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Singular limit laminations, Morse index, and positive scalar curvature

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SINGULAR LIMIT LAMINATIONS, MORSE INDEX, AND POSITIVE SCALAR CURVATURE

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Abstract. For any 3-manifold M^3 and any nonnegative integer $\mathbf{g}$, we give here examples of metrics on M each of which has a sequence of embedded minimal surfaces of genus $\mathbf{g}$ and without Morse index bounds (all our surfaces will be orientable). On any spherical space form $\mathbf{S}^3/\Gamma$ we construct such a metric with positive scalar curvature. More generally we construct such a metric with Scal > 0 (and such surfaces) on any 3-manifold which carries a metric with Scal > 0.

0. Introduction

For any 3-manifold M^3 and any nonnegative integer $\mathbf{g}$, we give here examples of metrics on M each of which has a sequence of embedded minimal surfaces of genus $\mathbf{g}$ and without Morse index bounds (all our surfaces will be orientable). On any spherical space form $\mathbf{S}^3/\Gamma$ we construct such a metric with positive scalar curvature. More generally we construct such a metric with Scal > 0 (and such surfaces) on any 3-manifold which carries a metric with Scal > 0; see Theorem 0.2 below. In all but one of our examples the Hausdorff limit will be a singular minimal lamination. The singularities being in each case exactly two points lying on a closed leaf (the leaf is a strictly stable sphere).

There are two prior examples of embedded minimal surfaces in 3-manifolds without Morse index bounds. In [CH1] it was shown that even in one dimension less (i.e., for simple closed geodesics on surfaces) there are examples of metrics without Morse index bounds. [CH1] also gave examples on any 3-manifold of a metric which has embedded minimal tori without such bounds. In [HaNoRu] examples were given of metrics on any M^3 that have embedded minimal spheres without bounds. As mentioned above in this paper we are not only interested in giving such examples for any genus and of metrics with positive scalar curvature but also in a particular type of degeneration of the surfaces.

We use in part ideas of Hass-Norbury-Rubinstein [HaNoRu] to achieve this (and in the process answer a question of theirs). As in [HaNoRu], but unlike the examples in [CH1], the surfaces will have no uniform curvature bounds. In fact, it follows easily (see appendix B of [CM4]) that if $\Sigma_i \subset M^3$ is a sequence of embedded minimal surfaces with uniformly bounded curvatures, then a subsequence converges to a smooth lamination. Moreover, with the right notion of being generic, the following seems likely (by [CH1] bumpy is not the right generic notion):

Conjecture: Let M^3 be a closed 3-manifold with a generic metric and $\Sigma_i \subset M$ a sequence of embedded minimal surfaces of a given genus. If any limit of the Σ_i's is a smooth (minimal) lamination, then the sequence Σ_i has a uniform Morse index bound.

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A codimension one *lamination* of M^3 is a collection $\mathcal{L}$ of smooth disjoint connected surfaces (called leaves) such that $\cup_{\Lambda \in \mathcal{L}} \Lambda$ is closed. Moreover, for each $x \in M$ there exists an open neighborhood U of x and a local coordinate chart, (U, Φ), with $\Phi(U) \subset \mathbf{R}^3$ such that in these coordinates the leaves in $\mathcal{L}$ pass through the chart in slices of the form $(\mathbf{R}^2 \times \{t\}) \cap \Phi(U)$.

A lamination is said to be minimal if the leaves are (smooth) minimal surfaces. If the union of the leaves is all of M, then it is a foliation.

There are two results that support this conjecture. The first concerns the corresponding conjecture in one dimension less (that is for geodesics on surfaces); see [CH2], [CH3]. The second concerns the conjecture for 3-manifolds with positive scalar curvature; see [CM3]. However, there are examples where the limit is not smooth as the following shows:

Theorem 0.1. On any 3-manifold, M^3, and for any nonnegative integer $\mathbf{g}$, there exists a metric and a sequence of embedded minimal surfaces of genus $\mathbf{g}$ with Morse index going to infinity and converging to a singular (minimal) lamination $\mathcal{L}$. This can be done so that the singular set of $\mathcal{L}$ consists of two points lying on a leaf which is a strictly stable 2-sphere.

For manifolds which carry a metric with positive scalar curvature we use a connected sum construction to show (cf. section 5 of Gromov-Lawson [GrLa] and theorem 4 of Schoen-Yau [ScYa]):

Theorem 0.2. (See fig. 8). Any 3-manifold which carries a metric with positive scalar curvature has for any nonnegative integer $\mathbf{g}$ a metric with positive scalar curvature and a sequence of embedded minimal surfaces of genus $\mathbf{g}$ as in Theorem 0.1.

As a consequence we get by [GrLa], [ScYa]:

Corollary 0.3. Any manifold of the form

$$\mathbf{S}^3/\Gamma_1 \# \cdots \# \mathbf{S}^3/\Gamma_k \# \mathbf{S}^2 \times \mathbf{S}^1 \# \cdots \# \mathbf{S}^2 \times \mathbf{S}^1 \,, \tag{0.4}$$

where $\mathbf{S}^3/\Gamma_i$ is a spherical space form, has for any nonnegative integer $\mathbf{g}$ a metric with positive scalar curvature and a sequence of embedded minimal surfaces of genus $\mathbf{g}$ as in Theorem 0.1.

The following is a different kind of example (different from [CH1]; however not bumpy) that illustrates why generic is needed in the above conjecture:

Theorem 0.5. In $\mathbf{S}^2 \times \mathbf{S}^1$ with the product metric, there is a sequence of embedded minimal tori with Morse index going to infinity. Moreover, these converge to the foliation by parallel $\mathbf{S}^2 \times \{t\}$.

The next four sections contain the proofs of the above three theorems. In Section 5 we show how to generalize Theorems 0.1 and 0.3 to where the singular set contains points on any given finite collection of disjoint embedded strictly stable 2-spheres. Finally, in Section 6 we return to a result shown in Section 1 and speculate on how the space of noncompact embedded minimal annuli limiting a strictly stable 2-sphere look like. Moreover, we speculate there on what the structure of this space of annuli might imply for structure of the singular set of a limit lamination for a generic metric.

Recall that if $\Sigma^2 \subset M$ is a closed minimal surface, then the *Morse index* of Σ is the index of the critical point Σ for the area functional, i.e., the number of negative eigenvalues (counted with multiplicity) of the second derivative of area. If Σ has a unit normal $\mathbf{n}$, the second derivative of area at Σ in the direction of a normal variation $u\,\mathbf{n}$ is $-\int_\Sigma u\,L\,u$ where $L\,u = \Delta\,u + [|A|^2 + \mathrm{Ric}_M(\mathbf{n},\mathbf{n})]\,u$; so the Morse index is the number of negative eigenvalues of L. (By convention, an eigenfunction ϕ with eigenvalue λ of L is a solution of $L\,\phi + \lambda\,\phi = 0$.) Σ is said to be *stable* if the index is zero. A metric on M^3 is *bumpy* if each closed minimal surface is a nondegenerate critical point, i.e., $L\,u = 0$ implies $u \equiv 0$. By a result of B. White bumpy metrics are generic; that is the set of bumpy metrics contain a countable intersection of open dense subsets. We use throughout the normalization of the curvature so that the round unit 3–sphere has sectional curvature 1 and scalar curvature 3.

Our interest in whether the Morse index is bounded for embedded minimal tori in a 3-manifold comes in part from its connection with the spherical space form problem; see [PiRu], [CM2].

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1. The metric and surfaces near the stable 2-sphere

Following [HaNoRu] (see also [HsLa]) we look at metrics on $\mathbf{S}^2 \times \mathbf{R}$ of the form

$$ds_0^2 = dr^2 + \lambda^2(r)\,(d\phi^2 + \sin^2\phi\,d\theta^2)\,. \tag{1.1}$$

Here (ϕ,θ) are spherical coordinates on $\mathbf{S}^2$ and $r \in \mathbf{R}$. Computing the scalar curvature of the warped product gives

$$\mathrm{Scal}_M = -2\frac{\lambda''}{\lambda} + \frac{1-(\lambda')^2}{\lambda^2}\,. \tag{1.2}$$

To find our minimal surfaces we consider on the infinite strip $[0,\pi]\times\mathbf{R}$ the degenerate metric

$$ds^2 = \lambda^2(r)\,\sin^2\phi\,(dr^2 + \lambda^2(r)\,d\phi^2) \tag{1.3}$$

and calculate the geodesics in this metric. Our minimal surfaces will be the preimages of simple closed geodesics in the metric (1.3) under the map $(\phi,\theta,r) \to (\phi,r)$. For completeness we will now see why these preimages are minimal. So let Σ be a surface of the form $\mathbf{S}^1 \times \gamma$ where $\gamma(t) = (\phi(t), r(t))$ is a curve in $[0,\pi]\times\mathbf{R}$ (below ϕ will be different from 0 and π for the curve γ so the preimage of each $\gamma(t)$ is indeed a circle). A surface is minimal if and only if the first variation of the area functional is zero for any smooth vector field perpendicular to it. Since the rotations $\theta \to \theta +$ constant preserve the metric (1.1) and Σ, it is sufficient to check that the first variation vanishes with respect to vector fields invariant for this family of isometries. Being perpendicular to Σ, these vector fields are of the form $v = v_\phi(\phi,r)\,\partial_\phi + v_r(\phi,r)\,\partial_r$. Thus checking first variation of the area for Σ is equivalent to check the first variation of the functional

$$F(\gamma) = \int_\gamma \mathrm{length}(\mathbf{S}^1 \times \{\gamma(t)\}) = \int_\gamma 2\pi\,\lambda(t)\,\sin(\phi(t)) \tag{1.4}$$

in the space of curves of $[0,\pi]\times\mathbf{R}$ with the metric $dr^2 + \lambda^2(r)\,d\phi^2$. Notice that $F(\gamma)$ is 2π times the length of γ in the metric (1.3) and hence the first variation of F vanishes if and only if γ is a geodesic in (1.3).

