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Heat content asymptotics with transmittal and transmission boundary conditions

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HEAT CONTENT ASYMPTOTICS WITH TRANSMITTAL AND TRANSMISSION BOUNDARY CONDITIONS

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Abstract. We study the heat content asymptotics on a Riemannian manifold with smoooth boundary defined by Dirichlet, Neumann, transmittal and transmission boundary conditions.
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1. Introduction

Let M be a compact m dimensional Riemannian manifold with smooth boundary ∂M. Let D be an operator of Laplace type on a vector bundle V over M. Let $\mathcal{B}$ be a suitable local boundary condition and let $D_{\mathcal{B}}$ be the associated realization. Let $\phi \in C^\infty(V)$ describe the initial temperature distribution. The subsequent temperature distribution $u := e^{-tD_{\mathcal{B}}}\phi$ for $t \geq 0$ is described by the equations:

$$(\partial_t + D)u = 0,\ u(x;0) = \phi,\ \text{and}\ \mathcal{B}u = 0. \tag{1.1}$$

The specific heat ρ is a section to the dual bundle V^*. Let

$$\beta(\phi, \rho, D, \mathcal{B})(t) := \textstyle\int_M u\rho$$

be the total heat energy content. As $t \downarrow 0$, there is a complete asymptotic expansion of the form

$$\beta(\phi, \rho, D, \mathcal{B})(t) \sim \textstyle\sum_{n\geq 0} \beta_n(\phi, \rho, D, \mathcal{B}) t^{n/2};$$

the *heat content coefficients* $\beta_n(\phi, \rho, D, \mathcal{B})$ are locally computable.

If $D_{\mathcal{B}}$ is self-adjoint, then let $\{\phi_i, \lambda_i\}$ be a discrete spectral resolution. Let $\gamma_i(\phi) := \int_M \phi\phi_i$ be the associated Fourier coefficients. Then:

$$\beta(\phi, \rho, D, \mathcal{B})(t) = \textstyle\sum_i e^{-t\lambda_i}\gamma_i(\phi)\gamma_i(\rho). \tag{1.2}$$

It is convenient to introduce a formalism to consider both Dirichlet and Robin boundary conditions at the same time. Suppose given a decomposition $\partial M = C_D \cup C_R$ of the boundary as the disjoint union of two closed (possibly empty) sets. Let S be an auxiliary endomorphism of $V|_{C_R}$ and let $\phi_{;m}$ be the covariant derivative of ϕ with respect to the inward unit normal, where we use the natural connection which is induced on V by D - see Section 2 for details. We define

$$\mathcal{B}_{DR} = \mathcal{B}_D \oplus \mathcal{B}_R \text{ where } \mathcal{B}_D\phi := \phi|_{C_D} \text{ and } \mathcal{B}_R\phi := (\phi_{;m} + S\phi)|_{C_R}$$

are the pure Dirichlet and Robin operators respectively. In Section 2, we review previous results for the boundary conditions $\mathcal{B}_{DR}$.

Transmittal and transfer boundary conditions will form the primary focus of this paper. Let $(M_\pm, g_\pm)$ be smooth compact m dimensional Riemannian manifolds. We assume that $\Sigma = \partial M_+ = \partial M_-$ is a smooth $m-1$ dimensional manifold and that the induced metrics agree, i.e. $g_+|_\Sigma = g_-|_\Sigma$. Let $D_\pm$ be operators of Laplace type on vector bundles $V_\pm$ over $M_\pm$. Let $\nu_\pm$ be the inward unit normals of $\Sigma \subset M_\pm$; note that $\nu_+ = -\nu_-$. Let $\phi := (\phi_+, \phi_-)$ and $\rho := (\rho_+, \rho_-)$.

Suppose that $V_+|_\Sigma = V_-|_\Sigma$ and that there is given an auxiliary endomorphism U of $V_\Sigma := V_\pm|_\Sigma$ serving as an impedance matching term. Let $\nabla^\pm$ be the natural connections defined by the operators $D_\pm$. Let

$$\begin{aligned} \mathcal{B}_1\phi : &= \{\phi_+|_\Sigma - \phi_-|_\Sigma\} \\ &\oplus \{(\nabla^+_{\nu_+}\phi_+)|_\Sigma + (\nabla^-_{\nu_-}\phi_-)|_\Sigma - U\phi_+|_\Sigma\}. \end{aligned} \tag{1.3}$$

Equivalently, ϕ satisfies the boundary conditions given in display (1.3) if and only if ϕ extends continuously across the interface Σ and if the normal derivatives match, modulo the impedance matching term U. In Section 3, we determine the invariants β_n for $n \le 3$ for these boundary conditions, see Theorems 3.1 and 3.2 for details. The *transmittal boundary operator* $\mathcal{B}_1 = \mathcal{B}_1(U)$ is of relevance in the presence of distributional sources [7, 10, 13] as they have been considered, e.g., in the brane world scenario.

We shall also be studying boundary conditions which are defined by the boundary operator $\mathcal{B}_2 = \mathcal{B}_2(S)$:

$$\mathcal{B}_2\phi := \left\{ \begin{pmatrix} \nabla^+_{\nu_+} + S_{++} & S_{+-} \\ S_{-+} & \nabla^-_{\nu_-} + S_{--} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \right\} \Bigg|_\Sigma \tag{1.4}$$

where

$$\begin{aligned} &S_{++} : V_+|_\Sigma \to V_+|_\Sigma, \qquad S_{+-} : V_-|_\Sigma \to V_+|_\Sigma, \\ &S_{-+} : V_+|_\Sigma \to V_-|_\Sigma, \qquad S_{--} : V_-|_\Sigma \to V_-|_\Sigma. \end{aligned}$$

If $S_{+-} = S_{-+} = 0$, then equation (1.4) decouples to define Robin boundary conditions. Note that we do not assume given an identification of $V_+|_\Sigma$ with $V_-|_\Sigma$; in particular, we can consider the situation when $\dim V_+ \neq \dim V_-$. In Section 4 we determine the heat content invariants β_n for $n \le 3$ for the *heat transfer boundary conditions* $\mathcal{B}_2$, see Theorem 4.3.

The boundary conditions defined by equations (1.3) and (1.4) can be thought of as living on the singular manifold $M := M_+ \cup_\Sigma M_-$. Both boundary conditions are relevant to heat transfer problems between two media of different conductivities. Which boundary condition is to be applied depends on the details of the surface of separation Σ between M_+ and M_-. Let K_+ and K_- be the thermal conductivities of M_+ and M_-. The flux of heat is continuous over the interface Σ,

$$(K_+\nabla^+_{\nu_+}\phi_+ + K_-\nabla^-_{\nu_-}\phi_-)\,|_\Sigma = 0. \tag{1.5}$$

If the contact between the two media M_+ and M_- is very intimate, in addition one assumes

$$\phi_+\,|_\Sigma = \phi_-|_\Sigma\,, \tag{1.6}$$

and boundary conditions of the type (1.3) are found. Otherwise, e.g., for surfaces pressed lightly together, in a linear approximation the flux of heat between M_+ and M_- is proportional to their temperature difference. In this case, equation (1.5) has to be augmented by

$$(K_+\nabla^+_{\nu_+}\phi_+)\,|_\Sigma = H(\phi_+ - \phi_-)|_\Sigma \tag{1.7}$$

where H is referred to as the surface conductivity. The boundary conditions (1.5) and (1.7) can be combined into the form of equation (1.4); see, for example, the discussion in [8].

