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Minimizing deformation of Legendrian submanifolds in the standard sphere

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by

Hông Vân Lê

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Hông-Vân Lê
Max-Planck-Institute for Mathematics in Sciences
Inselstraße 22-26
D-04103 Leipzig
email: hvle@ mis.mpg.de

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Abstract

In this note we study the moduli space of minimal Legendrian submanifolds in the standard sphere S^{2n-1}. We suggest to find new examples of minimal Legendrian submanifolds by solving a certain equation for a function on a nearby glued Legendrian submanifold, or by using certain evolution equation on the space of immersed Legendrian submanifolds . A new neccesary condition for a Lagrangian embedding into $\mathbf{C}P^n$ is given.

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Keyword: minimal Legendrian submanifolds.
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1 Introduction

It is well-known that we can study singularities of special Lagrangian submanifolds by considering the link of these singularities. The following obsevation tells us that these links are precisely minimal Legendrian submanifolds (or varieties, more generally) in the sphere with the standard contact structure.

1.1. Observation. (folklore) *There is a 1-1 correspondence between minimal Lagrangian cones in* $\mathbf{R}^{2n} = \mathbf{C}^n$ *and minimal Legendrian submanifolds in the sphere* S^{2n-1} *with the standard contact metric structure.*

Let us recall that the standard symplectic structure on $\mathbf{R}^{2n}$ is the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. The standard contact structure on S^{2n-1} is defined via the restriction of the 1-form $\alpha = \sum_{i=1}^n x_i dy_i$ to S^{2n-1}. The metric on $\mathbf{R}^{2n}$ is the Euclidean metric and the metric on S^{2n-1} is the induced metric. The correspondence between Lagrangian cones and Legendrian submanifolds follows from the fact that $\mathbf{C}^n \setminus \{0\}$ is the symplectization of S^{2n-1} (see also [Has2000]). The minimality correspondence is well-known in the theory of minimal submanifolds (see also [Si1968] and (3.6)).

It was observed [Has2000, L-W 2001c, ect] that the equation of a Legendrian and conformal harmonic mapping from a 2-dimensional simply connected domain into S^{2n-1} is an intergable equation. Based on this observation Haskins has discovered many examples of minimal Legendrian tori in the sphere S^5. But it is still unclear whether minimal Legendrian surfaces of higher genus in S^5 exist. Our knowledge of minimal Legendrian submanifolds in spheres of higher dimension is even poorer.

In this note we propose to find Legendrian minimal submanifolds by several ways. [1] We denote by *Leg* a Legendrian submanifold in S^{2n-1}. Since any Legendrian submanifold in a neighborhood of *Leg* can be described by a function f on *Leg*, so we shall search for a minimal Legendrian submanifold Leg_{min} nearby *Leg* by solving a certain differential equation for f. This equation is derived in Theorem 2.21.

In the section 3 we study the relation between Legendrian submanifolds in the contactization over a Kähler symplectic manifold M and Lagrangian submanifolds in M. This is motivated by the fact that the the standard

[1] Another approach based on a spectral characterization of minimal Legendrian submanifolds is proposed in [L-W 2001a].

contact sphere S^{2n-1} is the contactization of the Kähler symplectic manifolds $\mathbf{C}P^{n-1}$.

We prove that there is no embedded orientable Lagrangian surface of higher genus in $\mathbf{C}P^2$ (Prop. 3.22).

In sections 4 and 5 we want to construct new minimal Legendrian submanifolds by gluing the old ones. This could be done in two steps. First we glue 2 Legendrian submanifolds into one Legendrian submanifold (Proposition 4.2). Then we try to deform the glued one to a new minimal Legendrian submanifold. Since the linearization of of the equation (2.21.1) is self-adjoint, we cannot use the implicit function theorem for our gluing. Instead we propose an evolution equation for these submanifolds which is volume decreasing. We prove the short time existence of our equation (Proposition 5.17). Our evolution equation is a parabolic equation of 4-order.

