \times (-1/2, 1/2) = \Omega$, there exists a subsequence (not relabelled) and a $G \in L^2(\Omega)$ such that

$$G^{(h)} \rightharpoonup G \ \text{ in } L^2(\Omega). \tag{74}$$

The first task is to estimate the bending energy from below in terms of G; the second, to identify G in terms of the limiting deformation y.

We expand W around the identity: $W(I+A) = \frac{1}{2}Q_3(A) + \eta(A)$, where Q_3 is the quadratic form of linear elasticity theory introduced above, and $\eta(A)/|A|^2 \to 0$ as $|A| \to 0$. Letting $\omega(t) := \sup_{|A|\le t} |\eta(A)|$ we have

$$W(I+hA) \ge \tfrac{1}{2}\, Q_3(hA) - \omega(|hA|) \tag{75}$$

where $\omega(t)/t^2 \to 0$ as $t \to 0$. Define

$$\chi_h(x) := \begin{cases} 1 & x \in S_h' \cap \{|G^{(h)}(x)| \le h^{-1/2}\} \\ 0 & \text{otherwise.} \end{cases} \tag{76}$$

By the boundedness of $G^{(h)}$ in $L^2(S \times (-\frac{1}{2}, \frac{1}{2}))$ and the fact that $S_h' \supset \{x \in S : \operatorname{dist}(x, \partial S) \ge Ch\}$, $\chi_h \to 1$ boundedly in measure. Hence

$$\chi_h G^{(h)} \rightharpoonup G \ \text{ in } L^2(\Omega). \tag{77}$$

Now using the frame-indifference of W and (75),

$$\begin{aligned} \frac{1}{h^2}\int_\Omega W(\nabla_h y^{(h)})\,dx &\ge \frac{1}{h^2}\int_\Omega \chi_h W(\nabla_h y^{(h)})\,dx \\ &= \frac{1}{h^2}\int_\Omega \chi_h W((R^{(h)})^T \nabla_h y^{(h)})\,dx \\ &\ge \int_\Omega \tfrac{1}{2}\chi_h Q_3(G^{(h)}) - \frac{1}{h^2}\chi_h \omega(h|G^{(h)}|)\,dx. \end{aligned} \tag{78}$$

As regards the first term, since Q_3 is quadratic the function χ_h can be pulled inside Q_3, and since Q_3 is nonnegative definite (by the hypotheses on W), it is lower semicontinuous with respect to the convergence (77). The second term on the right converges to zero, because $|G^{(h)}|$ is bounded in $L^2(\Omega)$ and $h|G^{(h)}| \le h^{\frac{1}{2}}$ wherever $\chi_h \neq 0$, whence $|G^{(h)}|^2 \cdot \chi_h \omega(h|G^{(h)}|)/(h|G^{(h)}|)^2$ is the product of a bounded sequence in L^1 and a sequence tending to zero in L^∞. Putting these two facts together we obtain

$$\liminf_{h\to 0} \frac{1}{h^2}\int_\Omega W(\nabla_h y^{(h)})\,dx \ge \tfrac{1}{2}\int_\Omega Q_3(G)\,dx. \tag{79}$$

Finally we use the trivial bound

$$Q_3(A) \ge Q_2(A') \tag{80}$$

where here and below we use the convention that A' denotes the 3×3 matrix obtained from A by putting zeros in its third row and third column (cf. (67)). Consequently

$$\liminf_{h\to 0} \frac{1}{h^2}\int_\Omega W(\nabla_h y^{(h)})\,dx \ge \tfrac{1}{2}\int_{S\times(-\frac{1}{2},\frac{1}{2})} Q_2(G')\,dx. \tag{81}$$

To identify the weak limit G' in terms of y, we denote the matrix consisting of the first two columns of $G^{(h)}$ (respectively G) by $\tilde{G}^{(h)}$ (resp. $\tilde{G}$) and consider the finite difference quotient in x_3-direction

$$H^{(h)}(x', x_3) := \frac{\tilde{G}^{(h)}(x', x_3+z) - \tilde{G}^{(h)}(x', x_3)}{z} = (R^{(h)}(x'))^T \frac{\frac{1}{h}\nabla' y^{(h)}(x', x_3+z) - \frac{1}{h}\nabla' y^{(h)}(x', x_3)}{z}.$$

Let Ω' be any compact subset of Ω and let $|z| < \operatorname{dist}(\Omega', \partial\Omega)$. By (74)

$$H^{(h)} \rightharpoonup H := \frac{\tilde{G}(x', x_3+z) - \tilde{G}(x', x_3)}{z} \ \text{ in } L^2(\Omega').$$

By Theorem 4.1, $R^{(h)}$ converges boundedly in measure to $(\nabla' y, b) \in H^1(\Omega)$ and $b = y_{,1} \wedge y_{,2}$. It follows that

$$\frac{\frac{1}{h}\nabla' y^{(h)}(x', x_3 + z) - \frac{1}{h}\nabla' y^{(h)}(x', x_3)}{z} = R^{(h)} H^{(h)} \rightharpoonup (\nabla' y|b)H \text{ in } L^2(\Omega'). \tag{82}$$

To identify H note that the left hand side equals $\nabla'(\frac{1}{z}\int_{x_3}^{x_3+z} \frac{1}{h} y_{,3}^{(h)}(x', s)ds)$. Since $\frac{1}{h} y_{,3}^{(h)} \to b$ strongly in $L^2(\Omega; I\!R^3)$, the transverse average $\frac{1}{z}\int_{x_3}^{x_3+z} \frac{1}{h} y_{,3}^{(h)} ds$ converges strongly to $\frac{1}{z}\int_{x_3}^{x_3+z} b\, ds$, which moreover equals b, by the x_3-independence of b. Hence

$$\frac{\frac{1}{h}\nabla' y^{(h)}(x', x_3 + z) - \frac{1}{h}\nabla' y^{(h)}(x', x_3)}{z} \rightharpoonup \nabla' b \text{ in } W^{-1,2}(\Omega'). \tag{83}$$

Combining (82) and (83), we have $b \in W^{1,2}(\Omega')$ and $H = (\nabla' y, b)^T \nabla' b$. In particular H is independent of x_3 and hence

$$\tilde{G}(x', x_3) = \tilde{G}(x', 0) + x_3 H(x').$$

Hence by omitting the third row

$$G'(x', x_3) = G'(x', 0) + x_3 \mathrm{II}(x'), \quad \mathrm{II} = (\nabla' y)^T \nabla' b. \tag{84}$$

Since Ω' was arbitrary, the above identity holds in all of Ω. Consequently the right hand side of (81) becomes

$$\frac{1}{2}\int_\Omega Q_2(G')\, dx \;=\; \frac{1}{2}\int_\Omega Q_2(G'(x', 0))\, dx \;+\; \frac{1}{2}\int_\Omega x_3^2\, Q_2(\mathrm{II})\, dx, \tag{85}$$

the absence of a coupling term being due to the fact that $\int_{-1/2}^{1/2} x_3\, dx = 0$. Dropping the (nonnegative) first term and carrying out the x_3-integration yields (i). $\square$

Proof of Theorem 6.1 (ii) If $y \notin \mathcal{A}$ the assertion is trivial, so assume $y \in \mathcal{A}$; in particular $y \in W^{2,2}(S; I\!R^3)$, $y_{,1} \wedge y_{,2} =: b \in W^{1,2}(S; I\!R^3)$. Since S is strongly Lipschitz, we can extend y and b to maps in $W^{2,2}(I\!R^2; I\!R^3)$ respectively $W^{1,2}(I\!R^2; I\!R^3)$. Next we invoke a truncation result for Sobolev maps defined on $I\!R^n$ ([31], [42], [17]), which yields, for any $\lambda > 0$, the existence of $y^\lambda \in W^{2,\infty}(I\!R^2; I\!R^3)$ and $b^\lambda \in W^{1,\infty}(I\!R^2; I\!R^3)$ such that

$$||\nabla^2 y^\lambda||_{L^\infty}, \ ||\nabla b^\lambda||_{L^\infty} \le \lambda, \quad |S^\lambda| \le C\frac{\omega(\lambda)}{\lambda^2}, \tag{86}$$

where

$$S^\lambda = \{x \in I\!R^2 \,|\, y(x) \neq y^\lambda(x) \text{ or } b(x) \neq b^\lambda(x)\},$$
$$\omega(\lambda) = \int_{|\nabla^2 y| \ge \lambda/2} (|y|^2 + |\nabla y|^2 + |\nabla^2 y|^2)\, dx + \int_{|\nabla b| \ge \lambda/2} (|b|^2 + |\nabla b|^2)\, dx \to 0 \ \ (\lambda \to \infty).$$

In fact we may assume $b^\lambda \in C^1$, see e.g., [17], Thm. 1, p. 251.

