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A fully nonlinear conformal flow on
locally conformally flat manifolds

by

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A FULLY NONLINEAR CONFORMAL FLOW ON LOCALLY CONFORMALLY FLAT MANIFOLDS

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ABSTRACT. We study a fully nonlinear flow for conformal metrics. The long-time existence and the sequential convergence of flow will be established for locally conformally flat manifolds. As an application, we solve the σ_k-Yamabe problem for locally conformal flat manifolds when $k \neq n/2$.

1. INTRODUCTION

For a compact, connected smooth Riemannian manifold (M, g) of dimension $n \geq 3$, the Schouten tensor S_g is defined as

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right),$$

where Ric_g and R_g are the Ricci tensor and scalar curvature of g respectively. We are interested in deforming the metric in the conformal class $[g_0]$ of a fixed background metric g_0 to certain extremal metric with respect to the Schouten tensor along some curvature flow.

To introduce the flow, we recall σ_k-*scalar curvature* of g introduced in [22]

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where $g^{-1} \cdot S_g$ is defined locally by $(g^{-1} \cdot S_g)^i_j = g^{ik}(S_g)_{kj}$, and $\sigma_k(A)$ is the kth elementary symmetric function of the eigenvalues of $n \times n$ matrix A. When $k = 1$, σ_1-scalar curvature is just the scalar curvature R (upto a constant multiple). For $k > 1$, it is natural to consider σ_k in the positive cone

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall j \leq k\}.$$

A metric g is said to be in Γ_k^+ if $g^{-1} \cdot S_g(x) \in \Gamma_k^+$, $\quad \forall x \in M$.

We propose the following curvature flow

$$(1) \qquad \begin{cases} \dfrac{d}{dt}g &= -(\log \sigma_k(g) - \log r_k(g)) \cdot g, \\ g(0) &= g_0, \end{cases}$$

where $r_k(g)$ is given by

$$r_k(g) = \exp\left(\frac{1}{vol(g)} \int_M \log \sigma_k(g) \, dvol(g)\right).$$

The main goal of this paper is to prove the following existence and convergence result of flow (1) on locally conformally flat manifolds.

Theorem 1. *Suppose (M, g_0) be a compact, connected and locally conformally flat manifold. Assume that $g_0 \in \Gamma_k^+$ and smooth, then flow (1) exists for all time $0 < t < \infty$ and $g(t) \in C^\infty(M) \quad \forall t$. There exist two positive constants C, β depending only on g_0, k and n (independent of t), such that,*

$$(2) \qquad \|g\|_{C^2(M)} \le C; \quad and \; \forall k \ne n/2, \quad \lim_{t\to\infty} \|\sigma_k(g) - \beta\|_{L^2(M)} = 0,$$

where the norms are taken with respect to the background metric g_0. Furthermore, $\forall k \ne n/2$, for any sequence $t_n \to \infty$, there is a subsequence $\{t_{n_l}\}$ with $g(t_{n_l})$ converging in $C^{1,\alpha}$-norm ($\forall 0 < \alpha < 1$) to some smooth metric g and

$$(3) \qquad\qquad\qquad\qquad \sigma_k(g) = \beta.$$

If $k = 1$, flow (1) is a logarithmic version of the Yamabe flow introduced by Hamilton [12] in connection to the Yamabe problem [25]. The Yamabe problem is a problem of deforming scalar curvature to a constant along the conformal class, the final solution was obtained by Schoen [18] after works of Yamabe [25], Trudinger [21] and Aubin [1]. For the study of the Yamabe flow, we refer to [4] and [26].

The problem of finding constant σ_k-curvature in the conformal class for $k \ge 2$ is a fully nonlinear version of the Yamabe problem. This problem was considered by Viaclovsky in [22]. It is reduced to a nonlinear elliptic equation. The main difficulty in this elliptic approach is the lack of compactness. One way to attack the problem is the blow-up analysis as in [19] to rule out the standard sphere. In the case $k = n$, Viaclovsky obtained a sufficient condition for the solution of this nonlinear problem in [23]. In an important case $n = 4$ and $k = 2$, Chang, Gursky and Yang established the existence of a metric with $\sigma_2 > 0$ everywhere on any compact four-manifold under the positivity assumption on the Yamabe constant and $\int_M \sigma_2$ in [2]. In a subsequent paper [3], they deformed conformally the metric to a metric of constant σ_2-scalar curvature for general 4-dimensional manifolds using a priori estimates and a blow-up analysis. To do the blow-up analysis, one needs local estimates and classification of resulting entire solutions of the corresponding equation on $\mathbf{R}^n$. This type of local estimates for the fully nonlinear version of Yamabe problem has been established by us in [10] for general cases. Flow (1) is designed as a parabolic approach to the problem. As a consequence of our main result, we obtain the solution for the fully nonlinear version of Yamabe problem for locally conformally flat manifolds when $k \ne \frac{n}{2}$ (of course, the $k = 1$ is a result of Schoen [18]).

Corollary 1. *If (M, g_0) is a compact, connected and locally conformally flat manifold and $g_0 \in \Gamma_k^+$, $k \ne \frac{n}{2}$, then there is a smooth metric $g \in \Gamma_k^+$ in $[g_0]$ such that the σ_k-curvature of g is a positive constant.*

Remark 1. *If (M, g_0) is a compact, connected and locally conformally flat manifold with positive Ricci curvature, then (M, g_0) is conformally equivalent to a spherical space form. We note that $g_0 \in \Gamma_k^+$ for $k \geq n - 1$ implies the positivity of Ricci curvature. Hence, Corollary 1 for the cases $k = n - 1$ and $k = n$ is obvious.*

2. PRELIMINARIES

In this section, we collect and derive some properties of σ_k and its functionals.

Let $(\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$. The k-th elementary symmetric function is defined as

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

A real symmetric $n \times n$ matrix A is said to lie in Γ_k^+ if its eigenvalues lie in Γ_k^+.

