On moving Ginzburg-Landau filament vortices

by

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Abstract

In this note, we establish a quantization property for the heat equation of Ginzburg-Landau functional in $\mathbb{R}^4$ which models moving filament vortices. It asserts that if the energy is sufficiently small on a parabolic ball in $\mathbb{R}^4 \times \mathbb{R}_+$ then there is no filament vortices in the parabolic ball of $\frac{1}{2}$ radius. This extends a recent result of Lin-Rivièrè [LR3] in $\mathbb{R}^3$ but the problem is open for $\mathbb{R}^n$ with $n \geq 5$.

§1 Introduction

For $n \geq 2$ and $\epsilon > 0$, the heat equation for the Ginzburg-Landau functional on $\mathbb{R}^n$ is:

$$
\frac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon = \frac{1}{\epsilon^2} (1 - |u_\epsilon|^2) u_\epsilon, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \quad (1.1)
$$

$$
u_\epsilon(x,0) = g_\epsilon(x), \quad x \in \mathbb{R}^n
$$

Here $g_\epsilon : \mathbb{R}^n \to \mathbb{R}^2$ are given maps. Notice that (1.1) is the negative gradient flow for the Ginzburg-Landau functional

$$
E_\epsilon(v) = \int_{\mathbb{R}^n} \frac{1}{2} |Dv|^2 + \frac{1}{4\epsilon^2} (1 - |v|^2)^2. \quad (1.2)
$$

Asymptotic behaviors for minimizers of $E_\epsilon$ in dimension two was first studied by Bethuel-Brezis-Hélein [BBH] (see also Struwe [S1] and recent important works by Pacard-Rivièrè [PR] on steady solutions to (1.1)). Moreover, such static theories were also developed by Rivièrè [R1] [R2] and Lin-Rivièrè
[LR1] in high dimensions in connection with codimension two area minimizing currents, especially a crucial quantization property for steady solutions to the equation (1.1) was proved by Lin-Riviére [LR2] for \( n = 3 \) and late by Bethuel-Brezis-Orlandi [BBO] for all \( n \geq 3 \). The asymptotic for the equation (1.1) in dimension two was initiated by Lin [L1]-[L2] and also studied by Jerrard-Soner [JS]. Notice that in the context of understanding the limiting behavior for sequence of solutions \( u_\epsilon \) to either static or parabolic versions of the equation (1.1), one encounters the main difficulty from the possibilities that \( u_\epsilon \) may vanish on sets, called Ginzburg-Landau vortex, where the equation (1.1) degenerates and \( \frac{|u_\epsilon|}{|\nabla u_\epsilon|} \) have nontrivial topological obstructions as well. On the other hand, it is well-known that existences of vortices requires the Ginzburg-Landau energy at least of the order \( \log \frac{1}{\epsilon} \). In other words, \( g_\epsilon \) above is assumed to have \( E_\epsilon(g_\epsilon) = O(\log(\frac{1}{\epsilon})) \), this, combined with the energy inequality for (1.1), implies

\[
E_\epsilon(u_\epsilon(\cdot, t)) \leq O(\log \frac{1}{\epsilon}). \tag{1.3}
\]

From the analytic point of view, the size estimate for the bad set, which leads to vortices at the limit, \( B_\epsilon = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}_+: |u_\epsilon|(x, t) \leq \frac{1}{\epsilon}\} \) plays a critical role in obtain \( W^{1,p} \) compactness for suitable \( p \in (1, 2) \) (see, e.g. [BBH] and [PR]). To obtain sharp size estimates of \( B_\epsilon \), one often needs to obtain the so-called \( \eta \)-compactness property for \( u_\epsilon \) which rough says that if \( E_\epsilon(u_\epsilon) \) is of order \( \eta \log \frac{1}{\epsilon} \) for sufficiently small \( \eta > 0 \) then there is no interior bad points for \( u_\epsilon \), which was established for (i): minimizers of \( E_\epsilon \) by Riviére [R1] [R2] for \( n = 3 \) and by Lin-Riviére [LR1] \( n \geq 3 \); (ii) critical points of \( E_\epsilon \) by Lin-Riviére [LR2] for \( n = 3 \) and by Bethuel-Brezis-Orlandi [BBO] for all \( n \geq 3 \). Moreover, such \( \eta \)-compactness property was also proved for solutions to the equation (1.1) very recently by Lin-Riviére [LR3] in the case \( n = 3 \). It was believed that their result still holds for \( \mathbb{R}^n \) with \( n \geq 4 \). In this note, we confirm such a belief in the case that \( n = 4 \). More precisely,
we prove