For a unit speed geodesic in (1.3) (throughout this paper all geodesics will have unit speed)

$$r'' = -2\,\frac{\cos\phi}{\sin\phi}\,r'\,\phi' - \frac{\lambda'(r)}{\lambda(r)}\,(r')^2 + 2\,\lambda'(r)\,\lambda(r)\,(\phi')^2\,, \tag{1.5}$$

$$(r')^2 + \lambda^2\,(\phi')^2 = \lambda^{-2}\,\sin^{-2}\phi\,. \tag{1.6}$$

From (1.5) it follows that if $\lambda' \geq 0$, then provided $r' > 0$

$$\frac{d}{dt}\log r' \geq \frac{d}{dt}\log\sin^{-2}\phi - \frac{\lambda'(r)}{\lambda(r)}\,r'\,. \tag{1.7}$$

In particular (1.7) yields that if $r'(0) > 0$, then $r'(t) > 0$ for all $t > 0$. Namely, suppose that $r'(t_0) = 0$ and that $t_0 = \inf\{t > 0 \,|\, r'(t) = 0\}$, applying (1.7) yields a contradiction. It follows that if $r'(0) > 0$, then the geodesic is simple. Moreover, integrating (1.7) yields for $t_2 > t_1$

$$\frac{r'(t_2)}{r'(t_1)} \geq \frac{\sin^2\phi(t_1)}{\sin^2\phi(t_2)}\,\exp\left(C_1\,(r(t_1) - r(t_2)) + C_2\right)\,. \tag{1.8}$$

One may also easily check that if $\lambda'' > 0$, then the only curve where r is constant that is a geodesic is for $\{r = 0\}$ (this follows for instance since the only level set of r in (1.1) that is a minimal surface is $\{r = 0\}$). Finally, it follows from (1.6) and (1.8) that the boundary of the infinite strip is repelling. That is, the only geodesics that intersect the boundary are $\{r = 0\}$, $\{\phi = \pi\}$, and $\{\phi = 0\}$. Using this we can now show:

Proposition 1.9. For $\varepsilon > 0$ set $\lambda_\varepsilon(r) = \cosh(\varepsilon\, r)$. On $M = \mathbf{S}^2 \times_{\lambda_\varepsilon} \mathbf{R}$ (see fig. 1), $\mathbf{S}^2 \times \{0\}$ is the only closed minimal surface and there is a singular minimal lamination $\mathcal{L}$ on M with antipodal points on $\mathbf{S}^2 \times \{0\}$ as the only singularities of $\mathcal{L}$; see fig. 2. Moreover, there is a sequence of embedded minimal annuli Σ_i with $\Sigma_i \to \mathcal{L}$ and with Morse index going to infinity.

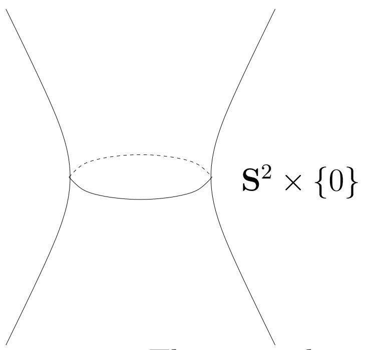

FIGURE 1. The strictly stable 2–sphere in the warped metric (1.1) with $\lambda(r) = \cosh(\varepsilon r)$

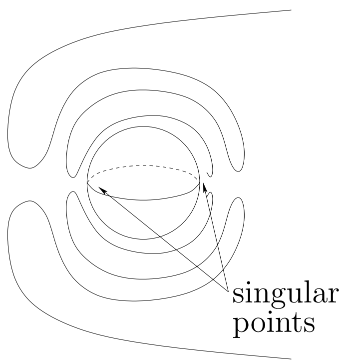

FIGURE 2. The singular lamination in half of a neighborhood of the strictly stable 2–sphere

Proof. To prove this proposition all we need is to find the corresponding geodesic lamination and simple geodesics on the infinite strip $[0, \pi] \times \mathbf{R}$ with the degenerate metric (see fig. 3)

$$ds^2 = \cosh^2(\varepsilon\, r)\,\sin^2\phi\,(dr^2 + \cosh^2(\varepsilon\, r)\,d\phi^2)\,. \tag{1.10}$$

Let $\gamma_\delta(t) = (\phi_\delta(t), r_\delta(t))$ be a geodesic in the metric (1.10) with $(\phi_\delta(0), r_\delta(0)) = (\pi/2, 0)$ and so that the angle between $\gamma'_\delta(0)$ and $\{r = 0\}$ is δ. We extract a sequence $\gamma_i = \gamma_{\delta_i}$ with $\delta_i \to 0$ and which converges in the Hausdorff sense. By the equations for geodesics above it follows that every γ_i is simple and that $\gamma_i \to \mathcal{G}$ as $i \to \infty$, where $\mathcal{G}$ is a geodesic lamination consisting of $\{r = 0\}$ and two infinite geodesics γ_∞ and $\gamma_{-\infty}$ which lie on each side of $\{r = 0\}$ and spiral into it. The surfaces Σ_i and the singular minimal lamination $\mathcal{L}$ can now be taken to be the preimages of the geodesics γ_i and of $\mathcal{G}$.

It follows (by a standard argument) that for any $r_0 > 0$ the Morse index of $T_{r_0}(\mathbf{S}^2 \times \{0\}) \cap \Sigma_\delta$ goes to infinity as $\delta \to 0$ (basically it follows easily, at least for r small, that the preimage of each "turn" in γ_δ, see fig. 2 and fig. 4, corresponds to a small neck that contributes to the index). Alternatively we can use the fact that Jacobi fields on the geodesics in (1.3) lift to Jacobi fields on the respective minimal surfaces in (1.2) and then reason as in [HaNoRu]. Finally, it follows easily from the maximum principle, as in the proof of proposition 1.8 of [CM3] (the sublevel sets $\{r \leq r_0\}$ are strictly mean convex for $r_0 > 0$), that $\mathbf{S}^2 \times \{0\}$ is the only closed minimal surface in M. $\square$

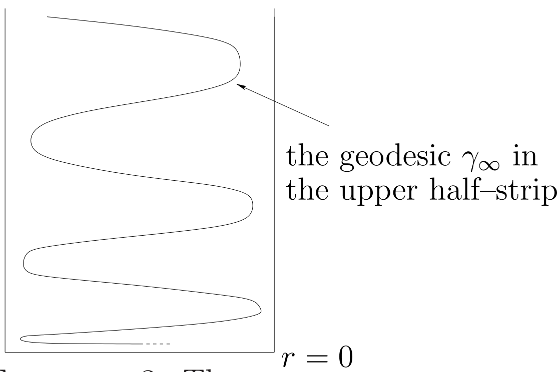

FIGURE 3. The upper half–strip with the degenerate metric (1.3) where $\lambda(r) = \cosh(\varepsilon r)$

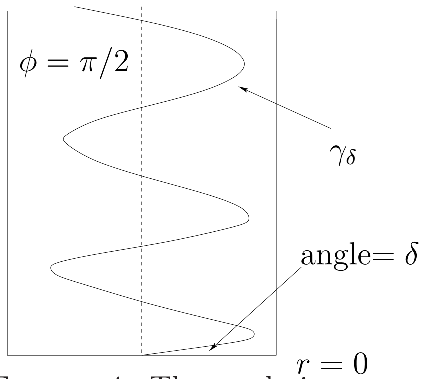

FIGURE 4. The geodesic γ_δ in the same half–strip

We will later need to deal with that the geodesics γ_i and γ_∞ (and hence also the corresponding minimal surfaces) cross in many points. In order to prove our theorems we will use that, by the next lemma, we can choose the Σ_i's and Σ_∞ so that in a neighborhood of some point of Σ_∞ the Σ_i's can be completed to a smooth minimal foliation.

