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ON TEMPORAL ASYMPTOTICS FOR THE P'TH POWER VISCOUS REACTIVE GAS

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Abstract. In this paper we investigate the long time behaviour of solutions to the system governing a heat-conductive, viscous reactive p'th power gas confined between two parallel plates. For initial-boundary value problems with the end points held at a prescribed temperature or insulated, we prove the global existence of physically relevant solutions and establish their rate of convergence to equilibria, for generic initial data. The estimates for different boundary conditions are presented in a unified manner.

1. Introduction.

The purpose of this paper is to describe the asymptotic behaviour of a viscous reactive Newtonian fluid, confined between two infinite parallel plates, and undergoing the dynamic combustion. We assume that the pressure $\mathcal{P}$, in terms of the absolute temperature θ and specific volume ξ, is given by

$$\mathcal{P} = \frac{\theta}{\xi^p}, \tag{1.1}$$

with the pressure exponent $p \geq 1$.

The complete system of governing equations, in the mass-Lagrangian form, expressing the balance of mass, momentum and energy, coupled with the description of the chemical reaction, is the following

$$\xi_t = v_x, \tag{1.2}$$

$$v_t = \left(-\mathcal{P} + \mu\frac{v_x}{\xi}\right)_x, \tag{1.3}$$

$$\theta_t = \left(-\mathcal{P} + \mu\frac{v_x}{\xi}\right)v_x + \left(\kappa\frac{\theta_x}{\xi}\right)_x + \delta f(\xi,\theta,z), \tag{1.4}$$

$$z_t = \left(\sigma\frac{z_x}{\xi^2}\right)_x - f(\xi,\theta,z). \tag{1.5}$$

Here the quantities ξ (specific volume), θ (temperature), v (velocity), and z (the concentration of the unburned fuel) are unknown functions of $(x,t) \in [0,1]\times[0,\infty)$, subject to physical constraints

$$\xi > 0, \quad \theta > 0, \quad z \geq 0, \tag{1.6}$$

while $\mu, \kappa, \sigma, \delta$, are positive constants, representing viscosity, conductivity, the species diffusion coefficient, and the reaction rate (the so-called Frank-Kamenetskii parameter), respectively. The intensity of the chemical reaction is given through the function f, depending on ξ, θ, z.

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We impose the following boundary conditions

$$v(0,t) = v(1,t) = 0, \quad z_x(0,t) = z_x(1,t) = 0, \tag{1.7}$$

along with either the Dirichlet temperature condition

$$\theta(0,t) = \theta(1,t) = \Theta, \tag{D}$$

where $\Theta > 0$ is a prescribed constant, or the Neumann condition

$$\theta_x(0,t) = \theta_x(1,t) = 0. \tag{N}$$

The initial data are given by:

$$\xi(x,0) = \xi_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x) \quad z(x,0) = z_0(x) \tag{1.8}$$

and subject to the physical constraints (1.6). The two initial-boundary value problems (1.1) – (1.8) along with the condition (D) or (N), will be referred to as $(IBVP)_D$ and $(IBVP)_N$ respectively.

Note that equations (1.3) and (1.4) may be written as conservation laws of momentum and energy of the system:

$$v_t = S_x, \tag{M}$$

$$\left(\theta + \frac{1}{2}v^2\right)_t = (Sv - q)_x + \delta f. \tag{E}$$

where $S = -\mathcal{P} + \mu v_x/\xi$ is the stress tensor and $q = -\kappa\theta_x/\xi$ the heat flux. The function f in (E) and (1.5) has typically the form

$$f(\xi, \theta, z) = \varepsilon \xi^{1-m} z^m \exp \frac{\theta - 1}{\varepsilon\theta}, \tag{1.9}$$

which is called the Arrhenius rate law for chemical reaction, with constants $\varepsilon > 0$ and $m \geq 1$. The model (1.1)–(1.5), (1.9) for the perfect gas case ($p = 1$) was introduced in [KP] and studied in [BB], [GZ], [CHT].

The first main result of this paper (Theorem I) concerns the global existence of physically relevant solutions to $(IBVP)_D$ and $(IBVP)_N$, for generic initial data. A key element of the proof is deriving the uniform upper and lower bounds on the density ξ of the fluid. The main idea here is due to Kazhikhov [K]. His method was later developed and extended to a variety of other boundary conditions or pressure laws [KS], [BB], [FP], [J], [N], [MY], [M], [W]. In our case we apply the analysis from [LW]. The form of the pressure $\mathcal{P}$ in (1.1) can be seen as a generalization of the equation of state for the prefect gas, as well a modification of the relation for the barotropic gas, where $\mathcal{P} = \xi^{-p}$. From this point of view, (1.1) is an interpolation between these models. For the analysis of existence issue for the barotropic fluid we refer to [M], [MY].

Our second concern is proving the convergence of the specific volume, velocity, temperature, and the concentration of species to their respective equilibrium values. The temporal asymptotics of a one-dimensional Newtonian fluid with different exponents $p \geq 1$ in (1.1) was studied in [LW]. In this work, however, the heat is not allowed to be generated by chemical reactions ($\delta = 0$), and the solution quantities converge exponentially fast. In general (Theorem II), the rate of convergence (exponential or power) depends on the parameter m in (1.9). More precisely, our analysis applies to all intensity functions f of the form

$$f(\xi, \theta, z) = z^m \tilde{f}(1/\xi, \theta, z),$$

where $m \geq 1$ is an integer and $\tilde{f}$ is continuous and defined on $(0, \infty) \times [0, \infty) \times [0, \infty)$. We additionally require that $\tilde{f}$ be positive in $(0, \infty) \times (0, \infty) \times [0, \infty)$ and be bounded on every set of the form $[\lambda, \Lambda] \times [0, \infty) \times [0, \infty)$.

The paper is organised as follows. In Section 2 we state our main theorems, while in Sections 3 and 4 we gather some preliminary global pointwise and integral estimates relevant for the further analysis. The key bounds are obtained in Sections 5 and 6. The final two Sections are devoted to establishing the convergence and its rate of the solutions to their equilibria. As in [LW], the estimates for different boundary conditions are presented in a unified manner.