2. Dirichlet and Robin boundary conditions

We begin by reviewing some of the basic invariance theory of operators of Laplace type. Let (M, g) be a compact Riemannian manifold of dimension m. We suppose the boundary Σ of M is smooth. We adopt the Einstein convention and sum over repeated indices. Let

$$D = -(g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B)$$

be an operator of Laplace type on $C^\infty(V)$. The operator D determines a natural connection ∇ and a natural endomorphism E such that we may express D invariantly in the form:

$$D = -\{Tr(\nabla^2) + E\};$$

see [9] for details. Let Γ be the Christoffel symbols of the metric. We may express the connection 1 form ω of ∇ and the endomorphism E:

$$\begin{aligned} &\omega_\delta = \tfrac{1}{2}g_{\nu\delta}(A^\nu + g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu) \text{ and} \\ &E = B - g^{\nu\mu}(\partial_\nu\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma). \end{aligned} \tag{2.1}$$

Note that the connection defined by the dual operator $\tilde{D}$ on the dual bundle V^* is the associated dual connection with connection 1 form given by $-\omega^*$; furthermore the associated endomorphism is E^*.

We shall let Roman indices a, b, etc. range from 1 to $m-1$ and index a local coordinate frame for the tangent bundle of the boundary. Let e_m be the inward unit normal and let indices i, j, etc. range from 1 to m and index this augmented frame for TM. Let L_{ab} be the second fundamental form, let R_{ijkl} be the Riemann curvature tensor with the sign convention that $R_{1221}=+1$ for the unit sphere in $\mathbb{R}^3$. Let Ω be the curvature of the induced connection on V. Let ':' and ';' denote multiple covariant differentiation with respect to the Levi-Civita connection of the boundary and of the interior, respectively; these two connections differ by the second fundamental form.

The invariants β_n may be decomposed as sums $\beta_n=\beta_n^{int}+\beta_n^{bd}$ of locally computable invariants given by integrals over the interior and over the boundary. Let $\tilde{D}$ and $\tilde{\mathcal{B}}$ be the dual operators on $C^\infty(V^*)$. The interior invariants are independent of the boundary condition and vanish if n is odd. For $n\leq 3$, we have:

$$\begin{array}{ll}\beta_0^{int}(\rho,\phi,D,\mathcal{B})=\int_M\phi\cdot\rho, & \beta_1^{int}(\rho,\phi,D,\mathcal{B})=0,\\ \beta_2^{int}(\rho,\phi,D,\mathcal{B})=-\int_M D\phi\cdot\rho, & \beta_3^{int}(\rho,\phi,D,\mathcal{B})=0.\end{array}$$

The heat content asymptotics β_n defined by the Dirichlet and Robin boundary operator $\mathcal{B}_{DR}$ have been studied previously [1, 2, 3, 4, 11, 12, 14, 15, 16]. There are also results available in the singular setting, see for example [5, 6]. We summarize the results for β_0, β_1, β_2, and β_3:

Theorem 2.1.

(1) $\beta_0(\phi,\rho,D,\mathcal{B}_{DR})=\int_M\phi\rho$.

(2) $\beta_1(\phi,\rho,D,\mathcal{B}_{DR})=-\frac{2}{\sqrt{\pi}}\int_{C_D}\phi\rho$.

(3) $\beta_2(\phi,\rho,D,\mathcal{B}_{DR})=-\int_M D\phi\cdot\rho+\int_{C_D}\{\frac12 L_{aa}\phi\rho-\phi\rho_{;m}\}$
$+\int_{C_R}\mathcal{B}_R\phi\cdot\rho$.

(4) $\beta_3(\phi,\rho,D,\mathcal{B}_{DR})=-\frac{2}{\sqrt{\pi}}\int_{C_D}\{-\frac23 D\phi\cdot\rho-\frac23\phi\tilde{D}\rho+\frac13\phi_{:a}\rho_{:a}$
$(-\frac13 E+\frac1{12}L_{aa}L_{bb}-\frac16 L_{ab}L_{ab}+\frac16 R_{amam})\phi\rho\}+\frac{4}{3\sqrt{\pi}}\int_{C_R}\mathcal{B}_R\phi\cdot\tilde{\mathcal{B}}_R\rho$.

3. The boundary operator $\mathcal{B}_1$

We postpone for the moment the discussion of β_3. Using the chiral symmetry and the homogeneity of the invariants, we see:

Theorem 3.1. *There exist universal constants so*

(1) $\beta_0(\phi,\rho,D,\mathcal{B}_1)=\int_{M_+}\phi_+\rho_++\int_{M_-}\phi_-\rho_-$.

(2) $\beta_1(\phi,\rho,D,\mathcal{B}_1)=\int_\Sigma\{a_1(\phi_+\rho_++\phi_-\rho_-)+a_2(\phi_+\rho_-+\phi_-\rho_+)\}$.

(3) $\beta_2(\phi,\rho,D,\mathcal{B}_1)=-\int_{M_+}D_+\phi_+\cdot\rho_+-\int_{M_-}D_-\phi_-\cdot\rho_-$
$+\int_\Sigma\{a_3(\phi_+\rho_+L_{aa}^++\phi_-\rho_-L_{bb}^-)+a_4(\phi_+\rho_+L_{aa}^-+\phi_-\rho_-L_{aa}^+)$

$$\begin{aligned}&+a_5(\phi_+\rho_-L^+_{aa}+\phi_-\rho_+L^-_{aa})+a_6(\phi_+\rho_-L^-_{aa}+\phi_-\rho_+L^+_{aa})\\&+a_7(\phi_{+;\nu_+}\rho_++\phi_{-;\nu_-}\rho_-)+a_8(\phi_{+;\nu_+}\rho_-+\phi_{-;\nu_-}\rho_+)\\&+a_9(\phi_+\rho_{+;\nu_+}+\phi_-\rho_{-;\nu_-})+a_{10}(\phi_+\rho_{-;\nu_-}+\phi_-\rho_{+;\nu_+})\\&+a_{11}(\phi_+\rho_++\phi_-\rho_-)U+a_{12}(\phi_+\rho_-+\phi_-\rho_+)U\big\}.\end{aligned}$$

(4) *We have:*

$$\begin{array}{llllll}a_1=-\frac{1}{\sqrt{\pi}}, & a_2=\frac{1}{\sqrt{\pi}}, & a_3=\frac18, & a_4=\frac18, & a_5=-\frac18, & a_6=-\frac18,\\ a_7=\frac12, & a_8=\frac12, & a_9=-\frac12, & a_{10}=\frac12, & a_{11}=-\frac14, & a_{12}=-\frac14.\end{array}$$