**Theorem A.** For $n = 4$ and $\epsilon > 0$. Let $u_\epsilon : R^4 \times R_+ \rightarrow R^2$ be solutions to the equation (1.1) satisfying $|u_\epsilon| \leq 1$ and $|Du_\epsilon| \leq \frac{C_0}{\epsilon^2}$. Then there exist $\epsilon_0 > 0$ and $\eta > 0$ depending only on $C_0$ such that if for $(x_0, t_0) \in R^4 \times R_+$, $0 < \rho < \sqrt{t_0}$, and $\epsilon \leq \epsilon_0$

$$
\frac{1}{\rho^2} \int_{t_0-\rho^2}^{t_0} \int_{R^4} \left( \frac{1}{2} |Du_\epsilon|^2 + \frac{(1 - |u_\epsilon|^2)^2}{4\epsilon^2} \right) e^{\frac{|x-x_0|^2}{4\epsilon^2}} \leq \eta \log \frac{\rho}{\epsilon},
$$

then

$$
|u_\epsilon(x_0, t_0)| \geq \frac{1}{2}.
$$

We would like to remark that the idea developed by Lin-Riviere [LR2] [LR3] was to interpolate between the Lorentz spaces $L^{2,1}$ and $L^{2,\infty}$ on generic two dimensional slices which therefore worked very well in $R^3$, but it seems unclear how to extend them to $R^n$ with $n \geq 4$. On the other hand, there is the interpolation technique between $L^1$ and $L^\infty$ developed by Bethuel-Brezis-Orlandi [BBO] available for the statics case in $R^n$ for all $n \geq 3$, where they made very clever uses of the energy monotonicity formulair for static solutions to the equation (1.1). Our method starts with the observation that there exists an energy monotonicity inequality for all time slice $R^n \times \{t\}$ when $n = 4$, which enables us to adapt the main ideas from [BBO] and some of those ideas from [LR3]. Notice that one can always view solutions to the equation (1.1) in $R^3 \times R_+$ as solutions to the equation (1.1) in $R^4 \times R_+$ which are independent of the fourth spatial variable. Hence, our proof also gives a somewhat different proof of a main theorem of [LR3].

The paper is organized as follows. In §2, we derive the needed elliptic type energy inequality in $R^4 \times \{t\}$; In §3, we recall the parabolic type energy monotonicity inequalities established by Struwe [S2] and Lin-Rivière [LR3] and extract a good time slice; In §4, we illustrate the main estimate by
performing an intrinsic Hodge decomposition of a suitable quantity on good
time slices and prove theorem A.

§2 Euclidean monotonicity at time slice for $n = 4$

This section is devoted to a slice monotonicity inequality (2.1) for $u_\epsilon : R^n \times R_+ \to R^2$ satisfying (1.1) when $n = 4$. For $n \geq 4$, $x \in R^n$, $r > 0$, and $t > 0$, we denote

$$E_\epsilon(x, r) = \int_{B_r(x)} \left( \frac{1}{2} |Du_\epsilon|^2 + \frac{n(1 - |u_\epsilon|^2)^2}{2(n-2)e^2} \right)(y) \, dy$$

as the Ginzburg-Landau energy of $u_\epsilon$ over $B_r(x)$ at time $t$. Then we have the following differential inequality.

**Lemma 2.1.** For $n \geq 4$. For $\epsilon > 0$, let $u_\epsilon : R^n \times R_+ \to R^2$ be a solution to (1.1). Then, for any $(x, t) \in R^n \times R_+$ and $r > 0$, one has

$$\frac{d}{dr} \left( r^{2-n} E_\epsilon(x, r) \right) + \frac{r^{3-n}}{3-n} \int_{B_r(x)} \left| \frac{\partial u_\epsilon}{\partial t} \right| \left| \frac{\partial u_\epsilon}{\partial r} \right| \geq r^{2-n} \int_{\partial B_r(x)} \left| \frac{\partial u_\epsilon}{\partial r} \right|^2 + \frac{(1 - |u_\epsilon|^2)^2}{2(n-2)e^2} + \frac{r^{3-n}}{3-n} \int_{\partial B_r(x)} \left| \frac{\partial u_\epsilon}{\partial t} \right| \left| \frac{\partial u_\epsilon}{\partial r} \right|$$

(2.1)

**Proof.** For simplicity, we assume that $x = 0$ and denote $u$ for $u_\epsilon$. Multiplying (1.1) by $x \cdot Du$, integrating over $B_r$ and using integration by parts, we obtain

$$\int_{B_r} u_\times x \cdot Du = \int_{B_r} \Delta u x \cdot Du - \frac{1}{4e^2} x \cdot D(1 - |u|^2)^2$$

$$= \int_{B_r} D \cdot (Du x \cdot Du) - Du \cdot D(x \cdot Du) - x \cdot D(1 - |u|^2)^2$$

$$= r \int_{\partial B_r} \left| \frac{\partial u}{\partial r} \right|^2 - \int_{B_r} |Du|^2$$

$$- \int_{B_r} x \cdot D(\frac{1}{2} |Du|^2 + \frac{(1 - |u|^2)^2}{4e^2})$$
Proposition 2.2

This yields

\[(n - 2)E_\epsilon(0, r) = \int_{B_r} u_t x \cdot Du + r \int_{\partial B_r} \left( \frac{1}{2} |Du|^2 - \frac{(1 - |u|^2)^2}{4\epsilon^2} \right)\]