An interesting consequence, which is related to the fact that in two dimensions $W^{1,2}$ embeds almost into L^∞, is that for all sufficiently large λ

$$f^\lambda(x) := \mathrm{dist}((\nabla' y^\lambda(x), b^\lambda(x)), \mathrm{SO}(3)) \le C w(\lambda)^{1/2}, \quad \forall x \in S. \tag{87}$$

To prove this, note first that $f^\lambda = 0$ on $S\backslash S^\lambda$, and that f^λ is Lipschitz with Lipschitz constant $\mathrm{Lip}\, f^\lambda = \sup_{x\neq y} \frac{|f^\lambda(x)-f^\lambda(y)|}{|x-y|} \leq C\lambda$. Next we claim that for a suitable constant δ (depending only on S), for $R := \frac{\delta}{\lambda}\omega(\lambda)^{1/2}$, and all $x_0 \in S$

$$B(x_0, R) \cap (S\backslash S^\lambda) \neq \emptyset. \tag{88}$$

Otherwise, $|B(x_0,R)\cap S| = |B(x_0,R)\cap S^\lambda| \leq |S^\lambda| \leq C\frac{\omega(\lambda)}{\lambda^2}$, which contradicts the fact that due to the Lipschitz property of S

$$|B(x_0,R)\cap S| \geq AR^2 = \frac{A\delta^2\omega(\lambda)}{\lambda^2},$$

as soon as $\delta < (C/A)^{1/2}$. This establishes (88). It follows that

$$f(x) \leq (\mathrm{Lip}\, f)\, R \leq C\delta(\omega(\lambda))^{1/2} \quad \forall x \in S,$$

establishing (87).

Now consider the trial function

$$y^{(h)}(x',x_3) = y^{\lambda_h}(x') + hx_3 b^{\lambda_h}(x') + h^2\frac{x_3^2}{2}d(x'), \tag{89}$$

with truncation scale $\lambda_h = c/h$ and with $d \in C_0^1(S, I\!R^3)$. Let $R(x') := (\nabla' y(x'), b(x')) \in \mathrm{SO}(3)$, and denote

$$R^T(\nabla' y^{(h)}, \frac{1}{h}y^{(h)}_{,3}) = R^T\Big((\nabla' y^{\lambda_h}, b^{\lambda_h}) + hx_3(\nabla' b^{\lambda_h}, d) + h^2\frac{x_3^2}{2}(\nabla' d, 0)\Big) =: I + A^{(h)}.$$

On the good set $S\backslash S^{\lambda_h}$, we have $R^T(\nabla' y^{\lambda_h}, b^{\lambda_h}) = I$ and

$$|A^{(h)}| \leq C(h\lambda_h + h + h^2) \leq C(c + h_0 + h_0^2) \quad \text{for all } h \leq h_0. \tag{90}$$

Denoting by χ_h the characteristic function of $S\backslash S^{\lambda_h}$, choosing $c > 0$ and h_0 sufficiently small and using that W is C^2 in a neighbourhood of the identity we obtain, for all $h \leq h_0$,

$$\frac{1}{h^2}\chi_h W(I + A^{(h)}) \left\{ \begin{array}{l} \leq \frac{C}{h^2}|A^{(h)}|^2 \leq 2C(|(\nabla' b, d)|^2 + h_0^2|\nabla' d|^2) \in L^1(\Omega), \\ \to \frac{1}{2}Q_3(x_3 R^T(\nabla' b, d)) \text{ a.e.} \end{array} \right.$$

Thus by dominated convergence

$$\frac{1}{h^2}\int_\Omega \chi_h W(\nabla' y^{(h)}, \frac{1}{h}y^{(h)}_{,3})\, dx = \frac{1}{h^2}\int_\Omega \chi_h W(I + A^{(h)})\, dx \to \frac{1}{2}\int_\Omega x_3^2 Q_3(R^T(\nabla' b, d))\, dx. \tag{91}$$

On the bad set S^λ, we have $\mathrm{dist}(I + A^{(h)}, \mathrm{SO}(3)) \leq C$, giving, due to the local boundedness of W, the estimate $W(I + A^{(h)}) \leq C$. Consequently

$$\frac{1}{h^2}\int_\Omega (1-\chi_h) W(I + A^{(h)}) \leq C\frac{|S^{\lambda_h}|}{h^2} = \frac{C}{c^2}\lambda_h^2|S^{\lambda_h}| \to 0 \quad (h \to 0). \tag{92}$$

Combining (91) and (92) and carrying out the integration over x_3,

$$\frac{1}{h^2}\int_\Omega W(\nabla' y^{(h)}, \frac{1}{h}y^{(h)}_{,3}) \to \frac{1}{24}\int_S Q_3(R^T(\nabla' b, d))\, dx' \quad (h \to 0). \tag{93}$$

It remains to construct a sequence whose energy converges to the above right hand side with $d \in W_0^{1,\infty}$ replaced by $d_{min}(x') := \operatorname{argmin} Q_3(R^T(x')(\nabla' b(x'), d)) \in L^2$, in which case the right hand side equals $I^0(y)$. We use the density of C_0^1 in L^2 and the continuity of the above right hand side in L^2 to pick a sequence $d^j \subset C_0^1$ such that

$$\frac{1}{24}\int_S Q_2(R^T(\nabla' b, d^j)) \leq I^0(y) + \frac{1}{j}.$$

Hence, if h_j is chosen sufficiently small, due to (93) the sequence (89) with $h = h_j$ and $d = d_j$ satisfies $\frac{1}{h_j^3}E^{h_j}(y^{(h_j)}) \leq I^0(y) + \frac{2}{j}$ and $y^{(h_j)} \to y$ in $W^{1,2}$, as required. □

As explained in the introduction, sequences that satisfy rather innocent looking boundary conditions – even those for which the minimizing membrane energy is identically zero – may necessarily have infinite bending energy in the limit. The complete and explicit characterization of all boundary conditions consistent with finite bending energy appears difficult. The theorem below gives a general result for clamped boundary conditions on part of ∂S. In particular it applies to the classical boundary value problem of uniaxial compression:

Example *Let S be the rectangular domain $(0, L) \times (0, w)$, so that Ω_h is the standard plate $(0, L) \times (0, w) \times (-\frac{h}{2}, \frac{h}{2})$, and consider the following longitudinal compression boundary condition on the unrescaled deformation $v : \Omega_h \to I\!R^3$, applied at the right and left end of the plate*

$$v(z)|_{z_1=0,\,L} = z \mp (a, 0, 0),$$

where $a \in (0, L/2)$ is fixed (and the remaining part of the boundary is left free). Equivalently, the rescaled deformation $y^{(h)} : \Omega = (0, L) \times (0, w) \times (-\frac{1}{2}, \frac{1}{2})$ defined by $y^{(h)}(x) = v(z(x))$, $z = (x_1, x_2, hx_3)$ (see Section 2) satisfies

$$y^{(h)}(x_1, x_2, x_3)|_{x_1=0,\,L} = (x_1 \mp a, x_2, hx_3). \tag{94}$$

Since we have assumed no particular convexity properties of W (such as those in [5]) away from SO(3), the infimum of the total energy at nonzero h may not be attained. We shall therefore consider deformations of the plate of sufficiently low energy. Specifically, we say that a sequence $y^{(h)} \subset W^{1,2}(\Omega; I\!R^3)$ with finite elastic energy is a *low energy sequence* if its scaled elastic energy differs from that of the infimum by a tolerance of $\omega(h)$, where the function $\omega(h) \to 0$ as $h \to 0$.

As above, our results could be stated in the formal language of Γ-convergence, but for simplicity we follow the direct approach.