There are a number of functorial properties that these invariants satisfy. Suppose that the bundles $V_\pm$ are equipped with Hermitian inner products, that the operators $D_\pm$ are formally self-adjoint, and that U is self-adjoint. We then have that D is self-adjoint, see [10] (equation (17)) for details. We may therefore apply the relations of display (1.2) to see:

$$\beta_n(\phi,\rho,D,\mathcal{B}_1)=\beta_n(\rho,\phi,D,\mathcal{B}_1).$$

More generally, if $\tilde D$ is the formal adjoint of D on $C^\infty(V^*)$ and if $\tilde{\mathcal{B}}_1$ are the dual boundary conditions, then we have

$$\beta_n(\phi,\rho,D,\mathcal{B}_1)=\beta_n(\rho,\phi,\tilde D,\tilde{\mathcal{B}}_1). \tag{3.1}$$

The expression $-\int_M D\phi\cdot\rho$ is not symmetric in ϕ and ρ. We use equation (3.1) and integrate by parts to see:

$$a_5=a_6,\ a_7-a_9=1,\ a_8=a_{10}. \tag{3.2}$$

Doubling the manifold yields additional information. Let M_0 be a smooth Riemannian manifold with smooth boundary Σ and let D_0 be a self-adjoint operator of Laplace type over M_0. Let $\{\tilde\phi_{D,i},\lambda_{D,i}\}$ and $\{\tilde\phi_{R,i},\lambda_{R,i}\}$ be the discrete spectral resolutions for D_0 with Dirichlet (D) and Robin (R) boundary conditions over M_0. Let $M^\pm:=M_0$ define the double. Extend the $\tilde\phi_{D,i}$ to be odd and the $\tilde\phi_{R,i}$ to be even:

$$\phi_{D,i}(x_\pm)=\pm\tfrac{1}{\sqrt2}\tilde\phi_{D,i}(x)\text{ and }\phi_{R,i}(x_\pm)=\tfrac{1}{\sqrt2}\tilde\phi_{R,i}(x).$$

Set $U=-2S$. It was shown in [10] that $\mathcal{B}_1\phi_{D,i}=0$ and $\mathcal{B}_1\phi_{R,i}=0$. Furthermore, $\{\tilde\phi_{D,i},\tilde\phi_{R,i}\}$ is a complete orthonormal basis for $L^2(V)$ which defines the spectral resolution of $D:=(D_0^+,D_0^-)$. Decompose $\phi=\phi_o+\phi_e$ and $\rho=\rho_o+\rho_e$ as the sum of even and odd functions and let $\tilde\phi_o$, $\tilde\phi_e$, $\tilde\rho_o$, and $\tilde\rho_e$ be the restrictions to $M_0=M_+$. We then have

$$\begin{array}{ll}\tilde\phi_o=\sum_i\gamma_{D,i}(\tilde\phi_o)\tilde\phi_{D,i}, & \phi_o=\sqrt2\sum_i\gamma_{D,i}(\tilde\phi_o)\phi_{D,i},\\ \tilde\rho_o=\sum_i\gamma_{D,i}(\tilde\rho_o)\tilde\phi_{D,i}, & \rho_o=\sqrt2\sum_i\gamma_{D,i}(\tilde\rho_o)\phi_{D,i},\\ \tilde\phi_e=\sum_i\gamma_{R,i}(\tilde\phi_e)\tilde\phi_{R,i}, & \phi_e=\sqrt2\sum_i\gamma_{R,i}(\tilde\phi_e)\phi_{R,i},\\ \tilde\rho_e=\sum_i\gamma_{R,i}(\tilde\rho_e)\tilde\phi_{R,i}, & \rho_e=\sqrt2\sum_i\gamma_{R,i}(\tilde\rho_e)\phi_{R,i}.\end{array}$$

Consequently by equation (1.2),

$$\begin{aligned} &\beta(\phi,\rho,D,\mathcal{B}_1)(t) = 2\beta(\tilde\phi_o,\tilde\rho_o,D_0,\mathcal{B}_D)(t)+2\beta(\tilde\phi_e,\tilde\rho_e,D_0,\mathcal{B}_R)(t)\\ &\beta_n(\phi,\rho,D,\mathcal{B}_1) = 2\beta_n(\tilde\phi_o,\tilde\rho_o,D_0,\mathcal{B}_D)+2\beta_n(\tilde\phi_e,\tilde\rho_e,D_0,\mathcal{B}_R). \end{aligned} \tag{3.3}$$

This relation continues to hold even if D_0 is not self-adjoint. Thus

$$\begin{array}{ll} 2a_1+2a_2=0, & 2a_1-2a_2=-\frac{4}{\sqrt{\pi}},\\ 2a_3+2a_4+2a_5+2a_6=0, & 2a_3+2a_4-2a_5-2a_6=1,\\ 2a_7+2a_8=2, & 2a_7-2a_8=0,\\ 2a_9+2a_{10}=0, & 2a_9-2a_{10}=-2,\\ -4a_{11}-4a_{12}=2, & 2a_{11}-2a_{12}=0. \end{array} \tag{3.4}$$

Take arbitrary metrics on $M_\pm$ and let $D_\pm$ be the scalar Laplacian. Take $\phi=1$ and $U=0$. Then $D\phi=0$ and $\mathcal{B}_1\phi=0$ so $e^{-tD_{\mathcal{B}_1}}\phi=\phi$. Thus $\beta_n(1,\rho,D,\mathcal{B}_1)=0$ for $n\geq 1$. Take $\rho_-=0$. The terms ρ_+, $\rho_+L^+_{aa}$, $\rho_+L^-_{aa}$, $\rho_{+;\nu_+}$ can then be specified arbitrarily. This yields:

$$a_1+a_2=0,\ a_3+a_6=0,\ a_4+a_5=0,\ a_9+a_{10}=0. \tag{3.5}$$

This allows for the determination of the multipliers $a_1,...,a_{12}$. However, in order to provide further checks and because it will be useful later, we give one final property. Let $N_\pm:=[0,1]$ be the interval. Let $M_\pm:=[0,1]\times S^1$ be the cylinder with the metrics

$$ds^2=dr^2+e^{2f_\pm(r)}d\theta^2$$

where the real functions $f_\pm$ vanish on $\partial\{[0,1]\}$. Let $f_{\pm,r}:=\partial_r f_\pm$. Let

$$\begin{aligned} &D_{\pm,N}:=-(\partial_r^2+f_{\pm,r}\partial_r)\text{ on }N_\pm\text{ and}\\ &D_{\pm,M}=-(\partial_r^2+f_{\pm,r}\partial_r+e^{-2f_\pm}\partial_\theta^2)\text{ on }M_\pm. \end{aligned}$$

Then $D_{\pm,M}$ is the scalar Laplacian on $M_\pm$. The second fundamental form vanishes on $N_\pm$ while $L^\pm=-f_{\pm,r}$ is the second fundamental form on $M_\pm$. The connection forms defined by these two operators differ:

$$\begin{array}{ll} \omega_r^N=\frac12 f_r\text{ on }V, & \omega_r^N=-\frac12 f_r\text{ on }V^*,\\ \omega_r^M=0\text{ on }V, & \omega_r^M=0\text{ on }V^*. \end{array}$$