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2 Moduli space of minimal Legendrian submanifolds.

In this section we shall study the local manifold structure of the moduli space of the minimal Legendrian submanifolds in the standard sphere. In a neigborhood of a Legendrian submanifold *Leg* we can describe any Legendrian submanifold via a smooth function f on *Leg*. We derive a scalar differential equation (Propostion 2.17 and 2.18) for f whose associated Legendrian submanifold is minimal.

2.1. Lemma. *The set $\Lambda(Leg)$ of all Legendrian submanifolds Leg' nearby a Legendrian submanifold Leg in a contact manifold (M^{2n-1}, α) is a Banach space which is modelled on the space of functions on Leg.*

Proof. This Lemma must be well-known for experts as a consequence of the Darboux-Weinstein theorem on the neighborhood of a Legendrian submanifold in a contact manifold. Here we provide another simple proof. We recall that the Reeb field R on a contact manifold (M^{2n-1}, α) is defined uniquely by the following two conditions: $R \in \ker d\alpha$ and $\alpha(R) = 1$.

Now we choose a metric g_α on Leg which is compatible with the contact form α, i.e.
i) the Reeb field R has the constant unit length,
ii) the Reeb field R is orthogonal to the contact hyperplan $\ker \alpha$,
iii) the metric g_α is compatible with the symplectic form $\omega = d\alpha$ restricted to the contact hyperplan, i.e. there exists an almost complex structure $J \in End(\ker \alpha)$ such that

$$\omega(V, W) = g(JV, W) \text{ for all } V, W \in \ker \alpha. \tag{2.1.1}$$

Clearly such a compatible metric exists. Further let us denote also by J the unique tensor in $End(TM^{2n-1})$ such that $JR = 0$ and $J\{\ker \alpha\} = \{\ker \alpha\}$, moreover the restriction of J to the contact hyperplan $\ker \alpha$ is the almost complex structure J which is defined in (2.1.1).

It is easy to see that the standard metric on the sphere S^{2n-1} is compatible with the standard contact structure, and the associated complex structure J on S^{2n-1} is the restriction (by the orthogonal projection) of the standard complex structure J on $\mathbf{R}^{2n} = \mathbf{C}^n$.

Any smooth submanfold L' in a normal tubular neighborhood $U(Leg) \subset M^{2n-1}$ can be identified with a normal vector field $V(x) \in \Gamma(N(Leg))$ on Leg via the exponential map $Exp : N(Leg) \to (M^{2n-1}, g_\alpha)$. We decompose the normal bundle $N(Leg)$ as

$$N(Leg) =< R > \oplus JT_*Leg.$$

So a normal vector field $V(x)$ can be written as $f(x)R + J(W(x))$, where $f(x) \in C^\infty(Leg)$ and $W(x) \in T_xLeg$.

With this identification we choose $U_0 \subset \Gamma(N(Leg))$ to be a neighboorhood of the zero normal vector field on Leg such that $Exp(V) \subset U(Leg)$ for all $V \in U_0$. So U_0 can be considered as a neighborhood of Leg in the space of all (n-1)-dimensional submanifolds in M^{2n-1}. Then the set of all Legendrian submanifolds in U_0 is the zero set of the map

$$F : U_0 \to \Omega^1(Leg)$$

$$(F(V))(x) = (Exp_x V)^*(\alpha). \tag{2.2}$$

We have

$$dF(V) = \mathcal{L}_V(\alpha) = d(\alpha(V)) + (V \rfloor d\alpha)$$

$$= df + 2V \rfloor \omega. \tag{2.3}$$

Thus V is an infinitesimal Legendrian deformation of Leg, if and only if

$$V = V_f := f(x)R + \frac{1}{2} J \nabla^{Leg} f, \tag{2.4}$$

where $\nabla^L f$ denotes the gradient vector field on Leg. Hence the formal tangent space of the moduli space of Legendrian submanifolds at Leg is the space of functions f on Leg. Finally we note that $dF(0)$ is surjective (by letting e.g. $f = 0$.) This implies the manifold structure of $\Lambda(Leg)$. □

