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## Some properties of the Schouten tensor and applications to conformal geometry

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# SOME PROPERTIES OF THE SCHOUTEN TENSOR AND APPLICATIONS TO CONFORMAL GEOMETRY

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## 1. Introduction

Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold, $n \geq 3$, and let the Ricci tensor and scalar curvature be denoted by $Ric$ and $R$, respectively. We define the Schouten tensor

$$A_g = \frac{1}{n-2}\left(Ric - \frac{1}{2(n-1)}Rg\right).$$

There is a decomposition formula (see [1]):

$$\text{Riem} = A_g \odot g + \mathcal{W}_g, \tag{1}$$

where $\mathcal{W}_g$ is the Weyl tensor of $g$, and $\odot$ denotes the Kulkarni-Nomizu product (see [1]). As Weyl tensor is conformally invariant, to study the deformation of conformal metric, we only need to understand the Schouten tensor. A study of $k$-th elementary symmetric functions of the Schouten tensor was initiated in [13], it was reduced to certain fully nonlinear Yamabe type equations. In order to apply elliptic theory of fully nonlinear equations, one often restricts Schouten tensor to be in certain cone $\Gamma_k^+$ defined as follows (according to Gårding [5]).

**Definition 1.** *Let $(\lambda_1, \cdots, \lambda_n) \in \mathbf{R}^n$. Let $\sigma_k$ denote the $k$th elementary symmetric function*

$$\sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

*and we let*

$$\Gamma_k^+ = \textit{component of } \{\sigma_k > 0\} \textit{ containing } \quad (1, \cdots, 1).$$

*Let $\bar{\Gamma}_k^+$ denote the closure of $\Gamma_k^+$. If $(M, g)$ is a Riemannian manifold, and $x \in M$, we say $g$ has positive (nonnegative resp.) $\Gamma_k$-curvature at $x$ if its Schouten tensor $A_g \in \Gamma_k^+$ ($\bar{\Gamma}_k^+$ resp.) at $x$. In this case, we also say $g \in \Gamma_k^+$ ($\bar{\Gamma}_k^+$ resp.) at $x$.*

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We note that positive $\Gamma_1$-curvature is equivalent to positive scalar curvature, and the condition of positive $\Gamma_k$-curvature has some geometric and topological consequences for the manifold $M$. For example, when $(M, g)$ is locally conformally flat with positive $\Gamma_1$-curvature, then $\pi_i(M) = 0, \forall 1 < i \leq \frac{n}{2}$ by a result of Schoen-Yau [11]. In this note, we will prove that positive $\Gamma_k$-curvature for any $k \geq \frac{n}{2}$ implies positive Ricci curvature.

**Theorem 1.** *Let $(M, g)$ be a Riemannian manifold and $x \in M$, if $g$ has positive (nonnegative resp.) $\Gamma_k$-curvature at $x$ for some $k \geq n/2$. Then its Ricci curvature is positive (nonnegative resp.) at $x$. Moreover, if $\Gamma_k$-curvature is nonnegative for some $k > 1$, then*

$$Ric_g \geq \frac{2k-n}{2n(k-1)} R_g \cdot g.$$

*In particular if $k \geq \frac{n}{2}$,*

$$Ric_g \geq \frac{(2k-n)(n-1)}{(k-1)} \binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}(A_g) \cdot g.$$

*Remark.* Theorem 1 is not true for $k = 1$. Namely the condition of positive scalar curvature gives no restriction on the lower bound of Ricci curvature .

**Corollary 1.** *Let $(M^n, g)$ be a compact, locally conformally flat manifold with nonnegative $\Gamma_k$-curvature everywhere for some $k \geq n/2$. Then $(M, g)$ is conformally equivalent to either a space form or a finite quotient of a Riemannian $\mathbf{S}^{n-1}(c) \times \mathbf{S}^1$ for some constant $c > 0$ and $k = n/2$. Especially, if $g \in \Gamma_k^+$, then $(M, g)$ is conformally equivalent to a spherical space form.*

When $n = 3, 4$, the result in Theorem 1 was already observed in [9] and [2]. Theorem 1 and Corollary 1 will be proved in the next section.