Lemma 1.11. Consider on $[0, \pi] \times \mathbf{R}$ the degenerate metric (1.3) where $\lambda \in C^{1,1}$, $\lambda(r) = \lambda(-r)$, $\lambda' \geq 0$ on $[0, +\infty[$, and $\lambda(r) = \cosh(\varepsilon r)$ for some $\varepsilon > 0$ in a neighborhood of 0. For any fixed $\rho > 0$, we can assume that the geodesics γ_i, γ_∞ constructed in Proposition 1.9 pass through $(\pi/2, \rho)$. Thus $\{\gamma_i\} \cup \{\gamma^+\}$ can be completed to a smooth geodesic foliation in a punctured ball centered at $(\pi/2, \rho)$.

Proof. For any given γ which starts at $\{\pi/2, 0\}$ and any integer N let

$\alpha(\gamma)$ be the angle between $\{r = 0\}$ and γ .

$r_N(\gamma)$ be the N–th crossing between $\{\phi = \pi/2, r > 0\}$ and γ .

Fix a γ_∞ and a sequence $\gamma_i \to \gamma_\infty$ given by Proposition 1.9. It is not difficult to see that for sufficiently large N's there exist i, j such that $r_N(\gamma_j) \leq \rho \leq r_N(\gamma_i)$. Since $\{\phi = \pi/2\}$ is a geodesic, any other geodesic γ crosses $\{\phi = \pi/2\}$ transversally. This easily implies that $r_N(\gamma)$ is a continuous function of the starting angle $\alpha(\gamma)$.

Thus, varying this angle between $\alpha(\gamma_j)$ and $\alpha(\gamma_i)$, we find a geodesic $\tilde{\gamma}_N$ starting at $(\pi/2, 0)$ with $r_N(\tilde{\gamma}_N) = \rho$. Clearly $\alpha(\tilde{\gamma}_N) \to 0$ as $N \to \infty$. Hence, we can extract a subsequence of $\{\tilde{\gamma}_N\}$ converging to a geodesic lamination as in Proposition 1.9. □

The next definition and proposition are needed only in the proof of Theorem 0.2.

Definition 1.12. Let $z \in \Omega \subset \mathbf{S}^3$ be an open subset of the round unit 3-sphere and suppose that $\mathcal{F}$ is a foliation by great spheres of Ω. We say that the foliation is *parallel* at z if $\sup_{y \in \Lambda} \mathrm{dist}\,(y, \Lambda') = \mathrm{dist}\,(z, \Lambda')$ where Λ, $\Lambda' \in \mathcal{F}$ and $z \in \Lambda$ (Λ is said to be the *central* leaf of $\mathcal{F}$).

This particular kind of foliation is needed in the proof of Theorem 0.2 to make the connected sum construction.

Proposition 1.13. On $\mathbf{S}^3$, there is a metric with Scal > 0 which has a singular lamination and a sequence of embedded minimal surfaces of genus 0 as in Theorem 0.1. We can choose the metric so that these minimal spheres can be completed in a neighborhood of a point x to a foliation by great spheres parallel at x. Moreover, in an open (nonempty) set disjoint from the minimal spheres the sectional curvature of the metric on $\mathbf{S}^3$ is constant 1.

Proof. Fix on $\mathbf{S}^2 \times \mathbf{R}$ a metric with positive scalar curvature of the form (1.1) where λ satisfies, for some positive constants $a, b, c, \delta, \varepsilon$,

$$\begin{cases} \lambda(r) = \lambda(-r)\,, & \\ \lambda(r) = c\cosh(\varepsilon r) & \text{in a neighborhood of } 0\,, \\ \lambda'(r) \geq 0 & \text{for } r \in [0, \infty[\,, \\ \lambda(r) = 1 & \text{for } r \in [a, \infty[\,, \\ \lambda(r) = \sin(r + \pi/2 - a) & \text{for } r \in]a - \delta, a[\,. \end{cases} \tag{1.14}$$

λ can be chosen $C^{1,1}$ and C^∞ on $\mathbf{R} \setminus \{a, -a\}$. In particular, endowing $[0, \pi] \times \mathbf{R}$ with the degenerate metric (1.3), by Lemma 1.11 there are geodesics γ^+, γ^- through $(\pi/2, a)$ and $(\pi/2, -a)$, respectively, and spiraling into $\{r = 0\}$. Moreover, again by Lemma 1.11, there is a sequence of geodesics γ_i passing through $(\pi/2, a)$ and $(\pi/2, -a)$ which converges to the lamination $\gamma^+ \cup \gamma^- \cup \{r = 0\}$. Define $\tilde{\lambda}$ by

$$\tilde{\lambda}(r) = \begin{cases} \lambda(r) & \text{for } r \in [-a, a]\,, \\ \sin(r + \pi/2 - a) & \text{for } r \in [a, a + \pi/2]\,, \\ \sin(r + \pi/2 + a) & \text{for } r \in [-a - \pi/2, -a]\,. \end{cases} \tag{1.15}$$

Clearly $\tilde{\lambda} \in C^\infty$. On $\mathbf{S}^2 \times [-a - \pi/2, a + \pi/2]$ identify each of the spheres $\mathbf{S}^2 \times \{a + \pi/2\}$ and $\mathbf{S}^2 \times \{-a - \pi/2\}$ to a point to get the smooth metric

$$dr^2 + \tilde{\lambda}^2(r)\,(d\phi^2 + \sin^2\phi\, d\theta^2) \tag{1.16}$$

in $\mathbf{S}^3$. This $\mathbf{S}^3$ is obtained (loosely speaking) by capping off a neck with two standard half–$\mathbf{S}^3$'s, S^+ and S^-.

On $[0,\pi]\times[-a-\pi/2,a+\pi/2]$ with the degenerate metric

$$\tilde\lambda^2(r)\sin^2\phi\,(dr^2+\tilde\lambda^2(r)\,d\phi^2) \tag{1.17}$$

the curve $\gamma^+\cap[0,\pi]\times[0,a]$ is a geodesic curve. Continuing it in $[0,\pi]\times[0,a+\pi/2]$ we find a geodesic which hits the boundary $[0,\pi]\times\{a+\pi/2\}\cup\{0,\pi\}\times[a,a+\pi/2]$. This lifts to a minimal surface Σ^+ on $\mathbf{S}^3$ with the metric (1.16). Note that the subset of Σ^+ lying in S^+ is a hemisphere.

We argue in the same way for γ^- and γ_i. Thus we find a sequence of minimal 2–spheres Σ_i converging to a singular lamination given by the union of Σ^+, Σ^- (lifting of γ^+ and γ^-) and the strictly stable 2–sphere $\{r=0\}$. Every Σ_i contains two hemispheres H_i^+ and H_i^-, lying in S^+ and S^-. All H_i^+'s intersect in the great circle given by $\{r=a,\phi=\pi/2\}$ (and by symmetry all H_i^-'s intersect in $\{r=-a,\phi=\pi/2\}$). Thus $\{\Sigma_i\}\cup\{\Sigma_\infty\}$ can be completed locally to a foliation by great spheres parallel at two points. □

2. Completing the metric and the surfaces; proof of Theorem 0.1

In this section we show how to complete the metric (and the minimal annuli) constructed near the strictly stable 2-sphere in the previous section. This will give Theorem 0.1, which is significantly easier to prove than Theorem 0.3 since we do not require any curvature control.

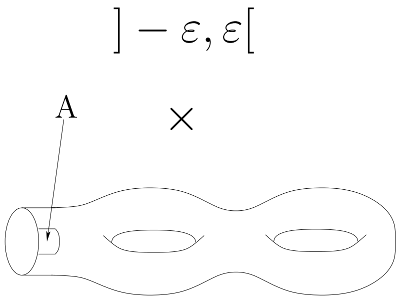

Figure 5. Metric on the product of an interval with a genus $\mathbf{g}$ surface with a cylindrical end

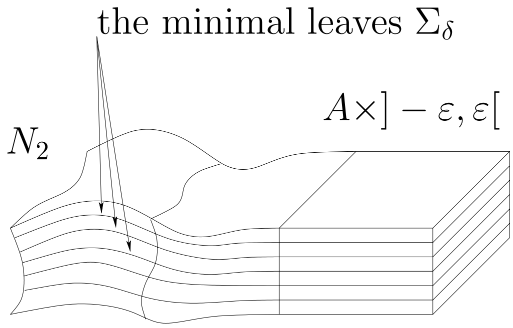

Figure 6. Gluing together the minimal foliation of N_2 and the minimal foliation of N_1

Proof. (Rough sketch of Theorem 0.1). Let $\Sigma_{\mathbf{g}}\setminus\{p\}$ be a punctured surface of genus $\mathbf{g}$ equipped with a metric which near the puncture p is isometric to a flat cylinder. Let N_1 be the metric product $(\Sigma_{\mathbf{g}}\setminus U)\times\,]-\varepsilon,\varepsilon[$ for some sufficiently small ε; see fig. 5. Then N_1 is foliated by the minimal surfaces $(\Sigma_{\mathbf{g}}\setminus U)\times\{t\}$. Let Σ_k, Σ_∞, $\Sigma_{-\infty}$ be the surfaces constructed in Proposition 1.9. In particular we can assume that they are lifting of the geodesics γ_k, γ_∞, $\gamma_{-\infty}$ of Lemma 1.11.