2.5. Corollary. *Each Legendrian normal vector field is defined uniquely by it Reeb component via the formula (2.4).*

We shall improve Lemma 2.1 in the case, when a contact manifold (M^{2n-1}, α) is the standard contact sphere. Namely we shall show in Corollary 2.15 an explicit expression for a normal vector field $V \in \Gamma(N(Leg))$ such that $Exp(V)$ is a Legendrian submanifold. The observation 1.1 indicates us to consider in this case the associated Lagrangian cone $CLeg$ in $\mathbf{R}^{2n}$. Let Lag denote a Lagrangian submanifold in $\mathbf{R}^{2n}$. We shall now look at the following sequence of mappings

$$T^*Lag \xrightarrow{i_1} T_*Lag \xrightarrow{i_2} NLag \xrightarrow{Exp} \mathbf{R}^{2n}.$$

Here the bundle isomorphism i_1 is induced from the Riemannian metric g on Lag (which is induced from the standard metric on $\mathbf{R}^{2n}$), $NLag$ is the normal bundle of Lag in $\mathbf{R}^{2n}$, i_2 is a bundle isomorphism with $i_2(v) = Jv$, and $Exp(v, x) = x + v$, where x and v are considered as vectors in $\mathbf{R}^{2n}$. The composition $(i_2)^{-1} \circ (i_1)^{-1}$ sends the natural symplectic form Ω from T^*Lag to $NLag$ which we also denote by Ω.

2.6. Proposition. *Let Lag be a Lagrangian submanifold in* $\mathbf{R}^{2n}$*. Then there is a neighboorhood* $N_\varepsilon Lag$ *of Lag in NLag such that the natural embedding* $Exp : (N_\varepsilon Lag, \Omega) \to (\mathbf{R}^{2n}, \omega)$ *is a symplectic embedding.*

Proof. Let (x_i) be a local coordinates around a point $x \in Lag$. We denote by π the natural projection $T^*Lag \to Lag$. Then a point $\xi \in T^*Lag$

has coordinates $((x^1, \cdots, x^n) = \pi(\xi), p^i(\xi) = (\xi, \partial x_i), i = \overline{1,n})$. The Louiville form α on T^*L is defines as follows:

$$\alpha(V)_\xi = (\xi, \pi_*(V)) = \sum_i p^i(\xi) dx^i(\pi_*(V)).$$

Thus we can write $\alpha = p^i dx^i$. The standard symplectic form Ω has the following expression : $\Omega = dx^i \wedge dp^i$.

Let g be a Riemannian metric on Lag. The dual coordinates (x^i, q^i) on T_*Lag can be defined as follows. Without misunderstanding we denote by the same π the natural projection from T_*Lag to Lag. Any point η on T_*Lag has coordinates $(x^1, \cdots, x^n) = \pi(\eta), q^i(\eta) =< \eta, \partial x_i >_g)$. Using these coordinates we write explicitly the map $i_1 : T_*Lag \to T^*Lag$ as follows

$$(x^1, \cdots, x^n, q^1, \cdots, q^n) \stackrel{i_1}{\mapsto} (x^1, \cdots, x^n, p^i = q^i).$$

Thus in the coordinates (x^i, q^i) on T_*Lag we can write

$$\Omega = dx^i \wedge dq^i. \tag{2.7}$$

We define an almost complex structure on T_*Lag as follows. For each $\xi \in T_*Lag$ with $\pi(\xi) = x$ we decompose $T_\xi T_*Lag$ into the vertical part $T^v_\xi T_*Lag$ and the horizontal part $T^h_\xi T_*Lag$ using the Levi-Civita connection on Lag. The vertical part $T^v_\xi T_*Lag$ is the tangential space to the fiber $T_\xi(\pi^{-1}(\xi))$, and therefore can be identified with T_xLag. We denote by L^h_ξ the horizontal lift $T_xL \to T^h_\xi T_*Lag$. Now for any vertical vector $V \in T^v_\xi T_*Lag = T_xLag$ we put

$$I_\xi(V) := -L^h_\xi(V). \tag{2.8}$$

Clearly the equation (2.8) defines a unique almost complex structure on T_*Lag which for the sake of simplicity we also denote by I.