Let A_{ij} be the $\{ij\}$-entry of an $n \times n$ matrix. Then for $0 \leq k \leq n$, the kth *Newton transformation* associated with A is defined to be

$$T_k(A) = \sigma_k(A)I - \sigma_{k-1}(A)A + \cdots + (-1)^k A^k.$$

We have (see [16])

$$T_k(A)_j^i = \frac{1}{k!} \delta_{j_1 \ldots j_k j}^{i_1 \ldots i_k i} A_{i_1 j_1} \cdots A_{i_k j_k},$$

where $\delta_{j_1 \ldots j_k j}^{i_1 \ldots i_k i}$ is the generalized Kronecker delta symbol. Here we use the summation convention. By definition,

$$\sigma_k(A) = \frac{1}{k!} \delta_{j_1 \ldots j_k}^{i_1 \ldots i_k} A_{i_1 j_1} \cdots A_{i_k j_k}, \quad T_{k-1}(A)_j^i = \frac{\partial \sigma_k(A)}{\partial A_{ij}}.$$

The following properties of σ_k and Γ_k^+ are well-known (e.g., see [7]).

Proposition 1. *We have*

1. *Each set Γ_k^+ is an open convex cone.*
2. *$T_{k-1}(A)$ is positive definite when $A \in \Gamma_k^+$.*
3. *$\log \sigma_k$ and $\sigma_k^{1/k}$ are concave.*

The following are some variational characterizations of $\sigma_k(g)$ (see [22]).

Proposition 2. *If (M,g) is locally conformally flat, then $T_k(S_g)$ is divergence free with respect to the metric g, i.e., for any orthonormal frame, $\forall j$*

$$(4) \qquad \sum_i \nabla_i(T_k(S_g)_j^i) = 0.$$

If $k \neq \frac{n}{2}$, any solution of equation (3) is a critical point of the functional

$$(5) \qquad \mathcal{F}_k(g) = vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) \, dvol(g)$$

in $[g_0]$.

Now we consider some properties of flow (1).

Lemma 1. *The flow (1) preserves the volume. When $k \neq \frac{n}{2}$ and g_0 is locally conformally flat, then*

$$(6) \qquad \frac{d}{dt}\mathcal{F}_k(g) = -\frac{n-2k}{2}vol(g)^{\frac{2k-n}{2}}\int_M (\sigma_k(g) - r_k(g))(\log \sigma_k(g) - \log r_k(g)).$$

Therefore, for $k < n/2$, $\frac{d}{dt}\mathcal{F}_k(g(t)) \leq 0$, and for $k > n/2$, $\frac{d}{dt}\mathcal{F}_k(g(t)) \geq 0$.

Proof: The volume is preserved, as

$$\begin{aligned}
\frac{d}{dt}vol(g) &= \int_M g^{-1}\frac{d}{dt}g \, dvol(g) \\
&= \int_M (\log \sigma_k(g) - \log r_k(g)) \, dvol(g) = 0.
\end{aligned}$$

On any locally conformally flat manifold, from the computation in [22],

$$\frac{d}{dt}\int_M \sigma_k(g) \, dvol(g) = \frac{n-2k}{2}\int_M \sigma_k(g)g^{-1} \cdot \frac{d}{dt}g \, dvol(g).$$

Since flow preserves the volume, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_k(g) &= \frac{n-2k}{2}vol(g)^{\frac{2k-n}{2}}\int_M \sigma_k(g)g^{-1} \cdot \frac{d}{dt}g \, dvol(g) \\
&= \frac{n-2k}{2}vol(g)^{\frac{2k-n}{2}}\int_M (\sigma_k(g) - r_k(g))g^{-1} \cdot \frac{d}{dt}g \, dvol(g) \\
&= -\frac{n-2k}{2}vol(g)^{\frac{2k-n}{2}}\int_M (\sigma_k(g) - r_k(g))(\log \sigma_k(g) - \log r_k(g))dvol(g).
\end{aligned}$$

∎

If $g = e^{-2u} \cdot g_0$, one may compute that (see [22])

$$\sigma_k(g) = e^{2ku}\sigma_k\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right).$$

Equation (1) can be written in the following form

$$(7) \qquad \begin{cases} 2\dfrac{du}{dt} &= \log \sigma_k\left(\nabla^2 u + du \otimes du - \dfrac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) + 2ku - \log r_k \\ u(0) &= u_0. \end{cases}$$

If $g = v^{-2}g_0$, equation (1) is equivalent to

$$(8) \qquad \begin{cases} 2\dfrac{dv}{dt} &= \log \sigma_k\left(\dfrac{\nabla^2 v}{v} - \dfrac{|\nabla v|^2}{2v^2}g_0 + S_{g_0}\right) + 2k\log v - \log r_k \\ v(0) &= v_0. \end{cases}$$

Here all covariant derivatives are taken with respect to the fixed metric g_0.

Since $g_0 \in \Gamma_k^+$, flow (1) is parabolic near $t = 0$, by the standard implicit function theorem we have the following short-time existence result.

Proposition 3. *For any $g_0 \in C^2(M)$ with $\sigma_k(g_0) \in \Gamma_k^+$, there exists a positive constant T^* such that flow (1) exists and is parabolic for $t \in [0, T^*)$, and $\forall T < T^*$,*

$$g \in C^{3,\alpha}([0,T] \times M), \forall 0 < \alpha < 1, \quad and \quad \sigma_k(g(t)) \in \Gamma_k^+.$$

3. A PRIORI ESTIMATES

In this section, we establish some a priori estimates for flow (1). Here the assumption of the locally conformally flatness is used. By the fundamental result of Schoen-Yau on positive mass theorem, if (M, g_0) is not conformally equivalent to the standard sphere, there is a precise asymptotic property for the developing mapping of (M, g_0) into the standard sphere [20]. Schoen applied this property together with Alexandrov's moving plane method developed in [8] to obtain the compactness in [19]. The similar approach was also taken by Ye in [26] to obtain a Harnack type inequality for the Yamabe flow. With some minor modification, the proof in [26] yields the following Harnack type estimates for flow (8) (see also [5] for further refinement of this type of argument for nonlinear parabolic equations).