To compensate for this difference, we let

$$U^N=\tfrac12(f_{+,r}+f_{-,r})\text{ on }\partial N\text{ and }U^M=0\text{ on }\partial M.$$

The volume forms also differ:

$$\operatorname{dvol}^N=dr\text{ on }N_\pm\text{ and }\operatorname{dvol}^M=e^{f_\pm}drd\theta\text{ on }M_\pm.$$

We let $\phi_\pm$ and $\rho_\pm$ be constants. We then have:

$$\begin{aligned} &e^{-tD^M_{\mathcal{B}_1}}\phi=e^{-tD^N_{\mathcal{B}_1}}\phi\text{ so}\\ &\beta_n(\phi,\rho,D^M,\mathcal{B}_1^M)=2\pi\beta_n(\phi,e^f\rho,D^N,\mathcal{B}_1^N). \end{aligned} \tag{3.6}$$

We take $f_+ = f$ and $f_- = 0$. On the cylinder, the only invariant that plays a role in the computation of $\beta_2(\phi,\rho,D,\mathcal{B}_1)$ is $L_{aa}^+ = -f_r$. On the interval, the only invariants that play a role are $U = \frac{1}{2}f_r$, the connection 1 form $\omega_r = \frac{1}{2}f_r$ on V, the dual connection one form $-\frac{1}{2}f_r$ on V^*, and the endomorphism $E = -\frac{1}{4}f_r^2 - \frac{1}{2}f_{rr}$; the interior invariants vanish as

$$D_M(\phi) = 0, \quad D_N(\phi) = 0, \quad \tilde{D}_M(\rho) = 0, \quad \tilde{D}_N(e^f\rho) = 0.$$

Note that on ∂N, $(e^f\rho_+)_{;\nu_+} = \frac{1}{2}f_r\rho_+$. Thus equation (3.6) implies:

$$\begin{aligned} & f_r(-a_3\phi_+\rho_+ - a_4\phi_-\rho_- - a_5\phi_+\rho_- - a_6\phi_-\rho_+) \\ = \; & \tfrac{1}{2}f_r\{a_7\phi_+\rho_+ + a_8\phi_+\rho_- + a_9\phi_+\rho_+ + a_{10}\phi_-\rho_+ \\ & \quad +a_{11}(\phi_+\rho_+ + \phi_-\rho_-) + a_{12}(\phi_+\rho_- + \phi_-\rho_+)\} \end{aligned}$$

and consequently

$$\begin{array}{ll} -2a_3 = a_7 + a_9 + a_{11}, & -2a_4 = a_{11}, \\ -2a_5 = a_8 + a_{12}, & -2a_6 = a_{10} + a_{12}. \end{array} \tag{3.7}$$

We solve the relations of displays (3.2), (3.4), (3.5), and (3.7) to complete the determination of β_0, β_1, and β_2 in this setting by determining the unknown coefficients to complete the proof of Theorem 3.1. $\square$

Let $\omega_a := \nabla_a^+ - \nabla_a^-$ on V and $\tilde{\omega}_a = -\omega_a^*$ on V^*; this is a chiral tensor that changes sign if we interchange the roles of $\pm$ or of V and V^*. We determine β_3 in this setting:

Theorem 3.2.

(1) *There exist universal constants so* $\beta_3(\phi,\rho,D,\mathcal{B}_1)$

$$\begin{aligned} &= \tfrac{1}{6\sqrt{\pi}}\int_\Sigma\{a_{20}(D_+\phi_+\cdot\rho_+ + \phi_+\cdot\tilde{D}_+\rho_+ + D_-\phi_-\cdot\rho_- + \phi_-\cdot\tilde{D}_-\rho_-) \\ &+a_{21}(D_+\phi_+\cdot\rho_- + \phi_+\cdot\tilde{D}_-\rho_- + D_-\phi_-\cdot\rho_+ + \phi_-\cdot\tilde{D}_+\rho_+) \\ &+a_{22}(\omega_a\nabla_a^+\phi_+\cdot\rho_+ - \omega_a\nabla_a^-\phi_-\cdot\rho_- - \omega_a\phi_+\cdot\tilde{\nabla}_a^+\rho_+ + \omega_a\phi_-\cdot\tilde{\nabla}_a^-\rho_-) \\ &+a_{23}(\omega_a\nabla_a^+\phi_+\cdot\rho_- - \omega_a\nabla_a^-\phi_-\cdot\rho_+ + \omega_a\phi_+\cdot\tilde{\nabla}_a^-\rho_- - \omega_a\phi_-\cdot\tilde{\nabla}_a^+\rho_+) \\ &+a_{24}(\nabla_{\nu_+}^+\phi_+\cdot\tilde{\nabla}_{\nu_+}^+\rho_+ + \nabla_{\nu_-}^-\phi_-\cdot\tilde{\nabla}_{\nu_-}^-\rho_-) \\ &+a_{25}(\nabla_{\nu_+}^+\phi_+\cdot\tilde{\nabla}_{\nu_-}^-\rho_- + \nabla_{\nu_-}^-\phi_-\cdot\tilde{\nabla}_{\nu_+}^+\rho_+) \\ &+a_{26}(\nabla_a^+\phi_+\cdot\tilde{\nabla}_a^+\rho_+ + \nabla_a^-\phi_-\cdot\tilde{\nabla}_a^-\rho_-) \\ &+a_{27}(\nabla_a^+\phi_+\cdot\tilde{\nabla}_a^-\rho_- + \nabla_a^-\phi_-\cdot\tilde{\nabla}_a^+\rho_+) \\ &+a_{28}U(\partial_{\nu_+}(\phi_+\rho_+) + \partial_{\nu_-}(\phi_-\rho_-)) \\ &+a_{29}U(\nabla_{\nu_-}^-\phi_-\cdot\rho_+ + \phi_-\cdot\tilde{\nabla}_{\nu_+}^+\rho_+ + \nabla_{\nu_+}^+\phi_+\cdot\rho_- + \phi_+\cdot\tilde{\nabla}_{\nu_-}^-\rho_-) \\ &+a_{30}(L_{aa}^+\partial_{\nu_+}(\phi_+\rho_+) + L_{aa}^-\partial_{\nu_-}(\phi_-\rho_-)) \\ &+a_{31}(L_{aa}^-\partial_{\nu_+}(\phi_+\rho_+) + L_{aa}^+\partial_{\nu_-}(\phi_-\rho_-)) \\ &+a_{32}(L_{aa}^+(\nabla_{\nu_+}^+\phi_+\cdot\rho_- + \phi_-\tilde{\nabla}_{\nu_+}^+\rho_+) + L_{aa}^-(\nabla_{\nu_-}^-\phi_-\cdot\rho_+ + \phi_+\tilde{\nabla}_{\nu_-}^-\rho_-)) \\ &+a_{33}(L_{aa}^-(\nabla_{\nu_+}^+\phi_+\cdot\rho_- + \phi_-\tilde{\nabla}_{\nu_+}^+\rho_+) + L_{aa}^+(\nabla_{\nu_-}^-\phi_-\cdot\rho_+ + \phi_+\tilde{\nabla}_{\nu_-}^-\rho_-)) \\ &+a_{34}\omega_a\omega_a(\phi_+\rho_+ + \phi_-\rho_-) + a_{35}\omega_a\omega_a(\phi_+\rho_- + \phi_-\rho_+) \end{aligned}$$