2. Main results.

For the convenient formulation of the existence theory for the initial-boundary value problems, we recall the standard definitions of the Sobolev norms:

$$\begin{aligned} ||u_0||^2_{W_2^1((0,1))} &= \int_0^1 \left(u_0^2 + (u_0)_x^2\right) \mathrm{d}x, \\ ||u||^2_{W_2^{2,1}((0,1)\times(t,T))} &= \int_t^T \int_0^1 \left(u^2 + u_{xx}^2 + u_t^2\right) \mathrm{d}x\mathrm{d}\tau, \\ ||u||^2_{W_2^{1,0}((0,1)\times(t,T))} &= \int_t^T \int_0^1 \left(u^2 + u_x^2\right) \mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

Theorem 0. *Consider the inital-boundary value problems given by* $(IBVP)_D$ *or* $(IBVP)_N$. *Let the initial data*

$$\xi_0, v_0, \theta_0, z_0 \in W_2^1((0,1))$$

satisfy the physical constraints (1.6) and be compatible with the relevant boundary conditions. Then there exists a unique local regular solution (ξ, v, θ, z) *on* $[0,1] \times [0, T_{loc})$, *for some* $T_{loc} > 0$, *with*

$$v, \theta, z \in W_2^{2,1}((0,1) \times (0, T_{loc})), \qquad \xi, \xi_t \in W_2^{1,0}((0,1) \times (0, T_{loc})).$$

The proof follows from the elementary fluid mechanics theory (see [AKM]), based on the energy method, and thus we omit it. We remark that the result presented in Theorem 0 is not sharp, however since our interest is to analyze the long time behaviour of solutions, we require them to be as regular, as stated.

Our first main theorem concerns the global in time existence of solutions with the regularity as in Theorem 0. From now on we adopt the convention that any constant that appears will depend at most on the norms of the initial data, $\min_{x\in[0,1]} \xi_0(x)$ and $\min_{x\in[0,1]} \theta_0(x)$. Also, we denote such generic "small" constants by λ, and "large" constants by Λ.

Theorem I. *Let the assumptions of Theorem 0 be satisfied. Then there exists a unique global regular solution* (ξ, v, θ, z) *on* $[0,1] \times [0, \infty)$*, such that:*

$$v, \theta, z \in W_2^{2,1}((0,1) \times (0,T)), \qquad \xi, \xi_t \in W_2^{1,0}((0,1) \times (0,T)).$$

Moreover there holds

$$\sup_{k \in \mathbf{N}} ||v,\, \theta,\, z||_{W_2^{2,1}((0,1)\times(k,k+1))} + \sup_{k\in\mathbf{N}} ||\xi,\, \xi_t||_{W_2^{1,0}((0,1)\times(k,k+1))} \leq \Lambda$$

and

$$\lambda \leq \xi(x,t) \leq \Lambda, \qquad \int_0^\infty \int_0^1 \theta_x^2 \, dx d\tau \leq \Lambda.$$

In order to prove the global in time existence of solutions, it is enough to provide a control on the magnitudes of norms of their traces. The relevant uniform in time estimates, allowing the prolongation of the local solution, will be stated in Lemmas 6.1 and 6.2.

Based on these estimates and other bounds, derived in Sections 5 and 6, we are able to establish the rate of convergence of solutions to the equilibrium states. Here, a distinction arises between the $(IBVP)_D$ and $(IBVP)_N$ with respect to the limiting temperature. More precisely, seting

$$\bar{\Theta} = \begin{cases} \Theta & \text{for } (IBVP)_D, \\ \displaystyle\int_0^1 \left(\theta_0 + \frac{1}{2}v_0^2 + \delta z_0\right) \mathrm{d}x & \text{for } (IBVP)_N, \end{cases}$$

we have the following result:

Theorem II. *Let* (ξ, v, θ, z) *be as in Theorem I. Then:*

(i) $$\max_{x\in[0,1]} \left(|\xi(x,t) - 1| + |v(x,t)| + |\theta(x,t) - \bar{\Theta}|\right) \leq \begin{cases} \Lambda e^{-\lambda t} & \textit{for } m = 1, \\ \Lambda t^{-1/2(m-1)} & \textit{for } m > 1, \end{cases}$$

(ii) $$\max_{x\in[0,1]} z(x,t) \leq \begin{cases} \Lambda e^{-\lambda t} & \textit{for } m = 1, \\ \Lambda t^{-1/3(m-1)} & \textit{for } m > 1. \end{cases}$$

3. Elementary pointwise estimates.

In this Section we demonstrate that the solution quantities ξ and z satisfy the physical constraints while for the temperature θ a weaker form of (1.6) is valid. Our proof can be seen as an application of a standard maximum principle to the parabolic structure of the governing equations (1.4) and (1.5).

All subsequant results, unless otherwise indicated, apply to both boundary value problems $(IBVP)_D$ and $(IBVP)_N$.

Lemma 3.1.
(i) $\xi(x,t) > 0$,
(ii) $\theta(x,t) \geq 0$,
(iii) $0 \leq z(x,t) \leq \max_{x\in[0,1]} z_0(x)$.

Proof. The equation (1.2) can be equivalently written as

$$\varrho_t + \varrho^2 v_x = 0,$$

where $\varrho = \xi^{-1}$ is the density of the gas. Since $\xi_0 > 0$, we get

$$\varrho(x,t) = \varrho(x,0)\exp\left\{\int_0^t \varrho(x,s)v_x(x,s)dx\right\} > 0,$$

which establishes (i) for all times $t < T_{loc}$, where T_{loc} is as in Theorem 0.

To prove (ii), define a nonpositive function $\theta_-(x,t) = \min\{\theta(x,t),0\}$. Integrating (1.4) in space and neglecting positive terms in the right hand side we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_- \mathrm{d}x + \int_0^1 \frac{\theta_- v_x}{\xi^p}\mathrm{d}x \geq 0.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |\theta_-|\mathrm{d}x \leq C(t)\int_0^1 |\theta_-|\mathrm{d}x,$$

where $C(t) = \max\limits_{x\in[0,1]} |v_x(x,t)/\xi^p(x,t)|$. Noting $\int_0^1 |\theta_-|(x,0)\mathrm{d}x = 0$ and the local integrability of $C(t)$, following from Theorem 0, we conclude (ii).

