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On a nonlinear elliptic equation arising in a free boundary problem

by

Guofang Wang and Dong Ye

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Abstract

Let $p^* = n/(n-2)$ and $n \geq 3$. In this paper, we first classify all non-constant solutions of

$$\begin{cases} -\Delta u &= u_+^{p^*} \quad \text{in } \mathbb{R}^n, \\ \displaystyle\int_{\mathbb{R}^n} u_+^{p^*} dx &< \infty. \end{cases}$$

We then establish a $\sup + \inf$ and a Moser-Trudinger type inequalities for the equation $-\Delta u = u_+^{p^*}$. Our results illustrate that this equation is much closer to the Liouville problem $-\Delta u = e^u$ in dimension two than the usual critical exponent equation, namely $-\Delta u = u^{\frac{n+2}{n-2}}$ is.

1 Introduction

The Liouville equation in dimension two

$$-\Delta u = Ke^u \tag{1}$$

and related problems have been extensively studied in the last twenty years. This equation arises in many mathematical and physical problems, for instance, in the problem of prescribing Gaussian curvature and Chern-Simons Higgs models. To understand the convergence or the blow-up phenomenon of its solutions, a crucial step is the classification of bounded energy solutions:

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^u dx < \infty, \tag{2}$$

which was obtained by Chen and Li in [9] (see other proofs in [11, 8, 15]). More precisely, all solutions of (1) are in the form

$$\phi_{\lambda,x_0}(x) = \frac{\ln(32\lambda^2)}{(4+\lambda^2|x-x_0|^2)^2} \quad \text{with } \lambda > 0 \text{ and } x_0 \in \mathbb{R}^2.$$

For the classification results of other related problems, see for instance [7, 20, 30]. The usual higher dimensional analogue of (2) is

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \quad (n \geq 3), \tag{3}$$

which is a limit equation of semilinear equations involving the critical exponent of the Sobolev inequality. Among them, the Yamabe equation is an important example. The classification of positive solutions of (3) was obtained by Caffarelli, Gidas and Spruck in [6] (see also in [9]).

In this paper, we consider the equation

$$-\Delta u = u_+^{p^*} \quad \text{in } \mathbb{R}^n \ (n \geq 3), \tag{4}$$

where $u_+ = \max\{0, u\}$ and $p^* = n/(n-2)$. This equation is a limit equation of the following equation

$$\begin{cases} -\Delta u &= M \dfrac{u_+^{p^*}}{\displaystyle\int_\Omega u_+^{p^*} dx} \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= c, \quad \text{an unknown constant.} \end{cases} \tag{5}$$

Equation (5) arises in the study of the free boundary problem, see [3, 31, 32]. The aim of this paper is to show that the equation (4) (or (5)) is much closer to (1) than (3) is. We present here a list of the similarities between these two equations:

1. Both equations (1) and (4) have a group of gauge which keep invariant the energy and the equation.
2. Classification of bounded energy solution in whole space.
3. Existence of a sup + inf type inequality.
4. Existence of a Moser-Trudinger type inequality.
5. Behaviors of blow-up solutions with or without Dirichlet boundary conditions.

In fact, for the equation $-\Delta u = e^u$, if we define $u_\lambda(x) = u(\lambda x) + 2\ln\lambda$, we see easily that

$$-\Delta u_\lambda = e^{u_\lambda} \quad \text{and} \quad \int_\Omega e^u dx = \int_{\Omega_\lambda} e^{u_\lambda} dx,$$

where $\Omega_\lambda = \Omega/\lambda = \{y \in \mathbb{R}^2, \lambda y \in \Omega\}$. For the general problem $-\Delta u = u^p$ with $p > 1$, the equation remains unchanged under the transformation $u(x) \mapsto u_\lambda(x) = \lambda^q u(\lambda x)$ with $q = 2/(p-1)$. But if we require further that

$$\int_\Omega u^p dx = \int_{\Omega_\lambda} u_\lambda^p dx,$$

the only possibility is then $p = n/(n-2)$.

It is clear that (4) has no positive solution, since p^* is subcritical with respect to the Sobolev exponent $\frac{n+2}{n-2}$, see [13]. Here we consider its bounded energy solutions:

$$-\Delta u = u_+^{p^*} \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \int_{\mathbb{R}^n} u_+^{p^*} dx < \infty. \tag{6}$$

The bounded energy condition is very natural for the corresponding variational problem of (5), see also the Moser-Trudinger inequality below.

Theorem 1 *Any non-trivial C^2 solution u of (6) is rotationally symmetric. Moreover, there are $\lambda \in (0,\infty)$ and $x_0 \in \mathbb{R}^n$ such that*

$$u(x) = \begin{cases} \lambda^{n-2}\phi(\lambda|x-x_0|) & \text{if } \lambda|x-x_0| \le r^*, \\ \omega_{n-1}^{-1} M_n^*(|x-x_0|^{2-n} - (\lambda^{-1}r^*)^{2-n})/(n-2) & \text{if } \lambda|x-x_0| > r^*, \end{cases} \tag{7}$$

where ω_{n-1} is the volume of the unit $(n-1)$-sphere, r^* denotes the first zero of the unique solution ϕ of

$$\begin{cases} \phi''(r) + \frac{n-1}{r}\phi'(r) + \phi^{p^*}(r) &= 0, \\ \phi(0) = 1, \quad \phi'(0) &= 0, \end{cases} \tag{8}$$

and

$$M_n^* = \omega_{n-1}\int_0^{r^*} \phi^{p^*}(r) r^{n-1} dr. \tag{9}$$

In particular, any solution satisfies

$$\int_{\mathbb{R}^n} u_+^{p^*} = M_n^*.$$

Without the boundedness of energy, one can easily construct other solutions.

The $\sup + \inf$ type inequality for the Liouville equation (1) in dimension two was established in [25] (see further developments in [4, 10]), while a $\sup \times \inf$ type inequality was established for positive solutions of equation (3) in [24], see also [17]. Here we present a $\sup + \inf$ type inequality for (4).

Theorem 2 *Let K be a compact subset of Ω, a bounded domain in $\mathbb{R}^n$. Then there exist two positive constants $C_1(n)$ and $C_2(K,T)$ such that*

$$\sup_K u + C_1 \inf_\Omega u \le C_2,$$

for any C^2 solution u of (4) in Ω satisfying

$$\int_\Omega u_+^{p^*} dx \le T < \infty.$$

In dimension two, we have the well-known Moser-Trudinger inequality. Namely, for any bounded domain Ω of $\mathbb{R}^2$ (or for any compact Riemannian surface), there is a constant $C(\Omega)$ such that for any $u \in H_0^1(\Omega)$,

$$\int_\Omega e^{4\pi u^2/\|\nabla u\|_2^2}\, dx \le C(\Omega),$$

which implies that

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - 8\pi \log \int_\Omega e^u dx \ge -C, \quad \forall\ u \in H_0^{1,2}(\Omega). \tag{10}$$

Thus we can minimize associate functional to get solutions of the equation $-\Delta u = Me^u / \int_\Omega e^u dx$ with Dirichlet boundary condition when $M < 8\pi$. Inequality (10) is a slightly weaker, but applicable form of the Moser-Trudinger inequality. We are interested in such type inequalities. See other Moser-Trudinger type inequalities for the Liouville equation in [27, 16, 26]. The Moser-Trudinger inequality has a higher dimensional analogue, which is the well-known Sobolev inequality. Here, we present another higher dimensional generalization of the Moser-Trudinger inequality, which looks more like the ordinary one.

