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Abstract:

The formulation of Seiberg-Witten maps from the point of view of consistent deformations of gauge theories in the context of the Batalin-Vilkovisky antifield formalism is reviewed. Some additional remarks on noncommutative Yang-Mills theory are made.

1 Introduction

Using arguments from string theory, noncommutative Yang-Mills theory has been shown [1] to be equivalent, by a redefinition of the gauge potentials and the parameters of the gauge transformation, to a Yang-Mills theory with standard gauge symmetries and with an effective action containing, besides the usual Yang-Mills term, higher dimensional gauge invariant operators.

By considering an expansion in the parameter of noncommutativity ϑ , noncommutative Yang-Mills theory can be understood as a consistent deformation of standard Yang-Mills theory in the sense that the action and gauge transformations are deformed simultaneously in such a way that the deformed action is invariant under the deformed gauge transformations. An appropriate framework for the analysis of such consistent deformations of gauge theories has been shown [2] to be the antifield-antibracket formalism (see [3, 4, 5, 6] in the Yang-Mills context, [7] for the generic case and [8, 9] for reviews).

By reformulating the question of existence of Seiberg-Witten maps in this context [10, 11, 12], the whole power of the theory of general (anti-)canonical transformations is available and Seiberg-Witten maps appear as "time-dependent" canonical transformations that map the gauge structure of the noncommutative theory to that of the commutative one. In the generic case, this leads to an appropriate "open" version of the gauge equivalence condition, valid only up to terms vanishing when the equations of motions hold. These features have been shown [12] to be crucial for the construction of a Seiberg-Witten map in the case of the noncommutative Freedman-Townsend model.

2 Equivalent formulations of Seiberg-Witten maps

Notations and conventions are as in [12], except that for later convenience, we denote the deformation parameter by ϑ instead of g . Consider a gauge theory determined by the minimal proper solution $\hat{S}[\hat{\phi}, \hat{\phi}^*; \vartheta]$ of the master equation,

$$\frac{1}{2}(\hat{S}, \hat{S}) = 0, \quad (1)$$

and suppose that $\hat{S}[\hat{\phi}, \hat{\phi}^*; \vartheta]$ admits formal power series expansion in the deformation parameter ϑ : $\hat{S}[\hat{\phi}, \hat{\phi}^*; \vartheta] = \sum_{s=0}^{\infty} \vartheta^s S^{(s)}[\hat{\phi}, \hat{\phi}^*]$. Then, the deformed theory defined by \hat{S} is a consistent deformation of the undeformed theory determined by the master action

$$S^{(0)}[\hat{\phi}, \hat{\phi}^*] = \hat{S}[\hat{\phi}, \hat{\phi}^*; \vartheta] \Big|_{\vartheta=0}, \quad \frac{1}{2}(S^{(0)}, S^{(0)}) = 0. \quad (2)$$

In what follows we also use expansions in antifield number; the antifield number of a local function¹ or of a local functional is denoted by a subscript, e.g. $\hat{S} = \sum_{k \geq 0} \hat{S}_k$.

In the context of the antifield formalism, the existence of a Seiberg-Witten map translates into the following four equivalent formulations.

- There exists a canonical field-antifield transformation² $\hat{\phi}[\phi, \phi^*; \vartheta], \hat{\phi}_*[\phi, \phi^*; \vartheta]$ such that

$$\hat{S}[\hat{\phi}[\phi, \phi^*; \vartheta], \hat{\phi}_*[\phi, \phi^*; \vartheta]; \vartheta] = S_0^{\text{eff}}[\phi; \vartheta] + \sum_{k \geq 1} S_k^{(0)}[\phi, \phi^*], \quad (3)$$

$$\iff \hat{S}[\hat{\phi}, \hat{\phi}_*; \vartheta] = S_0^{\text{eff}}[\phi[\hat{\phi}; \hat{\phi}^*; \vartheta]; \vartheta] + \sum_{k \geq 1} S_k^{(0)}[\phi[\hat{\phi}, \hat{\phi}^*; \vartheta], \phi^*[\hat{\phi}, \hat{\phi}^*; \vartheta]], \quad (4)$$

where $S_0^{\text{eff}}[\hat{\phi}; 0] = S_0^{(0)}[\hat{\phi}; 0]$.