Proposition 4. *Let (M, g_0) be a locally conformally flat manifold and $g_0 \in \Gamma_k^+$. If $g(t) = v^{-2} g_0$ is a solution of (8) with initial metric g_0 with $\sigma_k(g(t)) \in \Gamma_k^+$ in a time interval $[0, T^*)$, then*

$$(9) \qquad \sup_M \frac{|\nabla_{g_0} v|}{v} \leq C, \quad for \ x \in M, t \in [0, T^*),$$

where $C > 0$ is a constant depending only on g_0, k and n.

Proof: One only needs to consider the case that (M, g_0) is not conformally covered by $\mathbf{S}^n$. By [20], there is a conformal diffeomorphism Φ from the universal cover $\widetilde{M}$ of M onto a dense domain Ω of $\mathbf{S}^n$. The boundary $\partial\Omega$ of Ω is non-empty. Let $\pi : \widetilde{M} \to M$ be the covering map and $\tilde{g} = (\Phi^{-1})^* \pi^* g$ the pull-back metric. Set $\tilde{g} = \tilde{v}^{-2} g_{\mathbf{S}^n}$, where $g_{\mathbf{S}^n}$ is the standard metric of the unit sphere. By the conformal invariance, we have

$$(10) \qquad \begin{cases} 2\dfrac{d\tilde{v}}{dt} &= \tilde{v}^2 \sigma_k^{1/k} \left(\dfrac{\nabla^2 \tilde{v}}{\tilde{v}} - \dfrac{|\nabla \tilde{v}|^2}{2\tilde{v}^2} g_{S^n} + S_{g_{S^n}} \right) - r_k^{1/k}, & \text{in } \Omega, \\ \tilde{v}(0) &= \tilde{v}_0. \end{cases}$$

Here $\nabla \tilde{v}$ and $\nabla^2 \tilde{v}$ are derivatives with respect to the standard metric $g_{\mathbf{S}^n}$. A result of [20] implies

Asymptotic fact: $\forall T < T^*, \tilde{v}^{-1}(x, t) \to \infty$, uniformly $\forall t \leq T$, as $x \to \partial\Omega$.

Ye used the moving plane method of [8] to prove the Harnack inequality for solutions of Yamabe flow. The main ingredients in his proof are:

1. The **asymptotic fact**;
2. Certain growth conditions at ∞ of the transformed function w of $\tilde{v}$ under the stereographic projection from $\mathbf{S}^n$ to $\mathbf{R}^n$;
3. The invariance of the resulting parabolic equation for w in $\mathbf{R}^n$ under reflections and translations.

In fact, the explicit form of the equation for Yamabe flow were not used in the proof of the Harnack inequality in [26]. As all these ingredients are available for flow (8), the proof in [26] can be adapted to get the Harnack type inequality (9) for flow (8). We will not repeat the proof here.   ■

By Lemma 1, the volume is preserved under flow (1). For $u = \log v$ satisfying (7), we have

Corollary 2. *There exists a constant C depending only on u_0, (M, g_0), k and n, such that for any solution u of flow (7),*

$$(11) \qquad \qquad \|u\|_{C^1} \leq C.$$

Now, we consider the C^2-estimation for flow (1). We note that in the case of the Yamabe flow, C^2 and higher regularity estimates follow directly from C^1 estimates because the leading term is the heat equation. When $k \geq 2$, flow (1) is fully nonlinear. It is not elliptic until the establishment of C^2 estimates and the positivity of $\sigma_k(g)$ in the time interval considered. C^2 estimates are crucial to the global existence of flow (1).

Proposition 5. *Let $g = e^{-2u}g_0$ be a solution of flow (1) with $\sigma_k(g(t)) \in \Gamma_k^+$ on $M \times [0, T^*)$. Then there is a constant $c > 0$ depending only on v_0, g_0, k and n (independent of T^*) such that*

$$(12) \qquad \qquad |\nabla^2 g(t)| \leq c, \quad \forall t \in [0, T^*).$$

Proof: For any local frame $e_1, ..., e_n$, we denote $u_i = \nabla_i u$ and $u_{li} = \nabla_l \nabla_i u$ the first and second covariant derivatives with respect to the background metric g_0. The similar notation will be used for the higher order covariant derivatives.

We want to bound Δu. If $k \geq 2$ and $\sigma_k(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) \in \Gamma_k^+$, we know that

$$|u_{ij}| \leq c_1(\Delta u + |\nabla u| + 1).$$

Hence we only need to get an upper bound of Δu. Consider $G := (\Delta u + m|\nabla u|^2)$ on $M \times [0, T]$ for a given $T \in (0, T^*)$. Here m is a constant to be fixed later. Assume G achieves the maximum at $(x_0, t_0) \in M \times [0, T]$. Without loss of generality, we may assume that $G(x_0, t_0) \geq 1$. Since $|\nabla u|$ is bounded, we may also assume that at the point

$$G \geq \frac{1}{2} \mathrm{tr}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}).$$

By the Newton-MacLaurin inequality, at this point

$$(13) \qquad G \geq \frac{n}{2} \left(\frac{k!(n-k)!}{n!} \right)^{\frac{1}{k}} \sigma_k^{\frac{1}{k}} (\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}).$$

We have, at (x_0, t_0),

$$(14) \qquad \qquad G_t = \sum_l (u_{llt} + 2m u_{lt} u_l) \geq 0$$

and

$$(15) \qquad \qquad G_i = \sum_l (u_{lli} + 2m u_{li} u_l) = 0.$$

In what follows, we indicate c to be the constant (which may vary line form line) depending only on the quantities specified in the proposition. By (15) we have at (x_0, t_0),

$$(16) \qquad |\sum_l u_{lli}| \leq cG, \quad \text{for all } i.$$

Furthermore the matrix (G_{ij}) is semi-negative definite at this point. Let $W = (W_{ij})$ be a matrix defined by

$$W_{ij} = u_{ij} + u_i u_j - \frac{1}{2}|\nabla u|^2 \delta_{ij} + S(g_0)_{ij}$$

and $F = \log \sigma_k(W)$. Set $F^{ij} = \frac{\partial F}{\partial W_{ij}}$. We may assume that (W_{ij}) is diagonal at x_0, so (F^{ij}) is also diagonal at the point. Since (F^{ij}) is positive definite, we have

$$(17) \qquad 0 \geq \sum_{i,j} F^{ij} G_{ij} \;\; = \;\; \sum_{i,j,l} F^{ij}(u_{llij} + 2m u_{li}u_{lj} + 2u_{lij}u_l).$$