Theorem 3 *Let Ω be any bounded smooth domain in $\mathbb{R}^n$ and*

$$V(\Omega) = \left\{ v \in H^1(\Omega) \ s.t. \ v_{|\partial\Omega} = \ constant, \ \int_\Omega v_+^{p^*} dx = 1 \right\}.$$

Define

$$\mathcal{I}_{M,\Omega}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{M}{p^*+1}\int_\Omega u_+^{p^*+1} dx + M u_{|\partial\Omega} \tag{11}$$

for $u \in V(\Omega)$. Then

$$\inf_{u\in V(\Omega)} \mathcal{I}_{M,\Omega}(u) > -\infty \quad \text{if and only if} \quad M \le E_n^*,$$

where $E_n^* = (M_n^*)^{2/n}$ with M_n^* the constant given by (9).

Remark. Some similar functionals have been considered in [3, 28, 32]. Especially, in [32], a free energy formulation of Thereom 3 was provided.[1] Our proof is more direct and more precise.

The convergence of solutions of (4) (or (5)) is also more close to that of the Liouville equation. Here we obtain a Brezis-Merle type result for equation (4).

Theorem 4 *Let Ω be a bounded regular domain of $\mathbb{R}^n$ and u^k be a sequence of regular solutions satisfying*

$$\begin{cases} -\Delta u^k &= (u^k)_+^{p^*} \quad \text{in } \Omega, \\ \int_\Omega (u^k)_+^{p^*} dx &\le T < \infty. \end{cases} \tag{12}$$

Then passing to a subsequence (denoted also by u_k), we have one of the following possiblities

(1) u^k is bounded in $L^\infty_{loc}(\Omega)$;

(2) u^k tends to $-\infty$ uniformly on compact set of Ω;

(3) there exists a finite subset $\mathcal{S} = \{x_1, x_2, \ldots, x_m\} \subset \Omega$ such that u^k tends to $-\infty$ on compact set of $\Omega\backslash\mathcal{S}$. Moreover, $(u^k)_+^{p^*}$ converges to $\sum_i \alpha_i \delta_{x_i}$ in the sense of measure, with $\alpha_i \ge M_n^*$, $\forall\, 1 \le i \le m$.

Theorem 4 can be improved as in [19].

Theorem 5 *In Theorem 4, if case (3) holds, then $\alpha_i = M_n^* l_i$ with $l_i \in \mathbb{N}^*$.*

Theorem 5 is a generalization of the result obtained in [19] for the Liouville equation, see also [18, 21, 33]. Our results illustrate that as a higher dimensional analogue, equation (4) is more close to equation (2) than (3) is. There are other results to support our conclusion, see for instance [32, 29]. The peculiarity of the index p^* was noticed by many mathematicians, see for example [3], [1] and [22]. We believe that many results obtained for two dimensional problems will be naturally generalized to higher dimensional problems involving (6) or (5).

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[1] Our work is partly inspired by [32].

2 A classification of solutions

Proposition 1 *Any C^2 solution u of (6) satisfies* $\sup_{x\in\mathbb{R}^n} u < \infty$.

To prove the Proposition, we need several lemmas.

Lemma 1 *Let B_R be the ball of radius R, centered at the origin. Suppose that $u \in C^2(B_R)$ satisfies*

$$\begin{cases} -\Delta u &= u_+^{p^*} \quad \text{in } B_R \\ u(x_0) &= 1 \quad \text{for some } x_0 \in B_{R/2} \\ u &\le A \quad \text{in } B_R. \end{cases}$$

Then there exists a positive constant C depending only on A and R such that

$$u(x) \ge -C \quad \text{in } \overline{B}_{R/4}.$$

Proof. Let u_1 and u_2 be solutions of

$$\begin{cases} -\Delta u_1 &= u_+^{p^*} \quad \text{in } B_R \\ u_1 &= 0 \quad \text{on } \partial B_R, \end{cases} \tag{13}$$

and

$$\begin{cases} -\Delta u_2 &= 0 \quad \text{in } B_R \\ u_2 &= u \quad \text{on } \partial B_R. \end{cases} \tag{14}$$

It is easy to see $u = u_1 + u_2$ and $0 \le u_1 \le C(R)A^{p^*}$. Furthermore, $u_2 \le A$ by maximum principle and $\max_{\overline{B}_{R/2}} u_2 \ge 1 - C(R)A^{p^*}$. Applying Harnack's inequality to the nonnegative harmonic function $A - u_2$, we have

$$\min_{\overline{B}_{R/4}} u \ge \min_{\overline{B}_{R/4}} u_2 \ge C\left(\max_{\overline{B}_{R/2}} u_2 - A\right) + A \ge -C,$$

where C depends only on A and R. ■

Lemma 2 *There exist constants $C, \delta > 0$ such that for any u satisfying*

$$\begin{cases} -\Delta u &= u_+^{p^*} \quad \text{in } B_1 \\ \displaystyle\int_{B_1} u_+^{p^*}\,dx &< \delta, \end{cases} \tag{15}$$

we have

$$\max_{x\in\overline{B}_{1/4}} u(x) < C.$$

Proof. Here, we use a trick of Schoen [23]. Notice that the energy $\|u_+\|_{p^*}$ remains unchanged under the transformation $u \mapsto \lambda^{n-2}u(\lambda x)$. Suppose that the result is false, then there exists a sequence u^k satisfying $-\Delta u^k = (u^k)_+^{p^*}$ in B_1,

$$\int_{B_1} (u^k)_+^{p^*}\,dx \le \frac{1}{k} \quad \text{and} \quad \max_{x\in\overline{B}_{1/4}} u^k(x) \ge k$$

Consider $h^k(y) = (1/2 - r)^{n-2} u^k(y)$ with $r = |y|$ and $h^k(y_k) = \max_{\overline{B}_{1/2}} h^k(y)$, then

$$(1/2 - r_k)^{n-2} u_+^k(y_k) \geq (1/4)^{n-2} \max_{x \in \overline{B}_{1/4}} u^k(x) \geq (1/4)^{n-2} k \tag{16}$$

where $r_k = |y_k|$. Define $w^k(y) = \lambda_k^{n-2} u^k(y_k + \lambda_k y)$ in $B_{\mu_k/2}$, with

$$\sigma_k = (1/2 - r_k), \quad \mu_k^{n-2} = h^k(y_k) = \sigma_k^{n-2} u_+^k(y_k) \quad \text{and} \quad \lambda_k = \sigma_k/\mu_k.$$