- There exists a generating functional of “second type” $F[\phi, \hat{\phi}^*; \vartheta]$ in ghost number -1 , with

$$\hat{\phi}^A(x) = \frac{\delta^L F}{\delta \hat{\phi}_A^*(x)}, \quad \phi_A^*(x) = \frac{\delta^L F}{\delta \phi^A(x)}, \quad (5)$$

such that

$$\hat{S}\left[\frac{\delta^L F}{\delta \hat{\phi}^*}, \hat{\phi}_*; \vartheta\right] = S_0^{\text{eff}}[\phi; \vartheta] + \sum_{k \geq 1} S_k^{(0)}\left[\phi, \frac{\delta^L F}{\delta \phi}\right], \quad (6)$$

with initial condition $F = \int d^n x \hat{\phi}_A^* \phi^A + O(\vartheta)$.

- The differential condition

$$\frac{\partial \hat{S}}{\partial \vartheta} = \frac{\partial S_0^{\text{eff}}}{\partial \vartheta} + (\hat{S}, \hat{\Xi}) \quad (7)$$

holds. The associated field-antifield redefinition satisfies the differential equations

$$\frac{\partial \phi^A(x)}{\partial \vartheta} = (\phi^A(x), \Xi(\vartheta)), \quad \frac{\partial \phi_A^*(x)}{\partial \vartheta} = (\phi_A^*(x), \Xi(\vartheta)), \quad (8)$$

¹In this context, a local function is a formal power series in ϑ each term of which depends on the fields, the antifields and a finite number of their derivatives.

²Only canonical transformation that reduce to the identity at order 0 in the deformation parameter are considered here. Invertibility of these transformations in the space of formal power series is then guaranteed.

where $\Xi(\vartheta)$ is the same function of the unhatted fields and antifields than $\hat{\Xi}(\vartheta)$ is in terms of the hatted ones. The formal solution to this differential equation is given by

$$\phi^A(x) = [P \exp \int_0^\vartheta d\vartheta'(\cdot, \hat{\Xi}(\vartheta'))] \hat{\phi}^A(x), \quad \phi_A^*(x) = [P \exp \int_0^\vartheta d\vartheta'(\cdot, \hat{\Xi}(\vartheta'))] \hat{\phi}_A^*(x). \quad (9)$$

- The deformed and undeformed theories are weakly gauge equivalent in the following sense. In the case of an irreducible gauge theory, there exists a simultaneous redefinition³ of the original gauge fields $\hat{\varphi}^i = f^i[\varphi]$ and the parameters $\hat{\epsilon}^\alpha = g_\beta^\alpha[\varphi](\epsilon^\beta)$ of the irreducible generating set of nontrivial gauge transformations $\hat{R}_\alpha^i[\hat{\varphi}](\hat{\epsilon}^\alpha)$ such that

$$(\hat{\delta}_{\hat{\epsilon}} \hat{\varphi}^i)|_{\hat{\varphi}=f, \hat{\epsilon}=g} \approx \delta_\epsilon f^i. \quad (10)$$

Here \approx means terms that vanish when the equations of motions associated to $S_0^{\text{eff}}[\varphi] \equiv \hat{S}_0[f[\varphi]]$ hold, while $\hat{\delta}_{\hat{\epsilon}}, \delta_\epsilon$ are respectively given by

$$\hat{\delta}_{\hat{\epsilon}} = \sum_{k=0} \partial_{\mu_1} \dots \partial_{\mu_k} \left(\hat{R}_\alpha^i[\hat{\varphi}](\hat{\epsilon}^\alpha) \frac{\partial}{\partial(\partial_{\mu_1} \dots \partial_{\mu_k} \hat{\varphi}^i)} \right), \quad (11)$$

$$\delta_\epsilon = \sum_{k=0} \partial_{\mu_1} \dots \partial_{\mu_k} \left(R_\alpha^i[\varphi](\epsilon^\alpha) \frac{\partial}{\partial(\partial_{\mu_1} \dots \partial_{\mu_k} \varphi^i)} \right), \quad (12)$$

with $R_\alpha^i[\varphi](\epsilon^\alpha)$ being the irreducible generating set of nontrivial gauge transformations of the undeformed theory.

Finally, the existence of a Seiberg-Witten map is controlled by the BRST differential of the undeformed theory. In particular, if one can show that in a relevant subspace, the representatives of the cohomology of $s^{(0)}$ can be chosen to be antifield independent, i.e.,

$$(S^{(0)}, C) = 0 \implies C = C'_0 + (S^{(0)}, D), \quad (13)$$

the Seiberg-Witten map are guaranteed to exist and can be constructed as a succession of canonical transformations [12].