Let us choose a local corrdinate (x_i) around $x \in Lag$ such that this coordinate is normal w.r.t. to g, that is $g_{ij}(x) = \delta_{ij}$ and $\nabla_{\partial x_i}(\partial x_j)(x) = 0$ for all $i, j = \overline{1,n}$. In this normal coordinate we have

$$I_\xi(\partial x_i) = \partial q_i \tag{2.8.1}$$

for all $\xi \in T_xLag$. Combining (2.8.1) and (2.7) we get the following identity

$$\Omega_\xi(V, W) =< IV, W >_{\tilde{g}} \tag{2.9}$$

for any $V, W \in T_\xi T_* Lag$. Here $\tilde{g}$ is the unique metric on T_*L which is induced from g. Since both LHS and RHS of (2.9) does not depend on the choice of the origin of the normal coordinates, the identity (2.9) must be valid for all $\xi \in T_*Lag$. In particular we get that $\tilde{g}$ is a compatible metric and J is its associated almost complex structure on (T_*Lag, Ω).

Since J is a constant complex structure on $\mathbf{R}^{2n}$, we have

$$(2.10) \qquad < JY, \nabla_Z(JX) >=< Y, \nabla_Z X >$$

for all vector fields X, Y, Z on Lag. Here ∇ is the standard derivative on $\mathbf{R}^{2n}$. The equality (2.10) means that the diffeomorphism $i_2 : T_*Lag \to NLag$ is a bundle isomorphism which preserves the natural (induced) Levi-Civita connection ∇ on each bundle. Hence we have

$$(2.11) \qquad L^h_\xi(V) = L^h_{J\xi}(JV)$$

for all vertical vector $V \in T^v_\xi T_*Lag = T_{\pi(\xi)}Lag$. The induced almost complex structure $i_2^*(I)$ on $NLag$ has then the following expression

$$(2.12). \qquad i_2^*(I)JV = L^h_{J\xi}(JV).$$

The identities (2.12) and (2.9) imply that the induced symplectic form Ω on $NLag$ takes the form

$$(2.13) \qquad \Omega_{J\xi}(V, W) = - < L^h_{J\xi} JV, W >$$

for any vertical vector $V \in p^{-1}(J\xi) = N_x Lag$ and for any horizontal vector $W \in T^h_{J\xi} NLag$. From (2.9) it follows that the horizontal space and the vertical spaces are Lagrangian, therefore the equation (2.13) defines the symplectic form Ω on $NLag$ uniquely.

Now we note that the natural embedding $Exp : N_\varepsilon Lag \to \mathbf{R}^{2n}$ is an isometric embedding for a small neighborhood $N_\varepsilon Lag$ of Lag in $NLag$. Thus the horizontal lift $L^h_{J\xi} JV$ on $NLag$ is the parallel translation in $\mathbf{R}^{2n}$ of the vector $JV \in T_x Lag \subset T_x \mathbf{R}^{2n} = \mathbf{R}^{2n}$. Now we can rewrite (2.13) as follows

$$(2.13.1) \qquad \Omega_{J\xi}(V, W) =< JV, W > .$$

Clearly RHS of (2.13.1) is the symplectic structure $Exp^*(\omega)$ induced from $\mathbf{R}^{2n}$ and the LHS of (2.13.1) is the symplectic structure Ω. As a by-product we also obtain that the induced almost complex structure $i_2^*(I)$ on $NLag$

coincides with the natural complex structure J restricted to an open neighborhood $Exp(NLag)$. In particular an open neighborhood of T_*Lag has a natural Kähler structure with the complex structure I. $\Box$

Now we shall apply Proposition 2.6 to Lagrangian cones in $\mathbf{R}^{2n}$. For a normal vector field V over a cone $CLeg$ we denote by $Exp_{\mathbf{R}^{2n}}(V)$ the exponential map $Exp : CLeg \to \mathbf{R}^{2n} : x \mapsto Exp_x(V)$.