Now set $z_-(x,t) = \min\{z(x,t),0\}$ and $z_+(x,t) = \max\{z(x,t), \max_{y\in[0,1]} z_0(y)\}$. Multiplying (1.5) by z_-, integrating in space and recalling the boundary conditions (1.7) we receive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 z_-^2\mathrm{d}x = -\sigma\int_0^1 \frac{(z_-,x)^2}{\xi^2}\mathrm{d}x - \int_0^1 f z_-\mathrm{d}x.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 z_-^2\mathrm{d}x \leq -2\int_0^1 f z_-\mathrm{d}x = -2\int_0^1 \tilde{f} z^{m-1} z_-^2\mathrm{d}x \leq C(t)\int_0^1 z_-^2\mathrm{d}x,$$

where the locally uniform bound $C(t)$ follows from the assumed properties of the function $\tilde{f}$. Recalling $\int_0^1 z_-^2(x,0)\mathrm{d}x = 0$, we conclude that $\int_0^1 z_-^2\mathrm{d}x = 0$, which implies $z(x,t) \geq 0$. In order to prove the upper bound in (iii), we multiply (1.5) by z_+ and integrate in space, obtaining

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 z_+^2\mathrm{d}x = -\sigma\int_0^1 \frac{(z_+,x)^2}{\xi^2}\mathrm{d}x - \int_0^1 f z_+\mathrm{d}x \leq 0.$$

In view of the initial condition $\int_0^1 z_+^2(x,0)\mathrm{d}x = 0$ we conclude that $\int_0^1 z_+^2\mathrm{d}x = 0$ and the proof is complete. □

4. Entropy and energy bounds.

In this Section we identify the relevant thermodynamic quantities and establish some general integral bounds related to the solution quantities ξ, v, θ, z.

Note first, that without loss of generality, we may assume that $\int_0^1 \xi_0(x)\mathrm{d}x = 1$. Then it follows from (1.2) and (1.7) that

$$\int_0^1 \xi(x,t)\mathrm{d}x = 1.$$

Recall now (compare [LW]) that the entropy η of a solution to (M), (E) is a concave function given by

$$\eta(\theta,\xi) = \ln\theta + h(\xi),$$
$$\text{where } h(\xi) = \begin{cases} \ln\xi & \text{for } p=1 \\ \frac{1}{p-1}\left(1-\xi^{1-p}\right) & \text{for } p>1 \end{cases}, \tag{4.1}$$

and satisfying the following entropy identity:

$$\eta_t = \mu\frac{v_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2} + \delta\frac{f}{\theta} - \left(\frac{q}{\theta}\right)_x. \tag{4.2}$$

For the Neumann problem, the entropy flux, $-q/\theta$, is zero at the boundary. For the Dirichlet problem this is generally not true; we can, however, track the global entropy change for either temperature boundary condition noting that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\theta + \frac{1}{2}v^2 + \delta z - \bar\Theta\eta\right)\mathrm{d}x \\ &\qquad = -\int_0^1 q_x \mathrm{d}x - \bar\Theta\int_0^1 \left(\mu\frac{v_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2} + \delta\frac{f}{\theta} - \left(\frac{q}{\theta}\right)_x\right)\mathrm{d}x \\ &\qquad = \bar\Theta\int_0^1 \left(\mu\frac{v_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2} + \delta\frac{f}{\theta}\right)\mathrm{d}x. \end{aligned} \tag{4.3}$$

The above formula follows from equations (M), (E), (4.2), and the fact that both conditions (D) and (N) imply that the term $(1-\bar\Theta/\theta)q$ vanishes at the boundary.

Lemma 4.1. (i) $\displaystyle\int_0^1 v^2(x,t)\mathrm{d}x \le \Lambda$,

(ii) $\displaystyle\lambda \le \int_0^1 \theta(x,t)\mathrm{d}x \le \Lambda,$

(iii) $\displaystyle\int_0^t\int_0^1 \left(\mu\frac{v_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2} + \delta\frac{f}{\theta}\right)\mathrm{d}x\mathrm{d}\tau \le \Lambda.$

Proof. Integrating (4.3) over $[0,t]$, we get

$$\int_0^1 \left(\theta + \frac{1}{2}v^2 + \delta z - \bar\Theta\eta\right)\mathrm{d}x + \bar\Theta\int_0^t\int_0^1 \left(\mu\frac{v_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2} + \delta\frac{f}{\theta}\right)\mathrm{d}x\mathrm{d}\tau \le \Lambda. \tag{4.4}$$

Integrating (4.1) in space and then using Jensen's inequality we receive

$$\int_0^1 \eta\mathrm{d}x \le \ln\left(\int_0^1\theta\mathrm{d}x\right) + h\left(\int_0^1 \xi\mathrm{d}x\right) = \ln\left(\int_0^1 \theta\mathrm{d}x\right).$$

Thus, in view of (4.4) we see that

$$\begin{aligned} \int_0^1 \left(\theta + \frac{1}{2}v^2 + \delta z\right)\mathrm{d}x + \bar\Theta\int_0^t\int_0^1 \left(\mu\frac{v_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2} + \delta\frac{f}{\theta}\right)\mathrm{d}x\mathrm{d}\tau \\ \le \Lambda + \bar\Theta\ln\left(\int_0^1\theta\mathrm{d}x\right). \end{aligned} \tag{4.5}$$

In particular, $\int_0^1\theta\mathrm{d}x \le \Lambda + \bar\Theta\ln\left(\int_0^1\theta\mathrm{d}x\right)$, which yields (ii). Using (ii) in (4.5) we establish (i) and (iii). ☐

5. Pointwise bound on ξ and global L^2 bound on θ_x.

This Section is devoted to proving the bounds stated in Theorem I, which are crucial for establishing the convergence result of Theorem II. A central difficulty in the derivation of the pointwise uniform bound on the specific volume ξ is associated with the presence of an a priori unknown impulse $\int_0^1 S(1,\tau)\mathrm{d}\tau$, arising at the boundary. The requisite bound is obtained through an analysis of the momentum balance (M), in combination with estimates following from (4.2) and Lemma 4.1. Since the argument does not involve reaction terms, it can be carried out exactly as in the proof of Theorem I in [LW], and thus we omit it.

Theorem 5.1. $\lambda \le \xi(x,t) \le \Lambda$.

In the remaining part of this Section we show the other crucial bound, which is a global L^2 estimate on the temperature gradient. First, we note the following simple consequence of the equation (1.5).