Since $|\nabla u|$ is bounded, commutators of the covariant derivatives can be estimated as,

$$(18) \qquad |u_{lij} - u_{ijl}| \leq c, \quad |u_{llij} - u_{ijll}| \leq cG.$$

In view of (14)–(18) and the concavity of F, we have

$$
\begin{aligned}
0 \;\geq\;& \sum_{i,j} F^{ij} G_{ij} \\
\geq\;& \sum_{i,j,l} F^{ij}(u_{ijll} + 2m u_{li}u_{lj} + 2u_{ijl}u_l) - c\sum_i F^{ii} G \\
=\;& \sum_{i,j,l} F^{ij}\{w_{ijll} - (u_i u_j - \frac{1}{2}|\nabla u|^2 \delta_{ij} + S(g_0)_{ij})_{ll} + 2m u_{li}u_{lj} \\
& + 2m w_{ijl}u_l - 2m u_l(u_i u_j - \frac{1}{2}|\nabla u|^2 \delta_{ij} + S(g_0)_{ij})_l\} - c\sum_i F^{ii} G \\
\geq\;& \Delta F + 2m \sum_l F_l u_l + \sum_{i,j,l} F^{ii} u_{jl}^2 + 2(m-1)\sum_{i,l} F^{ii} u_{li}^2 - c\sum_i F^{ii} G \\
\geq\;& \Delta F + 2m \sum_l F_l u_l + \frac{1}{n} G^2 \sum_i F^{ii} + 2(m-1)\sum_{i,l} F^{ii} u_{li}^2 - c\sum_i F^{ii} G.
\end{aligned}
$$

(19)

Recall that $F = u_t - 2ku - \log r(g)$, (14) and (19) yield

$$
\begin{aligned}
0 \;\geq\;& (\Delta u_t + 2m \sum_l u_l u_{tl}) - 2k\Delta u - 4mk|\nabla u|^2 \\
& + \frac{1}{n}\sum_i F^{ii} G^2 + 2(m-1)\sum_i F^{ii} u_{ii}^2 - c\sum_i F^{ii} G \\
\geq\;& \{-2kG + 2(m-1)\sum F^{ii} u_{ii}^2\} + \sum_i F^{ii}(\frac{1}{n} G^2 - cG).
\end{aligned}
$$

(20)

We *claim* that for suitable large $m > 1$,

$$(21) \qquad -2kG + 2(m-1)\sum F^{ii}u_{ii}^2 \geq -c(1 + \sum_i F^{ii}G).$$

Since W and (F^{ij}) are diagonal at the point, $\lambda_i = W_{ii}$ and

$$W_i = (\lambda_1, \lambda_2, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots),$$

it is easy to check that $F^{ii} = \frac{\sigma_{k-1}(W_i)}{\sigma_k(W)}$. From (20) and the identity

$$\sum_i \sigma_{k-1}(W_i) = (n-k+1)\sigma_{k-1}(W),$$

if the *claim* is true,

$$(22) \qquad \frac{\sigma_{k-1}(W)}{\sigma_k(W)}(G^2 - cG) \leq c.$$

The Newton-MacLaurin inequality yields

$$\frac{\sigma_{k-1}(W)}{\sigma_k(W)} \geq \frac{kn}{(n-k+1)\text{tr}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})}.$$

Then the Proposition follows from (13) and (22).

Now we prove the *claim*. From (11) and the assumption that $G(x_0, t_0) \geq 1$, there is a constant $c > 0$ independent of T such that

$$\sum F^{ii}u_{ii}^2 \geq \sum F^{ii}w_{ii}^2 - c\sum F^{ii}G.$$

Since W is diagonal at the point, we have the identity (e.g., see [13])

$$\sum_i \sigma_{k-1}(W_i)w_{ii}^2 = \sigma_1(W)\sigma_k(W) - (k+1)\sigma_{k+1}(W).$$

In turn,

$$\begin{aligned}
\sum F^{ii}w_{ii}^2 &= \frac{1}{\sigma_k(W)}\sum_i \sigma_{k-1}(W_i)w_{ii}^2 \\
&= \frac{1}{\sigma_k(W)}(\sigma_1(W)\sigma_k(W) - (k+1)\sigma_{k+1}(W)).
\end{aligned}$$

When $\sigma_{k+1}(W) \leq 0$, the claim is automatically true. Hence we assume that $\sigma_{k+1}(W) > 0$. The Newton-MacLaurin inequality $(k+1)\sigma_{k+1} \leq \frac{n-k}{n}\sigma_1\sigma_k$ yields

$$\sum F^{ii}w_{ii}^2 \geq \frac{k}{n}\sigma_1(W).$$

Now the claim is verified if we choose $m > n + 1$. This completes the proof of the Proposition. ∎

4. A NONLINEAR EIGENVALUE PROBLEM

In this section, we consider a nonlinear eigenvalue problem

$$(23) \qquad \sigma_k^{1/k}\left(\nabla^2\phi + d\phi\otimes d\phi - \frac{|\nabla\phi|^2}{2} + S_{g_0}\right) = \lambda,$$

where covariant derivatives are taken with respect to the background metric. We say ϕ is admissible if $e^{-2\phi}g_0 \in \Gamma_k^+$. We refer to [15] and [24] for the treatment of other types of nonlinear eigenvalue problems.

Theorem 2. *Let $g_0 \in \Gamma_k^+$. Then there exists a function ϕ and a positive number λ such that ϕ is an admissible solution of equation (23). (ϕ, λ) is unique, in the sense that if there are two admissible (ϕ, λ) and (ϕ', λ') satisfying (23), then $\lambda = \lambda'$ and $\phi = \phi' + c$ for some constant c.*

In order to prove Theorem 2, we introduce an auxiliary equation

$$(24) \qquad \sigma_k^{1/k}\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2} + S_{g_0}\right) = he^u + f,$$

for some functions $f > 0$, $h \geq 0$.