Theorem 6.2 *Let S be a bounded Lipschitz domain and let $\Gamma \subset \partial S$ be a finite union of (nontrivial) closed intervals (i.e. maximally connected sets in ∂S). Consider*

$$\hat{y} \in W^{2,2}(S; I\!R^3) \cap C^1(\bar{S}; I\!R^3),\ \hat{b} \in W^{1,\infty}(S; I\!R^3). \tag{95}$$

Suppose that $y^{(h)} \subset W^{1,2}(\Omega; I\!R^3)$ is a low energy sequence, in the sense that

$$\frac{1}{h^2}\int_\Omega W(\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3})\,dx \leq \inf_{\substack{y \in W^{1,2}(\Omega; I\!R^3) \\ y\ satisfies\ \mathrm{BC}}} \frac{1}{h^2}\int_\Omega W(\nabla' y, \frac{1}{h} y_{,3})\,dx + \omega(h) \tag{96}$$

and suppose that the infimum on the right hand side of (96) remains bounded as $h \to \infty$*. Here* BC *refers to the boundary conditions*

$$BC: \quad y^{(h)}(x', x_3) = \hat{y}(x') + x_3 \hat{b}(x'), \quad x' \in \Gamma,\, x_3 \in (-\tfrac{1}{2}, \tfrac{1}{2}). \tag{97}$$

Then there exists a subsequence, not relabelled, such that $(\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3}) \to (\nabla' y, b)$ *in* $L^2(\Omega)$*, the limit map* y *is an isometry belonging to the class* $\mathcal{A}$ *introduced in Theorem 6.1 and is independent of* x_3*, and* $b = y_{,1} \wedge y_{,2}$*. The limiting bending energy of this sequence is*

$$\lim_{h \to 0} \frac{1}{h^2} \int_\Omega W(\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3}) dx = \frac{1}{24} \int_S Q_2(\nabla' y^T \nabla' b)\, dx' = I_0(y) \tag{98}$$

and y *satisfies the clamped boundary conditions*

$$y = \hat{y} \text{ and } b = \hat{b} \quad \text{on } \Gamma. \tag{99}$$

Moreover y *minimizes* I^0 *among all functions in* $\mathcal{A}$ *which satisfy (99).*

Remarks.

i) The boundary condition BC comprises (94) as a special case, as follows. Let $\Gamma = \{0, L\} \times [0, w]$, let $\hat{b}(x') = e_3$ and choose $\hat{y}$ to be a smooth extension of the map $\hat{y}(x') = x' + (a, 0)$ for $x_1 = 0$ and $x_2 \in (0, w)$, $\hat{y}(x') = x' - (a, 0)$ for $x_1 = 0$ and $x_2 \in (0, w)$.

ii) Note that we do not require that $\hat{y}$ is an isometry. It may happen that there is no map $y \in \mathcal{A}$ which satisfies (99). The proof given below shows in particular that this happens if and only if the right hand side of (96) blows up as $h \to 0$.

iii) In the literature sometimes $y^{(h)}$ is prescribed on an open set rather than on the boundary. Our approach can easily be adapted to this setting. In fact the verification of (99) is easier since we do not need to study traces and their convergence. For the construction of a low energy sequence one can use Proposition A.3 which is a minor variant of Proposition A.2. In particular Remark (iv) below applies to both settings.

iv) The sequence $y^{(h)}$ which we construct satisfies also $\operatorname{dist}((\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3}), \mathrm{SO}(3)) \to 0$ uniformly. If we assume in addition that $\hat{b} \in C^1(\bar{S}; I\!R^3)$ then we can choose $y^{(h)} \in C^1(\Omega; I\!R^3)$. This allows one to establish the upper bound in Pantz' approach [36] for general limit maps in $\mathcal{A}$ rather than just C^2 isometries.

v) If W satisfies a growth condition from above of the form $W(A) \le c_+ |A - I|^2$, the assumptions on the boundary data can be weakened to $\hat{y}, \hat{b} \in W^{1,2}(U; I\!R^3)$. Moreover in that case the proof is simplified. If $y \in \mathcal{A}$ and $b = y_{,1} \wedge y_{,2}$ satisfy (99) we can simply chose the trial function $\tilde{y}^{(h)}(x) = y(x') + h\, x_3\, b(x') + h^2\, (x_3^2/2)\, d(x')$ with $d \in W_0^{1,\infty}(S; I\!R^3)$. Then one passes to the limit $h \to 0$ using dominated convergence and minimizes out d at the last stage.

The above theorem entails an existence result for the limit problem.

Corollary 6.1 *Let S, Γ, $\hat{y}$ and $\hat{b}$ be as in Theorem 6.2.*

a) Let Q_2 be any quadratic form on $M^{2\times 2}$ which arises via (67), (68) from some $W : M^{3\times 3} \to I\!R$ satisfying hypotheses i), ii), iii) in Section 2. Let $\mathcal{A}_{BC}$ the set of maps in $\mathcal{A}$ which satisfy (99). Then I^0 attains its minimum in $\mathcal{A}_{BC}$ provided that this set is not empty.

b) There exists a minimizer of the Willmore functional $\frac{1}{12}\int_S |II|^2\, dx'$ among isometries $y \in \mathcal{A}_{BC}$ provided this set is not empty.

Existence results for minimizers of the Willmore functional among closed surfaces not required to be isometric to any reference surface, but of prescribed genus, were obtained by L. Simon [38], through a direct study of minimizing sequences and a careful study of 'bubbling' phenomena.

Proof of Corollary 6.1. a) is immediate from Theorem 6.2, and b) follows from a) by taking $W(F) = \mathrm{dist}^2(F, \mathrm{SO}(3))$.

Proof of Theorem 6.2. We first show that for every $y \in \mathcal{A}$ satisfying the 2D boundary conditions (99) there exists a sequence $\check{y}^{(h)} : S \times (-\frac{1}{2}, \frac{1}{2}) \to I\!R^3$ which satisfies the 3D boundary conditions (97), $\check{y}^{(h)} \to y$ in $W^{1,2}(\Omega; I\!R^3)$ and

$$\lim_{h\to 0} \frac{1}{h^2} \int_\Omega W(\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3}) dx = I_0(y). \tag{100}$$

Thus in particular the right hand side of (96) is bounded if there exists a $y \in \mathcal{A}$ satisfying (99).

Secondly we consider an arbitrary sequence $y^{(h)}$ which satisfies (97) and

$$\limsup_{h\to 0} \frac{1}{h^2} \int_\Omega W(\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3}) dx < \infty. \tag{101}$$

Then by the compactness result (Theorem 4.1) a subsequence of $\nabla_h y^{(h)}$ converges strongly in L^2 to $(\nabla' y, b) \in W^{1,2}$. Moreover by Theorem 6.1

$$\liminf_{h\to 0} \frac{1}{h^2} \int_\Omega W(\nabla' y^{(h)}, \frac{1}{h} y^{(h)}_{,3}) dx \geq I_0(y). \tag{102}$$

We will show that in addition the limit satisfies (99). Combining this with the construction of the $\check{y}^{(h)}$ one immediately deduces that the limit of a low energy sequence minimizes I^0 subject to (99).

Suppose now that $y \in \mathcal{A}$ satisfies (99) and recall that $b = y_{,1} \wedge y_{,2}$. To construct $\check{y}^{(h)}$ we use results from Appendix A on the truncation of $W^{1,2}$ and $W^{2,2}$ functions with prescribed boundary conditions. The notation will be as in the appendix: a superscript λ will denote the truncated function. For any truncation parameter $\lambda > 0$, define the following maps from S to $I\!R^3$

$$\begin{aligned} v_\lambda &= (y - \hat{y})^\lambda + \hat{y}, \\ q_\lambda &= (b - \hat{b})^\lambda + \hat{b}. \end{aligned} \tag{103}$$

By Proposition A.2,

$$||\nabla^2 (y - \hat{y})^\lambda||_{L^\infty(S)} \leq C\lambda, \; ||\nabla q_\lambda||_{L^\infty(S)} \leq C(\lambda + ||\nabla \hat{b}||_{L^\infty(S)}), \tag{104}$$

$$|S_\lambda| \leq \frac{C\omega(\lambda)}{\lambda^2}, \tag{105}$$

where

$$
\begin{aligned}
S_\lambda &= \{x' \in S \mid v_\lambda(x') \neq y(x') \text{ or } q_\lambda(x') \neq b(x')\}, && (106)\\
\omega(\lambda) &= \int_{\{x'\in S\,:\,|b|+|\nabla b|\geq \lambda/2\}} (|b|^2 + |\nabla b|^2)\,dx' \\
&+ \int_{\{x'\in S\,:\,|y|+|\nabla y|+|\nabla^2 y|\geq \lambda/2\}} (|y|^2 + |\nabla y|^2 + |\nabla^2 y|^2)\,dx' && (107)\\
&+ \int_{\{x'\in S\,:\,|\hat y|+|\nabla \hat y|+|\nabla^2 \hat y|\geq \lambda/2\}} (|\hat y|^2 + |\nabla \hat y|^2 + |\nabla^2 \hat y|^2)\,dx' \to 0 \ \ (\text{as } \lambda \to \infty). && (108)
\end{aligned}
$$

Arguing as in the proof of (87) we see that $\operatorname{dist}((\nabla' v_\lambda, q_\lambda), \mathrm{SO}(3)) \leq C(\omega(\lambda))^{1/2} + \bar\omega(C/\lambda)$, where $\bar\omega$ is a modulus of continuity of $\nabla'\hat y$.