Let $N_2=T_\nu(\Sigma_\infty)$ for some sufficiently small $\nu>0$. By Lemma 1.11 we can assume that part of N_2 has a smooth minimal foliation of the form $\left\{\mathbf{S}^1\times\,]-\delta,\delta[\,\times\{t\}\right\}_{t\in]-\varepsilon,\varepsilon[}$ where $\mathbf{S}^1\times\,]-\delta,\delta[\,\times\{0\}\subset\Sigma_\infty$ and, for a sequence σ_k, $\mathbf{S}^1\times\,]-\delta,\delta[\,\times\{\sigma_k\}\subset\Sigma_k$. The idea is now to glue N_1 together with N_2 along these two foliations while keeping the leaves minimal; see fig. 6. (In Lemma 2.3 below we will show how to do the gluing.) On the other side of $\mathbf{S}^2\times\{0\}$ we complete the metric in the same way except for this time letting the punctured surface have genus 0. This gives the desired embedded minimal surfaces and the limit lamination in a manifold with boundary which is topologically $\Sigma_{\mathbf{g}}\times\,]0,1[$. Since $\Sigma_{\mathbf{g}}\times\,]0,1[$ can be topologically embedded into $\mathbf{R}^3$ it is now easy to see that the metric can be completed to a metric on the given M with the desired property. $\square$

To make the construction outlined above precise we will need the following two lemmas:

Lemma 2.1. Let f be a smooth function on M^3 with 0 as a regular value and let $\Sigma_r = \{f = r\}$ be the level sets of f. In a tubular neighborhood of Σ_0 the metric can be written as

$$g = k^2(r,\theta)\, dr^2 + h(r,\theta) \tag{2.2}$$

where $f(r,\theta) = r$ and $h(r,\cdot)$ is the metric on Σ_r. Moreover, the level sets of f are minimal if and only if $\partial_r \det(h) = 0$.

Proof. The surface Σ_r is minimal if and only if $\mathrm{div}_{\Sigma_r}(\nabla f) = 0$. An easy computation shows that $2\det(h)\,\mathrm{div}_{\Sigma_r}(\nabla f) = k\,\partial_r \det(h)$ and hence gives the claim. $\square$

The next lemma shows that we can deform any metric on a product with a minimal foliation into the product metric with the product foliation, while keeping the leaves minimal; see fig. 6.

Lemma 2.3. Let g be a smooth metric of the form (2.2) on $\mathbf{S}^1\times\,]0,1+\varepsilon[\,\times\,]0,1[$ for some $\varepsilon > 0$ and assume that every slice $\mathbf{S}^1\times\,]0,1+\varepsilon[\,\times\{t\}$ is minimal. Then there is a smooth metric $\tilde g$ on $\mathbf{S}^1\times\,]0,3[\,\times\,]0,1[$ coinciding with g on $\mathbf{S}^1\times\,]0,1[\,\times\,]0,1[$ and with the product metric on $\mathbf{S}^1\times\,]2,3[\,\times\,]0,1[$ and such that every slice $\mathbf{S}^1\times\,]0,3[\,\times\{t\}$ is minimal.

Proof. By Lemma 2.1 it is sufficient to find a smooth positive function $\tilde k$ and a smooth family of 2-dimensional metrics $\tilde h(r,\cdot)$ (both functions of $(r,x,\theta)\in\mathbf{S}^1\times\,]0,3[\,\times\,]0,1[$) with $\partial_r \det(\tilde h) = 0$ and

- $\tilde k = k$, $\tilde h = h$ on $\mathbf{S}^1\times\,]0,1[\,\times\,]0,1[$;
- $\tilde k = \tilde h_{\theta\theta} = \tilde h_{xx} = 1$, $\tilde h_{x\theta} = 0$ on $\mathbf{S}^1\times\,]2,3[\,\times\,]0,1[$.

(Here and in what follows $h_{\theta\theta}$, h_{xx}, and $h_{x\theta} = h_{\theta x}$ denote the components of the metric tensor h in the coordinates (θ, x); the same convention is adopted for any other tensor.) The requirements on $\tilde k$ are trivial to satisfy; so we only need to construct $\tilde h$. To do that let $\eta :]0,3[\,\to\,]0,5/4[$ be a smooth function with

$$\eta(x) = \begin{cases} x & \text{for } x \in\,]0,1/2[\,, \\ 3/4 & \text{for } x\in\,]1,3[\,, \end{cases} \tag{2.4}$$

and set

$$h^{(1)}(\theta,x,r) = h(\theta,\eta(x),r) \quad \text{for } (\theta,x,r)\in\mathbf{S}^1\times\,]0,3[\,\times\,]0,1[\,. \tag{2.5}$$

Since $\det(h^{(1)}(\theta,x,r)) = \det(h(\theta,\eta(x),r))$ clearly $\det(h^{(1)}(\theta,x,r))$ is constant in r. Next choose a smooth function $\varphi :]0,3[\,\to [0,1]$ with

$$\varphi = \begin{cases} 1 & \text{on }]0,1[\,, \\ 0 & \text{on }]3/2,3[\,. \end{cases} \tag{2.6}$$

Set $h^{(2)}_{\theta\theta} = h^{(1)}_{\theta\theta}$ and

$$h^{(2)}_{x\theta}(\theta, x, r) = \varphi(x)\, h^{(1)}_{x\theta}(\theta, x, r)\,, \qquad h^{(2)}_{xx} = h^{(1)}_{xx} + \frac{(1-\varphi^2(x))\left[h^{(1)}_{x\theta}(\theta, x, r)\right]^2}{h^{(1)}_{\theta\theta}(\theta, x, r)}\,. \tag{2.7}$$

One easily checks that $h^{(2)}(\cdot, r)$ is a metric for all r and that $\det(h^{(2)})$ coincides everywhere with $\det(h^{(1)})$. Thus also $\det(h^{(2)})$ is constant in r. Note that for $x \in]3/2, 3[$ the metric $h^{(2)}$ is of the form

$$\begin{pmatrix} h^{(2)}_{\theta\theta}(\theta, x, r) & 0 \\ 0 & h^{(2)}_{xx}(\theta, x, r) \end{pmatrix}. \tag{2.8}$$

Now let $\Phi :]0, 3[\times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ be a smooth function with

$$\Phi(x, u, v) = \begin{cases} u & \text{for } x \in]0, 3/2[\,, \\ (uv)^{-1/2} & \text{for } x \in]2, 3[\,. \end{cases} \tag{2.9}$$

Set $h^{(3)}_{x\theta} = h^{(2)}_{x\theta}$ and

$$h^{(3)}_{\theta\theta}(\theta, x, r) \;=\; \Phi\left(x,\, h^{(2)}_{\theta\theta}(\theta, x, r),\, h^{(2)}_{xx}(r, x, \theta)\right), \tag{2.10}$$

$$h^{(3)}_{xx}(\theta, x, r) \;=\; \frac{h^{(2)}_{\theta\theta}(\theta, x, r)\, h^{(2)}_{xx}(r, x, \theta)}{h^{(3)}_{\theta\theta}(\theta, x, r)}\,. \tag{2.11}$$

Since Φ takes values in $\mathbf{R}^+$, $h^{(3)}$ is a well defined smooth metric. Moreover, we have the identity $h^{(3)}_{\theta\theta} h^{(3)}_{xx} = h^{(2)}_{\theta\theta} h^{(2)}_{xx}$ everywhere. Since $h^{(3)}$ coincides with $h^{(2)}$ for $x \in]0, 3/2]$ and $h^{(2)}$ is of the form (2.8) for $x \in]3/2, 3[$, this yields that $\det(h^{(3)}) = \det(h^{(2)})$ everywhere. Note that for $x \in]2, 3[$ we have $h^{(3)}_{\theta\theta} = h^{(3)}_{xx}$. Moreover, $\partial_r \det(h^{(3)}) = 0$ and hence $\partial_r h^{(3)}_{\theta\theta}(x, \theta, r) = 0$ for $x \in]2, 3[$. Thus $h^{(3)}$ is of the form

$$\begin{pmatrix} \overline{h}(\theta, x) & 0 \\ 0 & \overline{h}(\theta, x) \end{pmatrix}. \tag{2.12}$$

Clearly we can modify $\overline{h}$ for $x \in]5/2, 3[$ keeping it as above for $x \in]2, 5/2]$, positive and smooth on the whole $]2, 3[$ and forcing it to be identically 1 in a neighborhood of $x = 3$. This yields the desired metric. □

Proof. (of Theorem 0.1). Using Lemma 2.3 we can now easily carry out the gluing outlined in the rough sketch of Theorem 0.1 above. □

3. Connected sum construction; proof of Theorem 0.2

We prove Theorem 0.2 by using a connected sum construction. When M carries a metric with positive scalar curvature this gives a metric on M with positive scalar curvature and the desired degenerating sequence of minimal surfaces. For general metrics on general M this gives a different proof of Theorem 0.1.