Notice that for $y \in B_{\sigma_k/2}(y_k)$, we have $(1/2 - |y|) \geq \sigma_k - |y - y_k| \geq \sigma_k/2$, therefore $\mu_k^{n-2} \geq (\sigma_k/2)^{n-2} u^k(y)$ in $B_{\sigma_k/2}(y_k)$. Thus

$$\begin{cases} -\Delta w^k &= (w^k)_+^{p^*} & \text{in } B_{\mu_k/2} \\ \displaystyle\int_{B_{\mu_k/2}} (w^k)_+^{p^*} dx &= \displaystyle\int_{B_{\sigma_k/2}(y_k)} (u^k)_+^{p^*} dx \leq 1/k & \\ w^k(0) &= 1 & \\ w^k(x) &\leq 2^{n-2} & \text{in } B_{\mu_k/2}. \end{cases} \tag{17}$$

Since $\mu_k \to \infty$ by (16), from Lemma 1 and the standard elliptic theory, we can obtain a subsequence, noted always by w^k such that w^k converges in $C^2_{loc}(\mathbb{R}^n)$ to w, which satisfies

$$\begin{cases} -\Delta w &= 0 & \text{in } \mathbb{R}^n \\ w(0) &= 1 & \\ w(x) &\leq 2^{n-2} & \text{in } \mathbb{R}^n. \end{cases} \tag{18}$$

So w is a harmonic function bounded above in $\mathbb{R}^n$, hence $w \equiv 1$ in $\mathbb{R}^n$. We reach clearly a contradiction with the local uniform convergence of w^k to w and the convergence of w_+^k to 0 in $L^{p^*}_{loc}(\mathbb{R}^n)$. The lemma is thus established. ■

The proof of Proposition 1 follows readily from Lemma 1 and Lemma 2. By Proposition 1, we have the following representation formula:

Proposition 2 *Any C^2 solution u of (6) satisfies*

$$u(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |x - y|^{2-n} u_+^{p^*}(y) dy - c, \tag{19}$$

for some positive constant c. Moreover, for large x, u satisfies

$$u(x) = -c + c_0 |x|^{2-n} + o(|x|^{2-n}), \qquad \textit{where } c_0 = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} u_+^{p^*} dx. \tag{20}$$

Proof. Define

$$\omega(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |x - y|^{2-n} u_+^{p^*}(y) dy$$

Since $u_+ \in L^\infty \cap L^{p^*}(\mathbb{R}^n)$, by Hölder's inequality ω is well-defined. Obviously, $\omega \geq 0$ and $-\Delta\omega = u_+^{p^*}$ in $\mathbb{R}^n$. Hence $u - \omega$ is harmonic and bounded from above by Proposition 1, then there exists a constant c such that $u = \omega + c$.

Claim 1: $c < 0$. Suppose the contrary, we get $u \geq 0$, so u is a non trivial solution of $-\Delta u = u^{p^*}$ in $\mathbb{R}^n$, which is impossible since p^* is subcritical, see [13].

Claim 2: $\lim_{|x|\to\infty} \omega(x) = 0$. The proof is standard using the fact $u_+ \in L^\infty \cap L^{p^*}(\mathbb{R}^n)$, thus $u_+ \in L^q(\mathbb{R}^n)$ for any $p^* \leq q \leq \infty$.

By these two claims, we get that the support of u_+ is compact, since $\lim_{|x|\to\infty} u = c < 0$. Now it is easy to check that when $|x|$ tends to ∞

$$|x|^{n-2}\omega(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{supp(u_+)} \frac{|x|^{n-2} u_+^{p^*}}{|x-y|^{n-2}} dy \longrightarrow \frac{1}{(n-2)\omega_{n-1}} \int_{supp(u_+)} u_+^{p^*} dy. \qquad \blacksquare$$

Proof of Theorem 1. Define $f(t) = (t-c)_+^{p^*}$ with c the positive constant in Proposition 2. We notice that f is a C^1 function in $\mathbb{R}$ and is nonincreasing in a neighborhood of 0. Moreover, ω satisfies

$$\begin{cases} -\Delta\omega &= f(w) \quad \text{in } \mathbb{R}^n \\ \omega &> 0 \quad \text{in } \mathbb{R}^n \\ \lim\limits_{|x|\to\infty} \omega(x) &= 0. \end{cases} \tag{21}$$

The classical result of moving plane insures the symmetry of ω. $\blacksquare$

Remark. We see that our proof works also to classify all bounded energy solutions of $-\Delta u = u_+^p$ with $p \in [1, p^*)$. The rotational symmetry of solutions of $-\Delta u = u_+^p$ for more general $p > 1$ was proved in [14], under some additional assumptions that solutions are bounded from above and the Morse index $i(u)$ is finite.

Using Theorem 1, we can refine Lemma 2 as follows:

Proposition 3 *For any $\delta \in (0, M_n^*)$, there exists a constant C such that any solution of (15) satisfies* $\max_{x\in\overline{B}_{1/4}} u(x) < C$.

3 $\sup + \inf$ type inequalities

In this section, we prove Theorem 2. As in [19], one can reduce the proof of Theorem 2 to the following lemma.

Lemma 3 *There exist two positive constants C_1 and C_2 such that for any C^2 solution u of*

$$-\Delta u = u_+^{p^*} \quad \text{in } B_1, \quad \int_{B_1} u_+^{p^*} dx \leq T < \infty,$$

we have $u(0) + C_1 \inf_{B_1} u \leq C_2$.

Proof. Suppose it is false, then for any $C > 0$, we get a sequences u^k such that $-\Delta u^k = (u^k)_+^{p^*}$ in B_1 and

$$\int_{B_1} (u^k)_+^{p^*} dx \leq T < \infty, \quad u^k(0) + C \inf_{B_1} u^k \geq k.$$

Thus we have $u^k(0)$ tends to ∞ as $k \to \infty$. As in the proof of Lemma 2, we consider the sequence of functions $h^k(x) = (1-r)^{n-2} u^k(x)$, define $\mu_k^{n-2} = h^k(y_k) = \max_{\overline{B}_1} h^k$, $\sigma_k = (1 - |y_k|)$ and

$\lambda_k = \sigma_k/\mu_k$. If we denote $w^k(y) = \lambda_k^{n-2} u^k(y_k + \lambda_k y)$ in $B_{\mu_k/2}$, then we can get a subsequence (still denoted by u^k) which converges in $C^2_{loc}(\mathbb{R}^n)$ to a function w, satisfying

$$\begin{cases} -\Delta w &= w_+^{p^*} \quad \text{in } \mathbb{R}^n \\ \int_{\mathbb{R}^n} w_+^{p^*} dx &\leq T \\ w(0) &= 1 \\ w(x) &\leq 2^{n-2} \quad \text{in } \mathbb{R}^n. \end{cases} \tag{22}$$