Let $d \in C^1_0(S; I\!R^3)$ and consider the trial function,

$$
\check y^{(h)}(x) = v_{\lambda_h}(x') + h\, x_3\, q_{\lambda_h}(x') + h^2\, \frac{x_3^2}{2}\, d(x'), \tag{109}
$$

with λ_h chosen as c/h; note that $\check y^{(h)}$ satisfies BC. By the same arguments as were applied to (89) in the proof of Theorem 6.1, we infer $\frac{1}{h^3}E^{(h)}(\check y^{(h)}) \to \frac{1}{24}\int_S Q_3(R^T(\nabla' b, d))\,dx'$, and, by suitable choice of $d = d^{(h)}$ depending on h, $\frac{1}{h^3}E^{(h)}(y^{(h)}) \to I^0(y)$. This establishes (100) and finishes the first part of the proof.

We now show that the limit of an arbitrary sequence $y^{(h)}$ which has bounded scaled energy and satisfies (97) will satisfy (99). To this end we show that the difference quotient estimates obtained in Section 4 hold up to the boundary. This will allow us, after mollification, to obtain $W^{1,2}$ bounds (up to the boundary) for a very good approximation of $\nabla_h y^{(h)}$ and to pass to the limit in traces.

Let us first assume that an interval in Γ is contained in a flat part of the boundary with normal $(0,1)$. We will show that the limiting boundary conditions hold on that interval. To avoid additional notation we assume that Γ consists only of this interval and

$$
\begin{aligned}
S \supset U &:= (-1,1)\times(-t,0), \quad t>0\\
\partial S \cap \bar U &= (-1,1)\times\{0\}\\
\Gamma &= [a,b]\times\{0\}, \quad [a,b]\subset(-1,1).
\end{aligned}
$$

We consider the lattice of squares

$$
S_{a,h} = a + (-\frac{h}{2}, \frac{h}{2}) \times (-h, 0], \quad a \in (h\mathbb{Z})^2. \tag{110}
$$

Let

$$
U_\delta = (-1+\delta, 1-\delta) \times (-t+\delta, 0), \tag{111}
$$

where $\delta > 0$ is so small that $[a,b] \subset (-1+\delta, 1-\delta)$ and $\delta < t/2$. Using Theorem 3.1 we obtain a map $R^{(h)} : U_{\delta/2} \to \mathrm{SO}(3)$ which is constant on each $S_{a,h} \subset U_{\delta/2}$ and satisfies

$$
\int_{S_{a,h}\times(-\frac12,\frac12)} |\nabla_h y^{(h)} - R^{(h)}|^2\,dx \leq C \int_{S_{a,h}\times(-\frac12,\frac12)} W(\nabla_h y^{(h)})\,dx. \tag{112}
$$

We have already shown in Section 4 that for a subsequence

$$
\begin{aligned}
\nabla_h y^{(h)} &\to (\nabla' y, b) \text{ in } L^2(U; I\!R^{3\times3}), \quad y \in W^{2,2}(S; I\!R^3),\ b = y_{,1}\wedge y_{,2},\\
R^{(h)} &\to (\nabla' y, b) \text{ in } L^2(U; I\!R^{3\times3}).
\end{aligned}
$$

To obtain further information on the trace of $\nabla_h y^{(h)}$ and $R^{(h)}$ on $x_2 = 0$ we first repeat the arguments in (52)– (57) to obtain difference quotient estimates for tangential or downward translations. This yields

$$\int_{U_\delta} |R^{(h)}(x' + \zeta) - R^{(h)}(x')|^2 \, dx' \le C \int_{U\times(-\frac{1}{2},\frac{1}{2})} W(\nabla_h y^{(h)}) \, dx \le Ch^2, \tag{113}$$

whenever $|\zeta_1| \le h$, $-h \le \zeta_2 \le 0$. Consider a kernel

$$\eta(x') = \eta_1(x_1)\eta_2(x_2), \;\; \eta_i \in C_0^\infty((0,1)), \; \eta_i \ge 0, \; \int_{I\!R} \eta_i = 1 \tag{114}$$

and define the mollified function

$$G^{(h)}(x') = \int_{I\!R^2} h^{-2}\eta(\frac{z'}{h}) R^{(h)}(x' - z')dz'. \tag{115}$$

Now (113) implies that

$$||\nabla' G^{(h)}||_{L^2(U_\delta)} \le C, \quad ||G^{(h)} - R^{(h)}||_{L^2(U_\delta)} \le Ch. \tag{116}$$

Thus

$$G^{(h)} \rightharpoonup (\nabla' y, b) \quad \text{in}\, W^{1,2}(U_\delta; I\!R^{3\times 3}). \tag{117}$$

In particular the traces converge strongly in L^2

$$G^{(h)}(\cdot, 0) \to (\nabla' y, b)(\cdot, 0) \quad \text{in}\, L^2((-1+\delta, 1-\delta); I\!R^{3\times 3}). \tag{118}$$

Since $R^{(h)}(x_1, x_2) = R^{(h)}(x_1, 0)$ for $x_2 \in (-h, 0]$ we have

$$G^{(h)}(x_1, 0) = \int_{I\!R} h^{-1}\eta_1(\frac{z_1}{h}) R^{(h)}(x_1 - z_1, 0)dz_1 \tag{119}$$

and, using (113),

$$\int_{-1+\delta}^{1-\delta} |R^{(h)}(x_1 + \zeta_1, 0) - R^{(h)}(x_1, 0)|^2 \, dx_1 \le Ch \tag{120}$$

for $|\zeta_1| \le h$. This implies that $G^{(h)}(\cdot, 0) - R^{(h)}(\cdot, 0) \to 0$ in L^2 and thus

$$R^{(h)}(\cdot, 0) \to (\nabla' y, b) \quad \text{in } L^2((-1+\delta, 1-\delta); I\!R^{3\times 3}). \tag{121}$$

Finally we will use (112) for squares which touch the boundary (i.e. $a_2 = 0$) to relate $R^{(h)}e_3$ and $\hat{b}$. For any $f \in W^{1,2}((0,1)^3)$ we have

$$\int_{\partial(0,1)^3} |f - c|^2 \, d\mathcal{H}^2 \le C \int_{(0,1)^3} |\nabla f|^2 \, dx, \tag{122}$$

where $c = \int f$. With the change of variables

$$f(z) = \frac{1}{h} g(a_1 + h(z_1 - \tfrac{1}{2}), h(z_2 - 1), z_3 - \tfrac{1}{2}) \tag{123}$$

(122) implies that for $a = (a_1, 0)$

$$\frac{1}{h} \int_{(S_{a,h}\cap\partial S)\times(-\frac{1}{2},\frac{1}{2})} |\frac{1}{h} g - c|^2 \, d\mathcal{H}^2 \le \frac{1}{h^2} \int_{S_{a,h}\times(-\frac{1}{2},\frac{1}{2})} |(\nabla' g, \frac{1}{h} g_{,3})|^2 \, dx. \tag{124}$$

Apply this with

$$g(x) = y^{(h)}(x) - R^{(h)}(a) \begin{pmatrix} x' \\ hx_3 \end{pmatrix}. \tag{125}$$

For $x' \in \Gamma$ we have

$$y^{(h)}(x) = \hat{y}(x') + hx_3\hat{b}(x') \tag{126}$$

and thus

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\frac{1}{h}g(x', x_3) - c|^2\, dx_3 \geq \frac{1}{12}|\hat{b}(x') - R^{(h)}(a)e_3|^2. \tag{127}$$

Combining this with (124) and (112) we obtain

$$\frac{1}{h}\int_{S_{a,h}\cap\Gamma} |\hat{b} - R^{(h)}e_3|^2\, d\mathcal{H}^1 \leq \frac{C}{h^2}\int_{S_{a,h}} W(\nabla_h y^{(h)})\, dx'. \tag{128}$$

Summing over those squares $S_{a,h}$ which intersect the boundary $x_2 = 0$ we get

$$\frac{1}{h}\int_a^b |\hat{b}(x_1, 0) - R^{(h)}(x_1, 0)e_3|^2\, dx_1 \leq C \tag{129}$$

and together with (121) we finally deduce

$$\hat{b} = b \quad \text{on } \Gamma. \tag{130}$$

If (a subinterval of) Γ is not contained in a flat part of the boundary we can first flatten the boundary using the Lipschitz map (locally defined in a suitable orthonormal coordinate system) $\Phi(x_1, x_2) = (x_1, x_2 - f(x_1))$. We can then consider the partition $S_{a,h}$ in the local image $\Phi(S \cap \Phi^{-1}(U))$ (possibly using a smaller rectangle U than in the argument above). Since Theorem 3.1 holds in an arbitrary Lipschitz domain we can apply it in the domains $\Phi^{-1}(S_{a,h})$ and we obtain as before difference quotient estimates for the functions $R^{(h)} \circ \Phi^{-1}$ which are constant in $S_{a,h}$. Then we can conclude as above. □

7 Strong convergence of the rescaled nonlinear strain for low energy sequences

For sequences with finite bending energy the nonlinear strain $(\nabla_h y^{(h)T}\nabla_h y^{(h)})^{1/2}$ converges strongly to the identity by Theorem 4.1. For low energy sequences, we find below, using the positive definiteness of the limiting energy, that the asymptotic correction is of the form $he(x)$ and we find an explicit form for the linearized strain e.