2.14. Lemma. *Let Leg be a Legendrian submanifold in S^{2n-1} and let $CLeg$ denote the cone over Leg. Suppose that V is a homegeneous normal vector field on $CLeg$: $V(r,x) = rV(1,x)$. We identify Leg with the subset $\{(1,x)\} \subset CLeg$, $x \in Leg$. Then the cone $C(Leg + V_{|Leg}) = Exp_{\mathbf{R}^{2n}}(V)$ is Lagrangian, if and only if*

$$V_{|Leg}(x) = 2f \cdot J\partial r(x) + J\nabla^{Leg} f(x). \tag{2.14.1}$$

Proof. If the submanifold $Exp_{\mathbf{R}^{2n}}V$ is Lagrangian in $\mathbf{R}^{2n}$, then according to Proposition 2.6 the normal vector field V is also a Lagrangian submanifold in the normal bundle $N(CLeg)$ provided with the symplectic form Ω. Thus $V = (i_2 \circ i_1)(\gamma)$, where γ is a closed homogenous 1-form on $CLeg$. Using the Poincare lemma for the cone, it is easy to see that any such homogenous closed 1-form is in fact a differential of a function $f(r,x) = r^2 f(x)$, here $x \in Leg$. Now the equation (2.14.1) follows form the identity $V = (i_2 \circ i_1)df$ immediately . $\Box$

We denote by $Exp_{S^{2n+1}}$ the exponential map in S^{2n+1}. It is easy to see that the Reeb field R on (S^{2n-1}, α) is equal to $J(\partial r)$, where ∂r denote the unit radial vector field on $\mathbf{R}^{2n}$.

2.15. Corollary. *Let Leg be a Legendrian submanifold in S^{2n-1} of radius 1 and V be a normal vector field on Leg in S^{2n-1} such that $|V| < \pi/4$. Then $Exp_{S^{2n+1}}(V)$ is a Legendrian submanifold, iff there is a function f on Leg such that*

$$V := \tilde{V}_f = arcsin(\sqrt{4f^2 + |\nabla^{Leg} f|^2}) \cdot (2f \cdot J\partial r + J\nabla^{Leg} f). \tag{2.15.1}$$

Proof. It is easy to check that the cone $C(Exp_{S^{2n+1}}(\tilde{V}_f))$ coincides with the cone $C(Leg + V_f)$, where V_f has the same form as RHS of (2.14.1). Hence Corollary 2.15 follows from Lemma 2.14. $\Box$

For each $\theta \in S^1$ let us denote by ϕ_θ the special Lagrangian calibration $Re(e^{i\theta}dz^1 \wedge \cdots \wedge dz^n)$ on $\mathbf{R}^{2n} = \mathbf{C}^n$. We denote by β_θ the constant form $Im(e^{i\theta}dz^1 \wedge \cdots \wedge dz^n)$ on $\mathbf{R}^{2n}$.

It is known (see e.g. [HL1982]) that a submanifold $Leg \subset S^{2n-1}$ is a minimal Legendrian, if and only if there exists a constant $\theta \in S^1$ such that the cone $C(Leg)$ is ϕ_θ-calibrated submanifold in $\mathbf{R}^{2n}$ (equivalently, the restriction of β_θ to $C(Leg)$ vanishes). If $\theta = 0$, we shall call Leg a **γ-minimal submanifold**. The following theorem is a generalization of a theorem of Harvey and Lawson [Thm 2.3, Chapter III, HL1982] on the special Lagriangian minimality of a graph $\Gamma_f = \{(x, \nabla f(x))\} \subset \mathbf{R}^n \oplus \mathbf{R}^n = \mathbf{R}^{2n}$. Our formulation does not depend on a graph representation of a Legendrian submanifold. It can be also considered as a (nonlinear) analog of the McLean deformation theorem [McL1998], see also Remark 2.19.