Lemma 5.2. $\displaystyle\int_0^t\int_0^1 z_x^2 \mathrm{d}x\mathrm{d}\tau \le \Lambda.$

Proof. Multiply (1.5) by z and integrate in space, to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 z^2\mathrm{d}x + \sigma\int_0^1 \frac{z_x^2}{\xi^2}\mathrm{d}x \le 0.$$

Now integrating in time yields

$$\frac{1}{2}\int_0^1 z^2(x,t)\mathrm{d}x + \sigma\int_0^t\int_0^1 \frac{z_x^2}{\xi^2}\mathrm{d}x\mathrm{d}\tau \le \Lambda,$$

which completes the proof. □

The main difficulty in establishing the temperature gradient bound comes from the Dirichlet temperature condition, due to the a priori unknown energy flux $\int_0^1[q(0,\tau)-q(1,\tau)]\mathrm{d}\tau$, through the boundary. This is circumvented by identifying a thermodynamic quantity ω which serves as a Lyapunov function for solutions.

We begin by stating the preliminary estimates, proved in [LW].

Lemma 5.3. (i) *Define* $v_m(t) = \max_{x\in[0,1]} v(x,t)$. *Then* $\displaystyle\int_0^t v_m^2\mathrm{d}\tau \le \Lambda.$

(ii) $\displaystyle\int_0^t\int_0^1 \left(S^2v^2 + v^2v_x^2\right)\mathrm{d}x\mathrm{d}\tau \le \Lambda\left(1+\int_0^t\int_0^1 \theta^2v^2\mathrm{d}x\mathrm{d}\tau\right).$

(iii) *For every* $\varepsilon > 0$ *there exists a constant* Λ *such that*

$$\int_0^1 \xi_x^2(x,t)\mathrm{d}x + \int_0^t\int_0^1 \xi_x^2\mathrm{d}x\mathrm{d}\tau \le \Lambda + \varepsilon\int_0^t\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}\tau.$$

Theorem 5.4. $\displaystyle\int_0^t\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}\tau \le \Lambda.$

Proof. Let the constant $\gamma > 0$ be such that the quantity

$$\omega = \theta + \frac{1}{2}v^2 + \delta z - \bar{\Theta}\eta + \gamma,$$

satisfies $\omega \geq \theta/2$ (for $\theta \geq 0$ and $\lambda \leq \xi \leq \Lambda$, compare Theorem 5.1). Note that by (M), (E) and (4.2) we have

$$\omega_t = (Sv)_x + \delta\sigma\left(\frac{z_x}{\xi^2}\right)_x - \left(\left(1-\frac{\bar{\Theta}}{\theta}\right)q\right)_x - \bar{\Theta}\left(\mu\frac{v_x^2}{\xi\theta} + \kappa\frac{\theta_x^2}{\xi\theta^2} + \delta\frac{f}{\theta}\right)$$

$$\omega_x = \left(1-\frac{\bar{\Theta}}{\theta}\right)\theta_x + vv_x + \delta z_x - \bar{\Theta}h'(\xi)\xi_x.$$

Utilizing the above identities, Lemma 5.3 (ii) and (iv), and recalling the boundary conditions, we obtain from integration by parts and Young's inequality:

$$\begin{aligned}
\frac{1}{2}\int_0^1 \omega^2(x,t)\mathrm{d}x &\leq \Lambda + \int_0^t\int_0^1 \omega\omega_t \mathrm{d}x\mathrm{d}\tau \\
&\leq \Lambda + \int_0^t\int_0^1 \omega\left(Sv + \delta\sigma\frac{z_x}{\xi^2} - \left(1-\frac{\bar{\Theta}}{\theta}\right)q\right)_x \mathrm{d}x\mathrm{d}\tau \\
&= \Lambda - \int_0^t\int_0^1 \omega_x\left(Sv + \delta\sigma\frac{z_x}{\xi^2} + \frac{\kappa}{\xi}\left(1-\frac{\bar{\Theta}}{\theta}\right)\theta_x\right)\mathrm{d}x\mathrm{d}\tau \\
&\leq \Lambda - \lambda\int_0^t\int_0^1\left(1-\frac{\bar{\Theta}}{\theta}\right)^2\theta_x^2\mathrm{d}x\mathrm{d}\tau \\
&\qquad + \Lambda\int_0^t\int_0^1\left(S^2v^2 + v^2v_x^2 + \xi_x^2 + z_x^2\right)\mathrm{d}x\mathrm{d}\tau, \\
&\leq \Lambda - \lambda\int_0^t\int_0^1\left(1-\frac{\bar{\Theta}}{\theta}\right)^2\theta_x^2\mathrm{d}x\mathrm{d}\tau + \Lambda\int_0^t\int_0^1\left(\theta^2v^2 + \xi_x^2\right)\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{5.1}$$

But Lemma 3.1 (iii) gives

$$\begin{aligned}
\int_0^t\int_0^1\left(1-\frac{\bar{\Theta}}{\theta}\right)^2\theta_x^2\mathrm{d}x\mathrm{d}\tau &\geq \int_0^t\int_0^1\left(\frac{1}{2}-\frac{\bar{\Theta}^2}{\theta^2}\right)\theta_x^2\mathrm{d}x\mathrm{d}\tau \\
&\geq \frac{1}{2}\int_0^t\int_0^1\theta_x^2\mathrm{d}x\mathrm{d}\tau - \Lambda,
\end{aligned} \tag{5.2}$$

which, together with (5.1) and in view of $\omega \geq \theta/2$ implies

$$\begin{aligned}
\int_0^1\theta^2(x,t)\mathrm{d}x + \int_0^t\int_0^1\theta_x^2\mathrm{d}x\mathrm{d}\tau &\leq \Lambda + \Lambda\int_0^t\int_0^1\left(\theta^2v^2+\xi_x^2\right)\mathrm{d}x\mathrm{d}\tau \\
&\leq \Lambda + \Lambda\int_0^t v_m^2(\tau)\int_0^1\theta^2(x,\tau)\mathrm{d}x\mathrm{d}\tau + \Lambda\int_0^t\int_0^1\xi_x^2\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{5.3}$$

Now from Lemma 5.3 (i) and Gronwall's inequality applied to (5.3) we see that

$$\int_0^1\theta^2(x,t)\mathrm{d}x \leq \Lambda\left(1+\int_0^t\int_0^1\xi_x^2\mathrm{d}x\mathrm{d}\tau\right). \tag{5.4}$$

Substituting (5.4) into (5.3) and again noting Lemma 5.3 (i), yields

$$\int_0^1\theta^2\mathrm{d}x + \int_0^t\int_0^1\theta_x^2\mathrm{d}x\mathrm{d}\tau \leq \Lambda\left(1+\int_0^t\int_0^1\xi_x^2\mathrm{d}x\mathrm{d}\tau\right). \tag{5.5}$$

This estimate when combined with Lemma 5.3 (iii) completes the proof. □

6. Global integral bounds on ξ, v, θ, z.

In this Section we establish several uniform in time bounds of the norms of various combinations of the solution quantities and their derivatives. This will, in particular, imply the existence of a regular global solution to $(IBVP)_D$ or $(IBVP)_N$, as stated in Theorem I. Another consequence of the bounds in Lemmas 6.1 and 6.2 is the L^2 convergence of spacial derivatives of the solution to 0 (Lemma 6.3), from which we will deduce the pointwise convergence in Section 7.