According to Theorem 6.2, a low energy sequence satisfying certain boundary conditions has a limiting energy given by I^0 of (98). Here we avoid the discussion of boundary conditions by considering the more general situation of any sequence that has the limiting bending energy I^0.

Theorem 7.1 *Assume $\nabla_h y^{(h)} = (\nabla' y(h), \frac{1}{h}y^{(h)}_{,3})$ converges in $L^2(\Omega)$ to $(\nabla' y, b)$ and has limiting bending energy*

$$\lim_{h\to 0}\frac{1}{h^2}\int_\Omega W(\nabla_h y^{(h)})\, dx = I^0(y) \;<\; \infty. \tag{131}$$

Then $y \in \mathcal{A}$ and

$$\frac{[\nabla_h y^{(h)T} \nabla_h y^{(h)}]^{\frac{1}{2}} - I}{h} \to x_3 \Big(\widehat{\mathrm{II}(x')} + \frac{c_{min}(x') \otimes e_3 + e_3 \otimes c_{min}(x')}{2} \Big) \ \text{ in } L^2(\Omega), \tag{132}$$

where $\mathrm{II} = (\nabla' y)^T \nabla' (y_{,1} \wedge y_{,2})$ *is the second fundamental form of* y*,* $\hat{G}$ *denotes the 3×3 matrix obtained from* $G \in M^{2\times 2}$ *by the formula* $\hat{G} = \sum_{i,j=1}^2 G_{ij} e_i \otimes e_j$*, and* $c_{min} \in L^2(S; I\!R^3)$ *is the unique pointwise minimizer of the problem* $\min_c Q_3(\hat{\mathrm{II}} + c \otimes e_3)$*.*

To interpret this result physically, we confine ourselves for simplicity to the case when W is isotropic, whence Q_3 is given by (69). In this case, the elementary calculus problem defining c_{min} has the unique solution $c_{min}(x') = -\frac{\lambda}{2\mu+\lambda} H(x') e_3$, where $H(x') = \mathrm{trace}\, \mathrm{II}(x')$ is the mean curvature of the plate at x'. Hence, considering for simplicity the case when y is smooth, the strain of any sequence $y^{(h)}$ converging to y and achieving the minimum asymptotic bending energy $I^0(y)$ must agree to $o(h)$ with that of the prototypical such sequence

$$y^{(h)}(x', x_3) = y(x') + (h x_3 - \frac{\lambda}{2\mu + \lambda} H(x') \frac{h^2 x_3^2}{2} \Big) b(x'), \quad b = y_{,1} \wedge y_{,2},$$

which corresponds to the unrescaled sequence (see Section 2)

$$v^{(h)}(z', z_3) = y(z') + \Big(z_3 - \frac{\lambda}{2\mu + \lambda} H(z') \frac{z_3^2}{2} \Big) b(z').$$

As compared to the simple Cosserat ansatz (5), the fibers orthogonal to the mid surface are thus inhomogeneously stretched, depending on the mean curvature of the plate. More precisely, if, say, $H > 0$ (corresponding to a concavely bent plate such as (12) with $\theta'(x_1) < 0$), the fibers contract above the mid surface and elongate below it. This is intuitive from the lateral stretching of the material above the mid surface and its lateral compression below.

Proof. Note first that by finiteness of the limiting bending energy and Proposition 4.1, $y \in \mathcal{A}$ and $b = y_{,1} \wedge y_{,2}$. Recall from Section 4 the lattice of squares S_h' and the piecewise constant approximation $R^{(h)} : S_h' \to \mathrm{SO}(3)$ of $\nabla_h y^{(h)}$, and let $G^{(h)}, \chi_h$ be as in (73), (76). By (74), $G^{(h)} \rightharpoonup G$ in $L^2(\Omega)$; moreover by (84), the matrix G' obtained from G by omitting the third row and the third column is given by $G'(x', x_3) = G'(x', 0) + x_3 \mathrm{II}(x')$. By combining (131), (78), (79) and (85),

$$\begin{aligned} I^0(y) &= \limsup_{h\to 0} \frac{1}{h^2} \int_\Omega W(\nabla_h y^{(h)})\, dx \geq \limsup_{h \to 0} \int_\Omega \chi_h W(\nabla_h y^{(h)})\, dx \\ &\geq \limsup_{h\to 0} \frac{1}{2} \int_\Omega Q_3(\chi_h G^{(h)})\, dx \geq \frac{1}{2} \int_\Omega Q_3(G)\, dx \\ &= \frac{1}{2} \int_\Omega Q_2(G'(x', 0))\, dx' + \frac{1}{2} \int_\Omega Q_2(x_3 \mathrm{II}(x'))\, dx. \end{aligned} \tag{133}$$

Since Q_2 is nonnegative and positive definite on symmetric matrices, it follows first of all that all inequalities are equalities, $Q_2(G'(x', 0)) = 0$, and $\frac{1}{2}(G' + G'^T) = x_3 \mathrm{II}(x')$. Next, from (67) and the fact that $Q_3(A) = Q_3(\frac{1}{2}(A + A^T))$ for all $A \in M^{3\times 3}$ (which follows from the frame-indifference of W), $Q_2(x_3 \mathrm{II}(x')) = \min_c Q_3(x_3 \hat{\mathrm{II}} + \frac{1}{2}(c \otimes e_3 + e_3 \otimes c))$, which has a unique minimizer $\hat{c}_{min}$

because Q_3 is positive definite on symmetric matrices. Consequently, from the pointwise inequality $Q_3(\frac{1}{2}(G+G^T)) \geq Q_2(x_3\mathrm{II}(x'))$ and (133),

$$\frac{G+G^T}{2} = x_3\widehat{\mathrm{II}}(x') + \frac{\hat{c}_{min}(x)\otimes e_3 + e_3\otimes \hat{c}_{min}(x)}{2} = x_3\Big(\widehat{\mathrm{II}}(x') + \frac{c_{min}(x')\otimes e_3 + e_3\otimes c_{min}(x')}{2}\Big). \tag{134}$$

For the latter, we have used that $\hat{c}_{min}(x) = x_3 c_{min}(x')$, where c_{min} is given in the statement of the theorem. Next, since Q_3 is positive-definite on symmetric matrices (and therefore strictly weakly lower semicontinuous), we have from the fact that equality holds in the third inequality of (133)

$$\chi_h \frac{G^{(h)} + (G^{(h)})^T}{2} \to \frac{G+G^T}{2} \quad \text{in } L^2(\Omega). \tag{135}$$

On the set $\{x\in\Omega \,|\, \chi_h(x) = 1\}$ we have

$$G^{(h)} = \frac{R^{(h)T}\nabla_h y^{(h)} - I}{h}, \quad R^{(h)}(x)\in \mathrm{SO}(3), \quad |hG^{(h)}(x)| \leq h^{1/2},$$

whence

$$\begin{aligned} \nabla_h y^{(h)T}\nabla_h y^{(h)} &= (R^{(h)T}\nabla_h y^{(h)})^T(R^{(h)T}\nabla_h y^{(h)}) \\ &= I + h\,(G^{(h)T} + G^{(h)}) + h^2\, G^{(h)T}G^{(h)}, \end{aligned} \tag{136}$$

so that on the same set,

$$|(\nabla_h y^{(h)T}\nabla_h y^{(h)})^{\frac{1}{2}} - (I + \tfrac{1}{2}\,h\,(G^{(h)} + G^{(h)T}))| \leq C|hG^{(h)}|^2 \tag{137}$$

for sufficiently small $h>0$. Since, by (77), $\chi_h G^{(h)} \rightharpoonup G$ in $L^2(\Omega)$, we multiply (137) by $\frac{1}{h}\chi_h$ and get,

$$\chi_h \frac{[\nabla_h y^{(h)T}\nabla_h y^{(h)}]^{\frac{1}{2}} - I}{h} \to \frac{G+G^T}{2} \quad \text{in } L^2(\Omega). \tag{138}$$

It remains to remove the χ_h. We have for $A\in M^{3\times 3}$,

$$|(A^TA)^{1/2} - I| \leq \mathrm{dist}(A, \mathrm{SO}(3)) \leq C\,W(A)^{1/2}. \tag{139}$$

We have, using that all inequalities in (133) are equalities,

$$\limsup_{h\to 0} \int_\Omega (1-\chi_h) \left| \frac{[\nabla_h u^{(h)T}\nabla_h u^{(h)}]^{\frac{1}{2}} - I}{h} \right|^2 dx \leq \limsup_{h\to 0} \frac{C}{h^2}\int_\Omega (1-\chi_h) W(\nabla_h u^{(h)})\,dx = 0. \tag{140}$$

Thus by (138) we have,

$$\frac{[\nabla_h u^{(h)T}\nabla_h u^{(h)}]^{\frac{1}{2}} - I}{h} \to \tfrac{1}{2}\,(G+G^T) \quad \text{in } L^2(\Omega). \tag{141}$$

Combining this result with the form of $\frac{1}{2}(G+G^T)$ given in (134) we get (132). □

A Appendix: Two truncation theorems

In the proof of the geometric rigidity result in Section 3 we needed to approximate functions in $W^{1,2}(U,\mathbb{R}^m)$ by those in $W^{1,\infty}(U,\mathbb{R}^m)$.