2.16. Proposition. *Let f be a C^2 -function on a calibrated ϕ_0- submanifold Lag in $\mathbf{R}^{2n}$ such that $Exp_{\mathbf{R}^{2n}}(J\nabla^{Lag} f)$ is also a Lagrangian submanifold in $\mathbf{R}^{2n}$. Then the submanifold $Exp_{\mathbf{R}^{2n}}(J\nabla^{Lag} f)$ is ϕ_0-calibrated, if and only if*

$$Im(\det_{\mathbf{C}}(Id + \sqrt{-1}\nabla^{Lag}(\nabla^{Lag} f)) = 0. \tag{2.16.1}$$

Here we consider $\nabla^{Lag}(\nabla^{Lag} f)$ as an element in $End(T_x Lag)$ as follows

$$\nabla^{Lag}(\nabla^{Lag} f)(v) = \nabla^{Lag}_v(\nabla^{Lag} f).$$

Proof. We consider V as a map from Lag to $N(Lag)$. Then the image $V(Lag)$ is a submanifold in $N(Lag)$. The tangent plan $T_{V(x)}V(Lag) \subset T_{V(x)}N(Lag)$ is the image of the tangent plan $T_x Lag$ under the differential dV which sends $v \in T_x Lag$ to the vector $(L^h_{V(x)}v + \nabla^N_v V)$. Here ∇^N denotes the induced connection on the normal bundle $N(Lag)$. By identifying $T_x Lag$ with the horizontal lift in $T_{V(x)}NLag$ we can write

$$dV(v) = v + \nabla^N_v V.$$

Furthermore $\nabla^N_v V = \nabla^{Lag}_v(JV)$, since the diffeomorphism i_2 is a connection preserving diffeomorphism of the vector bundles. Hence for a normal vector field $V(x) = J\nabla^{Lag} f(x)$ we have

$$dV(v) = v + J\nabla^{Lag}_v(\nabla^{Lag} f).$$

We identify $T_{V(x)}NLag$ with the complexification of $T_x Lag$, namely $T^v_{V(x)}NLag = J(T^h_{V(x)}NLag) = \sqrt{-1}T_x Lag$. So we can rewrite

$$dV = Id + \sqrt{-1}\nabla^{Lag}\nabla^{Lag} f.$$

Now we remind that the image $Exp(J\nabla^{Lag} f)$ is a ϕ_0-calibrated submanifold, if and only if the form β_0 vanishes on it. Thus our statement follows immediately from Corollary 1.11, chapter III in [HL1982] which states that if A is a complex linear map which sends the n-vector $\xi_0 = e_1 \wedge \cdots \wedge e_n$ into $\lambda\xi$, where $\lambda \in \mathbf{R}$ and ξ is a unit n-vector, then $\lambda\beta(\xi) = Im\,\det_{\mathbf{C}} A$. $\square$

For any function f on Leg we denote by $\tilde{f}$ the function on $CLeg$ defined by $\tilde{f}(r, x) = r^2 f(x)$.

From Proposition 2.16 we get immediately.

2.17. Proposition. *Let Leg be a Legendrian γ-minimal submanifold in S^{2n-1}. Then there is a 1-1 correspondence between Legendrian γ-minimal submanifolds which is C^1-close to Leg and the space of solutions $f \in C^\infty(Leg)$ to the differential equation*

$$Im\,\det_C(Id + \sqrt{-1}(\nabla^{CLeg}\nabla^{CLeg}\tilde{f}))(1, x) = 0. \tag{2.17.1}$$

In (RHS) of (2.17.1) we can replace (or extend) $\nabla^{CLeg}\tilde{f}(r, x)$ as $r^2\nabla^{Leg} f(x) + 2rf(x)\partial r$. Further we have

$$\nabla^{CLeg}_{\partial x}(r^2\nabla^{Leg} f(x) + 2rf(x)\partial r)(1, x) = \nabla^{Leg}_{\partial x}(\nabla^L f(x))(1, x).$$

$$\nabla^{CLeg}_{\partial r}(r^2\nabla^{Leg} f(x) + 2rf(x)\partial r)(1, x) = \nabla^{Leg} f(x) + 2f(x)\partial r.$$