Applying Theorem 1, $w(x)$ is given by (7). Since $w(0) = 1$ and $w(x) \leq 2^{n-2}$, then $\lambda \in [1, 2]$. Hence for $C_1(n)$ and $R(n)$ large enough, we must have $w(0) + C_1 \inf_{\partial B_R} w < 0$. Moreover, by the local convergence of w^k to w, we deduce that for k sufficiently large, $w^k(0) + C_1 \inf_{\partial B_R} w^k < 0$. Using the definition of y_k and noting that u^k is super-harmonic, then (for k large enough),

$$u^k(0) + C_1 \inf_{B_1} u^k \leq u^k(y_k) + C_1 \inf_{B(y_k, \lambda_k R)} u^k = \lambda_k^{2-n} \left(w^k(0) + C_1 \inf_{B_R} w^k \right) < 0.$$

This contradicts the choice of u^k when $C \geq C_1$, the proof is done. ■

Remark. More precisely, we can take C_1 as any constant greater than $(n-2)(2r^*)^{n-2}\omega_{n-1}/M_n^*$. Otherwise, using the transformation, we can state that for the same C_1, C_2 and any $r \in (0, 1)$,

$$u(0) + C_1 \inf_{B_r} u \leq C_2 r^{2-n}.$$

Proof of Theorem 2. For $K \subset\subset \Omega$, $\exists\ \lambda_0 > 0$ such that for any $x \in K$, $B(x, \lambda_0) \subset \Omega$. Suppose that $x_0 \in K$ realize $\sup_K u(x)$, we define $v(x) = \lambda_0^{n-2} u(x_0 + \lambda_0 x)$ in B_1. Since

$$u(x_0) + C_1 \inf_{\Omega} u \leq u(x_0) + C_1 \inf_{B(x_0, \lambda_0)} u = \lambda_0^{2-n} \left(v(0) + C_1 \inf_{B_1} v \right),$$

we get $\sup_K u + C_1 \inf_\Omega u \leq C_2 \lambda_0^{2-n}$ by the above lemma. ■

4 Blow-up Analysis

Proof of Theorem 4. Our proof is inspired by that in [5]. Passing to a subsequence, we can assume that $(u^k)_+^{p^*}$ converges to a bounded nonnegative measure μ, in the sense of measure. Denote

$$\mathcal{S} = \{x \in \Omega \text{ s.t. } \mu(\{x\}) \geq M_n^*\}$$

and

$$\Sigma = \{x \in \Omega \text{ s.t. } \exists\ x_k \in \Omega \text{ satisfying } x_k \to x, u^k(x_k) \to \infty\}.$$

Step 1. $\Omega \setminus \mathcal{S} = \Omega \setminus \Sigma$, i.e. $\mathcal{S} = \Sigma$ and card$(\mathcal{S}) \leq T/M_n^*$.

If $x_0 \in \Omega$ such that $\mu(\{x_0\}) < M_n^*$, then there is $r_0 > 0$ such that $\mu(B(x_0, r_0)) < M_n^*$. Thus for k sufficiently large,

$$\int_{B(x_0, r_0)} (u^k)_+^{p^*} dx \leq \delta < M_n^*.$$

Applying Proposition 3 to $r_0^{n-2}u^k(x_0+r_0x)$, we get C satisfying $\max_{B(x_0,r_0/4)} u^k \le Cr_0^{2-n}$, hence $(u^k)_+$ is bounded in $L^\infty(B(x_0,r_0/4))$ and $x_0 \notin \mathcal{S}$. On the other hand, if $x_0 \in \Omega \setminus \Sigma$, we get $r_0 > 0$ such that $(u^k)_+$ is bounded in $L^\infty(B(x_0,r_0/4))$. Clearly this implies that

$$\lim_{r\to 0}\limsup_{k\to\infty}\int_{B(x_0,r)}(u^k)_+^{p^*}\,dx = 0.$$

This means that $\mu(\{x_0\}) = 0$, so $x_0 \notin \mathcal{S}$.

Step 2. $\mathcal{S} = \emptyset$ implies that case (1) or (2) occurs.

By Step 1, $(u^k)_+$ in bounded in $L^\infty_{loc}(\Omega)$, therefore $\mu \in L^1 \cap L^\infty_{loc}(\Omega)$. Define

$$\begin{cases} -\Delta v_k &= (u^k)_+^{p^*} \quad \text{in } \Omega \\ \quad v_k &= 0 \quad\quad\quad \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v &= \mu \quad \text{in } \Omega \\ \quad v &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{23}$$

We have then v_k converges uniformly to v on compact set of Ω, hence $w_k = u^k - v_k$ is a sequence of harmonic function bounded above on each compact subset of Ω. Using Harnack's principle, we get a subsequence (denoted also by w_k) such that

(i) either w_k is bounded in $L^\infty_{loc}(\Omega)$, which corresponds to case (1);

(ii) or w_k tends to $-\infty$ uniformly on compact set of Ω, corresponding to case (2).

Step 3. $\mathcal{S} \neq \emptyset$ implies that case (3) holds.

By Step 1, $(u^k)_+$ is bounded in $L^\infty_{loc}(\Omega\setminus\mathcal{S})$. Consider v_k, v defined by (23) and $w_k = u^k - v_k$. Analogously, v_k is bounded in $L^\infty_{loc}(\Omega \setminus \mathcal{S})$, and after passing to a subsequence, either w_k is bounded is $L^\infty_{loc}(\Omega \setminus \mathcal{S})$ or w_k tends uniformly to $-\infty$ on compact set of $\Omega \setminus \mathcal{S}$. Now we prove that the first case cannot occur. Suppose the contrary, we choose $x_1 \in \mathcal{S}$ and $r > 0$ such that $B(x_1,r) \cap \mathcal{S} = \{x_1\}$, thus there is a constant $C > 0$ such that $u^k \ge -C$ on $\partial B(x_1,r)$. Consider

$$\begin{cases} -\Delta z_k &= (u^k)_+^{p^*} \quad \text{in } B(x_1,r) \\ \quad z_k &= -C \quad\quad \text{on } \partial B(x_1,r) \end{cases} \quad \text{and} \quad \begin{cases} -\Delta z &= \mu \quad\quad \text{in } B(x_1,r) \\ \quad z &= -C \quad \text{on } \partial B(x_1,r). \end{cases} \tag{24}$$

Thus $z_k \to z$ a.e. in $B(x_1,r)$ and $z_k \le u^k$ in $B(x_1,r)$. Moreover, since $\mu(\{x_1\}) \ge M_n^*\delta_{x_1}$,

$$z(x) \ge \frac{M_n^*}{c(n)}\frac{1}{|x-x_1|^{n-2}} + O(1)$$

where $c(n)$ is a constant depending only on n. Therefore

$$\int_{B(x_1,r)} z_+^{p^*}\,dx = \infty \quad \text{because} \quad z_+^{p^*} \ge \frac{C}{|x-x_1|^n} \quad \text{near } x_1.$$

On the other hand, by Fatou's lemma,

$$\int_{B(x_1,r)} z_+^{p^*}\,dx \le \liminf \int_{B(x_1,r)} (z_k)_+^{p^*}\,dx \le \liminf \int_{B(x_1,r)} (u^k)_+^{p^*}\,dx \le T,$$

which gives a contradiction. Thus w_k tends to $-\infty$ on compact set of $\Omega \setminus \mathcal{S}$, so is u^k. ∎

Proof of Theorem 5. The proof is rather technical and very similar to that in [19], so we give here only the proof of crucial step.