We first recall a few estimates, proved in [LW].

Lemma 6.1. (i) $\displaystyle\int_0^1 \left(\xi_x^2 + v_x^2\right) \mathrm{d}x \le \Lambda,$

(ii) $\displaystyle\int_0^t \int_0^1 \left(\xi_x^2 + v_x^2 + v_{xx}^2 + \theta^2 \xi_x^2 + \theta^2 v_x^2 + \xi_x^2 v_x^2 + v_x^4\right) \mathrm{d}x\mathrm{d}\tau \le \Lambda.$

Our next goal is to prove the bounds involving the reaction equation (1.5) and the conservation of energy (1.4).

Lemma 6.2. (i) $\displaystyle\int_0^t \int_0^1 f \mathrm{d}x\mathrm{d}\tau \le \Lambda$ *and* $0 \le f(\xi, \theta, z)(x,t) \le \Lambda,$

(ii) $\displaystyle\int_0^1 \theta_x^2 \mathrm{d}x + \int_0^t \int_0^1 \left(\theta_{xx}^2 + \theta_x^2 \xi_x^2\right) \mathrm{d}x\mathrm{d}\tau \le \Lambda,$

(iii) $\displaystyle\int_0^1 z_x^2 \mathrm{d}x + \int_0^t \int_0^1 \left(z_{xx}^2 + z_x^2 \xi_x^2\right) \mathrm{d}x\mathrm{d}\tau \le \Lambda.$

Proof. To prove (i) note the following simple consequence of (1.4), (1.7) and Lemma 3.1 (iii):

$$\int_0^t \int_0^1 f \mathrm{d}x\mathrm{d}\tau = -\int_0^t \int_0^1 z_t \mathrm{d}x\mathrm{d}\tau = \int_0^1 z_0 \mathrm{d}x - \int_0^1 z(x,t)\mathrm{d}x \le \Lambda.$$

The boundedness of f is a straightforward consequence of the assumed properties of $\tilde{f}$ and the global pointwise bounds on ξ and z.

Next, from (E), integration by parts and Young's inequality, it follows for either boundary conditions (D) or (N) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \theta_x^2 \mathrm{d}x\right) &= 2\int_0^1 \theta_x \theta_{xt} \mathrm{d}x = -2\int_0^1 \theta_t \theta_{xx} \mathrm{d}x \\
&\le \Lambda \int_0^1 \left(|\theta v_x \theta_{xx}| + |v_x^2 \theta_{xx}| + |\theta_x \xi_x \theta_{xx}| + |f \theta_{xx}|\right) \mathrm{d}x - \lambda \int_0^1 \theta_{xx}^2 \mathrm{d}x \\
&\le \Lambda \int_0^1 \left(\theta^2 v_x^2 + v_x^4 + \theta_x^2 \xi_x^2 + f^2\right) \mathrm{d}x - \lambda \int_0^1 \theta_{xx}^2 \mathrm{d}x.
\end{aligned} \tag{6.1}$$

Note the following interpolation inequality

$$\max_{x \in [0,1]} \theta_x^2(x,t) \le \Lambda \int_0^1 \theta_x^2(x,t)\mathrm{d}x + \lambda \int_0^1 \theta_{xx}^2(x,t)\mathrm{d}x. \tag{6.2}$$

Thus, in view of Theorem 5.4:

$$\begin{aligned}
\int_0^t \int_0^1 \theta_x^2 \xi_x^2 \mathrm{d}x\mathrm{d}\tau &\le \int_0^t \max_{x \in [0,1]} \theta_x^2(x,\tau) \left(\int_0^1 \xi_x^2 \mathrm{d}x\right) \mathrm{d}\tau \\
&\le \Lambda + \lambda \int_0^t \int_0^1 \theta_{xx}^2 \mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{6.3}$$

Now, integrating (6.1) over $[0,t]$ and noting (i), we receive the boundedness of the first two summands in (ii). Recalling (6.3) concludes the proof of (ii).

To demonstrate (iii), we proceed in a similar manner; by (1.5), the boundary condition (1.7) and Young's inequality we obtain:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 z_x^2 \mathrm{d}x\right) &= 2\int_0^1 z_x z_{xt}\mathrm{d}x = -2\int_0^1 z_t z_{xx}\mathrm{d}x \\ &= -2\sigma\int_0^1 z_{xx}\left(\frac{z_x}{\xi^2}\right)_x \mathrm{d}x + 2\int_0^1 f z_{xx}\mathrm{d}x \\ &\le \Lambda\int_0^1 \left(z_x^2\xi_x^2 + f^2\right)\mathrm{d}x - \lambda\int_0^1 z_{xx}^2\mathrm{d}x. \end{aligned} \tag{6.4}$$

Again, since

$$\max_{x\in[0,1]} z_x^2(x,t) \le \Lambda\int_0^1 z_x^2(x,t)\mathrm{d}x + \lambda\int_0^1 z_{xx}^2(x,t)\mathrm{d}x,$$

in view of Lemma 5.2 it follows that

$$\begin{aligned} \int_0^t\int_0^1 z_x^2\xi_x^2\mathrm{d}x\mathrm{d}\tau &\le \int_0^t \max_{x\in[0,1]} z_x^2(x,\tau)\left(\int_0^1 \xi_x^2\mathrm{d}x\right)\mathrm{d}\tau \\ &\le \Lambda + \lambda\int_0^t\int_0^1 z_{xx}^2\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{6.5}$$

Upon integrating (6.4) in time and inserting (6.5), the boundedness of the first two summands in (iii) follows. Hence, by (6.5) the proof is complete. □

Note that due to the results of Section 5, we have actually concluded the proof of Theorem I. Here comes our first convergence result.