We will also consider the equation

$$\sigma_k(A_{\tilde{g}}) = constant, \tag{2}$$

for conformal metrics $\tilde{g} = e^{-2u} g$. This equation was studied in [13], where it was shown that when $k \neq n/2$, (2) is the conformal Euler-Lagrange equation of the functional

$$\mathcal{F}_k(g) = vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g)\, dvol(g), \tag{3}$$

when $k = 1, 2$ or for $k > 2$ when $M$ is locally conformally flat. We remark that in the even dimensional locally conformally flat case, $\mathcal{F}_{n/2}$ is a conformal invariant. Moreover, it is a multiple of the Euler characteristic, see [13].

This problem was further studied studied in [7], where the following conformal flow was considered:

$$\frac{d}{dt}g = -(\log \sigma_k(g) - \log r_k(g)) \cdot g,$$
$$g(0) = g_0,$$

where

$$\log r_k = \frac{1}{Vol(g)} \int_M \log \sigma_k(g) dvol(g).$$

Global existence with uniform $C^{1,1}$ a priori bounds of the flow was proved in [7]. It was also proved that for $k \neq n/2$ the flow is sequentially convergent in $C^{1,\alpha}$ to a $C^\infty$ solution of $\sigma_k = constant$. Also, if $k < n/2$, then $\mathcal{F}_k$ is decreasing along the flow, and if $k > n/2$, then $\mathcal{F}_k$ is increasing along the flow.

In Section 3, we will consider global properties of the functional $\mathcal{F}_k$, and give conditions for the existence of a global extremizer. We will also derive some conformal quermassintegral inequalities, which are analogous to the classical quermassintegral inequalities in convex geometry.

## 2. Curvature restriction

We first state a proposition which describes some important properties of the sets $\Gamma_k^+$.

**Proposition 1.** *(i) Each set $\Gamma_k^+$ is an open convex cone with vertex at the origin, and we have the following sequence of inclusions*

$$\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+.$$

*(ii) For any $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k^+$ ($\bar{\Gamma}_k^+$ resp.), $\forall 1 \leq i \leq n$, let*

$$(\Lambda|i) = (\lambda_1, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n),$$

*then $(\Lambda|i) \in \Gamma_{k-1}^+$ ($\bar{\Gamma}_{k-1}^+$ resp.). In particular,*

$$\Gamma_{n-1}^+ \subset V_{n-1}^+ = \{(\lambda_1, \cdots, \lambda_n) \in \mathbf{R}^n : \lambda_i + \lambda_j > 0, i \neq j\}.$$

The proof of this proposition is standard following from [5].

Our main results are the consequences of the following two lemmas. In this note, we assume that $k > 1$.

**Lemma 1.** *Let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n) \in \mathbf{R}^n$, and define*

$$A_\Lambda = \Lambda - \frac{\sum_{i=1}^n \lambda_i}{2(n-1)}(1, 1, \cdots, 1).$$

*If $A_\Lambda \in \bar\Gamma_k^+$, then*

$$\min_{i=1,\cdots,n} \lambda_i \geq \frac{(2k-n)}{2n(k-1)} \sum_{i=1}^n \lambda_i. \tag{4}$$

*In particular if $k \geq \frac{n}{2}$,*

$$\min_{i=1,\cdots,n} \lambda_i \geq \frac{(2k-n)(n-1)}{(n-2)(k-1)} \binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}(A_\Lambda).$$

*Proof.* We first note that, for any non-zero vector $A = (a_1, \cdots, a_n) \in \bar\Gamma_2^+$ implies $\sigma_1(A) > 0$. This can be proved as follow. As $A \in \bar\Gamma_2^+$, $\sigma_1(A) \geq 0$. If $\sigma_1(A) = 0$, there must be $a_i > 0$ for some $i$ since $A$ is a non-zero vector. We may assume $a_n > 0$. Let $(A|n) = (a_1, \cdots, a_{n-1})$, we have $\sigma_1(A|n) \geq 0$ by Proposition 1. This would give $\sigma_1(A) = \sigma_1(A|n) + a_n > 0$, a contradiction.

Now without loss of generality, we may assume that $\Lambda$ is not a zero vector. By the assumption $A_\Lambda \in \bar\Gamma_k^+$ for $k \geq 2$, so we have $\sum_{i=1}^n \lambda_i > 0$.