Proposition 4 *Let $R > 0$, u^k be a sequence of functions satisfying $-\Delta u^k = (u^k)_+^{p^*}$ in B_R such that $\max_{\overline{B}_R} u^k \to \infty$, $\max_{\overline{B}_R \setminus B_r} u^k \to -\infty$ with any $r \in (0, R)$. In addition, assume that*

$$\lim_{k\to\infty} \int_{B_R} (u^k)_+^{p^*} dx = \alpha \quad \textit{and} \quad u^k(x)|x|^{n-2} \le C_0 < \infty,$$

then $\alpha = M_n^*$. There exist also positive constants C, k_0 such that for any $k \ge k_0$, $u^k \le 0$ in $\overline{B}_R \setminus B_{C\delta_k}$ where $\delta_k^{2-n} = \max_{\overline{B}_R} u^k$.

Proof. Let u be any solution $-\Delta u = u_+^{p^*}$ satisfying $u(x)|x|^{n-2} \le C_0$ in B_R. By Lemma 3 and the transformation, we have for any $r > 0$

$$u(0) + C_1 \inf_{B_r} u \le \frac{C_2}{r^{n-2}}. \tag{25}$$

Furthermore, for $r \le R/2$, $v(x) = r^{n-2}u(rx)$ satisfies

$$-\Delta v = v_+^{p^*} \quad \text{and} \quad v(x) \le 2^{n-2}C_0 \quad \text{in } B_2 \setminus B_{1/2}.$$

Consider

$$\begin{cases} -\Delta w &= v_+^{p^*} \quad \text{in } B_2 \setminus B_{1/2} \\ \quad w &= 0 \quad\quad \text{on } \partial(B_2 \setminus B_{1/2}). \end{cases} \tag{26}$$

Clearly $\|w\|_\infty \le C$ and $\xi = v - w$ is a harmonic function bounded above by $2^{n-2}C_0$. By Harnack's principle, we have positive constant β such that

$$\sup_{\partial B_1} \left(2^{n-2}C_0 - \xi\right) \le \beta^{-1} \inf_{\partial B_1} \left(2^{n-2}C_0 - \xi\right).$$

We deduce then

$$\sup_{\partial B_1} v \le \beta \inf_{\partial B_1} v + C. \tag{27}$$

Associating (27) to (25), we obtain

$$\sup_{\partial B_r} u(x) \le \frac{C}{r^{n-2}} - \frac{\beta u(0)}{C_1}, \quad \forall\, r \le R/2. \tag{28}$$

Now we return to our sequence of functions u^k. Denote $u^k(x_k) = \delta_k^{2-n} = \max_{\overline{B}_R} u^k(x)$. By $u^k(x_k)|x_k|^{n-2} \le C_0$, we get $|x_k| \le C\delta_k$, we know also $\delta_k \to 0$. Applying (28) for the function $u^k(x_k + x)$ (defined on $B_{R/2}$ for k large), we get constants C and k_0 such that

$$\text{for any } k \ge k_0 \quad \text{and} \quad x \in \overline{B}_{R/4} \setminus B_{C\delta_k}, \quad u^k(x) \le 0. \tag{29}$$

On the other hand, we have $\max_{\overline{B}_R \setminus B_{R/4}} u^k \to -\infty$, so we can replace $\overline{B}_{R/4}$ by $\overline{B}_R$ in (29). Now it suffices to consider the sequence of functions $v_k(x) = \delta_k^{n-2} u^k(x_k + \delta_k x)$. by similar blow up argument as before, it is easy to get that v_k converges to $\varphi(|x|)$ given by (8), uniformly on compact set of $\mathbb{R}^n$. The rest of proposition is then easy to be done. ∎

5 An Optimal inequality

In this section, we will prove Theorem 3, an optimal Moser-Trudinger type inequality. Let Ω be any smooth domain in $\mathbb{R}^n$ and $V(\Omega)$, $\mathcal{I}_{M,\Omega}(u)$ be defined as in Theorem 3. Some similar functionals have been considered in [3, 28, 32]. Our proofs and results here are more direct and more precise. For studying the functional $\mathcal{I}_M$, we introduce two another ones:

$$\mathcal{J}(u) = \frac{\|\nabla u\|_2^2 \|u\|_{p^*}^{p^*-1}}{\|u\|_{p^*+1}^{p^*+1}} \quad \text{and} \quad \mathcal{K}_M(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{M}{p^*+1}\int_\Omega |u|^{p^*+1} dx \tag{30}$$

We denote

$$\alpha_0 = \inf_{H_0^1 \cap L^{p^*+1}(\Omega)} \mathcal{J}(u) \quad \text{and} \quad \beta_\Omega(M) = \inf_{u \in H_0^1(\Omega),\ \|u\|_{p^*}=1} \mathcal{K}_M(u).$$

Lemma 4 *We have*

(1) α_0 is a positive constant independent of Ω.

(2) $\beta_\Omega(M)$ is a decreasing function of M and $\beta_\Omega(M) > 0$ if and only if $M < M_1 = \alpha_0(p^*+1)/2$.

Proof. For the positivity of α_0, it suffices to note that $2/2^* + (p^*-1)/p^* = 1$, where $2^* = 2n/(n-2)$ is the critical exponent for Sobolev's embedding of $H^1(\Omega)$ into $L^q(\Omega)$, namely $1/2^* = 1/2 - 1/n$. By Hölder's inequality, $\|u\|_{p^*+1}^{p^*+1} \le \|u\|_{p^*}^{p^*-1}\|u\|_{2^*}^2$. Therefore,

$$\alpha_0 \ge \inf_{H^1(\mathbb{R}^n)} \mathcal{J}(u) \ge \inf_{H^1(\mathbb{R}^n)} \left(\frac{\|\nabla u\|_2}{\|u\|_{2^*}}\right)^2 > 0.$$

We observe that $\mathcal{J}(u)$ remains unchanged under the transformation $u(x) \mapsto \lambda^{n-2}u(x_0 + \lambda x)$ for any $\lambda \in \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. Associating with the density of $C_0^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n)$ and the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, we deduce that

$$\alpha_0 = \inf_{H_0^1 \cap L^{p^*+1}(\Omega)} \mathcal{J}(u) = \inf_{H^1(\mathbb{R}^n)} \mathcal{J}(u).$$