Therefore

\[\frac{d}{dr}(r^{2-n}E_\epsilon(0, r)) = (2 - n)r^{1-n}E_\epsilon(0, r) + r^{2-n} \int_{\partial B_r} \left( \frac{1}{2} |Du|^2 + \frac{n(1 - |u|^2)^2}{4(n-2)\epsilon^2} \right)\]

Observe that

\[-r^{1-n} \int_{B_r} u_t x \cdot Du \geq -r^{2-n} \int_{B_r} |u_t||\partial u| \epsilon\]

Hence

\[\frac{d}{dr}(r^{2-n}E_\epsilon(0, r) + \frac{r^{3-n}}{3-n} \int_{B_r} |u_t||\partial u|) \geq r^{2-n} \int_{\partial B_r} \left( \frac{\partial u}{\partial r} \right)^2 + \frac{(1 - |u|^2)^2}{2(n-2)\epsilon^2} + \frac{r^{3-n}}{3-n} \int_{\partial B_r} |\partial u| \partial u|\]

This completes the proof of (2.1). ■

Now we state the consequence of Lemma 2.1 for \(n = 4\), namely the following slice energy monotonicity inequality.

Proposition 2.2. For \(\epsilon > 0\), let \(u_\epsilon : R^4 \times R_+ \to R^2\) be a solution to (1.1). Then, for any \((x, t) \in R^4 \times R_+\) and \(0 \leq r \leq R < \infty\), it holds:

\[r^{-2}E_\epsilon(x, r) + \int_r^R \frac{dr}{r^2} \int_{\partial B_r(x)} \left( \frac{1}{2} |Du_\epsilon|^2 + (4\epsilon^2)^{-1}(1 - |u_\epsilon|^2)^2 \right)\]

\[\leq 2R^{-2}E_\epsilon(x, R) + 2 \int_{B_R(x)} |\partial u_\epsilon|^2. \quad (2.2)\]
In particular,
\[ \int_{B_{n}(x)} |y-x|^{-2} \frac{(1-|u_{r}(y)|)^{2}}{\epsilon^{2}} \leq 8R^{-2}E_{\epsilon}(x,R) + 8 \int_{B_{n}(x)} |\frac{\partial u_{\epsilon}}{\partial t}|^{2}. \tag{2.3} \]

**Proof.** It is clear that (2.2), with \( r \) tending to zero, gives (2.3). Therefore, it suffices to prove (2.2). First notice that, integrating (2.1) with \( n = 4 \) from \( r \) to \( R \), we have

\[
R^{-2}E_{\epsilon}(x,R) \geq r^{-2}E_{\epsilon}(x,r) + R^{-1} \int_{B_{n}(x)} \frac{\partial u}{\partial t} \left| \frac{\partial u}{\partial t} \right| - r^{-1} \int_{B_{n}(x)} \frac{\partial u}{\partial t} \left| \frac{\partial u}{\partial r} \right| \\
+ \int_{r}^{R} \int_{\partial B_{n}(x)} \left( \left| \frac{\partial u}{\partial r} \right|^{2} + \frac{(1-|u|^{2})^{2}}{4\epsilon^{2}} \right) - \int_{r}^{R} \int_{\partial B_{n}(x)} \left| \frac{\partial u}{\partial t} \right|^{2}.
\]

Now, we need to use the fact \( n = 4 \) for the following estimates.

\[
r^{-1} \int_{B_{n}(x)} \left| \frac{\partial u}{\partial t} \right| \left| \frac{\partial u}{\partial r} \right| \leq \frac{1}{2} r^{-2} \int_{B_{n}(x)} \left| \frac{\partial u}{\partial t} \right|^{2} + \frac{1}{2} \int_{B_{n}(x)} \left| \frac{\partial u}{\partial r} \right|^{2} \\
\leq \frac{1}{2} r^{-2} \int_{B_{n}(x)} \left| \frac{\partial u}{\partial t} \right|^{2} + \frac{1}{2} \int_{B_{n}(x)} \left| \frac{\partial u}{\partial t} \right|^{2}
\]

Applying the Young inequality again, we also have, for \( r \leq s \leq R \),

\[
s^{-1} \int_{\partial B_{n}(x)} \left| \frac{\partial u}{\partial t} \right| \left| \frac{\partial u}{\partial r} \right| \leq \frac{1}{2} s^{-2} \int_{\partial B_{n}(x)} \left| \frac{\partial u}{\partial t} \right|^{2} + \frac{1}{2} \int_{\partial B_{n}(x)} \left| \frac{\partial u}{\partial t} \right|^{2}
\]

so that

\[
\int_{r}^{R} s^{-1} \int_{\partial B_{n}(x)} \left| \frac{\partial u}{\partial t} \right| \left| \frac{\partial u}{\partial r} \right| \leq \frac{1}{2} \int_{r}^{R} \int_{\partial B_{n}(x)} \left| \frac{\partial u}{\partial t} \right|^{2} + \int_{B_{n}(x)} \left| \frac{\partial u}{\partial t} \right|^{2}.
\]

Putting these inequality together, we obtain

\[
R^{-2}E_{\epsilon}(x,R) \geq \frac{1}{2} r^{-2}E_{\epsilon}(x,r) - \int_{B_{n}(x)} \left| \frac{\partial u}{\partial t} \right|^{2} \\
+ \int_{r}^{R} \int_{\partial B_{n}(x)} \frac{1}{2} \left| \frac{\partial u}{\partial r} \right|^{2} + \frac{(1-|u|^{2})^{2}}{4\epsilon^{2}}.
\]
This implies (2.2).