Define

$$\Lambda_0 = (1, 1, \cdots, 1, \delta_k) \in \mathbf{R}^{n-1} \times \mathbf{R}$$

and we have $A_{\Lambda_0} = (a, \cdots, a, b)$, where

$$\delta_k = \frac{(2k-n)(n-1)}{2nk-2k-n},$$

$$a = 1 - \frac{n-1+\delta_k}{2(n-1)}, \quad b = \delta_k - \frac{n-1+\delta_k}{2(n-1)}$$

so that

$$\sigma_k(A_{\Lambda_0}) = 0 \quad \text{ and } \sigma_j(A_{\Lambda_0}) > 0 \text{ for } j \leq k-1. \tag{5}$$

It is clear that $\delta_k < 1$ and so that $a > b$. Since (4) is invariant under the transformation $\Lambda$ to $s\Lambda$ for $s > 0$, we may assume that $\sum_{i=1}^n \lambda_i = \operatorname{tr}(\Lambda_0) = n - 1 + \delta_k$ and $\lambda_n = \min_{i=1,\cdots,n} \lambda_i$. We write

$$A_\Lambda = (a_1, \cdots, a_n).$$

We claim that

$$\lambda_n \geq \delta_k. \tag{6}$$

This is equivalent to show

$$a_n \geq b. \tag{7}$$

Assume by contradiction that $a_n < b$. We consider $\Lambda_t = t\Lambda_0 + (1-t)\Lambda$ and

$$A_t := A_{\Lambda_t} = tA_{\Lambda_0} + (1-t)A_\Lambda = ((1-t)a + ta_1, \cdots, (1-t)a + ta_{n-1}, (1-t)b + ta_n).$$

By the convexity of the cone $\Gamma_k^+$ (see Proposition 1), we know

$$A_t \in \bar{\Gamma}_k^+, \qquad \text{for any } t \in (0,1].$$

Especially, $f(t) := \sigma_k(A_t) \geq 0$ for any $t \in [0,1]$. By the definition of $\delta_k$, $f(0) = 0$.

For any $i$ and any vector $V = (v_1, \cdots, v_n)$, we denote $(V|i) = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$ be the vector with the $i$-th component removed. Now we compute the derivative of $f$ at 0

$$f'(0) = \sum_{i=1}^{n-1} (a_i - a)\sigma_{k-1}(A_0|i) + (a_n - b)\sigma_{k-1}(A_0|n).$$

Since $(A_0|i) = (A_0|1)$ for $i \leq n-1$ and $\sum_{i=1}^n a_i = (n-1)a + b$, we have

$$f'(0) = (a_n - b)(\sigma_{k-1}(A_0|n) - \sigma_{k-1}(A_0|1)) < 0,$$

for $\sigma_{k-1}(A_0|n) - \sigma_{k-1}(A_0|1) > 0$. (Recall that $b < a$.) This is a contradiction, hence $\lambda_n \geq \delta_k$. It follows that

$$\min_{i=1,\cdots,n} \lambda_i \geq \delta_k = \frac{2k-n}{2n(k-1)} \sum_{i=1}^n \lambda_i.$$

Finally, the last inequality in the lemma follows from the Newton-MacLaurin inequality. ■

*Remark.* It is clear from the above proof that the constant in Lemma 1 is optimal.

We next consider the case $A_\Lambda \in \bar{\Gamma}_{\frac{n}{2}}^+$.

**Lemma 2.** *Let $k = n/2$ and $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{R}^n$ with $A_\Lambda \in \bar{\Gamma}_k^+$. Then either $\lambda_i > 0$ for any $i$ or*

$$\Lambda = (\lambda, \lambda, \cdots, \lambda, 0)$$

*up to a permutation. If the second case is true, then we must have $\sigma_{\frac{n}{2}}(A_\Lambda) = 0$.*

*Proof.* By Lemma 1, to prove the Lemma we only need to check that for $\Lambda = (\lambda_1, \cdots, \lambda_{n-1}, 0)$ with $A_\Lambda \in \bar{\Gamma}_k^+$,

$$\lambda_i = \lambda_j, \quad \forall i, j = 1, 2, \cdots, 2k-1.$$

We use the same idea as in the proof of the previous Lemma. Without loss of generality, we may assume that $\Lambda$ is not a zero vector. By the assumption $A_\Lambda \in \bar{\Gamma}_k^+$ for $k \geq 2$, we have $\sum_{i=1}^{n-1} \lambda_i > 0$. Hence we may assume that $\sum_{i=1}^{n-1} \lambda_i = n-1$. Define

$$\Lambda_0 = (1, 1, \cdots, 1, 0) \in \mathbf{R}^n$$

It is easy to check that

$$A_{\Lambda_0} \in \Gamma_{k-1}^+ \quad \text{and} \quad \sigma_k(A_{\Lambda_0}) = 0. \tag{8}$$