$$
\begin{aligned}
&+a_{36}(L^+_{aa}L^+_{bb}\phi_+\rho_+ + L^-_{aa}L^-_{bb}\phi_-\rho_-) + a_{37}L^+_{aa}L^-_{bb}(\phi_+\rho_+ + \phi_-\rho_-)\\
&+a_{38}(L^-_{aa}L^-_{bb}\phi_+\rho_+ + L^+_{aa}L^+_{bb}\phi_-\rho_-) + a_{39}L^+_{aa}L^-_{bb}(\phi_+\rho_- + \phi_-\rho_+)\\
&+a_{40}(L^+_{aa}L^+_{bb} + L^-_{aa}L^-_{bb})(\phi_+\rho_- + \phi_-\rho_+)\\
&+a_{41}(L^+_{ab}L^+_{ab}\phi_+\rho_+ + L^-_{ab}L^-_{ab}\phi_-\rho_-) + a_{42}L^+_{ab}L^-_{ab}(\phi_+\rho_+ + \phi_-\rho_-)\\
&+a_{43}(L^-_{ab}L^-_{ab}\phi_+\rho_+ + L^+_{ab}L^+_{ab}\phi_-\rho_-) + a_{44}L^+_{ab}L^-_{ab}(\phi_+\rho_- + \phi_-\rho_+)\\
&+a_{45}(L^+_{ab}L^+_{ab} + L^-_{ab}L^-_{ab})(\phi_+\rho_- + \phi_-\rho_+)\\
&+a_{46}U(L^+_{aa}\phi_+\rho_+ + L^-_{aa}\phi_-\rho_-) + a_{47}U(L^-_{aa}\phi_+\rho_+ + L^+_{aa}\phi_-\rho_-)\\
&+a_{48}U(L^+_{aa} + L^-_{aa})(\phi_+\rho_- + \phi_-\rho_+)\\
&+a_{49}U^2(\phi_+\rho_+ + \phi_-\rho_-) + a_{50}U^2(\phi_+\rho_- + \phi_-\rho_+)\\
&+a_{51}(E_+\phi_+\rho_+ + E_-\phi_-\rho_-) + a_{52}(E_-\phi_+\rho_+ + E_+\phi_-\rho_-)\\
&+a_{53}(E_+ + E_-)(\phi_+\rho_- + \phi_-\rho_+) + a_{54}(R^+_{ijji}\phi_+\rho_+ + R^-_{ijji}\phi_-\rho_-)\\
&+a_{55}(R^-_{ijji}\phi_+\rho_+ + R^+_{ijji}\phi_-\rho_-) + a_{56}(R^+_{ijji} + R^-_{ijji})(\phi_+\rho_- + \phi_-\rho_+)\\
&+a_{57}(R^+_{amma}\phi_+\rho_+ + R^-_{amma}\phi_-\rho_-)\\
&+a_{58}(R^-_{amma}\phi_+\rho_+ + R^+_{amma}\phi_-\rho_-)\\
&+a_{59}(R^+_{amma} + R^-_{amma})(\phi_+\rho_- + \phi_-\rho_+).
\end{aligned}
$$

(2)
$$
\begin{array}{lllll}
a_{20} = 4, & a_{21} = -4, & a_{22} = -1, & a_{23} = -1, & a_{24} = 4,\\
a_{25} = 4, & a_{26} = -2, & a_{27} = 2, & a_{28} = -2, & a_{29} = -2,\\
a_{30} = -1, & a_{31} = 1, & a_{32} = 1, & a_{33} = -1, & a_{34} = 1,\\
a_{35} = 0, & a_{36} = 0, & a_{37} = -\frac12, & a_{38} = 0, & a_{39} = \frac12,\\
a_{40} = 0, & a_{41} = \frac12, & a_{42} = 0, & a_{43} = \frac12, & a_{44} = 0,\\
a_{45} = -\frac12, & a_{46} = 1, & a_{47} = -1, & a_{48} = 0, & a_{49} = 1,\\
a_{50} = 1, & a_{51} = 1, & a_{52} = 1, & a_{53} = -1, & a_{54} = 0,\\
a_{55} = 0, & a_{56} = 0, & a_{57} = \frac12, & a_{58} = \frac12, & a_{59} = -\frac12.
\end{array}
$$

We use the relations of equation (3.1) to simplify the format at the outset and derive (1). We shall use the functorial properties involved to determine the unknown coefficients and prove (2).

We apply Theorem 2.1 and equation (3.3) to see:

$$
\begin{array}{ll}
2a_{20} + 2a_{21} = 0, & 2a_{20} - 2a_{21} = 16,\\
2a_{24} + 2a_{25} = 16, & 2a_{24} - 2a_{25} = 0,\\
2a_{26} + 2a_{27} = 0, & 2a_{26} - 2a_{27} = -8,\\
-4a_{28} - 4a_{29} = 16, & 2a_{28} - 2a_{29} = 0,\\
a_{30} + a_{31} + a_{32} + a_{33} = 0, & 2a_{30} + 2a_{31} - 2a_{32} - 2a_{33} = 0,\\
a_{36} + a_{37} + a_{38} + a_{39} + 2a_{40} = 0, & a_{36} + a_{37} + a_{38} - a_{39} - 2a_{40} = -1,\\
a_{41} + a_{42} + a_{43} + a_{44} + 2a_{45} = 0, & a_{41} + a_{42} + a_{43} - a_{44} - 2a_{45} = 2,\\
2a_{46} + 2a_{47} + 4a_{48} = 0, & 2a_{46} + 2a_{47} - 4a_{48} = 0,\\
8a_{49} + 8a_{50} = 16, & 2a_{49} - 2a_{50} = 0,\\
2a_{51} + 2a_{52} + 4a_{53} = 0, & 2a_{51} + 2a_{52} - 4a_{53} = 8,\\
2a_{54} + 2a_{55} + 4a_{56} = 0, & 2a_{54} + 2a_{55} - 4a_{56} = 0,\\
2a_{57} + 2a_{58} + 4a_{59} = 0, & 2a_{57} + 2a_{58} - 4a_{59} = 4.
\end{array}
$$