Assertions in (2) for the functional $\mathcal{K}_M(u)$ are easy consequences of the definition of α_0. ■

Remark. A natural question is to ask whether the constant α_0 can be achieved. We claim that the answer is negative when Ω is bounded. Suppose the contrary, then the infimum of $\mathcal{J}$ is achieved by u in $H_0^1(\Omega)$. Extending u by 0, it is also a minimizer with compact support in $H^1(\mathbb{R}^n)$. Note that $\mathcal{J}$ is invariant under the transformation $u \mapsto \xi\lambda^{n-2}|u|(\lambda x)$, for positive constants ξ and λ, without loss of generality, we may assume that $u \ge 0$ and $\|u\|_{p^*} = \|u\|_{p^*+1} = 1$. Taking $R > 0$ such that $supp(u) \subset B_R$, then u satisfies the Euler-Lagrange equation

$$\begin{cases} -2\Delta u &= g(u) = \alpha_0(p^*+1)u^{p^*} - \alpha_0(p^*-1)u^{p^*-1} \quad \text{in } B_R \\ u &\ge 0 \quad \text{in } B_R \\ u &= 0 \quad \text{on } \partial B_R. \end{cases} \tag{31}$$

The classical Pohozaev's identity gives

$$\left(1 - \frac{n}{2}\right)\int_{B_R} g(u)u dx + n\int_{B_R} G(u)dx = 2\int_{\partial B_R} R\left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma \ge 0, \tag{32}$$

where $G(u) = \int_0^u g(s)ds$. Thus,

$$\text{l.h.s. of (32)} = \left(1 - \frac{n}{2}\right) \alpha_0 \left[(p^* + 1) - (p^* - 1)\right] + n\alpha_0 \left(1 - \frac{p^* - 1}{p^*}\right) = 0,$$

which implies that $\partial u/\partial \nu = 0$ on ∂B_R. But this contradicts the fact $u \not\equiv 0$ in view of the Hopf lemma.

5.1 The B_1 case

Here we will prove Theorem 3 for the special case $\Omega = B_1$. In our proof, we need several times the following construction: for $v \in H_0^1(B_1)$ and $M \in \mathbb{R}$, set

$$v_\lambda(x) = \begin{cases} \lambda^{2-n} v(x/\lambda) & \text{if } 0 \le r = |x| \le \lambda, \\ \dfrac{M}{(n-2)\omega_{n-1}} \left(r^{2-n} - \lambda^{2-n}\right) & \text{if } \lambda \le r \le 1. \end{cases} \tag{33}$$

We remark that $v_{\lambda|_{B_1 \setminus B_\lambda}}$ is the unique minimizer of the functional

$$\frac{1}{2} \|\nabla u\|^2_{L^2(B_1 \setminus B_\lambda)} + M u|_{\partial B_1}$$

in $\Lambda_\lambda = \{u \in H^1(B_1 \setminus B_\lambda) \,|\, u_{|\partial B_\lambda} = 0, u_{|\partial B_1} = \text{ constant } \}$. By a direct calculation, we find

Lemma 5 *For any $v \in H_0^1(B_1)$, $\lambda \in (0, 1]$ and $M \in \mathbb{R}$, we have $\|(v_\lambda)_+\|_{p^*} = \|v_+\|_{p^*}$ and*

$$\mathcal{I}_{M,B_1}(v_\lambda) = \lambda^{2-n}(\mathcal{K}_M(v) - h(M)) + h(M), \tag{34}$$

where

$$h(M) = \frac{M^2}{2(n-2)\omega_{n-1}}.$$

For simplicity, we denote $\beta(M) = \beta_{B_1}(M)$, $V = V(B_1)$ and $I_M = I_{M,B_1}$. We show first the existence of a critical value $\overline{M}$ such that $\mathcal{I}_M(u)$ is bounded from below in V iff $M \le \overline{M}$, then we show that $\overline{M}$ is just E_n^*. The critical value $\overline{M}$ is determined as follows

Proposition 5 *There exists a unique constant $\overline{M} \in (0, M_1]$ such that $\beta(\overline{M}) = h(\overline{M})$. Moreover, we have*

(1) for any $M < \overline{M}$, $\inf_V \mathcal{I}_M(u) > -\infty$ and it is achieved by a nonnegative function.

(2) for any $M > \overline{M}$, $\inf_V \mathcal{I}_M(u) = -\infty$.

Proof. Clearly, $\beta(M)$ is achieved in V for any $M < M_1$, thus β is a decreasing continuous function in $(-\infty, M_1)$, so is $\beta - h$. Of course, $\beta(0) - h(0) = \beta(0)$ is positive, we have also $\lim_{M \uparrow M_1} \beta(M) = \beta(M_1)$.

Step 1: $\beta(M_1) \le h(M_1)$.

If it is false, there exists $c_0 > 0$ such that $\beta(M_1) = h(M_1) + c_0$. Take any $\sigma > 0$, we have $u \in H_0^1(B_1)$ such that $\|u\|_{p^*} = 1$ and $\mathcal{K}_{M_1+\sigma}(u) < 0$. By Schwarz symmetrization, we

can assume that u is a nonnegative decreasing radial function. Choose $\lambda \in (0,1)$ such that $c = \|(u-u(\lambda))_+\|_{p^*} = [M_1/(M_1+\sigma)]^{1/(p^*-1)}$, we define

$$\overline{v} = (u-u(\lambda))_+, \quad v = \overline{v}/\|\overline{v}\|_{p^*} \quad \text{and} \quad w(x) = \lambda^{n-2} v(\lambda x).$$

We have then v, $w \in H_0^1(B_1)$ with L^{p^*}-norm equal to 1, thus

$$\mathcal{K}_{M_1}(v) = \lambda^{2-n}\mathcal{K}_{M_1}(w) \geq \lambda^{2-n}\beta(M_1) = \lambda^{2-n}(h(M_1)+c_0). \tag{35}$$

Using the choice of λ, $\mathcal{K}_{M_1+\sigma}(\overline{v}) = c^2\mathcal{K}_{M_1}(v)$. Otherwise,

$$\begin{aligned}
& \mathcal{K}_{M_1+\sigma}(u) - \mathcal{K}_{M_1+\sigma}(\overline{v}) \\
= \; & \frac{1}{2}\int_{B_1\setminus B_\lambda} |\nabla u|^2 dx - \frac{M_1+\sigma}{p^*+1}\int_{B_1} u^{p^*+1}dx + \frac{M_1+\sigma}{p^*+1}\int_{B_1} \overline{v}^{p^*+1}dx \\
= \; & \frac{1}{2}\int_{B_1\setminus B_\lambda} |\nabla u|^2 dx - \frac{M_1+\sigma}{p^*+1}\int_{B_1} \left(u^{p^*+1} - (u-u(\lambda))_+^{p^*+1}\right) dx \\
\geq \; & \frac{1}{2}\int_{B_1\setminus B_\lambda} |\nabla u|^2 dx - (M_1+\sigma)\int_{B_1} u_+^{p^*} u(\lambda) dx \\
= \; & \frac{1}{2}\int_{B_1\setminus B_\lambda} |\nabla u|^2 dx - (M_1+\sigma)u(\lambda) \\
\geq \; & -h(M_1+\sigma)\left(\lambda^{2-n}-1\right).
\end{aligned} \tag{36}$$