Lemma 6.3. $\lim_{t\to+\infty}\int_0^1 \left(\xi_x^2 + v_x^2 + \theta_x^2 + z_x^2\right)(x,t)\mathrm{d}x = 0.$

Proof. It is sufficient to show that the following functions and their derivatives are integrable in time: $\int_0^1 \xi_x^2(x,t)\mathrm{d}x$, $\int_0^1 v_x^2(x,t)\mathrm{d}x$, $\int_0^1 \theta_x^2(x,t)\mathrm{d}x$, $\int_0^1 z_x^2(x,t)\mathrm{d}x$. The integrability of the mentioned functions is stated in Theorem 5.4, Lemma 6.1, and Lemma 5.2. The integrability of the derivatives of $\int_0^1 \theta_x^2(x,t)\mathrm{d}x$ and $\int_0^1 z_x^2(x,t)\mathrm{d}x$ is a consequence of (6.3), (6.4) and Lemma 6.1. To deal with the remaining two derivatives, we note that by (M) and (1.2) there holds:

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 v_x^2\mathrm{d}x\right)\right| &= \left|2\int_0^1 v_x v_{xt}\mathrm{d}x\right| \\ &= \left|2\int_0^1 v_t v_{xx}\mathrm{d}x\right| \le \Lambda\int_0^1 \left(\theta_x^2 + \theta^2\xi_x^2 + \xi_x^2 v_x^2 + v_{xx}^2\right)\mathrm{d}x, \\ \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \xi_x^2\mathrm{d}x\right)\right| &= \left|2\int_0^1 \xi_x\xi_{xt}\mathrm{d}x\right| = \left|2\int_0^1 \xi_x v_{xx}\mathrm{d}x\right| \le \Lambda\int_0^1 \left(\xi_x^2 + v_{xx}^2\right)\mathrm{d}x. \end{aligned}$$

From Lemma 6.1, the proof is complete. □

7. Pointwise convergence results.

We start by strengthening the result of Lemma 3.1 (ii), which justifies the physical relevance of the solution θ.

Lemma 7.1. $\lambda \leq \theta(x,t) \leq \Lambda$.

Proof. First, note that for the $(IBVP)_D$ and $(IBVP)_N$ there holds, respectively:

$$|\theta(x,t) - \Theta| \leq \int_0^1 |\theta_x| \mathrm{d}x \leq \left(\int_0^1 \theta_x^2 \mathrm{d}x \right)^{1/2}, \tag{7.1}$$

and

$$\left| \theta(x,t) - \int_0^1 \theta \mathrm{d}x \right| \leq \left(\int_0^1 \theta_x^2 \mathrm{d}x \right)^{1/2}. \tag{7.2}$$

Thus, in view of Lemma 6.3 and Lemma 4.1 (ii), we see that the proof is completed by showing that θ is bounded away from 0 on a bounded time interval $[0,T]$. Define $w(x,t) = \theta^{-1}(x,t)$. By (1.4) we receive

$$w_t - \kappa \left(\frac{w_x}{\xi} \right)_x = \frac{w}{\xi^p} v_x - \mu \frac{w^2}{\xi} v_x^2 - 2\kappa \frac{w_x^2}{w\xi} - \delta w^2 f \leq \frac{w}{\xi^p} v_x - \mu \frac{w^2}{\xi} v_x^2. \tag{7.3}$$

Now define

$$w_+(x,t) = \max\left\{ w(x,t) - \bar{\Theta}^{-1}, 0 \right\}.$$

Fix a natural number N and note that multiplying (7.3) by a nonnegative factor w_+^{N-1}, by Young's inequality we get

$$\begin{aligned} (w_+)_t w_+^{N-1} - \kappa \left(\frac{(w_+)_x}{\xi} \right)_x w_+^{N-1} &\leq \frac{v_x}{\xi^p} w w_+^{N-1} - \mu \frac{v_x^2}{\xi} w^2 w_+^{N-1} \\ &\leq \lambda \left(|v_x| w w_+^{(N-1)/2} \right)^2 + \Lambda \left(w_+^{(N-1)/2} \right)^2 - \mu \frac{v_x^2}{\xi} w^2 w_+^{N-1} \leq \Lambda w_+^{N-1}, \end{aligned} \tag{7.4}$$

Integrating by parts and recalling the initial conditions (D) or (N) we have

$$\int_0^1 \left(\frac{(w_+)_x}{\xi} \right)_x w_+^{N-1} \mathrm{d}x = \left. \frac{(w_+)_x}{\xi} w_+^{N-1} \right|_0^1 - (N-1) \int_0^1 \frac{(w_+)_x^2}{\xi} w_+^{N-2} \mathrm{d}x \leq 0,$$

and thus, integrating (7.4) in space, by Hölder's inequality, we obtain

$$\frac{1}{N} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^1 w_+^N \mathrm{d}x \right) \leq \Lambda \int_0^1 w_+^{N-1} \mathrm{d}x \leq \Lambda \left(\int_0^1 w_+^N \mathrm{d}x \right)^{(N-1)/N}.$$

Hence, for every N there holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^1 w_+^N \mathrm{d}x \right)^{1/N} = \frac{1}{N} \left(\int_0^1 w_+^N \mathrm{d}x \right)^{1/N-1} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^1 w_+^N \mathrm{d}x \right) \leq \Lambda,$$

and we see that the following bound is true for every large number N and every $t \in [0,T]$:

$$\left(\int_0^1 w_+^N(x,t) \mathrm{d}x \right)^{1/N} \leq \Lambda(1+T). \tag{7.5}$$

Since the constant Λ in (7.5) is independent of N, we have:

$$\max_{t\in[0,T]}\max_{x\in[0,1]} w_+(x,t) \leq \Lambda(1+T),$$

which yields the desired lower bound on θ. □

We are now in a position to prove the pointwise convergence of solutions.

Theorem 7.2. $\lim_{t\to+\infty}\max_{x\in[0,1]}\big(|\xi(x,t)-1| + |v(x,t)| + |\theta(x,t)-\bar{\Theta}| + z(x,t)\big) = 0.$

Proof. From (1.7) we have

$$|v(x,t)| \leq \int_0^1 |v_x|\mathrm{d}x \leq \left(\int_0^1 v_x^2\mathrm{d}x\right)^{1/2}. \tag{7.6}$$

Also,

$$|\xi(x,t)-1| = \left|\xi(x,t) - \int_0^1 \xi\mathrm{d}x\right| \leq \left(\int_0^1 \xi_x^2\mathrm{d}x\right)^{1/2}. \tag{7.7}$$

Thus the pointwise convergence of ξ and v ma be immediately concluded, in view of Lemma 6.3.