§3 Parabolic monotonicity and extracting a good time

In this section, we gather two of the necessary parabolic energy monotonicity inequality which was first proved by Struwe [S2] in the context of heat flow for harmonic maps, and slightly variance of which was established by Lin-Rivièrè [LR3] and then extract a good time slice. The formula below are valid for all $n \geq 2$.

**Lemma 3.1** (Energy monotonicity) Let $u_\epsilon : \mathbb{R}^n \to \mathbb{R}^2$ be solutions to the equation (1.1) and $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$. Then for any $0 < \rho \leq \sqrt{t_0}$

$$
\frac{d}{d\rho} \int_{t_0-\rho^2}^{t_0} \int_{\mathbb{R}^n} \left( \frac{1}{2} |Du_\epsilon|^2 + \frac{(1 - |u_\epsilon|^2)^2}{4\epsilon^2} e^{\frac{|x-x_0|^2}{2\epsilon^2}} \right) dx dt
$$

$$
= \frac{1}{\rho^{n+2}} \int_{t_0-\rho^2}^{t_0} \int_{\mathbb{R}^n} \left[ \frac{1}{2(2t_0-t)} |x-x_0| \cdot Du_\epsilon + 2(t-t_0) \frac{\partial u_\epsilon}{\partial t} \right]^2
$$

$$
\frac{(1 - |u_\epsilon|^2)^2}{2\epsilon^2} e^{\frac{|x-x_0|^2}{2\epsilon^2}} e^{\frac{|x|^2}{4(t_0-t)}}
$$

(3.1)

**Proof.** It follows exactly same lines of the proof of Lemma 2.1 of [LR3]. We omit it here.

We also need the following identity which indicates how the energy decays along the spatial infinity.

**Lemma 3.2.** Under the same notations as Lemma 3.1. For any $t_0 > 0$ and $0 < \rho \leq \sqrt{t_0}$. Then the following holds:

$$
\int_{t_0-\rho^2}^{t_0} \int_{\mathbb{R}^n} \left[ (1 + \frac{|x|^2}{4(t_0-t)})(\frac{1}{2} |Du_\epsilon|^2 + \frac{(1 - |u_\epsilon|^2)^2}{4\epsilon^2} e^{\frac{|x|^2}{2\epsilon^2}}) \right] dx dt
$$

$$
\leq \rho^2 \int_{\mathbb{R}^n \times \{t_0-\rho^2\}} \left[ \frac{1}{2} |Du_\epsilon|^2 + \frac{(1 - |u_\epsilon|^2)^2}{4\epsilon^2} e^{\frac{|x|^2}{2\epsilon^2}} \right] dx
$$

$$
+ \int_{t_0-\rho^2}^{t_0} \frac{x}{4(t_0-t)} \cdot Du_\epsilon \cdot |x \cdot Du_\epsilon + 2(t-t_0) \frac{\partial u_\epsilon}{\partial t} | e^{\frac{|x|^2}{4(t_0-t)}}
$$

(3.2)
Proof. It again follows from the same argument as that of Lemma 2.2 of [LR3].

Now we describe the extraction of a good time slice as follows. We follow closely from §2.2 of [LR3] and the reader may refer to [LR3] for more details. For simplicity, we assume that \((x_0, t_0) = (0, 0)\) and the equation (1.1) holds in \(R^4 \times R^-\). Assume that (1.4) holds for some \(\rho > 0\). Then, by integration of (3.1) from \(\epsilon\) to \(\rho\) and the Fubini’s theorem, there exists a \(\rho_1 = \rho \epsilon \in (\epsilon, \rho)\) such that:

\[
\frac{1}{\rho^4} \int_{-\rho^2}^{0} \int_{R^4} j_{\epsilon}(u_\epsilon)e^{\frac{t|\xi|^2}{4}} \leq \eta \quad (3.3)
\]

Here

\[
j_{\epsilon}(u_\epsilon) \equiv \frac{1}{2\epsilon^2} |x \cdot Du_\epsilon + 2\epsilon \frac{\partial u_\epsilon}{\partial t}|^2 + \frac{(1 - |u_\epsilon|^2)^2}{2\epsilon^2} \quad (3.4)
\]

so that

\[
\frac{1}{\rho^4} \inf_{\rho_1 \leq \rho \leq \rho_1^2} \int_{R^4 \times \{-\rho^2\}} j_{\epsilon}(u_\epsilon)e^{\frac{t|\xi|^2}{4}} \leq 2\eta \quad (3.5)
\]