The first inequality follows form the convexity of function $f(t) = t_+^{p^*+1}$ and the second one from the remark below (33). Combining (35) and (36), we get $-c^2(h(M_1)+c_0) \geq -h(M_1+\sigma)$, e.g.

$$\frac{(M_1+\sigma)^2}{2(n-2)\omega_{n-1}}\left(\frac{M_1+\sigma}{M_1}\right)^{2/(p^*-1)} \geq \frac{M_1^2}{2(n-2)\omega_{n-1}} + c_0,$$

which is impossible when σ is small enough. Thus $\beta(M_1) \leq h(M_1)$ and $\overline{M}$ exists uniquely in $(0, M_1]$.

Step 2: For any $M > \overline{M}$, the infimum of $\mathcal{I}_M$ on V is $-\infty$.

By the definition of $\overline{M}$, there exists $v \in V$ such that $\mathcal{K}_M(v) < h(M)$. If we take v_λ defined by (33), we see that $v_\lambda \in V$ and $\mathcal{I}_M(v_\lambda)$ tends to $-\infty$ when $\lambda \to 0$.

Step 3: For any $M < \overline{M}$, the infimum of $\mathcal{I}_M$ on V is achieved by a nonnegative function.

Let v^k be a minimizing sequence of $\mathcal{I}_M$ in V. Denote ${v^k}_{|\partial B_1} = c_k$. Considering the function $F(c) = \mathcal{I}_M(u+c)$, we get easily (for any domain Ω)

Lemma 6 *F is a concave function in $\mathbb{R}$ and the maximum is realized uniquely by c such that $\|(u+c)_+\|_{p^*} = 1$.*

In view of this lemma and $\|v_+^k\|_{p^*} = 1$, we have

$$\begin{aligned}
\mathcal{I}_M(v^k) \geq \mathcal{I}_M(v^k - c_k) \; & = \mathcal{K}_M((v^k-c_k)_+) + \frac{1}{2}\int_{B_1} |\nabla (v^k-c_k)_-|^2 dx \\
& \geq \frac{1}{2}\|\nabla(v^k-c_k)_-\|_2^2 + \delta\|\nabla(v^k-c_k)_+\|_2^2 \\
& \geq \min(1/2,\delta)\|\nabla v^k\|_2^2.
\end{aligned} \tag{37}$$

In the second inequality, we have used $M < M_1$ and $\|(v^k - c_k)_+\|_{p^*} \leq 1$.

If $c_k \geq 0$, we can replace v^k by v^k_+ to reduce the energy. If $c_k < 0$, we can replace first v^k by $c_k + (v^k - c_k)_+$ to reduce the energy, so we can assume that $v^k \geq c_k$ in B_1. Denote now v^k_* the Schwarz symmetrization of v_k, clearly $\mathcal{I}_M(v^k_*) \leq \mathcal{I}_M(v^k)$ ($v^k - c_k$ is positive and $v^k_* = (v^k - c_k)_* + c_k$). So we can assume that v^k is a radially decreasing function. Suppose $supp(v^k_+) = B_\lambda$ with $\lambda < 1$, define $w^k(x) = \lambda^{n-2} v^k(\lambda x)$ and define w^k_λ by (33) with $v = w^k(x)$. It is clear that $v^k(x) = w^k_\lambda(x)$ in B_λ. Using again the remark below (33) and $\mathcal{K}_M(w^k) \geq \beta(M) \geq h(M)$,

$$\mathcal{I}_M(v^k) \geq \mathcal{I}_M(w^k_\lambda) = \lambda^{2-n}\Big(\mathcal{K}_M(w^k) - h(M)\Big) + h(M) \geq \mathcal{K}_M(w^k) = \mathcal{I}_M(w^k).$$

This means that we can substitute v^k by w^k, and we get again a nonnegative function in V.

Thus we obtain a minimizing sequence of $\mathcal{I}_M$ with nonnegative functions v^k. Estimate (37) and $\|v^k\|_{p^*} = 1$ mean that v^k is bounded in $H^1(B_1)$. This proves the step and the proof of the Proposition is completed. ■

Remark. For $M < \overline{M}$, by the standard elliptic theory, we can conclude by the Euler equation that the nonnegative minimizer of $\mathcal{I}_M$ is a positive, radially decreasing function in B_1.

Proposition 6 *We have $\overline{M} = E^*_n$, and for any $M \leq E^*_n$, $\inf_V \mathcal{I}_M(u) \geq \dfrac{ME^*_n}{2(n-2)\omega_{n-1}}$.*

Proof. Let ϕ be the unique positive solution of

$$\begin{cases} -\Delta\phi &= E^*_n \phi^{p^*} \quad \text{in } B_1 \\ \phi &= 0 \quad \text{on } \partial B_1 \\ \displaystyle\int_{B_1} \phi^{p^*} dx &= 1. \end{cases} \tag{38}$$

Claim 1: $\overline{M} \leq E^*_n$. Consider the family of functions ϕ_λ given by (33) with $v = \phi$ and $M = E^*_n$. We have

Lemma 7 *$\mathcal{I}_{E^*_n}(\phi_\lambda)$ is independent of $\lambda \in (0, 1]$ and $\mathcal{I}_{E^*_n}(\phi) = \mathcal{K}_{E^*_n}(\phi) = h(E^*_n)$.*

Proof. Notice that ϕ_λ is a regular family w.r.t. to λ and $-\Delta\phi_\lambda = E^*_n(\phi_\lambda)^{p^*}_+$ in B_1. Hence, we have

$$\begin{aligned} \left(\mathcal{I}_{E^*_n}(\phi_\lambda)\right)'_\lambda &= \int_{B_1} \nabla\phi_\lambda \nabla(\phi_\lambda)' dx + E^*_n \int_{B_1} (\phi_\lambda)^{p^*}_+ (\phi_\lambda)' dx + E^*_n (\phi_\lambda)'|_{\partial B_1} \\ &= \left(\int_{\partial B_1} \frac{\partial \phi_\lambda}{\partial r} d\sigma + E^*_n\right) \times (\phi_\lambda)'|_{\partial B_1} \\ &= 0. \end{aligned} \tag{39}$$

Using (34), the independence of λ gives then $\mathcal{I}_{E^*_n}(\phi) = \mathcal{K}_{E^*_n}(\phi) = h(E^*_n)$. ■

Lemma 6 implies that $\beta(E_n^*) \leq \mathcal{K}_{E_n^*}(\phi) = h(E_n^*)$, which, in turn, implies that $\overline{M} \leq E_n^*$ by Proposition 5.