For the convergence of z, note that by (1.5) and (1.7) there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 z\mathrm{d}x\right) = -\int_0^1 f\mathrm{d}x \leq 0. \tag{7.8}$$

On the other hand, by (1.5), the assumed properties of $\tilde{f}$, and Young's inequality we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 z\mathrm{d}x\right) + \int_0^1 f\mathrm{d}x \geq \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 z\mathrm{d}x\right) + \lambda\left(\int_0^1 z\mathrm{d}x\right)^m,$$

and thus for large times t there holds:

$$\int_0^1 z(x,t)\mathrm{d}x \leq \begin{cases} \Lambda e^{-\lambda t} & \text{for } m=1, \\ \Lambda t^{-1/(m-1)} & \text{for } m>1. \end{cases} \tag{7.9}$$

Since

$$\left|z(x,t) - \int_0^1 z\mathrm{d}x\right| \leq \left(\int_0^1 z_x^2\mathrm{d}x\right)^{1/2},$$

by Lemma 6.3 we receive the convergence of z.

Finally, the pointwise convergence of θ to the equilibrium temperature $\bar{\Theta}$ can be proved for $(IBVP)_D$ directly from (7.1) and Lemma 6.3. For the Neumann problem $(IBVP)_N$ we first note that the quantity $\int_0^1 \big(\theta + v^2/2 + \delta z\big)\,\mathrm{d}x$ is constant in time, and thus

$$\big|\theta(x,t)-\bar{\Theta}\big| \leq \left|\theta(x,t) - \int_0^1 \theta\mathrm{d}x\right| + \frac{1}{2}\int_0^1 v^2(x,t)\mathrm{d}x + \delta\int_0^1 z(x,t)\mathrm{d}x. \tag{7.10}$$

By (7.2), Lemma 6.3 and already established pointwise convergence of v and z, the proof is complete. □

8. Proof of Theorem II.

In this Section we establish the rate of convergence of solutions to their equilibrium values. Here, the identity (4.3) supplies a natural Lyapunov function $\mathcal{A}$, upon which we build our proof. Noting the pointwise convergence in Theorem 7.2, the result follows through a Taylor expansion associated with $\mathcal{A}$.

Define the following nonnegative quantities:

$$\mathcal{V}(t) := \int_0^1 \left(v_x^2 + \theta_x^2\right)(x,t)\mathrm{d}x, \quad \mathcal{D}(t) := \int_0^1 \left(\mu\frac{\xi_x}{\xi} - v\right)^2 (x,t)\mathrm{d}x,$$
$$\mathcal{A}(t) := \int_0^1 \left(\theta + \frac{1}{2}v^2 + \delta z - \bar{\Theta}\eta + \gamma\right)(x,t)\mathrm{d}x,$$

where $\gamma = \bar{\Theta}(\ln\bar{\Theta} - 1)$.

Lemma 8.1. *There exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{A} + \varepsilon_1\mathcal{D} + \varepsilon_2\mathcal{V}\right) + \lambda\left(\mathcal{A} + \varepsilon_1\mathcal{D} + \varepsilon_2\mathcal{V}\right) \leq \int_0^1 z\mathrm{d}x.$$

Proof. The proof will be divided into several steps.

STEP 1. Observing the boundedness of θ, it follows from the Taylor expansion of ln, that

$$\lambda(\theta - \bar{\Theta})^2 \leq (\theta - \bar{\Theta}\ln\theta) + \gamma \leq \Lambda(\theta - \bar{\Theta})^2. \tag{8.1}$$

Analogously, using the boundedness of ξ, and the concavity of h:

$$\lambda\int_0^1 (\xi - 1)^2\mathrm{d}x \leq -\int_0^1 h(\xi)\mathrm{d}x \leq \Lambda\int_0^1 (\xi - 1)^2\mathrm{d}x. \tag{8.2}$$

Adding (8.1) and (8.2) yields:

$$\begin{aligned}\lambda\int_0^1 &\left((\theta - \bar{\Theta})^2 + (\xi - 1)^2 + v^2 + z\right)\mathrm{d}x \\ &\leq \mathcal{A} \leq \Lambda\int_0^1 \left((\theta - \bar{\Theta})^2 + (\xi - 1)^2 + v^2 + z\right)\mathrm{d}x.\end{aligned} \tag{8.3}$$

Since

$$\frac{1}{2}\mu\left(\frac{\xi_x}{\xi}\right)^2 \leq v^2 + \left(\mu\frac{\xi_x}{\xi} - v\right)^2, \tag{8.4}$$

and recalling (7.1) for the Dirichlet condition, or (7.10) and (7.2) for (N), together with (7.7), (7.6) and (8.3) we receive

$$\begin{aligned}\mathcal{A}(t) &\leq \Lambda\int_0^1 \left(\theta_x^2 + \xi_x^2 + v^2 + z\right)\mathrm{d}x \\ &\leq \Lambda\left[\mathcal{D}(t) + \int_0^1 \left(\theta_x^2 + v^2 + z\right)\mathrm{d}x\right] \leq \Lambda\left(\mathcal{D}(t) + \mathcal{V}(t) + \int_0^1 z\mathrm{d}x\right).\end{aligned} \tag{8.5}$$

STEP 2. We are now going to establish the following bound:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t) + \lambda\mathcal{D}(t) \leq \Lambda\mathcal{V}(t). \tag{8.6}$$

The balance of momentum (M) can be rewritten in the form:

$$\left(\mu\frac{\xi_x}{\xi} - v\right)_t = \left(\frac{\theta}{\xi^p}\right)_x .$$

Multiplying by $\mu\xi_x/\xi - v$ and integrating over $[0,1]$, we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t) &= 2\int_0^1 \left(\frac{\theta_x}{\xi^p} - p\frac{\theta\xi_x}{\xi^{p+1}}\right)\left(\mu\frac{\xi_x}{\xi} - v\right)\mathrm{d}x \\ &\le -\lambda\int_0^1 \theta\left(\mu\frac{\xi_x}{\xi} - v\right)^2 \mathrm{d}x + \Lambda\int_0^1 \left(\mu\frac{\xi_x}{\xi} - v\right)\left(\frac{\theta_x}{\xi^p} - \frac{p\theta}{\mu\xi^p}v\right)\mathrm{d}x,\end{aligned}$$

which in view of Lemma 7.1 and using Young's inequality implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t) + \lambda\mathcal{D}(t) \le \Lambda\int_0^1 \left(\frac{\theta_x}{\xi^p} - \frac{p\theta}{\mu\xi^p}v\right)^2 \mathrm{d}x \le \Lambda\int_0^1 \left(\theta_x^2 + v^2\right)\mathrm{d}x.$$

Now (8.6) follows, by the above inequality and (7.6).