Denote

\[
E = \frac{1}{\rho^4} \int_{-\rho^2}^{0} \int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|\xi|^2}{4\rho^4}} \quad (3.6)
\]

where

\[
e_\epsilon(u_\epsilon) \equiv \frac{1}{\epsilon^2} |Du_\epsilon|^2 + \frac{(1 - |u_\epsilon|^2)^2}{4\epsilon^2}
\]

Then (3.1) implies

\[
E \leq \inf_{\rho_1 \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{-\rho^2}^{0} \int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|\xi|^2}{4\rho^4}} + \int_{-\rho^2}^{0} \rho^{-5} \int_{R^4} j_{\epsilon}(u_\epsilon)e^{\frac{|\xi|^2}{4\rho^4}}
\]

\[
\leq \inf_{\rho_1 \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{-\rho^2}^{0} \int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|\xi|^2}{4\rho^4}} + \frac{4}{\rho^4} \int_{-\rho^2}^{0} \int_{R^4} j_{\epsilon}(u_\epsilon)e^{\frac{|\xi|^2}{4\rho^4}}
\]

\[
\leq \inf_{\rho_1 \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|\xi|^2}{4\rho^4}} + 4\eta
\]

As in [LR3], we may assume

\[
E \gg C\eta \quad (3.7)
\]
so that

\[
\inf_{\frac{\rho}{2} \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{R^4} e_\rho(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}} \leq E \leq 2 \inf_{\frac{\rho}{2} \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{R^4} e_\rho(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}} \quad (3.8)
\]

Therefore, there exists a \( \rho_0 \in [\frac{\rho}{2}, \rho_1] \) such that

\[
\max\left\{ \frac{1}{\rho_0^4} \int_{-\rho_0^3}^{0} \int_{R^4} j_\rho(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}}, \frac{1}{\rho_0^4} \int_{R^4 \times \{-\rho_0^3\}} j_\rho(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}} \right\} \leq C\eta \quad (3.9)
\]

\[
\frac{1}{\rho_0^4} \int_{-\rho_0^3}^{0} \int_{R^4} e_\rho(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}} \leq \frac{C}{\rho_0^4} \int_{-\rho_0^3}^{0} \int_{R^4} e_\rho(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}} \quad (3.10)
\]

\[
\frac{1}{\rho_0^4} \int_{R^4 \times \{-\rho_0^3\}} e_\rho(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}} \leq \frac{C}{\rho_0^4} \int_{R^4 \times \{-\rho_0^3\}} e_\rho(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}} \quad (3.11)
\]

These inequalities, combined with Lemma 3.2, also yield

\[
\frac{1}{\rho_0^4} \int_{R^4 \times \{-\rho_0^3\}} \frac{|x|^2}{|y|} e_\rho(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}} \leq CE \quad (3.12)
\]

Observe that (3.9) and (3.11) also imply

\[
\int_{R^4 \times \{-\rho_0^3\}} \frac{\partial u_\epsilon}{\partial t}^2 e^{-\frac{|x|^2}{4\rho_0^2}} \leq CE \quad (3.13)
\]

In particular, for any \( \lambda >> 1 \) to be chosen later, one has

\[
\int_{B_{2\lambda\rho_0}(x) \times \{-\rho_0^3\}} \frac{\partial u_\epsilon}{\partial t}^2 \leq Ce^{4\lambda^2} E \quad (3.14)
\]

Hence, applying the monotonicity inequality (2.3) for \( u_\epsilon \) at \( t = -\rho_0^3 \), we obtain the following key inequality:

\[
\int_{B_{2\lambda\rho_0}(x) \times \{-\rho_0^3\}} |y - x|^{-2 \left(1 - \frac{|u_\epsilon|^2}{\epsilon^2}\right)} \leq Ce^{4\lambda^2} E, \forall x \in B_{2\lambda\rho_0} \quad (3.15)
\]

On the other hand, (3.9) also yields:

\[
\frac{1}{\rho_0^4} \int_{B_{2\lambda\rho_0} \times \{-\rho_0^3\}} \frac{1 - |u_\epsilon|^2}{\epsilon^2} \leq Ce^{4\lambda^2} \eta \quad (3.16)
\]
Notice that (3.12) implies that
\[ \frac{1}{\rho_0^4} \int_{(\mathbb{R}^4 \setminus B_{\lambda \rho_0}) \times \{-\rho_0^2\}} e_c(u_e) e^{-\frac{|x|^2}{4\rho_0^2}} \leq \frac{C}{\lambda^2} E. \]  
(3.17)

This, combined with suitable choice of \( \lambda \gg 1 \) according to the Fubini's theorem, gives
\[ \frac{1}{\rho_0} \int_{\partial B_{\lambda \rho_0}} e_c(u_e) e^{-\frac{|x|^2}{4\rho_0^2}} \leq \frac{C}{\lambda^3} E \]  
(3.18)

\[ \frac{1}{\rho_0^2} \int_{B_{\lambda \rho_0} \times \{-\rho_0^2\}} e_c(u_e) e^{-\frac{|x|^2}{4\rho_0^2}} \geq \frac{E}{3}. \]  
(3.19)

Together with the inequalities (3.9)–(3.18), we can proceed on the estimate of \( E \) by estimating the left hand side of (3.19) in §4 below.