Proposition 6. *Let $g_0 \in \Gamma_k^+$, suppose $\frac{1}{L} \leq f \leq L$ for some constant $L > 0$ and $h \geq 0$. If u is a C^4 admissible solution of (24) and $\max_M u = \gamma$ for some constant γ, then for each $0 < \alpha < 1$, $l \geq 2$ integer, there is a constant C depending only on l, α, g_0, L, γ, $\|h\|_{C^l}$, $\|f\|_{C^l}$ such that $\|u\|_{C^{l+1,\alpha}(M)} \leq C$.*

Proof of Proposition 6. We will repeat the arguments in [10] for the a priori estimates. In fact, stronger local estimates hold for the solutions of (24) following the same lines of the proofs in [10]. Here we only concentrate on the global estimates.

We first obtain a C^1 bound. Since $\max_M u = \gamma$, we only need to bound the gradient of u. Let $W = (\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})$ and let w_{ij} be the entries of W. Set $H = |\nabla u|^2$ and assume that H achieves its maximum at x_0. After appropriate choice of the normal coordinates at x_0, we may assume that W is diagonal at the point. Since x_0 is the maximum point of H, we have $H_i(x_0) = 0$, i.e.,

$$(25) \qquad \sum_{l=1}^n u_{il}u_l = 0.$$

We may assume that $H(x_0) \geq A_0^2$ and $|S_{g_0}| \leq A_0^{-1}|\nabla u|^2$ for some large fixed number A_0 to be chosen later.

Since x_0 is the maximum point of H, the matrix $(H_{ij}) = (2u_{lij}u_l + 2u_{il}u_{jl})$ is semi-negative definite. Set $F^{ij} = \frac{\partial \sigma_k^{1/k}}{\partial w_{ij}}$. (F^{ij}) is a diagonal matrix at x_0 as W is diagonal.

Again, as in the proof of Proposition 5, we denote C (which may vary from line to line) as a constant depending only on the quantities mentioned in this proposition. Since $F = (he^u + f)^{1/k}$, $\sum_l F_l u_l \leq C(H + 1)$. We have

$$(26) \qquad 0 \geq F^{ij}H_{ij} = F^{ij}(2u_{lij}u_l + 2u_{il}u_{jl}).$$

Using (25), after commuting the covariant derivatives, the first term in (26) can be estimated as follows,

$$
\begin{aligned}
\sum_{i,j,l} F^{ij} u_{ijl} u_l &\geq \sum_{i,j,l} F^{ij} u_{ijl} u_l - C|\nabla u|^2 \sum_i F^{ii} \\
&= \sum_{i,j,l} F^{ij} \{ w_{ijl} u_l - (u_i u_j - \frac{|\nabla u|^2}{2} \delta_{ij})_l u_l \} - C|\nabla u|^2 \sum_i F^{ii} \\
&= \sum_l F_l u_l - 2 \sum_{i,j,l} F^{ij} u_{il} u_j u_l + \sum_{i,k,l} F^{ii} u_{kl} u_k u_l - C|\nabla u|^2 \sum_i F^{ii} \\
&\geq -C|\nabla u|^2 \sum_i F^{ii}.
\end{aligned}
\tag{27}
$$

Here we have used the homogeneity $\sum_i F^{ii} w_{ii} = F$.

By Lemma 1 in [10], for A_0 sufficiently large (depending only on k, n, and $\|g_0\|_{C^3}$)

$$
\sum_{i,j,l} F^{ij} u_{il} u_{jl} \geq A_0^{-\frac{3}{4}} |\nabla u|^4 \sum_{i \geq 1} F^{ii}.
\tag{28}
$$

(27) and (28), together with (26) yield the desired C^1 estimate.

With the C^1 bound, a C^2 bound can be obtained following the same lines of proof of Proposition 5 in the previous section. We may assume $k \geq 2$, since C^2 bound for $k = 1$ follows from linear elliptic theory. Since u is admissible, in this case we only need to get an upper bound for Δu as in the proof of Proposition 5. We estimate the maximum of $G = (\Delta u + |\nabla u|^2)$. We note that $F = \sigma_k^{1/k}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$ is also concave. At any maximum point $y_0 \in M$, by the similar computations as in (19) and (20) in the proof of Proposition 5, we have,

$$
0 \geq \Delta F + 2 \sum_l F_l u_l + \sum_{i,k,l} F^{ii} u_{kl}^2 - C(1+G) \sum_i F^{ii}.
\tag{29}
$$

We estimate the terms on the right hand side. As $F = (he^u + f)^{1/k}$, we have

$$
\sum_l F_l u_l \geq -C, \quad \sum_{l \geq 1} F_{ll} \geq -CG.
$$

By the facts $\sum F^{ii} \geq 1$ for $F = \sigma_k^{\frac{1}{k}}$ and $G(y_0) \geq 1$, the above yields

$$
\begin{aligned}
0 &\geq -C \sum F^{ii} G + \sum_{i,k,l} F^{ii} u_{kl}^2 \geq -C \sum F^{ii} G + \frac{1}{n} \sum_i F^{ii} (\Delta u)^2 \\
&\geq \sum F^{ii} \left\{ -CG + \frac{1}{n} G^2 \right\}.
\end{aligned}
\tag{30}
$$

It follows from (30) that at y_0, $G \leq C$.