STEP 3. Recalling (M), integrating by parts and using Young's inequality gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 v_x^2\mathrm{d}x\right) &= 2\int_0^1 v_x v_{xt}\mathrm{d}x = -2\int_0^1 v_t v_{xx}\mathrm{d}x \\ &\le \Lambda\int_0^1 \left(|\theta_x v_{xx}| + |\theta\xi_x v_{xx}| + |\xi_x v_x v_{xx}|\right)\mathrm{d}x - \lambda\int_0^1 v_{xx}^2\mathrm{d}x \\ &\le \Lambda\int_0^1 \left(\theta_x^2 + \theta^2\xi_x^2 + \xi_x^2 v_x^2\right)\mathrm{d}x - \lambda\int_0^1 v_{xx}^2\mathrm{d}x.\end{aligned}$$

Thus, together with (6.1), we receive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(t) + \lambda\int_0^1 \left(v_{xx}^2 + \theta_{xx}^2\right)\mathrm{d}x \\ &\qquad \le \Lambda\left[\mathcal{V}(t) + \int_0^1 \left(v_x^4 + \xi_x^2 + \theta_x^2\xi_x^2 + \xi_x^2 v_x^2 + f^2\right)\mathrm{d}x\right].\end{aligned} \tag{8.7}$$

Noting the boundedness of $\int_0^1 \left(\xi_x^2 + v_x^2\right)\mathrm{d}x$ (by Theorem 6.1(i)), the interpolation inequality (6.2) and the analogous one below:

$$\max_{x\in[0,1]} v_x^2(x,t) \le \Lambda\int_0^1 v_x^2(x,t)\mathrm{d}x + \lambda\int_0^1 v_{xx}^2(x,t)\mathrm{d}x,$$

in view of Lemma 6.2 (i), we see that the integral on the right hand side of (8.7) is estimated by

$$\lambda\int_0^1 \left(v_{xx}^2 + \theta_{xx}^2\right)\mathrm{d}x + \Lambda\left(\mathcal{V}(t) + \int_0^1 \left(\xi_x^2 + f\right)\mathrm{d}x\right).$$

Hence, recalling (8.4) and (7.6), we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(t) \le \Lambda\left(\mathcal{D}(t) + \mathcal{V}(t) + \int_0^1 f\mathrm{d}x\right). \tag{8.8}$$

STEP 4. By (4.3), Theorem 5.1 and Lemma 7.1 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t) + \lambda\mathcal{V}(t) \le -\bar{\Theta}\delta\int_0^1 \frac{f}{\theta}\mathrm{d}x \le -\lambda\int_0^1 f\mathrm{d}x. \tag{8.9}$$

Now, bringing together (8.5), (8.6), (8.8) and (8.9), the theorem follows. ☐

Proof of Theorem II. We first deal with convergence of ξ, v and θ. We treat separately the two cases $m = 1$ and $m > 1$. If $m = 1$ then by Lemma 8.1, (7.9), and the nonnegativity od $\mathcal{A}$ (see (8.3)), we receive immediately that

$$\mathcal{D} + \mathcal{V} \le \Lambda\left(\mathcal{A} + \varepsilon_1 \mathcal{D} + \varepsilon_2 \mathcal{V}\right) \le \Lambda e^{-\lambda t}.$$

By (7.6), (7.7), (8.4) and (7.1), (7.2), (7.10), the result follows.

If $m > 1$, then by Lemma 8.1 and (7.9)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{A} + \varepsilon_1 \mathcal{D} + \varepsilon_2 \mathcal{V}\right) + \lambda\left(\mathcal{A} + \varepsilon_1 \mathcal{D} + \varepsilon_2 \mathcal{V}\right) \le \Lambda t^{-1/(m-1)},$$

and thus

$$\mathcal{D} + \mathcal{V} \le \Lambda\left(\mathcal{A} + \varepsilon_1 \mathcal{D} + \varepsilon_2 \mathcal{V}\right) \le \Lambda e^{-\lambda t}\left(\Lambda + \int_1^t e^{\lambda s} s^{-\alpha} \mathrm{d}s\right), \tag{8.10}$$

with $\alpha = 1/(m-1)$. Note that for large s we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{\lambda s} s^{-\alpha}\right) = \left(\lambda - \alpha s^{-1}\right) s^{-\alpha} e^{\lambda s} > 0.$$

Hence, for large times t we receive

$$\begin{aligned} e^{-\lambda t}\int_1^t e^{\lambda s} s^{-\alpha}\mathrm{d}s &= e^{-\lambda t}\int_1^{t/2} e^{\lambda s} s^{-\alpha}\mathrm{d}s + e^{-\lambda t}\int_{t/2}^t e^{\lambda s} s^{-\alpha}\mathrm{d}s \\ &\le e^{-\lambda t/2}\left(\frac{t}{2}\right)^{-\alpha} + e^{-\lambda t}\left(\frac{t}{2}\right)^{-\alpha}\int_{t/2}^t e^{\lambda s}\mathrm{d}s \le \Lambda\left(\frac{t}{2}\right)^{-\alpha}. \end{aligned}$$

Recalling (8.10), we conclude

$$\mathcal{D} + \mathcal{V} \le \Lambda t^{-1/(m-1)}.$$

As in the case $m = 1$, this estimate implies the pointwise convergence rate of the solution, as stated in (i).

Now we turn to proving (ii). Using an interpolation inequality from [BIN] (Chap. XVIII) we see that for any constant $\varepsilon \in (0,1)$ there holds

$$\max_{x\in[0,1]} z(x,t) \le \varepsilon\left(\int_0^1 z_x^2 \mathrm{d}x\right)^{1/2} + \Lambda\varepsilon^{-2}\int_0^1 z\mathrm{d}x. \tag{8.11}$$

Now, in view of Lemma 6.2 (iii) and (7.9), the estimate (ii) is obtained by substituting $\varepsilon = e^{-\lambda t/3}$ in (8.11) for the case $m = 1$, and $\varepsilon = t^{-1/3(m-1)}$, when $m > 1$. □

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