Proposition A.1 *Let $n,m\geq 1$, and let $1\leq p<\infty$. Suppose $U\subset\mathbb{R}^n$ is a bounded Lipschitz domain. Then there exists a constant $C(U,n,m,p)$ with the following property. For each $u\in W^{1,p}(U,\mathbb{R}^m)$ and each $\lambda>0$, there exists $v:U\to\mathbb{R}^m$ such that*

$$\begin{aligned}
&(i) \quad \|\nabla v\|_{L^\infty(U)}\leq C\lambda,\\
&(ii) \quad |\{x\in U\,|\,u(x)\neq v(x)\}|\leq \frac{C}{\lambda^p}\int_{\{x\in U\,:\,|\nabla u(x)|>\lambda\}}|\nabla u|^p\,dx,\\
&(iii) \quad \|\nabla u-\nabla v\|^p_{L^p(U)}\leq C\int_{\{x\in U\,:\,|\nabla u(x)|>\lambda\}}|\nabla u|^p dx.
\end{aligned} \tag{142}$$

Proof. Note first that (iii) is an immediate consequence of (i) and (ii). Indeed

$$\begin{aligned}
\int_U|\nabla u-\nabla v|^p\,dx &= \int_{u\neq v}|\nabla u-\nabla v|^p\,dx\leq 2^p\int_{u\neq v}\Big(|\nabla u|^p+|\nabla v|^p\Big)\,dx\\
&\leq 2^p\int_{u\neq v}\Big(\lambda^p+|\nabla v|^p\Big)\,dx+2^p\int_{|\nabla u|>\lambda}|\nabla u|^p\,dx\leq C\int_{|\nabla u|>\lambda}|\nabla u|^p\,dx.
\end{aligned}$$

It remains to establish assertions (i) and (ii). This will be done in three steps, passing from simple domains to general domains.

Step 1. The proposition holds for $U=(0,1)^{n-1}\times(0,H)$. (Only this case was needed in the application to plate theory; see the proof of Theorem 4.1 in Section 4.) The proof is very similar to that of the corresponding result in $\mathbb{R}^n$. Since the result (although not the constant C) is invariant under anisotropic dilations we may assume U is the unit cube $\mathrm{Q}=(-1,1)^n$. We follow the proof in Evans and Gariepy [17], Sections 6.6.3 and 6.6.2, except we define,

$$R^\lambda=\left\{x\in\Omega\,:\,\frac{1}{\mathrm{Q}\cap B(x,r)}\int_{\mathrm{Q}\cap B(x,r)}|\nabla u(z)|dz<\lambda\ \ \forall\, r\leq 2\sqrt{n}\right\}. \tag{143}$$

(Note that Evans and Gariepy use the integrand $|u|+|\nabla u|$ instead). The main point is that the Poincaré inequality still applies on $\mathrm{Q}\cap B(x,r)$ for $r\leq 2\sqrt{n}$ (cf. Evans and Gariepy [17], p. 253, proof of Claim #2 in the proof of Theorem 2, Section 6.6.2).

Step 2. The proposition holds for a standard Lipschitz domain, i.e. a domain of the form $U=\{x',x_n)\,:\,x'\in(0,1)^{n-1},\ f(x')<x'<f(x')+H\}$, with a constant C depending only on H and the Lipschitz constant L of f.

To see this consider the obvious bilipschitz homeomorphism ϕ from $(0,1)^{n-1}\times(0,H)$ to U given by $\phi(y)=(y',f(y')+y_n)$, and for given $u:U\to\mathbb{R}^m$ consider the pullback

$$\tilde u(y):=u(\phi(y)),\quad y\in(0,1)^{n-1}\times(0.H).$$

Then

$$|\nabla\tilde u|=|(\nabla u)(\phi(\cdot))\nabla\phi|\leq L|(\nabla u)(\phi(\cdot))|.$$

Applying Step 1 with u and λ replaced by $\tilde{u}$ and $\tilde{\lambda} := L\lambda$ gives a map $\tilde{v}$ satisfying

$$\begin{aligned} |\{y \in (0,1)^{n-1} \times (0,H) \,:\, \tilde{u}(y) \neq \tilde{v}(y)\}| &\leq \frac{C(H)}{\tilde{\lambda}^p} \int_{\{y\in(0,1)^{n-1}\times(0,H)\,:\,|\nabla\tilde{u}|>\tilde{\lambda}\}} |\nabla \tilde{u}|^p dy \\ &\leq \frac{C(H,L)}{\lambda^p} \int_{\{x\in U\,:\,|\nabla u|>\lambda\}} |\nabla u|^p dx. \end{aligned}$$

Finally let $v(x) := \tilde{v}(\phi^{-1}(x))$, then the asserted estimates are immediate.

Step 3. The proposition holds for a general bounded Lipschitz domain.

By assumption U can be covered by open sets U_i, $i = 1, \ldots, I$, such that either $V_i := U_i \cap \Omega$ is a standard Lipschitz domain (up to a rigid rotation and translation) or V_i is a cube contained in U. It follows from Steps 1 and 2 and the invariance of the assertion of the Theorem under translation, rotation and dilation that there exist Lipschitz functions $v_i : V_i \to I\!R^m$ such that

$$\operatorname{Lip} v_i \leq C\lambda, \quad |\{x \in V_i \,:\, v_i(x) \neq u(x)\}| \leq \frac{C}{\lambda^p} \int_{\{x\in V_i\,:\,|\nabla u|>\lambda\}} |\nabla u|^p \, dx. \tag{144}$$

Now consider a partition of unity $\{\phi_i\}$ subordinate to the cover $\{U_i\}$, i.e., $\phi_i \in C_0^\infty(U_i)$, $\sum_i \phi_i = 1$ in U, $0 \leq \phi \leq 1$. By trivial arguments each v_i can be extended to a Lipschitz function on $I\!R^n$ with Lipschitz constant bounded above by $\operatorname{Lip} v_i$ times a constant depending only on the target dimension m. For ease of notation we appeal to Kirzbraun's Theorem which says that this constant can in fact be chosen equal to 1. Let

$$v = \sum_i \phi_i v_i.$$

Since $v - u = \sum_i (v_i - u)$ we have

$$\begin{aligned} |\{x \in U \,:\, v(x) \neq u(x)\}| &\leq \sum_i |\{x \in U \,:\, \phi_i(v_i - u) \neq 0\}| \\ &\leq \sum_i |\{x \in V_i \,:\, v_i \neq u\}| \leq \frac{C}{\lambda^p} \int_{\{x\in U\,:\,|\nabla u|>\lambda\}} |\nabla u|^p dx. \end{aligned}$$

Moreover

$$|\nabla v| \leq \sum_i \phi_i |\nabla v_i| + \Big|\sum_i v_i \otimes \nabla \phi_i\Big| \leq C\lambda + \Big|\sum_i v_i \otimes \nabla\phi_i\Big|. \tag{145}$$

Now $\sum_i \nabla\phi_i = \nabla \sum_i \phi_i = 0$. Hence for $x \in U_j$

$$\Big|\phi_j \sum_i v_i \otimes \nabla\phi_i\Big| = \Big|\phi_j \sum_i (v_i - v_j) \otimes \nabla\phi_i\Big| \leq C \sum_{\{i\,:\,V_i\cap V_j\neq\emptyset\}} |v_i - v_j|. \tag{146}$$

Let $\alpha := \min\{|V_i \cap V_j| \,:\, V_i \cap V_j \neq \emptyset\} > 0$. Assume first that the following inequality holds (with C as in (144))

$$\frac{C}{\lambda^p} \int_{\{x\in U\,:\,|u|>\lambda\}} |\nabla u|^p \, dx < \frac{\alpha}{4}. \tag{147}$$

Then there exists $x \in V_i \cap V_j$ such that $v_i(x) = v_j(x) = u(x)$. Hence

$$\sup_{V_i\cap V_j} |v_i - v_j| \leq (\operatorname{Lip} v_i + \operatorname{Lip} v_j) \operatorname{diam} U \leq C\lambda$$

whenever $V_i \cap V_j \neq 0$. Combining this estimate with (145) and (146), we infer $|\nabla v| \leq C\lambda$. On the other hand, if (147) fails then

$$\frac{1}{\lambda^p}\int_{\{x\in U\,:\,|\nabla u|>\lambda\}} |\nabla u|^p\,dx \geq \frac{\alpha}{4C}.$$

Therefore the assertion of the Theorem holds with $v = 0$ since

$$|\{x \in U\ :\ u \neq v\}| \leq |U| \leq \frac{4|U|C}{\alpha}\frac{1}{\lambda^p}\int_{\{x\in U\,:\,|\nabla u|>\lambda\}} |\nabla u|^p\,dx.$$

The proof of the proposition is complete. □

In the $\Gamma-$convergence arguments in Theorems 6.1 and 6.2 we needed to truncate $W_0^{2,p}$ functions, in order to cover the general case of stored-energy functions W not required to satisfy any growth condition from above; readers only interested in the case of $W's$ with quadratic growth may skip the result below, which was then not needed.