That is, $A_{\Lambda_0} \in \bar\Gamma_k^+$. If $\lambda$'s are not all the same, we have

$$\sum_{i=1}^{n-1} (\lambda_i - 1) = 0,$$

and

$$\sum_{i=1}^{n-1} (\lambda_i - 1)^2 > 0.$$

Now consider $\Lambda_t = t\Lambda_0 + (1-t)\Lambda$ and

$$A_t := A_{\Lambda_t} = tA_{\Lambda_0} + (1-t)A_\Lambda = (\frac{1}{2} + t(\lambda_1 - 1), \cdots, \frac{1}{2} + t(\lambda_{n-1} - 1), -\frac{1}{2}).$$

From the assumption that $A \in \bar\Gamma_k^+$, (8) and the convexity of $\bar\Gamma_k^+$, we have

$$A_t \in \bar\Gamma_k^+ \quad \text{for } t > 0. \tag{9}$$

For any $i \neq j$ and any vector $A$, we denote $(A|ij)$ be the vector with the $i$-th and $j$-th components removed. Let $\widetilde\Lambda = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ be $n-1$-vector, $\Lambda^* = (\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ be $n-2$-vector. It is clear that $\forall i \neq j, \quad i, j \leq n-1$,

$$\sigma_{k-1}(A_0|i) = \sigma_{k-1}(\widetilde\Lambda) > 0,$$

$$\sigma_{k-2}(A_0|ij) = \sigma_{k-2}(\Lambda^*) > 0.$$

Now we expand $f(t) = \sigma_k(A_t)$ at $t = 0$. By (8), $f(0) = 0$. We compute

$$\begin{aligned} f'(0) &= \sum_{i=1}^{n-1} (\lambda_i - 1)\sigma_{k-1}(A_0|i) \\ &= \sigma_{k-1}(\widetilde\Lambda) \sum_{i=1}^{n-1} (\lambda_i - 1) = 0 \end{aligned}$$

and

$$\begin{aligned} f''(0) &= \sum_{i \neq j} (\lambda_i - 1)(\lambda_j - 1)\sigma_{k-2}(A_0|ij) \\ &= \sigma_{k-2}(\Lambda^*) \sum_{i \neq j} (\lambda_i - 1)(\lambda_j - 1) \\ &= -\sigma_{k-2}(\Lambda^*) \sum_{i=1}^{n-1} (\lambda_i - 1)^2 < 0, \end{aligned}$$

for $\sigma_{k-2}(A_0|ij) = \sigma_{k-2}(\Lambda^*) > 0$ for any $i \neq j$ and $\sum_{i \neq j} (\lambda_j - 1) = (1 - \lambda_i)$. Hence $f(t) < 0$ for small $t > 0$, which contradicts (9). ∎

*Remark.* From the proof of Lemma 2, there is a constant $C > 0$ depending only on $n$ and $\frac{\sigma_{\frac{n}{2}}^{\frac{2}{n}}(A_\Lambda)}{\sigma_1(A_\Lambda)}$ such that

$$\min_i \lambda_i \geq C\sigma_{\frac{n}{2}}^{\frac{2}{n}}(A_\Lambda).$$

*Proof of Theorem 1.* Theorem 1 follows directly from Lemmas 1 and 2. ■

**Corollary 2.** *Let $(M, g)$ is a $n$-dimensional Riemannian manifold and $k \geq n/2$, and let $N = M \times \mathbf{S}^1$ be the product manifold. Then $N$ does not have positive $\Gamma_k$-curvature. If $N$ has nonnegative $\Gamma_k$-curvature, then $(M, g)$ is an Einstein manifold, and there are two cases: either $k = n/2$ or $k > n/2$ and $(M, g)$ is a torus.*