The higher regularity estimates follow from the Evans-Krylov theorem (e.g., [6] and [14]). $\blacksquare$

Proof of Theorem 2. First we want to prove that for small $\lambda > 0$ the following equation has a unique smooth admissible solution

$$(31) \qquad \tilde{F}(u) =: \sigma_k^{1/k}\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} + S_{g_0}\right) - e^u = \lambda.$$

Since $\frac{\partial \tilde{F}}{\partial u} < 0$, the uniqueness for the solutions of (31) follows from the Maximum principle. For the same reason, the kernel of the linearized operator of $\tilde{F}$ is trivial at any admissible v. We note that for any admissible u and $\forall 0 \leq t \leq 1$, since $g_0 \in \Gamma_k^+$, $u_t = tu$ is also admissible. The linearized operator of $\tilde{F}$ at $u_0 = 0$ is $L(\rho) = \text{tr}\{T_{k-1}(S_{g_0})\nabla^2_{g_0}\rho\} - h\rho$. By (4), it is self-adjoint with respect to the metric g_0. Since the index of elliptic operator is invariant under homotopy, so the kernel of the adjoint operator of the linearized operator of $\tilde{F}$ at u is trivial. That is, the linearized operator of $\tilde{F}$ is invertible at every admissible solution u. This fact will be used later in the proof.

We now want to show the existence using the continuity method. Since $g_0 \in \Gamma_k^+$, there is a constant $C_1 > 1$ such that $C_1^{-1} < \sigma_k^{1/k}(S_{g_0}) < C_1$. Thus, for small $\lambda > 0$, one can find two constants $\underline{\delta} < 0 < \overline{\delta}$ such that

$$e^{\underline{\delta}} + \lambda < \sigma_k^{1/k}(S_{g_0}) < e^{\overline{\delta}} + \lambda.$$

Let $v = \underline{\delta}$, we have $\tilde{F}(v) = f$ for some smooth positive function $f \geq \lambda$. For $0 \leq t \leq 1$, let us consider the equation

$$(32) \qquad \tilde{F}(u) = t\lambda + (1-t)f.$$

By the Maximum principle, for any solution u of (32), $\min u \geq \min v = \underline{\delta}$. Also for $\tilde{v} = \overline{\delta}$, as $\tilde{F}(\tilde{v}) < \lambda$, again by the Maximum principle, $\max u \geq \max \tilde{v} = \overline{\delta}$. That is, u is bounded. By Proposition 6 we have the uniform a priori estimates for solution u of (32). This implies the closeness. The openness follows from the standard implicit function theorem since the linearized operator of $\tilde{F}$ is invertible. Therefore, the existence of the unique solution of (31) is established for $\lambda > 0$ small.

Set

$$\Lambda = \{\lambda > 0 \,|\, (31) \text{ has a solution}\}.$$

Since $\Lambda \neq \emptyset$, we define

$$\lambda^* = \sup_{\lambda \in \Lambda} \lambda.$$

We claim λ^* is finite. For any admissible solution u of (31), by the Newton-MacLaurin inequality,

$$
\begin{aligned}
\lambda \; &< \; e^u + \lambda = \sigma_k^{1/k}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}) \\
(33) \qquad &\leq \; \frac{\left(\frac{n!}{k!(n-k)!}\right)^{\frac{1}{k}}}{n}\sigma_1(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}) \\
&= \; \frac{\left(\frac{n!}{k!(n-k)!}\right)^{\frac{1}{k}}}{n}(\Delta u - \frac{n-2}{2}|\nabla u|^2) + cR_0,
\end{aligned}
$$

where R_0 is the scalar curvature function of (M, g_0) and c is a constant depending only on n and k. Integrating the above inequality over M, we get $\lambda \leq c\frac{\int_M R_0 dvol(g_0)}{Vol(g_0)}$. We conclude that $\lambda^* \leq c\frac{\int_M R_0 dvol(g_0)}{Vol(g_0)}$.

For any sequence $\{\lambda_i\} \subset \Lambda$ with $\lambda_i \to \lambda^*$, and let u_{λ_i} be the corresponding solution of (31) with $\lambda = \lambda_i$, $i = 1, 2, 3, \cdots$. We claim that $\max_M u_{\lambda_i} \to -\infty$ as $i \to \infty$. Suppose $\max_M u_{\lambda_i} \geq -C_0$. From equation (31), at any maximum point x_0 of u_{λ_i}, $\max_M u_{\lambda_i} \leq -C$ for some constant C depending only on n, k, g_0. Then Proposition 6 implies that u_{λ_i} (by taking a subsequence) converges to a smooth function u_0 in C^4, such that u_0 satisfies (31) for $\lambda = \lambda_1$. As remarked at the beginning of the proof, the linearized operator of equation (31) is invertible. By the standard implicit function theorem, we have a solution of (31) for $\lambda = \lambda_1 + \varepsilon$ for $\varepsilon > 0$ small, this is a contradiction. Hence $\max_M u_{\lambda_i} \to -\infty$ as $i \to \infty$. Now let $w_{\lambda_i} = u_{\lambda_i} - \max_M u_{\lambda_i}$. It is clear that w_{λ_i} satisfies,

$$\sigma_k^{1/k}\left(\nabla^2 w_{\lambda_i} + dw_{\lambda_i} \otimes dw_{\lambda_i} - \frac{|\nabla w_{\lambda_i}|^2}{2}g + S_g\right) = e^{\max_M u_{l_i}}e^w + \lambda_i$$

with $\max_M w_{\lambda_i} \to 0$. By Proposition 6 again, w_{λ_i} converges to a smooth function ϕ in C^4 and ϕ satisfies (23) with $\lambda = \lambda^*$.

Finally we prove the uniqueness. For each admissible u, let

$$W = (\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})$$

and

$$a_{ij}(W) = \frac{\partial \sigma_k(W)}{\partial w_{ij}}.$$

For any smooth functions u_0 and u_1, let $v = u_1 - u_0$, $u_t = tu_1 + (1-t)u_0$ and $W_t = \Delta u_t + du_t \otimes du_t - \frac{|\nabla u_t|^2}{2}g + S_g$. By (4), the following identity holds

$$(34) \qquad \sigma_k(W_1) - \sigma_k(W_0) = \sum_{ij}\nabla_j[(\int_0^1 a_{ij}(W_t)dt)\nabla_i v] + \sum_l b^l \nabla_l v,$$

for some bounded functions b^l, $l = 1, ..., n$. If $u_0 = \phi$ and $u_1 = \phi'$ are two admissible solutions of (23) for some λ and λ' respectively, then $(\int_0^1 a_{ij}(W_t)dt)$ is positive definite. Therefore, $\phi = \phi' + c$ for some constant c by (4) and the Maximum principle. The proof of Theorem 2 is complete. ∎

5. Existence and convergence

First, we want to use the a priori estimates established in the section 3 to obtain the long time existence of flow. These estimates are independent of time t, as long as $u(t, x)$ stays in the positive cone Γ_k^+. Therefore, to establish the long time existence, we need to show σ_k stays strictly positive for all t.