The connected sum is done using in part arguments of [GrLa] and [ScYa]. We use the low–tech argument of Gromov and Lawson to construct an explicit neck connecting two domains in a round 3-sphere. (This explicit construction is used when we glue together minimal surfaces.) We also use a more high–tech argument of Schoen and Yau to show that such a metric exists on any 3-manifold which carries a metric with positive scalar curvature. (The result of Schoen and Yau that we use says that if a 3-manifold carries a metric of positive scalar curvature, then the punctured manifold (punctured at a point) has a metric with Scal > 0 and a cylindrical end.)

Consider again a warped product metric on $\mathbf{S}^2 \times \mathbf{R}$ of the form (1.1) where $\lambda = \lambda(r)$ is given by

$$\lambda(r) = \begin{cases} -\sin r & \text{for } r \in [-\pi, -\varepsilon[\,, \\ \lambda_{GL}(r) & \text{for } r \in [-\varepsilon, \varepsilon]\,, \\ \sin r & \text{for } r \in]\varepsilon, \pi]\,. \end{cases} \tag{3.1}$$

Note that the resulting metric is a metric on the 3-sphere that is metrically the connected sum of two round unit metrics on the 3-sphere by a neck given by the function λ_{GL}.

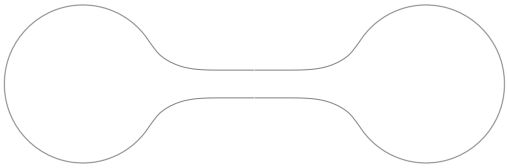

FIGURE 7. The connected sum of two round $\mathbf{S}^3$'s. The resulting metric has positive scalar curvature

By section 5 of [GrLa] (see also [ScYa]) λ_{GL} can be chosen so that the connected sum still has positive scalar curvature for all $\varepsilon > 0$; see fig. 7. (For completeness we show in Appendix A how to choose λ_{GL} so that the scalar curvature of the warped product is positive.) Call x and y the two points in the two copies of $\mathbf{S}^3$ about where we do the connected sum.

Suppose next that we have two one parameter families of minimal surfaces (one in each copy of $\mathbf{S}^3$). Suppose that one of these families goes through x and the other goes through y and so that near x, respectively, y the families of minimal surfaces are foliations by great 2-spheres. We show in Lemma 3.2 below that when we take the connected sum of the two $\mathbf{S}^3$'s by a neck as above, then we can glue the minimal surfaces in one of the two 3-spheres together with the minimal surfaces in the other 3-sphere keeping the surfaces minimal through the neck. In Lemma 3.13 below we then show that we can find a metric on $\mathbf{S}^3$ with positive scalar curvature and with a family of embedded minimal tori going through a point as a foliation by great spheres on a round unit $\mathbf{S}^3$. Taking the connected sum of $\mathbf{g}$ copies of this metric on $\mathbf{S}^3$ with the metric on $\mathbf{S}^3$ and minimal spheres constructed in Proposition 1.13 will then prove Theorem 0.2 when $M = \mathbf{S}^3$. Finally, taking the connected sum (using theorem 4 of [ScYa]) with a general M^3 we get Theorem 0.2.

In Lemma 3.2 we are able to glue only foliations which are parallel (see Definition 1.12).

Lemma 3.2. Let Ω_1, Ω_2 be two open subsets of the round unit $\mathbf{S}^3$ with $x_i \in \Omega_i$. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are foliations by great spheres parallel at x_1 and x_2 respectively. In $\Omega_1 \# \Omega_2$ we can connect the central leaves and the ones nearby keeping them minimal and Scal > 0.

Proof. Let (ϕ, θ, r) be spherical coordinates on $\mathbf{S}^3$ centered at $x_1 \in \Omega_1$. The standard metric is $dr^2 + \sin^2 r\,(d\phi^2 + \sin^2\phi\, d\theta^2)$. Endow the square $[0,\pi] \times [0,\pi]$ with the degenerate metric $\sin^2 r \sin^2\phi\,(dr^2 + \sin^2 r\, d\phi^2)$. Clearly the geodesics passing through $(\pi/2, \pi/2)$ lift to great

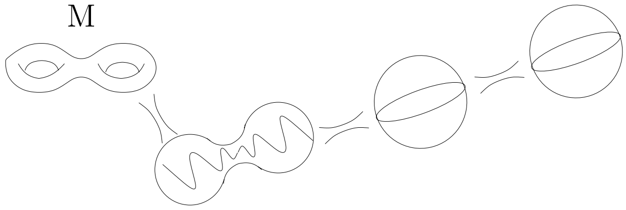

FIGURE 8. Connected sum of one–parameter families of tori in **g** 3–spheres with the desired degeneration and M

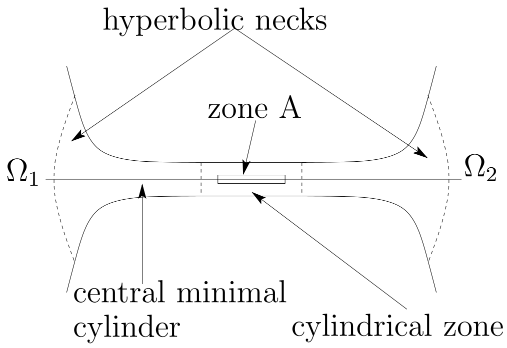

FIGURE 9. The connected sum construction

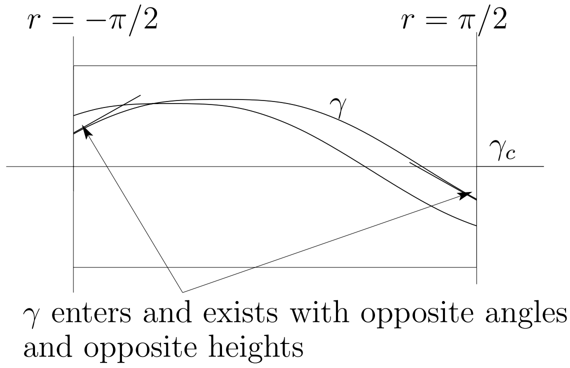

FIGURE 10. The rectangle Ap (corresponding to zone A). The metric (3.7) is the standard metric on $\mathbf{S}^2$

spheres parallel at x_1, with $\{\phi = \pi/2\}$ the central leaf containing x_1. We do the same at $x_2 \in \Omega_2$.

The construction outlined in Appendix A shows that we can replace the balls $B_\varepsilon(x_1) \subset \Omega_1$ and $B_\varepsilon(x_2) \subset \Omega_2$ with two hyperbolic necks and then connect the two necks with a cylinder $\mathbf{S}^2 \times]-K, K[$. More precisely, the construction gives a metric on $\mathbf{S}^2 \times]-K_1 - \varepsilon, K_1 + \varepsilon[$ of the form

$$dr^2 + \lambda^2(r)\,(d\phi^2 + \sin^2\phi\, d\theta^2) \tag{3.3}$$

where, see fig. 9,

- $\lambda(r) = \lambda(-r)$ and $\lambda'(r) \geq 0$ for $r \geq 0$;
- $\lambda(r) = \sin(r - (K_1 - \varepsilon))$ for $r \in [K_1, K_1 + \varepsilon]$;
- λ has a hyperbolic behavior on $[K, K_1]$ and is constant R on $[0, K]$.

Note that we can make the cylindrical tube as long as we want (in particular we can assume that $K > \pi R/2$). The coordinates (ϕ, θ) have been chosen in such a way that the leaves of the minimal foliations are lifting of two families of geodesic segments in $[0, \pi] \times]-K_1 - \varepsilon, K_1 + \varepsilon[$ with the corresponding degenerate metric (1.3). We can continue our geodesic segments throughout the whole strip $[-\pi, \pi] \times]-K_1 - \varepsilon, K_1 + \varepsilon[$. They do not hit the boundary lines $\{\phi = 0\}$, $\{\phi = \pi\}$ and they give two one–parameter families of geodesics $\mathcal{G}_1$ and $\mathcal{G}_2$, which lift to mininal surfaces in the metric (3.3). These minimal surfaces are all cylinders: Their boundaries are two circles lying on $\mathbf{S}^2 \times \{K_1 + \varepsilon\}$ and $\mathbf{S}^2 \times \{-K_1 - \varepsilon\}$.

Note the following:

(i) The two central leaves are lifting of the "central" geodesic $\gamma_c = \{\phi = \pi/2\}$, hence they naturally connect.
(ii) Both $\mathcal{G}_i$ are symmetric around γ_c, i.e., if $\gamma = \{(\phi(t), r(t))\}_{t\in]a,b[}$ lies in $\mathcal{G}_i$, then so does $\{(\pi - \phi(t), r(t))\}_{t\in]a,b[}$.
(iii) If $\{(\phi(t), r(t))\}_{t\in]a,b[}$ lies in $\mathcal{G}_1$, then $\{(\phi(t), -r(t))\}_{t\in]a,b[}$ lies in $\mathcal{G}_2$.