Take arbitrary metrics on $M_\pm$ and let $D_\pm$ be the scalar Laplacian. Take $\phi = 1$ and $U = 0$. Then ϕ satisfies transmittal boundary conditions. Thus $\beta_n(1,\rho,D,\mathcal{B}_1) = 0$ for $n \geq 1$. Take $\rho_- = 0$. This yields:

$$\begin{array}{lll} a_{30}+a_{32}=0, & a_{31}+a_{33}=0, & \\ a_{36}+a_{40}=0, & a_{38}+a_{40}=0, & a_{37}+a_{39}=0, \\ a_{41}+a_{45}=0, & a_{43}+a_{45}=0, & a_{42}+a_{44}=0, \\ a_{51}+a_{53}=0, & a_{52}+a_{53}=0, & a_{54}+a_{56}=0, \\ a_{55}+a_{56}=0, & a_{57}+a_{59}=0, & a_{58}+a_{59}=0. \end{array}$$

Let $D_\pm(\varepsilon) = D_\pm - \varepsilon$. Then $\tilde{D}_\pm(\varepsilon) = \tilde{D}_\pm - \varepsilon$ and $E_\pm(\varepsilon) = E_\pm + \varepsilon$. As $e^{-tD_{\mathcal{B}_1}(\varepsilon)} = e^{t\varepsilon}e^{-tD_{\mathcal{B}_1}}$, $\beta(\phi,\rho,D(\varepsilon),\mathcal{B}_1)(t) = e^{t\varepsilon}\beta(\phi,\rho,D,\mathcal{B}_1)(t)$ and hence $\partial_\varepsilon\beta_n|_{\varepsilon=0} = \beta_{n-2}$. Thus studying the coefficients of the terms $\{\phi_+\rho_+, \phi_+\rho_-\}$ leads to the relations:

$$\begin{aligned} &\tfrac{1}{6\sqrt{\pi}}\{-2a_{20}+a_{51}+a_{52}\} = a_1 = -\tfrac{1}{\sqrt{\pi}}, \\ &\tfrac{1}{6\sqrt{\pi}}\{-2a_{21}+2a_{53}\} = a_2 = \tfrac{1}{\sqrt{\pi}}. \end{aligned}$$

We use separation of variables to generate additional relationships among the coefficients. First, we study flat metrics. Let (r,θ) be the usual parameters on $M_\pm := [0,1]\times S^1$. Let

$$D_\pm := -(\partial_r^2 + \partial_\theta^2 + 2\varepsilon_\pm\partial_\theta)$$

where $\varepsilon_\pm = \varepsilon_\pm(\theta)$. Let $N_\pm := [0,1]$ and let $\bar{D}_\pm := -\partial_r^2$. Let $\phi_\pm$ and $\rho_\pm$ be constant. Let U_0 be constant. Separation of variables and an application of equation (3.3) and Theorem 2.1 then yields:

$$\begin{aligned} &e^{-tD_{\mathcal{B}_1}}\phi = e^{-t\bar{D}_\pm}\phi \text{ so} \\ &\beta_3(\phi,\rho,D,\mathcal{B}_1)_M = 2\pi\beta_3(\phi,\rho,\bar{D},\mathcal{B}_1)_N \\ &\qquad = 4\pi\beta_3(\tilde{\phi}_e,\tilde{\rho}_e,\bar{D},\mathcal{B}_R)_{N_+} + 4\pi\beta_3(\tilde{\phi}_o,\tilde{\rho}_o,\bar{D},\mathcal{B}_D)_{N_+} = 0. \end{aligned}$$

As $\nabla_\theta\phi_\pm = \varepsilon_\pm\phi_\pm$, $\nabla_\theta\rho_\pm = -\varepsilon_\pm\rho_\pm$, and $E_\pm = -\varepsilon_\pm^2 - \partial_\theta\varepsilon_\pm$, we have:

$$\begin{aligned} 0 = {} & a_{22}(\varepsilon_+-\varepsilon_-)(2\varepsilon_+\phi_+\rho_+ - 2\varepsilon_-\phi_-\rho_-) \\ & +a_{23}(\varepsilon_+-\varepsilon_-)^2(\phi_+\rho_- + \phi_-\rho_+) \\ & +a_{26}(-\varepsilon_+^2\phi_+\rho_+ - \varepsilon_-^2\phi_-\rho_-) + a_{27}\varepsilon_+\varepsilon_-(-\phi_+\rho_- - \phi_-\rho_+) \\ & +a_{34}(\varepsilon_+-\varepsilon_-)^2(\phi_+\rho_+ + \phi_-\rho_-) + a_{35}(\varepsilon_+-\varepsilon_-)^2(\phi_+\rho_- + \phi_-\rho_+) \\ & +a_{51}(-\varepsilon_+^2\phi_+\rho_+ - \varepsilon_-^2\phi_-\rho_-) + a_{52}(-\varepsilon_-^2\phi_+\rho_+ - \varepsilon_+^2\phi_-\rho_-) \\ & +a_{53}(-\varepsilon_+^2 - \varepsilon_-^2)(\phi_+\rho_- + \phi_-\rho_+). \end{aligned}$$

The terms $\{\varepsilon_+^2\phi_+\rho_+,\ \varepsilon_-^2\phi_+\rho_+,\ \varepsilon_+\varepsilon_-\phi_+\rho_+, \varepsilon_+^2\phi_+\rho_-, \varepsilon_+\varepsilon_-\phi_+\rho_-\}$ are then studied to conclude that:

$$\begin{array}{ll} 0 = 2a_{22} - a_{26} + a_{34} - a_{51}, & 0 = a_{34} - a_{52}, \\ 0 = -2a_{22} - 2a_{34}, & 0 = a_{23} + a_{35} - a_{53}, \\ 0 = -2a_{23} - a_{27} - 2a_{35}. & \end{array}$$

We can also get information from the divergence terms. We now let $\phi=(1,1)$ and $\rho=(\rho_+(\theta),0)$. We work modulo $O(\varepsilon^2)$ and use the fact that $a_{20}+a_{21}=0$ to show:

$$\begin{aligned}0=\textstyle\int_\Sigma\big\{&(\partial_\theta\rho_+)\{-a_{22}(\varepsilon_+-\varepsilon_-)-a_{23}(\varepsilon_+-\varepsilon_-)+a_{26}\varepsilon_++a_{27}\varepsilon_-\}\\&+\rho_+\{-a_{51}\partial_\theta\varepsilon_+-a_{52}\partial_\theta\varepsilon_--a_{53}\partial_\theta(\varepsilon_++\varepsilon_-)\}\big\}\\-a_{22}-a_{23}&+a_{26}+a_{51}+a_{53}=0,\\a_{22}+a_{23}&+a_{27}+a_{52}+a_{53}=0.\end{aligned}$$

Next, let $N_\pm:=[0,1]$ and $M_\pm:=[0,1]\times S^1\times S^1$ have the metrics:

$$ds^2=dr^2+e^{2f_{1,\pm}(r)}d\theta_1^2+e^{2f_{2,\pm}(r)}d\theta_2^2$$

where $f_{i,\pm}$ vanishes on the boundary. Let $f_{i,\pm,r}:=\partial_rf_{i,\pm}$ and let

$$\begin{aligned}D^N_\pm&:=-\{\partial_r^2+\textstyle\sum_if_{i,\pm,r}\partial_r\},\\D^M_\pm&:=-\{\partial_r^2+\textstyle\sum_i(f_{i,\pm,r}\partial_r+e^{-2f_{i,\pm}}\partial_\theta^2)\}.\end{aligned}$$