Lemma 2. *We have*

$$(35) \qquad \frac{d}{dt}\log\sigma_k(g) = \frac{1}{2\sigma_k(g)}tr\{T_{k-1}(S_g)\nabla_g^2\log\sigma_k(g)\} + \log\sigma_k(g) - \log r_k(g).$$

Proof: It is easy to check, see for example [16],

$$\frac{d}{dt}\sigma_k(g) = k\sigma_k(g)g \cdot \frac{d}{dt}(g^{-1}) + \text{tr}\{T_{k-1}(S_g)g^{-1}\frac{d}{dt}S_g\}.$$

Under the conformal change $g = e^{-2u}g_0$, the Schouten tensor is changed as follows

$$S_g = \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}.$$

Hence, we have

$$\frac{d}{dt}S_g = \nabla_g^2 \frac{du}{dt} = -\frac{1}{2}\nabla_g^2(g^{-1}\frac{d}{dt}g).$$

Now we have

$$\frac{d}{dt}\log\sigma_k(g) = -\frac{1}{2}\sigma_k^{-1}\text{tr}\{T_{k-1}(S_g)g^{-1}\nabla_g^2(g^{-1}\frac{d}{dt}g)\} + \log\sigma_k(g) - \log r_k(g)$$

$$= \frac{1}{2\sigma_k(g)}\text{tr}\{T_{k-1}(S_g)\nabla_g^2\log\sigma_k(g)\} + \log\sigma_k(g) - \log r_k(g).$$

∎

Corollary 3. *Suppose $g(t)$ is a solution of equation (1) with initial metric g_0 on a time interval $[0, T^*)$, then there is a positive constant $c > 0$ depending on g_0, k, n (independent of t), such that $\forall t \in [0, T^*)$,*

$$(36) \qquad\qquad\qquad \sigma_k(g(t)) \geq ce^{-\frac{e^t}{c}}.$$

Therefore, flow (1) exists for all $t > 0$.

Proof: By Propositions 4 and 5, $\log r_k(g) \leq C$ for some constant C independent of t. Let $f = e^{-t}\log\sigma_k(g) - Ce^{-t}$. By Lemma 2 f satisfies a parabolic inequality

$$\frac{d}{dt}f \geq \frac{1}{2\sigma_k(g)}\text{tr}\{T_{k-1}(S_g)\nabla_g^2 f\}.$$

Now the corollary follows from the maximum principle. Again by Propositions 4, 5, and what we just proved, flow (1) stays uniformly parabolic in $[0, T)$ for any fixed $T < \infty$. By Krylov Theorem for fully nonlinear parabolic equations [14], $g(t) \in C^{2,\alpha_T}, \forall 0 \leq t \leq T$ for some $\alpha_T > 0$. In turn, $\|g\|_{C^l([0,T]\times M)} \leq C_{l,T}$ for all $l \geq 2$. Now the long time existence follows from the standard parabolic PDE theory. ∎

The next proposition will also be used in our proof of Theorem 1.

Proposition 7. *If $g_0 \in \Gamma_+^k$, then there is no $C^{1,1}$ function $u \in \overline{\Gamma_+^k}$ such that $\sigma_k(g) = 0$ for $g = e^{-2u}g_0$ in the viscosity sense.*

Proof of Proposition 7. By Theorem 2, there is a smooth admissible solution ϕ of (23) for some $\lambda > 0$. Assume by contradiction that there is a $C^{1,1}$ solution u of

$$\sigma_k\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g + S_g\right) = 0.$$

Let $u_0 = u$ and $u_1 = \phi$ and $v = u_1 - u_0$, then (34) still holds for v in weak sense (since $u \in C^{1,1}$). Also, $(\int_0^1 a_{ij}(W_t)dt)$ is uniformly positive definite everywhere since $a_{ij}(W_t)$ is

semi-positive definite for all t and uniformly positive definite for $\delta \leq t \leq 1$, $\forall \delta > 0$. By Maximum principle again, $u = \phi + c$ for some constant c. That is,

$$0 = \sigma_k \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g + S_g \right) = \sigma_k(W_1) = \lambda > 0.$$

This is a contradiction. ∎

Proof of Theorem 1. The a priori estimates and the longtime existence are proved in Proposition 5 and Corollary 3. We now prove the convergence for $k \neq n/2$. Since the volume is preserved under flow (1), from (6) we have $\forall T$

$$\int_0^T \int_M (\sigma_k(g) - r_k(g))(\log \sigma_k(g) - \log r_k) dvol(g) dt \leq \frac{2V^{\frac{n-2k}{2}}(g_0)}{|n - 2k|} |\mathcal{F}_k(g(T)) - \mathcal{F}_k(g_0)|.$$

By Proposition 5, $\sigma_k(g)$ and $\mathcal{F}_k(g)$ are uniformly bounded. We conclude that

$$(37) \qquad \int_0^\infty \int_M (\sigma_k(g) - r_k(g))^2 dvol(g) dt < \infty.$$