3 Remarks on noncommutative $U(N)$ Yang-Mills theory

We assume the space-time manifold to be \mathbf{R}^n with coordinates $x^\mu, \mu = 1, \dots, n$. The Weyl-Moyal star-product is defined through

$$f * g(x) = \exp(i\wedge_{12}) f(x_1)g(x_2)|_{x_1=x_2=x}, \quad \wedge_{12} = \frac{\vartheta}{2} \theta^{\mu\nu} \partial_\mu^{x_1} \partial_\nu^{x_2}, \quad (14)$$

for a real, constant, antisymmetric matrix $\theta^{\mu\nu}$. The parameter ϑ has mass dimensions -2 . A minimal proper solution of the master equation for noncommutative $U(N)$ Yang-Mills theory is given by:

$$\hat{S} = \int d^n x \text{Tr} \left(-\frac{1}{4\kappa^2} \hat{F}^{\mu\nu} * \hat{F}_{\mu\nu} + \hat{A}^{*\mu} * \hat{D}_\mu \hat{C} + \frac{1}{2} \hat{C}^{*} * [\hat{C} * \hat{C}] \right), \quad (15)$$

where fields, ghost fields, and their conjugated antifields are $u(N)$ valued and Tr denotes ordinary matrix trace. In particular, $\hat{A}^{*\mu} = \hat{A}_B^{*\mu} g^{BA} T_A$, $\hat{C}^{*} = \hat{C}_B^* g^{BA} T_A$, with T_A being

³A square bracket means a local dependence on the fields and their derivatives, while the round bracket means that this dependence is linear and homogeneous.

generators of Lie algebra $u(N)$ and $g_{AB} = \text{Tr} T_A T_B$ being an invariant metric on the algebra. We denote by $[A * B] = A * B - (-)^{|A||B|} B * A$ the graded star commutator and by $\{A * B\} = A * B + (-)^{|A||B|} B * A$ the graded star anticommutator. Noncommutative Yang-Mills theory is a particular case of a consistent deformation of standard Yang-Mills theory in the sense explained in the previous section. The deformation parameter is ϑ , the parameter of "noncommutativity".

In the noncommutative Yang-Mills case, the differential condition (7) can be explicitly solved by

$$\frac{\partial S_0^{\text{eff}}}{\partial \vartheta} = \frac{1}{\kappa^2} \int d^n x \text{Tr} \frac{i\theta^{\alpha\beta}}{2} \left(-\frac{1}{2} \hat{F}^{\mu\nu} \{ \hat{F}_{\alpha\mu} * ; \hat{F}_{\beta\nu} \} + \frac{1}{8} \{ \hat{F}^{\rho\sigma} * ; \hat{F}_{\rho\sigma} \} \hat{F}_{\alpha\beta} \right), \quad (16)$$

$$\hat{\Xi} = \int d^n x \text{Tr} \frac{i\theta^{\alpha\beta}}{2} \left(-\frac{1}{2} \hat{A}^{*\mu} \{ \hat{F}_{\alpha\mu} + \partial_\alpha \hat{A}_\mu * ; \hat{A}_\beta \} + \frac{1}{2} \hat{C}^* \{ \hat{A}_\alpha * ; \partial_\beta \hat{C} \} \right), \quad (17)$$

The evolution equations $\partial \hat{A} / \partial \vartheta = -(\hat{A}, \hat{\Xi})$ and $\partial \hat{C} / \partial \vartheta = -(\hat{C}, \hat{\Xi})$ then reproduce the original differential equations of [1].

Linearity in antifields of the generating functional $\hat{\Xi}$ implies that the generating functional F of second type can also be chosen linear in antifields,

$$F = \int d^n x \text{Tr} \left(\hat{A}^{*\mu} f_\mu + \hat{C}^* h \right) \quad (18)$$

where $f_\mu = f_\mu^A[A; \vartheta] T_A$ and $h = h^A[A, C; \vartheta] T_A$, with h linear and homogeneous in the ghosts C^A and their derivatives). Linearity in antifields then implies that equation (6) reduces to

$$\int d^n x \text{Tr} -\frac{1}{4\kappa^2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} [f; \vartheta] = S_0^{\text{eff}}[A; \vartheta] \quad (19)$$