*Proof.* This follows from Lemmas 1 and 2. ■

*Proof of Corollary 1.* From Theorem 1, we know that the Ricci curvature $Ric_g$ is nonnegative. Now we deform it by the Yamabe flow considered by Hamilton, Ye [15] and Chow [4] to obtain a conformal metric $\widetilde{g}$ of constant scalar curvature. The Ricci curvature $Ric_{\widetilde{g}}$ is nonnegative, for the Yamabe flow preserves the non-negativity of Ricci curvature, see [4]. Now by a classification result given in [12, 3], we have $(M, \widetilde{g})$ is isometric to either a space form or a finite quotient of a Riemannian $\mathbf{S}^{n-1}(c) \times \mathbf{S}^1$ for some constant $c > 0$. In the latter case, it is clear that $k = n/2$, otherwise it can not have nonnegative $\Gamma_k$-curvature. ■

Next, we will prove that if $M$ is locally conformally flat with positive $\Gamma_{n-1}$-curvature, then $g$ has positive sectional curvature. If $M$ is locally conformally flat, then by (1) we may decompose the full curvature tensor as

$$\text{Riem} = A_g \odot g,$$

**Proposition 2.** *Assume that $n = 3$, or that $M$ is locally conformally flat. Then Schouten tensor $A_g \in V_{n-1}^+$ if and only if $g$ has positive sectional curvature.*

*Proof.* Let $\pi$ be any 2-plane in $T_p(N)$, and let $X$,$Y$ be an orthonormal basis of $\pi$. We have that

$$\begin{aligned} K(\sigma) &= \text{Riem}(X, Y, X, Y) = A_g \odot g(X, Y, X, Y) \\ &= A_g(X, X)g(Y, Y) - A_g(Y, X)g(X, Y) + A_g(Y, Y)g(X, X) - A_g(X, Y)g(Y, X) \\ &= A_g(X, X) + A_g(Y, Y). \end{aligned}$$

From this it follows that

$$\min_{\sigma \in T_p N} K(\sigma) = \lambda_1 + \lambda_2,$$

where $\lambda_1$ and $\lambda_2$ are the smallest eigenvalues of $A_g$ at $p$. ■

**Corollary 3.** *If $(M, g)$ is locally conformally flat with positive $\Gamma_{n-1}$-curvature, then $g$ has positive sectional curvature.*

*Proof.* This follows easily from Propositions 1 and 2. ∎

## 3. Extremal metrics of $\Gamma_k$-curvature functionals

We next consider some properties of the functionals $\mathcal{F}_k$ associated to $\sigma_k$. These functionals were introduced and discussed in [13], see also [7]. Further variational properties in connection to 3-dimensional geometry were studied in [9].

We recall that $\mathcal{F}_k$ is defined by

$$\mathcal{F}_k(g) = vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g)\, dvol(g).$$

We denote $\mathcal{C}_k = \{g \in [g_0] | g \in \Gamma_k^+\}$, where $[g_0]$ is the conformal class of $g_0$.

We now apply our results to show that if $g_0 \in \Gamma^+_{\frac{n}{2}}$, then there is an extremal metric $g_e$ which minimizes $\mathcal{F}_m$ for $m < n/2$, and if $m > n/2$, there is an extremal metric $g_e$ which maximizes $\mathcal{F}_m$.

**Proposition 3.** *Suppose $(M, g_0)$ is locally conformally flat and $g_0 \in \Gamma_k^+$ for some $k \geq \frac{n}{2}$, then $\forall m < \frac{n}{2}$ there is an extremal metric $g_e^m \in [g_0]$ such that*

$$\inf_{g \in \mathcal{C}_m} \mathcal{F}_m(g) = \mathcal{F}_m(g_e^m), \tag{10}$$

*and $\forall m > \frac{n}{2}$, there is extremal metric $g_e^m \in [g_0]$ such that*

$$\sup_{g \in \mathcal{C}_m} \mathcal{F}_k(g) = \mathcal{F}_k(g_e^m), \tag{11}$$

*In fact, any solution to $\sigma_m(g) = constant$ is an extremal metric.*

*Proof.* First by Corollary 1, $(M, g_0)$ is conformal to a spherical space form. For any $g \in \mathcal{C}_m$, from [7] we know there is a conformal metric $\widetilde{g}$ in $\mathcal{C}_m$ satisfying that $\sigma_m(\widetilde{g})$ is constant and

(a). if $m > n/2$, then $\mathcal{F}_m(g) \leq \mathcal{F}_m(\widetilde{g})$.
(b). if $m < n/2$, then $\mathcal{F}_m(g) \geq \mathcal{F}_m(\widetilde{g})$.