Together (ii) and (iii) give

$$\text{if } \{(\phi(t), r(t))\}_{t\in]a,b[} \text{ lies in } \mathcal{G}_1, \text{ then } \{(\pi - \phi(t), -r(t))\}_{t\in]a,b[} \text{ lies in } \mathcal{G}_2 \,. \tag{3.4}$$

For $\varepsilon > 0$ sufficiently small we can modify the metric in $\mathbf{S}^2 \times [-K, K]$ so that:

(a) It has $\mathrm{Scal} > 0$ and is of the form
$$k^2(r,\phi)\, dr^2 + R^2\, (d\phi^2 + g^2(r,\phi)\, d\theta^2)\,. \tag{3.5}$$
(b) $k(r,\phi) = k(-r,\phi) = k(r, \pi - \phi)$ and the same is true for g.
(c) In an ε–neighborhood of $\{\phi = \pi/2, r \in [-\pi R/2, \pi R/2]\}$ ("zone A" in fig. 9) the metric is
$$\sin^2\phi\, dr^2 + R^2\, (d\phi^2 + d\theta^2)\,. \tag{3.6}$$
(d) The cylinder $\{\phi = \pi/2\}$ remains a minimal surface.

That this modification is possible can be shown in the same way as Lemma 3.13 (cf. the second step of the proof). We give the details of this at the end. We first show how in the new metric the two foliations connect nearby $\{\phi = \pi/2\}$.

By (a) we can apply the discussion of Section 1. The families $\mathcal{G}_1$ and $\mathcal{G}_2$ become two new families of curves $\mathcal{G}_1'$ and $\mathcal{G}_2'$, which are geodesics in the modified metric

$$R^2\, g^2(\phi, r)\, (k^2(\phi, r)\, dr^2 + R^2\, d\phi^2)\,. \tag{3.7}$$

$\mathcal{G}_1'$ coincides with $\mathcal{G}_1$ for $r > K$, whereas $\mathcal{G}_2'$ coincides with $\mathcal{G}_2$ for $r < -K$. Moreover, γ_c is a geodesic also for (3.7) and lies in both $\mathcal{G}_1'$ and $\mathcal{G}_2'$. By continuity of the dependence on the initial data, all the curves of $\mathcal{G}_1'$ which in $\{r > K_1\}$ start sufficiently near γ_c never leave its ε–neighborhood (and are all graphs of functions of r).

In the rectangle $Ap =]\pi/2 - \varepsilon, \pi/2 + \varepsilon[\times] - \pi R/2, \pi R/2[$ the metric (3.7) is given by

$$R^2 \sin^2\phi\, dr^2 + R^4\, d\phi^2\,. \tag{3.8}$$

Note that (3.8) is the metric on the round 2–sphere of radius R^2 and $\gamma_c \cap Ap$ is half of a great circle.

Now take a $\gamma \in \mathcal{G}_1'$ which intersects $\{r = \pi R/2\}$ transversally and leaves Ap crossing $\{r = -\pi R/2\}$. Also $\gamma \cap Ap$ is half of a great circle. It is easy to check that the crossings of γ with $\{r = \pi R/2\}$ and $\{r = -\pi R/2\}$ are two antipodal points. Thus if γ crosses $\{r = \pi R/2\}$ at $\phi = \phi_0$ with angle δ, it crosses $\{r = -\pi R/2\}$ at $\phi = \pi - \phi_0$ with angle $-\delta$ (see fig. 10). By (b), the families $\mathcal{G}_1'$ and $\mathcal{G}_2'$ satisfy condition (3.4). Thus there is a geodesic in $\mathcal{G}_2'$ which crosses $\{r = -\pi R/2\}$ at $\phi = \pi - \phi_0$ with angle $-\delta$. This geodesic connects with γ.

<u>The modified metric</u>: We complete the proof by showing how to construct the modified metric. Straightforward computations give that for a metric of the form (3.5)

$$\mathrm{Scal} = -\left(\frac{k_{\phi\phi}}{kR^2} + \frac{g_{\phi\phi}}{gR^2} \right) - \left(\frac{g_{rr}}{gk^2} + \frac{g_\phi k_\phi}{gkR^2} - \frac{g_r k_r}{gk^3} \right). \tag{3.9}$$

Fix a bump function $\varphi : [0, K] \to [0, 1]$ which is 0 in a neighborhood of K and is 1 in a neighborhood of $[0, \pi R/2]$. Let C be a constant such that $|\varphi'|, |\varphi''| \leq C$.

It is easy to check that for any $\varepsilon > 0$ we can find functions $\tilde{g}, \tilde{k} : [\pi/2, \pi] \to [0, 1]$ such that

(α) $\tilde{k}(\phi) = \sin\phi$ in a neighborhood I of $\pi/2$ and is 1 outside another neighborhood.
(β) $\tilde{g}(\phi) = \sin\phi$ *outside* I and is 1 in a smaller neighborhood of $\pi/2$.
(γ) $|\tilde{g} - 1| \leq \varepsilon$ where $\tilde{g}$ differs from sine; $|\tilde{k} - 1|, |\tilde{k}'|, |\tilde{g}'| \leq \varepsilon$ and $\tilde{k}'', \tilde{g}'' \leq \varepsilon$ everywhere.

The functions k and g are then given by

$$g(\phi, r) = \varphi(r)\, \tilde{g}(\phi) + (1 - \varphi(r)) \sin\phi, \qquad k(\phi, r) = \varphi(r)\, \tilde{k}(\phi) + (1 - \varphi(r)) \tag{3.10}$$

on $[\pi/2, \pi] \times [0, K]$ and we extend them by symmetry to $[0, \pi] \times [-K, K]$. The resulting metric is smooth and coincides with the product outside a neighborhood of $\{\phi = \pi/2, r \in [-\pi R/2, \pi R/2]\}$ in $\mathbf{S}^2 \times [-K, K]$. Clearly, k and g satisfy (b), (c), and, by Lemma 2.2, (d).

To complete the proof we need to show that the scalar curvature is positive where the metric differs from the standard product. It is easy to check that $|\partial_r k|, \partial_\phi k|, |\partial_r g|, |\partial_\phi g|, |\partial_{rr} k| \leq C\varepsilon$. Thus, for ε small,

$$\left| \frac{g_{rr}}{gk^2} + \frac{g_\phi k_\phi}{gkR^2} - \frac{g_r k_r}{gk^3} \right| \leq 2C\varepsilon(1 + R^{-4} + R^{-2}). \tag{3.11}$$

Moreover, if ε is sufficiently small, (α), (β), and the inequalities $\tilde{k}'', \tilde{g}'' \leq \varepsilon$ give that

$$-\frac{g_{\phi\phi}}{gR^2} - \frac{k_{\phi\phi}}{kR^2} \geq \frac{1}{2R^2}. \tag{3.12}$$

Since ε can be chosen arbitrarily this completes the proof. $\square$

On $\mathbf{S}^2 \times \mathbf{S}^1$ with the product metric, the great circles on $\mathbf{S}^2$ times $\mathbf{S}^1$ give a one parameter family of minimal (intrinsically flat) tori. The next lemma shows that we can deform this example into a one parameter family of embedded minimal tori on $\mathbf{S}^3$ with a metric with positive scalar curvature and so that in a neighborhood of some point the metric has constant sectional curvature 1 and the tori pass through as parallel great 2-spheres. (The proof of this lemma is postponed to Appendix B.)

Lemma 3.13. On $\mathbf{S}^3$ there exists a metric with Scal > 0 and a family of minimal tori $\{T_\delta\}_{\delta \in]-1,1[}$ such that in a neighborhood of two antipodal points x and y the metric coincides with the round unit metric and $\{T_\delta\}$ with a foliation by great 2–spheres parallel at x and y.