§4 An intrinsic Hodge decomposition to estimate \( u_e \times du_e \)

This section is devoted to the proof of theorem A. The main technical part is to obtain \( L^2 \)-estimate of \( u_e \times du_e \) on \( B_{\lambda \rho_0} \times \{-\rho_0^2\} \). To do it, we need an intrinsic Hodge decomposition of \( u_e \times du_e \) at \( t = -\rho_0^2 \). We adapt ideas from both [BBO] and [LR3] for this purpose.

From now on, we work on \( t = -\rho_0^2 \) and denote \( u \) as \( u_e \).

First, we define \( H : B_{\lambda \rho_0} \rightarrow \mathbb{R}^2 \) by the auxiliary Neumann problem:
\[ \frac{\partial}{\partial x_i} \left( e^{-\frac{|x|^2}{4\rho_0^2}} \frac{\partial H}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left( e^{-\frac{|x|^2}{4\rho_0^2}} u \times \frac{\partial u}{\partial x_i} \right), \text{ in } B_{\lambda \rho_0} \]  
(4.1)

\[ \frac{\partial H}{\partial r} = u \times \frac{\partial u}{\partial r}, \text{ on } \partial B_{\lambda \rho_0} \]  
(4.2)

Observe that
\[ \left| \frac{\partial}{\partial x_i} \left( e^{-\frac{|x|^2}{4\rho_0^2}} u \times \frac{\partial u}{\partial x_i} \right) \right| = e^{-\frac{|x|^2}{4\rho_0^2}} \left| \frac{-2\rho_0^2 \frac{\partial u}{\partial r} + x \cdot Du}{2\rho_0^2} \right| \times u \]
\[ \leq e^{-\frac{|x|^2}{4\rho_0^2}} \left| -\frac{2\rho_0^2 \frac{\partial u}{\partial r} + x \cdot Du}{2\rho_0^2} \right| \]
\[ \leq 2\rho_0 \ e^{-\frac{|x|^2}{4\rho_0^2}} (j_c(u_e))^\frac{1}{2} \]
so that we can establish the following estimate for $DH$.

**Lemma 4.1** Under the same notations as above. The following holds: there exists a $\lambda > 0$ such that

$$
\frac{1}{\rho_0^2} \int_{B_{\lambda \rho_0}} |DH|^2 e^{-|x|^2 \over 4\rho_0^2} \leq C_{\lambda} \rho_0^{-2} \int_{B_{\lambda \rho_0}} j_\lambda(u_\epsilon) e^{-|x|^2 \over 4\rho_0^2} + C_{\lambda} \rho_0 \int_{\partial B_{\lambda \rho_0}} |\partial u_\epsilon| \partial_r e^{-|x|^2 \over 4\rho_0^2} \tag{4.3}
$$

In particular, one has

$$
\frac{1}{\rho_0^2} \int_{B_{\lambda \rho_0}} |DH|^2 e^{-|x|^2 \over 4\rho_0^2} \leq C_{\lambda} \eta + CE^2 \tag{4.4}
$$

**Proof.** First, notice that (4.4) is the consequence of (4.3) and the inequalities (3.9) and (3.18). Secondly, the proof of (4.3) can be obtained by copying lines of arguments of Lemma 2.4 of [LR3](page 836-839). We omit it here. \[\Box\]

Observe that (4.1) and (4.2) can be rewritten into:

$$
\frac{\partial}{\partial x_i} \left( e^{-|x|^2 \over 4\rho_0^2} \left( \frac{\partial H}{\partial x_i} - u \times \frac{\partial u_\epsilon}{\partial x_i} \right) I_{B_{\lambda \rho_0}} \right) = 0 \tag{4.5}
$$

in the sense of distributions on $R^4$, here $I_{B_{\lambda \rho_0}}$ denotes the characteristic function of the ball $B_{\lambda \rho_0}$.