Therefore for any sequence $t_n \to \infty$, there is a subsequence $\{t_{n_l}\}$) such that $r_k(g_{t_{n_L}}) \to \beta$, for some nonnegative constant β, and $\int_M (\sigma_k(g_{t_{n_l}}) - r_k(g_{t_{n_l}}))^2 dvol(g_{t_n}) \to 0$. Again, from Proposition 5 (by taking subsequence) $u(t_{n_l})$ converges to some $C^{1,1}$ function u_∞ in $C^{1,\alpha}$ norm for any $0 < \alpha < 1$. By Lebesgue domination theorem,

$$(38) \qquad \sigma_k(g_\infty) = \beta, \quad \text{almost everywhere,}$$

where $g_\infty = e^{-2u_\infty} g_0$. Since $u_\infty \in C^{1,1}$, u_∞ satisfies the equation (38) in viscosity sense. By Lemma 1 $\mathcal{F}_k(g(t))$ is monotonic, β is independent of the choice of the sequence. It follows (37) again that

$$\lim_{t \to \infty} \|\sigma_k(g) - \beta\|_{L^2(M)} = 0.$$

Finally, β is positive by (38) and Corollary 7. We conclude that $u_\infty \in C^\infty(M)$ by the Evans-Krylov Theorem. We remark that the positivity of β for $k > n/2$ also follows from Lemma 1, Propositions 4 and 5, since $r_k(g_t)$ is uniformly bounded from below. ∎

6. DISCUSSIONS

We discuss some implications of the flow (1). Let $\mathcal{F}_k(g)$ be the functional defined as in (5). When $n = 4$ and $R_g > 0$, Gursky proved in [11] that there is a constant C (independent of (M, g)) such that

$$\mathcal{F}_2(g) \leq C(\mathcal{F}_1(g))^2,$$

and the equality holds if and only if (M, g) is conformally equivalent to the round sphere. We note that the inequality is also a consequence of the Yamabe problem. Since $\mathcal{F}_2(g)$ is a conformal invariant when $n = 4$, the inequality can also be proved using Schoen's solution of the Yamabe problem in [18] and the Newton-MacLaurin inequality for elementary symmetric functions.

We now turn to higher dimensional cases. If (M, g_0) is locally conformally flat and $n = 2m$, and assume $g_0 \in \Gamma_{m-1}$. We note that $\mathcal{F}_m(g_0)$ is a conformal invariant. By Lemma 1 and Theorem 1, there is a $g \in [g_0]$ such that $vol(g) = vol(g_0)$, $\mathcal{F}_{m-1}(g) \leq \mathcal{F}_{m-1}(g_0)$ and $\sigma_{m-1}(g) = \beta$. We may assume $vol(g) = vol(g_0) = 1$. From the Newton-MacLaurin inequality, there is a constant $C(m, n) > 0$ such that

$$\sigma_m(g) \leq C(m, n)\sigma_{m-1}^{\frac{m}{m-1}}(g) = C(m, n)\beta^{\frac{m}{m-1}},$$

and the equality holds if and only if the Schouten tensor $S(g)$ is a multiple of g. Integrating over M, we get

(39)
$$\mathcal{F}_m(g) \leq C(m, n)\mathcal{F}_{m-1}^{\frac{m}{m-1}}(g).$$

If the equality holds, (M, g) is a spherical space form. If $g_0 \in \Gamma_k$ and $k > m/2$, then the same argument works, namely, there is a constant $C(m, k, n) > 0$ such that

(40)
$$\mathcal{F}_k(g) \leq C(m, k, n)\mathcal{F}_m^{\frac{k}{m}}(g),$$

and the equality holds if and only if (M, g) is a spherical space form. Notice that in this case $\mathcal{F}_k$ is increasing along flow (1).

We now recall the classical quermassintegral inequality. Let $W_k(\Omega)$ be the k-th cross-sectional integral of a convex body $\Omega \subset R^{n+1}$. The classical quermassintegral inequality (e.g., see [17]) asserts that $\forall 1 \leq l < k \leq n + 1$:

$$W_k^{1/k}(\Omega) \leq C(k, l, n)W_l^{1/l}(\Omega),$$

and the equality holds if and only if Ω is a round ball.

Let u is the support function of Ω and u_{ij} its second order covariant derivatives with respect to an orthonormal frame of the standard sphere. The quermassintegral inequality can be expressed as

(41)
$$\left(\int_{S^n} \sigma_k(u_{ij} + u\delta_{ij})\right)^{1/k} \leq C(k, l, n)\left(\int_{S^n} \sigma_l(u_{ij} + u\delta_{ij})\right)^{1/l}.$$

This inequality was generalized to k-convex functions on S^n in [9].

We note that there is a similarity of inequalities (39) and (40) with inequality (41). This comparison leads us to speculate the following conformal version of quermassintegral inequality for $\mathcal{F}_k(g)$: For any $1 \leq l < k \leq n$ there is a constant $C(k, l, n) > 0$ such that

(42)
$$(\mathcal{F}_k(g))^{1/k} \leq C(k, l, n)(\mathcal{F}_l(g))^{1/l}, \quad \forall g \in \Gamma_k^+,$$

and the equality holds if and only if (M, g) is a spherical space form.

Flow (1) is a natural geometric equation. There are many related questions remaining to be answered. We conclude this paper with some remarks.

Remark 2. *When $k = \frac{n}{2}$, the right hand side in (6) is trivial. For this reason, we are unable to get the sequential convergence for flow (1) in this case. It is interesting to find a variational characterization of metrics of constant σ_k-scalar curvature for this case. When $n = 4$ and $k = 2$, such a variational characterization can be found in [2].*

Remark 3. *Though the sequential convergence of flow (1) is sufficient for Corollary 1, it is of interest to know the uniqueness of g_∞ and global convergence of flow (1). We would also like to raise the question of the global existence and convergence of the flow (1) for general case without the assumption of locally conformally flatness.*

Remark 4. *With some modifications in our proof, one can establish the similar a priori estimates for the following flow $(0 \le l < k \le n)$ on locally conformally flat manifolds:*

$$
(43) \qquad
\begin{cases}
\dfrac{d}{dt}g & = & -(\log(\frac{\sigma_k(g)}{\sigma_l(g)}) - \log r_{k,l}(g)) \cdot g, \\[2mm]
g(0) & = & g_0,
\end{cases}
$$

with $r_{k,l}(g)$ given by

$$
r_{k,l}(g) = \exp\left(\frac{1}{vol(g)} \int_M \log(\frac{\sigma_k(g)}{\sigma_l(g)}) \, dvol(g) \right).
$$

Therefore the global existence result is valid for (43).

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