Proof. (of Theorem 0.2). Let M_{tor} be the metric on $\mathbf{S}^3$ given by Lemma 3.13 and let M_{sing} be the metric on $\mathbf{S}^3$ given by Proposition 1.13. By Lemma 3.2 $\#_{i=1}^{\mathbf{g}} M_{\mathrm{tor}} \# M_{\mathrm{sing}}$ gives a metric on $\mathbf{S}^3$ and a sequence of embedded minimal surfaces of genus $\mathbf{g}$ with the desired properties. (Here all necks are attached at points where the sectional curvatures are constant.) By theorem 4 of Schoen-Yau, [ScYa], there exists a metric on M^3 with positive scalar curvature and a cylindrical end. Connecting this metric with the metric on the 3-sphere constructed above completes the proof. (The last neck connects the cylindrical end with an open set of the 3-sphere where the sectional curvatures are constant.) □

4. Metrics on $\mathbf{S}^2 \times \mathbf{S}^1$

Proof. (of Theorem 0.5). This is essentially proven in [HaNoRu] although not recorded there. Namely, similarly to Proposition 1.9 consider the degenerate metric

$$ds^2 = \sin^2\phi\,(dr^2 + d\phi^2) \tag{4.1}$$

on the cylinder $[0,\pi] \times \mathbf{S}^1$. Geodesics in this metric lift to minimal surfaces on the product $\mathbf{S}^2 \times \mathbf{S}^1$ and simple closed geodesics lift to embedded minimal tori. By lemma 2.1 of [HaNoRu] geodesics in (4.1) are periodic (in r) and as the angle that they make with the geodesic $\{r = 0\}$ goes to zero the period in r goes to zero. Moreover, it follows easily that the period is continuous as a function of the angle. Combining these facts is easily seen to give that there are simple closed geodesics on the cylinder with arbitrarily small period in r and that these converge to the foliation of the cylinder by the parallel geodesics $\{r = \text{ constant}\}$. Lifting these simple closed geodesics to $\mathbf{S}^2 \times \mathbf{S}^1$ gives the desired sequence of embedded minimal tori. □

Remark 4.2. Arbitrary close to the product metric on $\mathbf{S}^2 \times \mathbf{S}^1$ we can also find a metric which has a sequence of embedded minimal tori converging to a singular lamination of the type of Theorem 0.1. Indeed we choose on $\mathbf{S}^2 \times \mathbf{R}$ a metric of the form (1.1) where $\lambda(r)$ is symmetric, equal to $\cosh(\varepsilon r)$ for $r \in]-1,1[$ and constant on $]-\infty,2] \cup [2,\infty[$. Consider on the strip $[0,\pi] \times \mathbf{R}$ the degenerate metric whose geodesics lift to minimal surfaces on $\mathbf{S}^2 \times \mathbf{R}$. By Lemma 1.11 for any given $r_0 > 2$ there is a sequence of geodesics γ_i which all pass through $(\pi/2, r_0)$, $(\pi/2, 0)$ and $(\pi/2, -r_0)$ and which converges to a lamination consisting of $\{r = 0\}$ and two infinite geodesics γ_∞ and $\gamma_{-\infty}$ spiraling into it. We now identify the lines $\{r = r_0\}$ with $\{r = -r_0\}$ on the strip and the spheres $\{r = r_0\}$ and $\{r = -r_0\}$ in $\mathbf{S}^2 \times \mathbf{R}$. Thus we obtain a smooth metric on $\mathbf{S}^2 \times \mathbf{S}^1$ and a degenerate metric on $[0,\pi] \times \mathbf{S}^1$ whose geodesics lift to minimal surfaces in $\mathbf{S}^2 \times \mathbf{S}^1$. Because of the symmetry of our construction the geodesics γ_i generate simple closed geodesics in $[0,\pi] \times \mathbf{S}^1$ and γ_∞ and $\gamma_{-\infty}$ smoothly glue themselves forming an infinite geodesic spiraling into $\{r = 0\}$ from both sides. These geodesics lift to the desired minimal surfaces in $\mathbf{S}^2 \times \mathbf{S}^1$.

5. More than one strictly stable 2-sphere with singularities

The proof of Theorem 0.1 easily generalizes to show that for any given integer $n > 0$ we can find a limit lamination which is singular at n pairs of points, where the pairs of points lie on n disjoint strictly stable 2-spheres. That is:

Theorem 5.1. On any 3-manifold, M^3, and for any nonnegative integer $\mathbf{g}$, and any positive integer n there exists a metric on M and a sequence of embedded minimal surfaces of genus $\mathbf{g}$ with Morse index going to infinity and which converges to a singular (minimal) lamination $\mathcal{L}$. This can be done so that the singular set of $\mathcal{L}$ consists of pairs of points lying on n leaves which are strictly stable 2-spheres.

Proof. Let M_δ be a δ-tubular neighborhood of the strictly stable 2-sphere in the metric on $\mathbf{S}^2 \times \mathbf{R}$ given by Proposition 1.9. Using Lemma 2.3 glue n copies of M_δ together along the minimal leaves of the foliation near the boundary coming from the Σ_i's and Σ_∞ while keeping the leaves minimal. The desired metric can now be obtained by completing this metric to a metric on M using Lemma 2.3 as in the proof of Theorem 0.1. □

Likewise we can easily generalize Theorem 0.3 to:

Theorem 5.2. If M^3, $\mathbf{g}$ are as in Theorem 0.3, and n is a positive integer, then there is a metric with $\mathrm{Scal}_M > 0$ which has a singular lamination and a sequence of embedded minimal surfaces of genus $\mathbf{g}$ as in Theorem 5.1.

Proof. We use the same ideas of the proof of Proposition 1.13 to glue n hyperbolic necks and 2 halves of standard $\mathbf{S}^3$. Thus we produce a metric on $\mathbf{S}^3$ with positive scalar curvature and with a sequence of embedded minimal spheres which converge to a singular lamination containing n strictly stable 2–spheres and which pass through two points as parallel great spheres. We can use the connected sum construction of Section 3 to complete the proof. □

6. The space of minimal annuli limiting a strictly stable 2-sphere

Recall the following theorem from [CH2] (here T_1M is the unit tangent bundle):

Theorem 6.1. [CH2]. Let M^2 be an orientable surface and $\gamma \subset M$ be a simple closed and strictly stable geodesic. Then there are four "circles" of noncompact geodesics limiting on γ. That is, on each side of γ in M, and for each orientation of γ there is a C^1 map $\mathbf{S}^1 \to T_1M$ which gives a bijection between the circle $\mathbf{S}^1$ and the set of geodesics ℓ with $\overline{\cap_{t>0}\ell|[t,\infty[} = \gamma$ which limit on γ from the given side of M with the given orientation.

Motivated by this theorem one is tempted to ask:

Question 1: Let M^3 be an orientable 3-manifold and $\Gamma \subset M$ a strictly stable embedded 2-sphere. Does there exist a map from the space of noncompact embedded minimal annuli in T limiting Γ and into $\mathbf{S}^2 \times \mathbf{S}^1 \times \mathbf{Z}/2\mathbf{Z}$?

A particular case of the reverse of this question is:

Proposition 6.2. Let $M = \mathbf{S}^2 \times \mathbf{R}$ with a metric (1.1) where $\lambda'(0) = 0$, and $\lambda''(0) > 0$. In a neighborhood of the strictly stable 2-sphere $\Gamma = \mathbf{S}^2 \times \{0\}$ there are at least two "2-spheres" of $\mathbf{S}^1$ invariant noncompact minimal annuli limiting on Γ. That is, on each side of Γ in M, there is a continuous map from $\mathbf{S}^2 \times \mathbf{S}^1$ to the set of minimal annuli Σ which are the preimages of geodesics σ in (1.3) with $\overline{\cap_{t>0}\sigma|[t,\infty[} = \{r = 0\}$ from the given side.

Proof. For each $x \in \mathbf{S}^2$ we can use spherical coordinates (ϕ, θ) centered at x and consider the corresponding degenerate metric (1.3) on the strip. By Lemma 1.11 for every $\rho > 0$ we can find a geodesic passing through $(\pi/2, \rho)$ with a given fixed orientation and spiraling into $\{r = 0\}$. This gives a circle worth of annuli. Varying x gives the claim. □

A weaker question is:

Question 2: Let M^3, Γ be as in Question 1. What is the tangent space of the set of all noncompact embedded minimal annuli limiting Γ? In particular, for each such minimal annulus, what can be said about the dimension of the space of Jacobi fields that come from a variation of such annuli?

In view of [CH2] and [CM3] it seems plausible that if the answer to Question 1 is yes, then the following should be the case:

Question 3: Let M^3 be an orientable 3-manifold with a generic metric with positive scalar curvature. Is every singular minimal lamination, which is the limit of a sequence of embedded minimal surfaces of a given fixed genus, singular along at most one strictly stable 2-sphere (which is a leaf of the lamination)?

Appendix A. The connecting neck

For completeness we show here how to connect sum two round unit $\mathbf{S}^3$'s by a thin neck so that the resulting metric has positive scalar curvature everywhere (cf. section 5 of [GrLa]). Fix one of the spheres and a point x on it. Choose spherical coordinates centered at x. In these coordinates the metric is given by $dr^2 + \sin^2 r\,(d\phi^2 + \sin^2\phi\, d\theta^2)$. Starting from $r = \varepsilon$ we will replace $\sin r$ with a function λ and modify the metric as $dr^2 + \lambda^2(r)\,(d\phi^2 + \sin^2\phi\, d\theta^2)$. Shift the coordinate r so that the replacement of sin (and hence the neck) starts at $\{r = 0\}$: thus our function λ is given by $\sin(\varepsilon + r)$ on $\{r > 0\}$. Hence, $\lambda(0) = \sin\varepsilon$ and $\lambda'(0) = \cos\varepsilon$. Our goal is to continue λ in $C^{1,1}$, while keeping $\mathrm{Scal} > 0$ and reaching $\lambda'(-K) = 0$ for some $K > 0$ (keeping λ positive, so the metric is not degenerate). For $r < -K$ let λ be constant: hence our metric turns out to be the product of a half–line with the round 2–sphere of radius $\lambda(-K)$.