We set $U_M=U_0$ constant. As the connection forms defined by D^N and D^M are different, we set $U_N=U_0+\frac12\sum_i(f_{i,+,r}+f_{i,-,r})$. Let $\phi_\pm$ and $\rho_\pm$ be constant. The argument used to establish equation (3.6) then generalizes immediately to yield:

$$\beta_3(\phi,\rho,D_M,\mathcal{B}_{1,M})=(2\pi)^2\beta_3(\phi,e^{\sum_if_i}\rho,D_N,\mathcal{B}_{1,N}).\tag{3.8}$$

We compute on ∂M:

$$\begin{aligned}\Gamma^\pm_{mab}&=\tfrac12\partial_mg^\pm_{ab}=e^{2f_{a,\pm}}f_{a,\pm,r}\delta_{ab},\\\Gamma^\pm_{abm}&=-\tfrac12\partial_mg^\pm_{ab}=-e^{2f_{a,\pm}}f_{a,\pm,r}\delta_{ab},\\\Gamma^\pm_{amb}&=-\Gamma^\pm_{abm}=e^{2f_{a,\pm}}f_{a,\pm,r}\delta_{ab},\\\Gamma^{\pm\ \ b}_{am}&=g^{bb}\Gamma^\pm_{amb}=f_{a,\pm,r}\delta_{ab},\\L^\pm_{ab}&=(\nabla_a\partial_b,\partial_m)=\Gamma^\pm_{abm}=-e^{2f_{a,\pm}}f_{a,\pm,r}\delta_{ab}\,,\\R^\pm_{amma}&=((\nabla^\pm_a\nabla^\pm_m-\nabla^\pm_m\nabla^\pm_a),\partial_m,\partial_a)=-(\nabla^\pm_m\Gamma^{\pm\ \ b}_{am}\partial_b,\partial_m)\\&=-(\nabla^\pm_mf_{a,\pm,r}\partial_a,\partial_a)=-f_{a,\pm,rr}-f^2_{a,\pm,r}.\end{aligned}$$

(We do not compute $R^\pm_{ijji}$ as we have shown $a_{54}=a_{55}=a_{56}=0$ so these terms do not appear). Let $f_\pm=\sum_if_{i,\pm}$. We compute on ∂N:

$$\begin{array}{ll}\omega^\pm_r=\frac12f_{\pm,r}, & \tilde\omega^\pm_r=-\frac12f_{\pm,r},\\U_N=U_0+\frac12(f_{+,r}+f_{-,r}), & E^\pm_N=-\frac12f_{\pm,rr}-\frac14f^2_{\pm,r}\,,\\\nabla^\pm_\nu\rho_\pm=\frac12f_\pm\phi_\pm, & \nabla^\pm_\nu(e^{f_\pm}\rho)=\frac12f_\pm\rho_\pm,\\D^N_\pm\phi_\pm=0, & \tilde D^N_\pm\{e^{f_\pm}\rho_\pm\}=0.\end{array}$$

We use equation (3.8) to derive the following equations from the coefficients of the indicated monomials:

$$\begin{array}{ll}
-a_{57} = -\frac{1}{2}a_{51} & (f_{1,+,rr}\phi_+\rho_+) \\
-a_{58} = -\frac{1}{2}a_{52} & (f_{1,-,rr}\phi_+\rho_+) \\
-a_{59} = -\frac{1}{2}a_{53} & (f_{1,+,rr}\phi_+\rho_-) \\
a_{36} + a_{41} - a_{57} = \frac{1}{4}a_{24} + \frac{1}{2}a_{28} + \frac{1}{4}a_{49} - \frac{1}{4}a_{51} & (f_{1,+,r}f_{1,+,r}\phi_+\rho_+) \\
a_{38} + a_{43} - a_{58} = \frac{1}{4}a_{49} - \frac{1}{4}a_{52} & (f_{1,-,r}f_{1,-,r}\phi_+\rho_+) \\
a_{37} + a_{42} = \frac{1}{2}a_{28} + \frac{1}{2}a_{49} & (f_{1,+,r}f_{1,-,r}\phi_+\rho_+) \\
2a_{36} = \frac{1}{2}a_{24} + a_{28} + \frac{1}{2}a_{49} - \frac{1}{2}a_{51} & (f_{1,+,r}f_{2,+,r}\phi_+\rho_+) \\
2a_{38} = \frac{1}{2}a_{49} - \frac{1}{2}a_{52} & (f_{1,-,r}f_{2,-,r}\phi_+\rho_+) \\
a_{37} = \frac{1}{2}a_{28} + \frac{1}{2}a_{49} & (f_{1,+,r}f_{2,-,r}\phi_+\rho_+) \\
a_{40} + a_{45} - a_{59} = \frac{1}{4}a_{29} + \frac{1}{4}a_{50} - \frac{1}{4}a_{53} & (f_{1,+,r}f_{1,+,r}\phi_+\rho_-) \\
a_{39} + a_{44} = \frac{1}{4}a_{25} + \frac{1}{2}a_{29} + \frac{1}{2}a_{50} & (f_{1,+,r}f_{1,-,r}\phi_+\rho_-) \\
2a_{40} = \frac{1}{2}a_{29} + \frac{1}{2}a_{50} - \frac{1}{2}a_{53} & (f_{1,+,r}f_{2,+,r}\phi_+\rho_-) \\
a_{39} = \frac{1}{4}a_{25} + \frac{1}{2}a_{29} + \frac{1}{2}a_{50} & (f_{1,+,r}f_{2,-,r}\phi_+\rho_-) \\
-a_{46} = a_{28} + a_{49} & (U_0 f_{1,+,r}\phi_+\rho_+) \\
-a_{47} = a_{49} & (U_0 f_{1,-,r}\phi_+\rho_+) \\
-a_{48} = \frac{1}{2}a_{29} + a_{50} & (U_0 f_{1,+,r}\phi_+\rho_-)
\end{array}$$

We now let $\rho = \rho(r)$. Equation (3.8) continues to hold and yields:

$$\begin{array}{ll}
-a_{31} = \frac{1}{2}a_{28} & (f_{1,-,r}\phi_+\partial_{\nu_+}\rho_+) \\
-a_{33} = \frac{1}{2}a_{25} + \frac{1}{2}a_{29} & (f_{1,+,r}\phi_+\partial_{\nu_-}\rho_-)
\end{array}$$

We combine the relations given above to complete the determination of β_3 by determining the constants $a_{20} - a_{59}$ and complete the proof of Theorem 3.2. □

4. The boundary condition $\mathcal{B}_2$

Again, we begin our discussion by studying β_0, β_1, and β_2. Note that terms such as $\phi_+\rho_-$ cannot occur since we do not assume an identification of $V_+|_\Sigma$ with $V_-|_\Sigma$.