Claim 2: $E_n^* \leq \overline{M}$. Since $\overline{M} \leq E_n^*$, then for any $M < \overline{M}$, the minimizer is clearly the unique positive function $w_M \in H^1(B_1)$ satisfying

$$\begin{cases} -\Delta w_M &= M(w_M)^{p^*} \quad \text{in } B_1 \\ w_M &= c_M \quad \text{on } \partial B_1 \\ \displaystyle\int_{B_1} (w_M)^{p^*} dx &= 1. \end{cases} \tag{40}$$

Indeed, $w_M = \xi\phi(\lambda x)$ with some convenient $\lambda \leq 1$ and $\xi > 0$, we prove then

Lemma 8 *For any $M \in (0, E_n^*)$, $\mathcal{I}_M(w_M) \geq \dfrac{ME_n^*}{2(n-2)\omega_{n-1}} > h(M)$.*

Define $G(M) = \frac{1}{M}\mathcal{I}_M(w_M) - \frac{M}{2(n-2)\omega_{n-1}}$, then

$$\begin{aligned} G'(M) = \; & -\frac{1}{2M^2}\int |\nabla w_M|^2 dx - \frac{1}{2(n-2)\omega_{n-1}} \\ & + \frac{1}{M}\int \nabla w_M \nabla \eta dx - \int (w_M)^{p^*}\eta dx + \eta|_{\partial B_1}, \end{aligned} \tag{41}$$

where $\eta = (w_M)'$. The sum of the three last terms is zero by equation (40), thus $G'(M) \leq -\frac{1}{2(n-2)\omega_{n-1}}$ in $(0, E_n^*]$, so $G(M) \geq \frac{E_n^* - M}{2(n-2)\omega_{n-1}}$ for $M \leq E_n^*$ since $G(E_n^*) = 0$. Consequently

$$\mathcal{I}_M(w_M) \geq \frac{ME_n^*}{2(n-2)\omega_{n-1}} > \frac{M^2}{2(n-2)\omega_{n-1}}, \quad \forall\, M \in (0, E_n^*). \tag{42}$$

Furthermore, for any $M \leq \overline{M}$, $\beta(M) \geq \inf_V \mathcal{I}_M$ by definition. Thus if $\overline{M} < E_n^*$, we get

$$\beta(\overline{M}) = \lim_{M \uparrow \overline{M}} \beta(M) \geq \lim_{M \uparrow \overline{M}} \inf_V \mathcal{I}_M = \lim_{M \uparrow \overline{M}} \mathcal{I}_M(w_M) = \mathcal{I}_{\overline{M}}(w_{\overline{M}}) > h(\overline{M}),$$

which contradicts clearly the defintion of $\overline{M}$, this completes the proof. ■

For completing the Proof of Theorem 3 for the unit disk, we need to check the case $M = E_n^*$. As $\inf_V \mathcal{I}_{E_n^*} \geq \lim_{M \uparrow E_n^*} (\inf_V \mathcal{I}_M) \geq \lim_{M \uparrow E_n^*} h(M) = h(E_n^*)$ and $\inf_V \mathcal{I}_{E_n^*} \leq \mathcal{I}_{E_n^*}(\phi) = h(E_n^*)$, we conclude immediately that $\inf_V \mathcal{I}_{E_n^*} = h(E_n^*)$ is achieved by ϕ given by (38).

Corollary 1 *We have $E_n^* \in [\alpha_0, \alpha_0(p^*+1)/2]$ and $\inf_V \mathcal{I}_{E_n^*} = \frac{(E_n^*)^2}{2(n-2)\omega_{n-1}}$.*

5.2 The general domain case

Let $v_\lambda(x) = \lambda^{2-n} v(x/\lambda)$, we get $v_\lambda \in V(B_\lambda)$ iff $v \in V(B_1)$ and $\mathcal{I}_{M,B_\lambda}(v_\lambda) = \lambda^{2-n}\mathcal{I}_{M,B_1}(v)$, hence $\inf_{V(B_\lambda)} \mathcal{I}_{M,B_\lambda} = \lambda^{2-n} \inf_V \mathcal{I}_{M,B_1}$. Thus Theorem 3 holds true for any B_λ ($\lambda > 0$).

Let $\Omega \in \mathbb{R}^n$ be a bounded domain, up to a translation, we can suppose there exist R_1, $R_2 > 0$ such that $B_{R_1} \subset\subset \Omega \subset\subset B_{R_2}$.

For any $M \leq E_n^*$ and $v \in V(\Omega)$, if $c = v|_{\partial\Omega} > 0$, we have $\mathcal{I}_M(v) \geq \mathcal{K}_M(v_+) \geq 0$ since $M \leq M_1$ and $v_+ \in H_0^1(\Omega)$, $\|v_+\|_{p^*} = 1$. If $c < 0$, denote $\overline{v} = v\chi_\Omega + c\chi_{B_{R_2}\setminus\Omega}$, then $\overline{v} \in V(B_{R_2})$ and $\mathcal{I}_{M,\Omega}(v) = \mathcal{I}_{M,R_2}(\overline{v})$. So we get $\inf_{V(\Omega)} \mathcal{I}_M > -\infty$.

For any $M > E_n^*$, if $M > M_1$, by the definition of M_1 and the construction of (33), it is easy to see that the infimum of $\mathcal{I}_M$ is $-\infty$. If $M \in (E_n^*, M_1]$, we take a sequence $v^k \in V(B_{R_1})$ such that $\mathcal{I}_{M,B_{R_1}}(v^k)$ tends to $-\infty$ as k tends to ∞. If $c_k = v^k|_{\partial B_{R_1}}$ is positive, using similar argument as above, we get $\mathcal{I}_M(v^k) \geq 0$. Thus $c_k \leq 0$ for great k, in this case we denote $\overline{v}^k = v^k\chi_{B_{R_1}} + c_k\chi_{\Omega\setminus B_{R_1}}$. Obviously $\overline{v}^k \in V(\Omega)$ and $\mathcal{I}_{M,\Omega}(\overline{v}^k) = \mathcal{I}_{M,B_{R_1}}(v^k)$ tends to $-\infty$. ■

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G. Wang (gwang@mis.mpg.de)
Max-Planck-Institute for Mathematics,
Inselstr. 22-26,
D-04103 Leipzig, Germany

D. Ye (dong.ye@math.u-cergy.fr)
Département de Mathématiques, site Saint-Martin
Université de Cergy-Pontoise, BP 222
95302 Cergy-Pontoise cedex, France
and
C.M.L.A., E.N.S. de Cachan, 61 avenue du Président Wilson
94235 Cachan cedex, France