A classification result of [13], [14] which is analogous to a result of Obata for the scalar curvature, shows that $\widetilde{g}$ has constant sectional curvature. Therefore $\widetilde{g}$ is the unique critical metric unless $M$ is conformally equivalent to $\mathbf{S}^n$, in which case any critical metric is the image of the standard metric under a conformal diffeomorphism. The clearly implies the conclusion of the Proposition. ∎

Next we consider the case $k < n/2$. We have

**Proposition 4.** *Suppose* $(M, g_0)$ *is locally conformally flat and* $g_0 \in \Gamma_k^+$ *for some* $k < \frac{n}{2}$. *Suppose furthermore that the space of solutions to the equation* $\sigma_k = C$ *is compact, where* $C$ *is a fixed constant. Then there is an extremal metric* $g_e^k \in [g_0]$ *such that*

$$\inf_{g \in \mathcal{C}_k} \mathcal{F}_k(g) = \mathcal{F}_k(g_e^k).$$

*Proof.* Since the space of solutions is assumed to be compact, there exists a critical metric $g_e^k$ which has least energy. If the functional assumed a value strictly lower than $\mathcal{F}_k(g_e^k)$, then by [7], the flow would decrease to another solution of $\sigma_k = constant$, which is a contradiction since $g_e^k$ has minimal energy. ∎

*Remark.* An explicit example of this situation is given by the locally conformally flat manifold $M = \mathbf{S}^1(T) \times \mathbf{S}^{n-1}$, where $T$ is the radius of the $S^1$ factor. A moving planes argument shows that all solutions of $\sigma_k = constant$ must be symmetric, and therefore the equation reduces to an ODE. The solutions were analyzed in [13]; depending on $T$ there are finitely many solutions, so the solution space is clearly compact.

Recently, it was announced in [10] that, based on the local estimates established in [6], if $M$ is locally conformally flat, and not conformally equivalent to $S^n$, then the solution space is compact. It follows for $k < n/2$ there always exists a global minimizer of $\mathcal{F}_k$.

We conclude with conformal quermassintegral inequalities, which were speculated in [7], and verified there for some special cases when $(M, g)$ is locally conformally flat and $g \in \Gamma^+_{\frac{n}{2}-1}$ or $g \in \Gamma^+_{\frac{n}{2}+1}$ using the flow method. In the case of $k = 2, n = 4$, the inequality was proved in [8] without the locally conformally flat assumption.

**Proposition 5.** *Suppose* $(M, g_0)$ *is locally conformally flat and* $g_0 \in \Gamma_k^+$ *for some* $k \geq \frac{n}{2}$, *then for any* $1 \leq l < \frac{n}{2} \leq k \leq n$ *there is a constant* $C(k, l, n) > 0$, *such that for any* $g \in [g_0]$ *and* $g \in \Gamma_k^+$

$$(\mathcal{F}_k(g))^{1/k} \leq C(k, l, n)(\mathcal{F}_l(g))^{1/l}, \tag{12}$$

*with equality if and only if* $(M, g)$ *is a spherical space form.*

*Proof.* By Proposition 3, we have a conformal metric $g_e$ of constant sectional curvature satisfies such that

$$\inf_{g \in \mathcal{C}_l} \mathcal{F}_l(g) = \mathcal{F}_l(g_e)$$

and

$$\sup_{g \in \mathcal{C}_k} \mathcal{F}_k(g) = \mathcal{F}_k(g_e).$$

Hence, we have for any $g \in \Gamma_k^+$

$$\begin{aligned} \frac{(\mathcal{F}_k(g))^{1/k}}{(\mathcal{F}_l(g))^{1/l}} &\leq \frac{(\mathcal{F}_k(g_e))^{1/k}}{(\mathcal{F}_l(g_e))^{1/l}} \\ &= \frac{(l!(n-l)!)^{1/l}}{(k!(n-k)!)^{1/k}}. \end{aligned}$$

When the equality holds, $g$ is an extremal of $\mathcal{F}_l$, hence a metric of constant sectional curvature by [13]. ∎

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