Lemma 4.1. *There exist universal constants so*

(1) $\beta_0(\phi, \rho, D, \mathcal{B}_1) = \int_{M_+} \phi_+\rho_+ + \int_{M_-} \phi_-\rho_-$.

(2) $\beta_1(\phi, \rho, D, \mathcal{B}_2) = \int_\Sigma b_1(\phi_+\rho_+ + \phi_-\rho_-)$.

(3) $\beta_2(\phi, \rho, D, \mathcal{B}_2) = -\int_{M_+} D_+\phi \cdot \rho_+ - \int_{M_-} D_-\phi_- \cdot \rho_-$
$+ \int_\Sigma \{b_2(\phi_+\rho_+L^+_{aa} + \phi_-\rho_-L^-_{bb}) + b_3(\phi_+\rho_+L^-_{aa} + \phi_-\rho_-L^+_{aa})$
$+ b_4(\phi_{+;\nu_+}\rho_+ + \phi_{-;\nu_-}\rho_-) + b_5(\phi_+\rho_{+;\nu_+} + \phi_-\rho_{-;\nu_-})$
$+ b_6(S_{++}\phi_+ \cdot \rho_+ + S_{--}\phi_- \cdot \rho_-) + b_7(S_{+-}\phi_- \cdot \rho_+ + S_{-+}\phi_+ \cdot \rho_-)\}$.

Taking $S_{+-}=0$ and $S_{-+}=0$ forces the boundary conditions given in equation (1.4) to decouple and defines Robin boundary conditions on M_+ and on M_- separately. We use Theorem 2.1 to see:

$$b_1=0,\quad b_2=0,\quad b_3=0,\quad b_4=1,\quad b_5=0,\quad b_6=1. \tag{4.1}$$

We use an argument similar to that used to establish display (3.5) to determine b_7. Let $D_\pm$ be the scalar Laplacians on manifolds $M_\pm$. Let $\phi_+=\phi_-=1$, let $S_{++}=S_{--}=1$, and let $S_{+-}=S_{-+}=-1$. Then $\mathcal{B}_2\phi=0$ and $D\phi=0$ so $\beta_n=0$ for $n\geq 0$. Thus:

$$b_6-b_7=0. \tag{4.2}$$

In view of the remarks noted above, we see that:

Lemma 4.2. *There exist universal constants so*

$$\begin{aligned}
&\beta_3(\phi,\rho,D,\mathcal{B}_2)=\tfrac{4}{3\sqrt{\pi}}\textstyle\int_\Sigma \mathcal{B}_2\phi\cdot\tilde{\mathcal{B}}_2\rho\\
&\quad+b_{10}(S_{+-}S_{-+}\phi_+\cdot\rho_+ + S_{-+}S_{+-}\phi_-\cdot\rho_-)\\
&\quad+b_{11}(S_{--}S_{-+}\phi_+\cdot\rho_- + S_{++}S_{+-}\phi_-\cdot\rho_+)\\
&\quad+b_{12}(S_{-+}S_{++}\phi_+\cdot\rho_- + S_{+-}S_{--}\phi_-\cdot\rho_+)\\
&\quad+b_{13}(S_{-+}\phi_{+;\nu_+}\cdot\rho_- + S_{+-}\phi_{-;\nu_-}\cdot\rho_+)\\
&\quad+b_{14}(S_{-+}\phi_+\cdot\rho_{-;\nu_-} + S_{+-}\phi_-\cdot\rho_{+;\nu_+})\\
&\quad+b_{15}(L^+_{aa}S_{-+}\phi_+\cdot\rho_- + L^-_{aa}S_{+-}\phi_-\cdot\rho_+)\\
&\quad+b_{16}(L^-_{aa}S_{-+}\phi_+\cdot\rho_- + L^+_{aa}S_{+-}\phi_-\cdot\rho_+).
\end{aligned}$$

Let $D_\pm$ be the scalar Laplacians on manifolds $M_\pm$. Let $\phi_+=1$, $\phi_-=1$, let $S_{++}=a$, $S_{+-}=-a$, $S_{--}=b$, and $S_{-+}=-b$. Then ϕ satisfies transmittal boundary conditions and is harmonic so $\beta_n=0$ for $n\geq 0$. Consequently taking $\rho_-=0$ yields the equations:

$$b_{10}-b_{12}=0,\ b_{11}=b_{14}=b_{15}=b_{16}=0. \tag{4.3}$$

We work with $m=1$ and $D_\pm=-\partial_x^2$. Suppose that $\phi_\pm=a_\pm x+b_\pm$ is such that ϕ satisfies $\mathcal{B}_2\phi=0$, i.e.

$$\varepsilon a_+ + S_{++}b_+ + S_{+-}b_-=0 \text{ and } \varepsilon a_- + S_{--}b_- + S_{-+}b_+=0 \tag{4.4}$$

where $\varepsilon(0)=+1$ and $\varepsilon(1)=-1$. We choose S_* and b_* arbitrarily and use equation (4.4) to determine a_*. Let $\rho_-=0$. Since $\beta_n=0$ for $n>0$,

$$\begin{aligned}
&b_{10}S_{+-}S_{-+}b_+ + b_{12}S_{+-}S_{--}b_- + b_{13}\varepsilon S_{+-}a_-=0,\\
&(b_{10}-b_{13})S_{+-}S_{-+}b_+ + (b_{12}-b_{13})S_{+-}S_{--}b_-=0 \text{ so}
\end{aligned}$$

$$b_{10}=b_{13},\quad\text{and}\quad b_{12}=b_{13}. \tag{4.5}$$

We double the manifold to complete our determination. Suppose given an operator D_0 of Laplace type on a manifold M_0 with boundary Σ. Let an initial condition ϕ_0 be given and let u_0 solve equation (1.1) with the boundary operator $\mathcal{B}_R$. Let $M_\pm := M_0$ and $D_\pm := D_0$. Then $u_\pm := u_0$ and $\phi_\pm := \phi_0$ solves equation (1.1) with the boundary operator $\mathcal{B}_2$ defined by equation (1.4) with $S_{++} = S_{--} = 0$ and $S_{+-} = S_{-+} = S$. Thus

$$\beta_n(\phi_0, \rho_+ + \rho_-, D_0, \mathcal{B}_R) = \beta_n(\phi, \rho, D, \mathcal{B}_2).$$

We may now conclude $b_{10} = b_{13} = 0$. This proves

Theorem 4.3.

(1) $\beta_0(\phi, \rho, D, \mathcal{B}_2) = \int_M \phi\rho$.
(2) $\beta_1(\phi, \rho, D, \mathcal{B}_2) = 0$.
(3) $\beta_2(\phi, \rho, D, \mathcal{B}_2) = -\int_M D\phi \cdot \rho + \int_\Sigma \mathcal{B}_2\phi \cdot \rho$.
(4) $\beta_3(\phi, \rho, D, \mathcal{B}_2) = \frac{4}{3\sqrt{\pi}} \int_\Sigma \mathcal{B}_2\phi \cdot \tilde{\mathcal{B}}_2\rho$.

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