Proposition A.2 *Let $1 \leq p \leq \infty$, $\lambda > 0$. Let S be a bounded Lipschitz domain in $I\!R^n$ and let Γ be a closed subset of ∂S which satisfies*

$$\mathcal{H}^{n-1}(B(\bar{x}, r) \cap \Gamma) \geq c r^{n-1}, \quad \forall \bar{x} \in \Gamma,\ 0 < r < r_0, \tag{148}$$

where $c > 0$.

(i) Suppose $u \in W^{1,p}(S)$ with

$$u = 0 \quad \text{on } \Gamma \tag{149}$$

in the sense of traces. Then there exists $u^\lambda \in W^{1,\infty}(S)$ such that

$$u^\lambda = 0 \quad \text{on } \Gamma$$

and

$$\begin{aligned} \|u^\lambda\|_{W^{1,\infty}} &\leq C(p, S)\,\lambda, \\ |\{x \in S : u^\lambda(x) \neq u(x)\}| &\leq \frac{C(p)}{\lambda^p}\int_{\{|u|+|\nabla u|\geq\lambda/2\}} (|u| + |\nabla u|)^p\,dx. \end{aligned} \tag{150}$$

In particular,

$$\lim_{\lambda\to 0}\Big(\lambda^p\ \text{meas}\{x \in S : u^\lambda(x) \neq u(x)\}\Big) = 0. \tag{151}$$

Moreover we can achieve $u^\lambda \in C^1(\bar{S})$.

(ii) Suppose $u \in W^{2,p}(S)$ with

$$u = \nabla u = 0 \quad \text{on } \Gamma.$$

Then there exists $u^\lambda \in W^{2,\infty}(S)$ such that

$$u^\lambda = \nabla u^\lambda = 0 \quad \text{on } \Gamma$$

and

$$\begin{aligned} \|u^\lambda\|_{W^{2,\infty}} &\leq C(p,S)\,\lambda, \\ |\{x\in S: u^\lambda(x)\neq u(x)\}| &\leq \frac{C(p)}{\lambda^p}\int_{\{|u|+|\nabla u|+|\nabla^2 u|\geq \lambda/2\}}(|u|+|\nabla u|+|\nabla^2 u|)^p\,dx. \end{aligned} \tag{152}$$

In particular,

$$\lim_{\lambda\to 0}\Big(\lambda^p \ \mathrm{meas}\{x\in S: u^\lambda(x)\neq u(x)\}\Big)=0. \tag{153}$$

Remarks.

i) For $S = I\!R^n$ this result was obtained by Liu [31] and Ziemer [42], building on earlier work of Calderon and Zygmund. The main point here is to preserve the boundary condition.

ii) A corresponding result holds for $W^{k,p}(S)$. We have limited ourselves to $k=1$ and $k=2$ to avoid more heavy notation. For $k=1$ and $\Gamma=\partial S$ the argument is simpler since one can use Kirszbraun's theorem on the extension of Lipschitz functions (see, [15]).

iii) Condition (148) states that the $\mathcal{H}^{n-1}$ measure of the rescaled sets $\frac{1}{r}(-\bar{x}+\Gamma\cap B(\bar{x},r))$ is uniformly bounded from below. In fact for $1<p\leq n$ it suffices to assume that the Riesz capacity $R_{1,p}$ is uniformly bounded from below since in this case one still has a (local) Poincaré inequality; for $p<n$ see e.g. [42], Corollary 4.5.3. Lewis [30] calls such sets Γ locally uniformly fat and establishes a number of interesting properties including a Hardy inequality (which is stronger the the local Poincaré inequality) for $1<p\leq n$. For $p>n$ no condition on Γ (beyond compactness) is needed since in this case u (and, for $k=2$, also ∇u) are C^α and a Poincaré inequality holds in $B(0,1)$ as long as we fix the value at one point.

Proof. The proof follows closely the presentation in Ziemer [42]. We only consider assertion (ii) since the proof of (i) is simpler. We first extend u to a function in $W^{2,2}(I\!R^n)$ with compact support (see e.g. [39]). Let

$$a=|u|+|\nabla u|+|\nabla^2 u|, \tag{154}$$

and let Ma be the maximal function of a:

$$Ma(x)=\sup_{r>0}⨍_{B(x,r)} a(y)\,dy. \tag{155}$$

Consider the good set

$$A^\lambda=\{x\in I\!R^n: Ma(x)<\lambda \text{ and } x \text{ is a Lebesgue point of } u,\nabla u \text{ and } \nabla^2 u\}. \tag{156}$$

We have that $\mathrm{meas}\,(I\!R^n\setminus A^\lambda)\leq \lambda^{-p}\|Ma\|^p_{L^p}\leq \lambda^{-p}\|a\|^p_{L^p}$ for $p\geq 1$. In fact, a covering argument (see Evans and Gariepy [17]) gives the stronger estimate,

$$\lambda^p\ \mathrm{meas}\,(I\!R^n\setminus A^\lambda)\leq C\int_{\{a>\lambda/2\}}|a|^p\,dx\to 0 \quad \text{as } \lambda\to 0. \tag{157}$$

By the Poincaré inequality we have for a.e. $x \in A^\lambda$ (see, e.g. Ziemer [42], Theorem 3.4.1)

$$\left(\fint_{B(x,r)} |u(y) - u(x) - \nabla u(x)(y-x)|^p \, dy \right)^{\frac{1}{p}} \le C\, r^2\, Ma(x) \le C\, r^2\, \lambda. \tag{158}$$

Removing if necessary a set of measure zero from A^λ, we assume from now on that (158) holds for every $x \in A^\lambda$. We claim that for $x, z \in A^\lambda$,

$$\begin{aligned} |u(z) - u(x) - \nabla u(x)(z-x)| &\le C\,\lambda\,|z-x|^2, \\ |\nabla u(z) - \nabla u(x)| &\le C\,\lambda\,|z-x|. \end{aligned} \tag{159}$$

This follows from Ziemer [42], Theorem 3.5.7. We recall the argument since we will use similar reasoning below. Replacing u by $\tilde u(\xi) = u(\frac{x+z}{2} + \delta\,\xi)$ where $\delta = |x-z|$ we may assume that $|z-x| = 1,\ z = -x$. Let

$$P_x(y) = u(x) + \nabla u(x)(y-x) \tag{160}$$

and apply (158) for x and z with $r=1$. Since the intersection $B(x,1) \cap B(z,1)$ contains the ball $B(0,\frac{1}{2})$ we conclude from the triangle inequality that,

$$\left(\int_{B(0,\frac{1}{2})} |P_z(y) - P_x(y)|^p \, dy \right)^{\frac{1}{p}} \le C\lambda. \tag{161}$$

This implies that the coefficients of $P_x - P_z$ are bounded by $C\lambda$, i.e.,

$$\begin{aligned} |\nabla u(z) - \nabla u(x)| &\le C\lambda, \\ |u(z) - \nabla u(z) z - u(x) - \nabla u(x) x| &\le C\lambda, \end{aligned} \tag{162}$$

and this proves (159). We next claim that for $x \in A^\lambda$

$$|u(x)| \le C\,\lambda\, d(x)^2, \quad |\nabla u(x)| \le C\,\lambda\, d(x), \tag{163}$$

where $d(x) = \operatorname{dist}(x, \Gamma)$.