Define $\delta \in C^\infty(R_+, R_+)$ by $\delta(r) = r^2$ for $0 \leq r \leq 2\lambda \rho_0$, $\delta(r) = (4\lambda \rho_0)^2$ for $r \geq 4\lambda \rho_0$ and $(2\lambda \rho_0)^2 \leq \delta(r) \leq (4\lambda \rho_0)^2$ for $r \in [2\lambda \rho_0, 4\lambda \rho_0]$. Let $g_{ij}(x) = e^{-|x|^2 \over 4\rho_0^2} \delta_{ij}$ be the new conformal metric on $R^4$, which is readily seen to be bilipschitzly equivalent to the standard metric on $R^4$. Denote $d_g^*$ as the adjoint of $d$ with respect to $g$ and $\Delta_g \equiv d_g^*d + dd_g^*$ as the Laplace-Beltrami operator with respect to $g$. Notice that (4.5) is equivalent to

$$
d_g^*((dH - u \times du) I_{B_{\lambda \rho_0}}) = 0, \text{ in } R^4 \tag{4.6}
$$
Therefore, by the classical Hodge decomposition theory (see, e.g., Iwaniec-Martin [IW]), there exists a 2-form $\alpha \in H^1_\partial (R^4, R^2)$ such that

$$d^\ast g \alpha = (dH - u \times du) I_{B_{\lambda \rho}} , \quad d\alpha = 0$$  \hspace{1cm} (4.7)$$

$$\|D\alpha\|_{L^2_\partial(R^4)} \leq C(\|Du\|_{L^2_{B_{\lambda \rho}}}) + \|D H\|_{L^2_\partial(B_{\lambda \rho})})$$  \hspace{1cm} (4.8)$$

Here $H^1_\partial$ (or $L^2_\partial$ respectively) denotes $H^1$ (or $L^2$ respectively) with respect to $g$. Notice that

$$\|Df\|_{L^2_\partial(R^4)}^2 = \int_{R^4} |D f|^2 (x) e^{\frac{-4|u|^2}{\lambda \rho}}$$

In order to estimate $D\alpha$ in $L^2_\partial$, we modify the approach of [BBO] as follows. Let $\beta \in (0, \frac{1}{2})$ be determined later, and $f : R_+ \rightarrow [0, \frac{1}{1-\beta}]$ be a smooth function such that $f(t) = \frac{1}{t}$ for $t \geq 1 - \beta$, $f(t) = 1$ for $t \leq 1 - 2\beta$, and $|f'| \leq 4$. Define on $R^4$ the function $a$ such that $a(x) = f^2(|u|(x))$ on $B_{\lambda \rho}$ and $a(x) = 1$ elsewhere, so that $0 \leq a - 1 \leq 4\beta$ holds on $R^4$. Observe that $f^2(|u|^2)u \times du = f(|u|)u \times d(f(|u|)u)$. Therefore, (4.7) implies

$$d (ad^\ast g \alpha) = I_{B_{\lambda \rho}} d(f(|u|)u \times d(f(|u|)u)
 \quad + f(|u|)u \times du \wedge d[x] \sigma^\partial_{B_{\lambda \rho}} - d(I_{B_{\lambda \rho}} adH)
 \quad = \omega_1 + \omega_2 + \omega_3$$  \hspace{1cm} (4.9)$$

where $\sigma^\partial_{B_{\lambda \rho}}$ denotes the surface measure of $\partial B_{\lambda \rho}$ with respect to the metric $g$. Observe that if $|u| \geq 1 - \beta$ then $d(f(|u|)u \times d(f(|u|)u) = d(\frac{|u^2|}{|u|}) \times d(\frac{|u|}{|u|}) = 0$, otherwise we have $1 \leq \beta^{-2}(1 - |u|^2)^2$ so that

$$|\omega_1|(x) \leq C e^{-2} \leq C \beta^{-2} \frac{(1 - |u(x)|^2)^2}{e^2}, \forall x \in B_{\lambda \rho}$$  \hspace{1cm} (4.10)$$

Using the fact that $d\alpha = 0$, we get

$$\Delta_g \alpha = dd^\ast g \alpha = d(ad^\ast g \alpha) + d((1-a)ad^\ast g \alpha) = \omega_1 + \omega_2 + \omega_3 + d((1-a)ad^\ast g \alpha)$$  \hspace{1cm} (4.11)$$
Denote $G(x, y) = G(|x - y|)$ as the fundamental solution of $\Delta_g$ on $R^4$. Then it follows from the bilipschitz equivalence between $g$ and the euclidean metric on $R^4$ that there exists a $C > 0$ such that

$$Ce^{-4\lambda^2|x - y|^{-2}} \leq G(x, y) \leq Ce^{4\lambda^2|x - y|^{-2}}, |D_y G(x, y)| \leq C e^{4\lambda^2|x - y|^{-3}}$$

(4.12)

Let $\alpha_i = G \ast \omega_i$ for $1 \leq i \leq 3$. Then $\alpha_4 = \alpha - \sum_{i=1}^{3} \alpha_i$ solves

$$\Delta_g \alpha_4 = d((1 - a) d^*_g \alpha)$$

(4.13)

Direct calculations, using $|a - 1| \leq 4\beta$ and smallness of $\beta$, yield

$$\|D\alpha_4\|_{L^2(R^4)} \leq C \beta \sum_{i=1}^{3} \|D\alpha_i\|_{L^2(R^4)}$$

(4.14)