$$(\partial_\mu h + [f_\mu * ; h])^A = \gamma f_\mu^A, \quad (20)$$

$$\frac{1}{2} [h * ; h]^A + \gamma h^A = 0, \quad (21)$$

where γ is the gauge part of the BRST differential of the commutative theory:

$$\begin{aligned} \gamma = & \sum_{k=0} [\partial_{\rho_1} \dots \partial_{\rho_k} (D_\mu C)^B \frac{\partial^L}{\partial (\partial_{\rho_1} \dots \partial_{\rho_k} A_\mu^B)} - \\ & - \frac{1}{2} \partial_{\rho_1} \dots \partial_{\rho_k} (f_{DE}^B C^D C^E) \frac{\partial^L}{\partial (\partial_{\rho_1} \dots \partial_{\rho_k} C^B)}]. \end{aligned} \quad (22)$$

Note that for a generating functional F of general form, i.e., not necessarily linear in the antifields, equations (20) and (21) would contain equations of motion terms [12].

Equation (20) is the Seiberg-Witten equation (3.3) of [1] under the form $\hat{\delta}_{\hat{\lambda}} \hat{A} = \delta_\lambda \hat{A}$, with the identifications $\hat{A}_\mu \leftrightarrow f_\mu$, $\hat{\lambda} \leftrightarrow h$ and $\lambda \leftrightarrow C$. When solving (20)-(21), it is useful to solve first the BRST version of the integrability condition (21) before solving the Seiberg-Witten equation (20), because it contains as unknown functions only the noncommutative gauge parameter h as a function of $\theta^{\mu\nu}$, C^A , A_μ^A and their derivatives.

Remarks:

1. The existence of the Seiberg-Witten map can be inferred a priori from the knowledge of the local BRST cohomology of commutative Yang-Mills theory [13]. The point is that in the infinitesimal noncommutative deformation, only differentiated ghosts

appear in the antifield dependent terms while the local cohomology in ghost number 0 of the BRST differential $s^{(0)}$ of commutative Yang-Mills theory can be shown to depend only on undifferentiated antifields and undifferentiated ghosts. This implies the existence of the Seiberg-Witten generating functional to first order in the deformation parameter ϑ . Furthermore, one can show that this reasoning can be iterated, which allows to prove the existence of the Seiberg-Witten map along the lines of section 2.5 of [12].

2. Besides the expansion in the deformation parameter ϑ , it is often useful to consider an expansion in the homogeneity of the fields. The BRST differential of the noncommutative theory then reduces to lowest order to the BRST differential $s^{[0]}$ of N free commutative abelian fields. In both expansions, the contracting homotopy that allows to invert $s^{(0)}$ respectively $s^{[0]}$ in the relevant subspace can be explicitly constructed by a simple change of generators that consists essentially in replacing the derivatives of the gauge potentials by the symmetrized derivatives of the gauge potentials and the symmetrized covariant derivatives of the field strengths. This last symmetrization includes the first index of the field strength in order to get rid of the redundancies due to the Bianchi identities (see e.g. [14, 13] for details). This explicit form of the contracting homotopy operator allows to construct the generating functional $\hat{\Xi}$ of equation (17) and to solve the gauge equivalence conditions recursively (see also [15]).
3. A natural question to ask is whether the whole noncommutative deformation is trivial in the sense that it can be undone by a field redefinition. This question can also be addressed using local BRST cohomology. Indeed, in the Yang-Mills case, it turns out that the infinitesimal deformation of the action corresponds to a non trivial BRST cohomology class in ghost number 0 implying the non triviality of the noncommutative deformation. In noncommutative Chern-Simons theory however, results on the local BRST cohomology of standard Chern-Simons theory imply that the whole noncommutative deformation is trivial. This has been shown directly in [16].
4. After gauge fixing, which corresponds merely to another canonical transformation in the antifield formalism, the appropriate formulation to control the Seiberg-Witten map during perturbative renormalization is the functional differential equation (7), which should be promoted to an analogous equation for the generating functional for 1PI Green's function, in the same way than the master equation (1) gets promoted to the Zinn-Justin equation and controls the gauge invariance. Since it is well known how the second term on the right hand side of (7) renormalizes (see e.g. [17]), the question reduces to the renormalization of the higher dimensional operator $\partial S_0^{\text{eff}}/\partial\vartheta$ given in (16).

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