We make the same construction for the other unit sphere and then glue the two cylindrical parts. This gives a $C^{1,1}$ metric which can be smoothed in a standard way to a metric with $\mathrm{Scal} > 0$.

We will construct λ and $-K$ so to have $\lambda'' \geq 0$ on $[-K, 0]$. Thus

$$0 < \lambda'(r) \leq \lambda'(0) = \cos\varepsilon \quad \text{for } r \in]-K, 0]\ , \tag{A.1}$$

$$\lambda \text{ is invertible on }]-K, 0] \text{ with } \lambda^{-1} = \alpha\ . \tag{A.2}$$

By (A.1), $(\lambda')^2 \leq (1-\eta)$ on $[-K, 0]$ for some $\eta > 0$. Constructing λ in this way we will have by (1.2)

$$\mathrm{Scal}_M \geq -2\frac{\lambda''}{\lambda} + \frac{\eta}{\lambda^2}\,. \tag{A.3}$$

Thus we need to find λ satisfying

$$\frac{\eta}{4\lambda} \geq \lambda'' \geq 0 \quad \text{on } [-K, 0]\,, \quad \lambda'(-K) = 0\,, \quad \lambda'(0) = \cos\varepsilon\,, \quad \lambda(0) = \sin\varepsilon\,. \tag{A.4}$$

To do this we solve backward in time the ODE $\lambda'' = \eta/(4\lambda)$ and prove that there is $K > 0$ large enough so that $\lambda'(-K) = 0$ somewhere and λ is positive on $[-K, 0]$. Indeed set $-K = \inf\{t|\lambda(t) > 0 \text{ and } \lambda'(t) > 0\}$. We claim that if $-K > -\infty$, then $\lambda(-K) > 0$. If not, then we get the contradiction

$$\begin{aligned} \lambda'(0) \quad &\geq \quad \lambda'(0) - \lambda'(-K) = \int_{-K}^{0} \lambda''(t)\, dt = \int_{-K}^{0} \frac{dt}{4\lambda(t)} \\ &\overset{(A.2)}{=} \quad \frac{1}{4}\int_0^{\lambda(0)} \frac{d\tau}{\tau\, \lambda'(\alpha(\tau))} \overset{(A.1)}{\geq} \frac{1}{4\cos\varepsilon}\int_0^{\lambda(0)} \frac{d\tau}{\tau} = \infty\,. \end{aligned} \tag{A.5}$$

Thus, either $-K = -\infty$ or it is finite and $\lambda'(-K) = 0$. In the first case we would have

$$\lambda'(0) \geq \lambda'(0) - \lim_{x\to\infty} \lambda'(x) = \int_{-\infty}^{0} \lambda''(t)dt = \int_{-\infty}^{0} \frac{dt}{4\lambda(t)} \geq \frac{1}{4\lambda(0)} \int_{-\infty}^{0} dt = \infty\,. \tag{A.6}$$

This gives a contradiction; thus $-K > -\infty$ and $\lambda'(-K) = 0$.

Appendix B. Proof of Lemma 3.13

Proof. A metric with Scal > 0 on $\mathbf{S}^3$ containing totally geodesic tori.

We first exhibit a metric on $\mathbf{S}^3$ with positive scalar curvature containing a neighborhood of a totally geodesic torus (given by a great circle times $\mathbf{S}^1$) in $\mathbf{S}^2 \times \mathbf{S}^1$ with the product metric. The induced metric on a tubular neighborhood T of such a totally geodesic torus is

$$dx^2 + \cos^2 x\, dy^2 + dz^2\,, \tag{B.1}$$

where (x, z) are coordinates on the torus. Choose two functions $f, k : [-\pi/2, \pi/2] \to [0, 1]$ and $a \in (0, \pi)$ such that

- both are positive on $]-\pi/2, -\pi/2[$;
- f coincides with cosine on $]-\pi/2, -2a]$ and is 1 on $]-a, \pi/2[$;
- k coincides with cosine on $[-2a, \pi/2]$ and is constant in a neighborhood of $-\pi/2$;
- $f'' \leq 0$ and $k''/k \leq 1/4$.

All these conditions can be satisfied provided a is sufficiently small. We now take $M = [-\pi/2, \pi/2] \times \mathbf{S}^1 \times \mathbf{S}^1$ with the metric $dx^2 + g^2(x)dy^2 + f^2(x)dz^2$. Note that the scalar curvature of this metric is $-k''/k - f''/f$. Define on M the equivalence relation $(-\pi/2, x, y) \approx (-\pi/2, x, z)$ and $(\pi/2, y, x) \approx (\pi/2, z, x)$. $M/\approx$ is obtained by gluing two solid tori along their boundary (exchanging parallels and meridians) and thus it is a 3–sphere. The metric on $M/\approx$ is smooth and has positive scalar curvature.

Deforming parts of minimal tori into parts of great spheres.

The standard metric on $\mathbf{S}^3$ is

$$\cos^2\phi\, \cos^2\theta\, dr^2 + d\phi^2 + \cos^2\phi\, d\theta^2 \quad \left(= \cos^2\phi\,(\cos^2\theta\, dr^2 + d\theta^2) + d\phi^2\right), \tag{B.2}$$

where $\{r = \text{ constant}\}$ give a one parameter families of great spheres parallel in $(0, 0, 0)$ (see Definition 1.12).

The product metric on $\mathbf{S}^2 \times \mathbf{S}^1$ is given by

$$\cos^2\phi\, dr^2 + d\phi^2 + d\theta^2\,. \tag{B.3}$$

(Here (ϕ, r) are spherical coordinates on $\mathbf{S}^2$ and θ is the standard coordinate on $\mathbf{S}^1$.) Note that the level sets $\{r = \text{ constant}\}$ is a one parameter family of totally geodesic tori. We modify (B.3) in a neighborhood of $(0, 0, 0)$ so that in a smaller neighborhood the metric is (B.2), the scalar curvature is everywhere positive and $\{r = c\}$ is a minimal torus (for all c sufficiently small). Notice that we can do the same modification around the point $(0, \pi, 0)$.

We first take care of the term in front of $d\theta^2$. We can find a function k which is 1 outside a neighborhood of $(0, 0, 0)$, coincides with $\cos\phi$ in a smaller neighborhood and does not depend on r if r is sufficiently small. Moreover, for every η we can find such a k so that:

(a) $|k - 1|$ and the norm of all first and second partial derivatives of k but $\partial_{\phi\phi}k$ are less than η.

(b) $\partial_{\phi\phi}k \leq \eta$.

Since for r sufficiently small k does not depend on r, by Lemma 2.1 the leaves $\{r = \text{ constant}\}$ are still minimal in the modified metric (for r small). Moreover, the scalar curvature of the metric $\cos^2\phi\, dr^2 + d\phi^2 + k^2(\phi, r, \theta)\, d\theta^2$ is

$$1 - \frac{\partial_{\phi\phi} k}{k} - \frac{\partial_{rr} k}{k\, \cos^2\phi} - \tan\phi\, \frac{\partial_\phi k}{k}\,. \tag{B.4}$$

Thus k can be chosen so that the scalar curvature remains positive. In a neighborhood of $(0,0,0)$ our new metric is $\cos^2\phi\, dr^2 + d\phi^2 + \cos^2\phi\, d\theta^2$. Similarly, we can further modify the metric in a smaller neighborhood so to adjust the term in front of dr^2.

We conclude the proof by constructing the function k. Take a smooth cut–off function $\varphi :]-\delta, \delta[\to [0,1]$ which is 0 in a neighborhood of $-\delta$ and δ, and 1 in a neighborhood of 0. For some constant $C > 1$ we will have $|\varphi'|, |\varphi''| \leq C$. Next choose a function $\tilde{k} : (-\delta, \delta) \to [0,1]$ equal to 1 in a neighborhood of $-\delta$ and δ, equal to cos in a neighborhood of 0 and such that $|k-1|, |k'| \leq \eta/C$ and $k'' \leq \eta/C$. (This is possible since $\cos(0) = 1$ and $(\cos)'(0) = -\sin(0) = 0$.) Set

$$k(r,\phi,\theta) = [1 - \varphi(\theta)] \,+ \varphi(\theta)\, [(1 - \varphi(r)) \,+\, \varphi(r)\tilde{k}(\phi)]\,. \tag{B.5}$$

Clearly, k is 1 in a neighborhood of the boundary of $[-\delta, \delta]^3$ and does not depend on r if r is sufficiently small. Moreover, $|k-1| \leq \eta/C$ and

$$\partial_r k(r,\phi,\theta) = \varphi(\theta)\, \varphi'(r)\, (\tilde{k}(\phi) - 1)\,. \tag{B.6}$$

Hence $|\partial_r k| \leq |\varphi'||\tilde{k} - 1| \leq \eta$. We argue similarly for all first and second partial derivatives except for $\partial_{\phi\phi} k$. Finally, $\partial_{\phi\phi} k(r,\phi,\theta) = \varphi(\theta)\, \varphi(r)\, \tilde{k}''(\phi) \,\leq \eta$. □

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