The main difficulty comes from estimates of $D\alpha_1$ which can be done as follows, due to the monotonicity inequality (3.15) and (3.16). Indeed, by the maximum principle, we have $\|\alpha_1\|_{L^\infty(R^4)} = \|\alpha_1\|_{L^\infty(B_{\lambda \rho_0})}$ and, by (4.10), (4.12), and (3.15),

$$\|\alpha_1\|_{L^\infty(B_{\lambda \rho_0})} \leq \sup_{x \in B_{\lambda \rho_0}} \int_{B_{\lambda \rho_0}} G(x - y) |\omega_1|(y)$$

$$\leq C \lambda \beta^{-2} \sup_{x \in B_{\lambda \rho_0}} \int_{B_{\lambda \rho_0}} |x - y|^{-2} (1 - |u(y)|^2)^2$$

$$\leq C \lambda \beta^{-2} E$$

(4.15)

This, combined with (3.16), implies

$$\|D\alpha_1\|_{L^2(R^4)}^2 \leq \|\omega_1\|_{L^1(R^4)} \|\alpha_1\|_{L^\infty(R^4)} \leq C \lambda \beta^{-2} \lambda^2 \eta E$$

(4.16)

For $\alpha_3$, using integration by parts and (4.4), we have

$$\|D\alpha_3\|_{L^2(R^4)}^2 \leq C \|DH\|_{L^2(B_{\lambda \rho_0})}^2 \leq C \lambda \eta \rho_0^2 + \frac{C \rho_0^2 E}{\lambda^2}.$$
For \( \alpha_2 \), we can modify the Lemma A1 of appendix in [BBO] to conclude that

\[
\|D\alpha_2\|_{L^2(R^+)}^2 \leq C\lambda \rho_0 \|Du\|_{L^2(\partial B_{\lambda \rho_0})}^2
\]

hence, combined with (3.18), gives

\[
\|D\alpha_2\|_{L^2(R^+)}^2 \leq \frac{C\rho_0^2}{\lambda^2} E
\]

Putting these estimates for \( \alpha_i \) for \( 1 \leq i \leq 4 \) and Lemma 4.1 together, we then obtain

\[
1 \rho_0^2 \int_{B_{\lambda \rho_0}} |u \times du|^2 e^{\frac{|u|^2}{4\rho_0^2}} \leq CE + C\lambda \beta^{-2} \eta E
\]

This, combined with the fact that \( 4|u|^2|du|^2 = 4|u \times du|^2 + |D|u|^2|^2 \) and the following estimate (see (2.67) of [LR3] page 845)

\[
1 \rho_0^2 \int_{B_{\lambda \rho_0}} |Du|^2 e^{\frac{|u|^2}{4\rho_0^2}} \leq C\eta^\frac{1}{2} E + C\eta^\frac{3}{2}
\]

implies

\[
\frac{1}{\rho_0^2} \int_{B_{\lambda \rho_0}} |Du|^2 e^{\frac{|u|^2}{\rho_0^4}}
\]

\[
= \frac{1}{\rho_0^2} \int_{B_{\lambda \rho_0}} |Du|^2 e^{\frac{|u|^2}{\rho_0^4}} + \frac{1}{\rho_0^2} \int_{B_{\lambda \rho_0}} |u|^2 |Du|^2 e^{\frac{|u|^2}{\rho_0^4}}
\]

\[
\leq C \rho_0^2 \int_{B_{\lambda \rho_0}} |Du|^2 e^{\frac{|u|^2}{\rho_0^4}}
\]

\[
+ \frac{4}{\rho_0^2} \int_{B_{\lambda \rho_0}} (|u \times du|^2 + |D|u|^2|^2) e^{\frac{|u|^2}{\rho_0^4}}
\]

\[
\leq C\lambda \rho_0^2 \int_{B_{\lambda \rho_0}} |Du|^2 e^{\frac{|u|^2}{\rho_0^4}} + (C\lambda \eta + C\eta^\frac{1}{2})
\]

\[
+ (\lambda^{-1} + C\lambda^{-2} + C\lambda\beta^{-2} \eta + C\eta^\frac{3}{2}) E
\]

\[
\leq (\lambda^{-1} + C\lambda^{-2} + C\lambda\beta^{-2} \eta + C\eta^\frac{3}{2}) E + (C\lambda \eta + C\eta^\frac{3}{2})
\]

(4.22)
Therefore, for any given $\delta > 0$, we can first choose a sufficiently large $\lambda > 1$ and a sufficiently small $\beta$ and then choose much smaller $\eta$ so that

$$E \leq C\delta$$  \hspace{1cm} (4.23)

so that, using the monotonicity inequality (3.1) again,

$$\frac{1}{\epsilon^6} \int_{-\epsilon^2}^{0} \int_{B_\epsilon(0)} \frac{(1 - |u_\epsilon|^2)^2}{\epsilon^2} \leq \delta.$$  \hspace{1cm} (4.24)

This, combined with the fact that $|Du_\epsilon| \leq C\epsilon^{-1}$, yields $|u_\epsilon(0,0)| \geq \frac{1}{2}$. Therefore, the proof of theorem A is complete.  

References