To see this let $\bar x \in \Gamma$ be a point with $|x - \bar x| = d(x)$. By assumption

$$\mathcal{H}^{n-1}(B(x, 2d(x)) \cap \Gamma) \ge \mathcal{H}^{n-1}(B(\bar x, d(x)) \cap \Gamma) \ge c\, d^{n-1}(x). \tag{164}$$

With the rescaling

$$\tilde u(\xi) = \frac{1}{d(x)^2} u(x + d(x)\xi), \quad \tilde\Gamma = \frac{1}{d(x)}(-x + \Gamma), \tag{165}$$

it is sufficient to show (163) for $x = 0$, $d(x) = 1$, $u = \tilde u$ with $\tilde u = \nabla \tilde u = 0$ on Γ since (163) is invariant under this rescaling. Now $H^{n-1}(\tilde\Gamma) \ge c$ so we can apply the Poincaré inequality (see e.g. [42], Cor. 5.12.8 and Cor. 4.5.3 and use that for $p > 1$ positive $\mathcal{H}^{n-1}$ measure implies positive $B^{1,p}$ capacity).

$$\int_{B(0,2)} |\tilde u|^p \, dx \le C \int_{B(0,2)} |\nabla \tilde u|^p \, dx \le C \int_{B(0,2)} |\nabla^2 \tilde u|^p \, dx \le C\lambda^p. \tag{166}$$

Combining this with (158) applied with $u = \tilde u$, $x = 0$ and $r = 2$ we find that

$$\left(\int_{B(0,2)} |P_x(y)|^p \, dy \right)^{\frac{1}{p}} \le C\lambda. \tag{167}$$

This yields the desired estimates for the coefficients of P_x and thus (163).

Now define the extension u^λ in two steps. If meas$\,(I\!R^n \setminus A^\lambda) = 0$, we can take $u^\lambda = u$. If meas$\,(I\!R^n \setminus A^\lambda) > 0$ then there exists a closed subset $\tilde{A}^\lambda$ of $A^\lambda \cap S$ such that meas$\,(I\!R^n \setminus \tilde{A}^\lambda) \leq 2\,$meas$\,(I\!R^n \setminus A^\lambda)$. Let $B^\lambda = \tilde{A}^\lambda \cup \Gamma$ and define on B^λ the function

$$v(x) = \begin{cases} u(x) & \text{if } \; x \in \tilde{A}^\lambda \\ 0 & \text{if } \; x \in \Gamma. \end{cases} \tag{168}$$

Combining (159) – (163) we see that for $x, y \in B^\lambda$,

$$|v(z) - P_x(z)| \leq C\lambda |z - x|^2, \quad |\nabla P_z - \nabla P_x| \leq C\lambda |z - x|, \tag{169}$$

where

$$P_x(z) = \begin{cases} u(x) + \nabla u(x)(z - x) & \text{if } \; x \in \tilde{A}^\lambda, \\ 0 & \text{if } \; x \in \Gamma \end{cases} \tag{170}$$

Note also that the definition of $\tilde{A}^\lambda$ immediately implies that

$$|v| + |\nabla v| \leq \lambda \;\; \text{on } B^\lambda. \tag{171}$$

We will show that (169) implies that v has an extension $\tilde{v} : I\!R^n \to I\!R$ which satisfies $\tilde{v}|_{B^\lambda} = v$ and

$$|\tilde{v}(y) - P_x(y)| \leq C\lambda |y - x|^2 \;\; \forall x \in B^\lambda, \; \forall y \in I\!R^n. \tag{172}$$

Then Theorem 3.6.2 of Ziemer [42] guarantees that there exists $u^\lambda \in W^{2,\infty}(I\!R^n)$ such that

$$u^\lambda = \tilde{v} = v \;\; \text{on } B^\lambda, \tag{173}$$

and $\|u^\lambda\|_{W^{2,\infty}} \leq C\lambda$. In fact one can define u^λ by mollification,

$$u^\lambda(x) = \int_{I\!R^n} \rho^{-n}(x)\, \varphi\!\left(\frac{x - y}{\rho(x)}\right) \tilde{v}(y)\, dy, \tag{174}$$

where ρ is a smooth approximation of the distance function (i.e., $\rho \in C^\infty(I\!R^n \setminus B^\lambda)$, $c\,\text{dist}(x, B^\lambda) \leq \rho(x) \leq C\,\text{dist}(x, B^\lambda)$, $|D^\alpha \rho| \leq C_\alpha \rho^{1-|\alpha|}$) and $\varphi \in C_0^\infty$ has the property $\varphi \star P = P$ for all polynomials P of degree one (see [42], Lemmas 3.6.1 and 3.5.6 for the existence of ρ and φ). Note that Ziemer's construction only extends v to a neighborhood of B^λ of size 1. We may, however, assume without loss of generality that diam$\,S < 1$ so that his construction suffices.

It remains to construct the extension $\tilde{v}$. We assume for simplicity that

$$|v(z) - P_x(z)| \leq |z - x|^2, \quad |\nabla P_z - \nabla P_x| \leq |z - x| \tag{175}$$

for $x, z \in B^\lambda$. The general situation is easily recovered by scaling. We define, for $y \in I\!R^n$,

$$\tilde{v}(y) = \sup_{x \in B^\lambda} P_x(y) - M|y - x|^2, \tag{176}$$

where $M > 1$ will be chosen later. It follows from (175) that $\tilde{v} = v$ on B^λ and we have the trivial bound,

$$\tilde{v}(y) \geq P_x(y) - M|y - x|^2 \quad \forall x \in B^\lambda, \; \forall y \in I\!R^n. \tag{177}$$

To prove an upper bound we first note that (175) and the closedness of B^λ imply that the supremum in the definition of $\tilde{v}$ is attained at $\bar{x}(y)$. Taking into account that $v(z) = P_z(z)$ for $z \in B^\lambda$ and that P_x is affine we have for $x, \bar{x} \in B^\lambda$,

$$\begin{aligned} |P_{\bar{x}}(y) - P_x(y)| &= |P_{\bar{x}}(\bar{x}) - P_x(\bar{x}) + \nabla P_{\bar{x}}(y-\bar{x}) - \nabla P_x(y-\bar{x})|, \\ &\le |x-\bar{x}|^2 + |x-\bar{x}||y-\bar{x}|, \\ &\le \frac{3}{2}|x-\bar{x}|^2 + \frac{1}{2}|y-\bar{x}|^2. \end{aligned} \tag{178}$$

Together with the trivial estimate $|x-\bar{x}| \le |x-y| + |y-\bar{x}|$, this gives,

$$\begin{aligned} \tilde{v}(y) &= P_{\bar{x}}(y) - M|y-\bar{x}|^2, \\ &\le P_x(y) + (\frac{7}{2} - M)|y-\bar{x}|^2 + 3|y-x|^2. \end{aligned} \tag{179}$$

Taking $M = 4$ and using (177) we arrive at the desired assertion, $|\tilde{v}(y) - P_x(y)| \le 4|y-x|^2, \ \ \forall x \in B^\lambda, \ \forall y \in I\!R^n$.

To see that in (i) we can choose the functions u^λ of class C^1 we first note that for each $\varepsilon > 0$ there exist a C^1 function v^λ such that $|\text{meas}\{u^\lambda \neq v^\lambda\}| < \varepsilon$ and $||\nabla v^\lambda||_{L^\infty} \le C||\nabla u^\lambda||_{L^\infty}$, where C only depends on n (see e.g. [17], Chapter 6.6.1, Thm. 1). In particular $|v^\lambda - u^\lambda| \le \delta := C\lambda\varepsilon^{1/n}$ since the set where the two functions do not agree cannot contain a large ball. Let ρ be the smooth distance function from Γ and define $w^\lambda = (\eta \circ \rho)v^\lambda$. Here $\eta : I\!R \to [0,1]$ is a smooth function which vanishes on $[0, \delta^{1/2})$, is identically 1 on $(2\delta^{1/2}, \infty)$ and satisfies $\eta' \le 2\delta^{-1/2}$. If we choose ε small enough (and replace λ by λ/C) then w^λ has all the desired properties. □

Proposition A.3 *Let S be as in Proposition A.2 and let T be an open subset of S which satisfies*

$$\mathcal{H}^n(B(\bar{x}, r) \cap T) \ge cr^n, \quad \forall \bar{x} \in \bar{T}, \ 0 < r < r_0. \tag{180}$$

Then the assertions of Proposition A.2 hold if the boundary conditions $u = 0$ on Γ and $u = \nabla u = 0$ on Γ are replaced by

$$u = 0 \quad \text{on } T \tag{181}$$

and

$$u = \nabla u = 0 \quad \text{on } T, \tag{182}$$

respectively.

Proof. The proof the same as Proposition A.2. To derive (163) we now apply a Poincaré inequality for functions which vanish on a set of positive measure. □

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