Computing the linearization of the equation (2.17.1) we get immediately

2.18. Corollary. *A function f on a minimal Legendrian submanifold $Leg^{n-1} \subset S^{2n-1}$ is a tangent vector to deformations of the link of a ϕ_0-calibrated cone in $\mathbf{C}^n$, only if f is an eigenfunction of the Laplacian operator on Leg^{n-1} corresponding to the eigenvalue $2n$.*

2.19. Remark. This corollary shows that even after linearization our deformation problem for Legendrian submanifolds is very different from the deformation problem of special Lagrangian submanifolds [McL1998].

Let us consider an example for Proposition 2.17 with $n = 2$. As we know all minimal Legendrian spheres in the standard sphere S^5 are geodesic ([Has 2000, L-W 2001c]). So non-trivial (orientable) examples have genus at least 1. Let us consider a γ-minimal torus T^2. One classical example of such a torus is the Clifford torus

$$T^2 = \{(\exp i\theta_1, \exp i\theta_2, \exp i\theta_3) \mid \sum_{i=1}^{3} \theta_i = 0\},$$

with the induced flat metric. Any function f on T^2 can be considered as a periodic function with two real variables (x, y). In this case the equation (2.17.1) can be rewritten as follows

$$\triangle f(x)(1 + f_{xy}) = -f(x)(2 + \det(Hessf)) + 2f_x f_{xy} f_y. \qquad (2.17.2)$$

We shall generalize Proposition 2.17. For any oriented Legendrian submanifold $Leg \subset S^{2n-1}$ we denote by $U_{\pi/4}(Leg)$ the injective radius neighborhood of Leg in S^{2n-1}. Suppose that we have chosen a unitary basic in $\mathbf{C}^n$. Then we can define a function $\theta_{Leg}(x)$ on Leg as follows.

$$\theta_{Leg}(x) = \det(T_x CLeg(x)). \qquad (2.20)$$

Here the RHS of (2.20) denotes the determinant of a unitary matrix which transforms the chosen unitary basis to an oriented orthonormal basis in the Lagrangian plan $T_x CLeg \subset \mathbf{R}^{2n} = \mathbf{C}^n$. Harvey and Lawson showed that this "angle" function θ_{Leg} measures the mean curvature of the Lagrangian cone $CLeg$ in $\mathbf{R}^{2n}$, namely the Maslov 1-form $d\theta_{CLeg}$ is symplectically dual to the mean curvature (see also (3.3).) Now using Corollary 2.15 and repeating the argument in the proof of Proposition 2.16 we get

2.21. Theorem. *Let Leg be an oriented Legendrian submanifold in S^{2n-1} and L_1 a minimal Legendrian submanifold in $U_{\pi/4}(Leg)$ such that Leg' (geodesically) 1-1 projects on Leg. Then $Leg' = Exp_{S^{2n}}(\tilde{V}_f)$, where the associated function $\tilde{f}$ satisfies the following equation*

$$Im \det_C[\exp(i\theta_{Leg}(x))(I + \sqrt{-1}(\nabla^{CLeg}\nabla^{CLeg}\tilde{f}))](1, x) = constant. \qquad (2.21.1)$$

So we can search for a minimal Legendrian surface of higher genus in S^5 by first gluing two Legendrian torus and then we have to solve the equation (2.21.1) a for a function f on the glued Legendrian surface. A Legendrian gluing construction is given in the section 4.

3 Legendrian submanifolds in the contactization of a Kähler symplectic manifold.

In this section we shall study the relation between minimal Legendrian submanifolds in the compact contactization $Cont(M^{2n}, \omega)$ of a symplectic Kähler manifold $(M^{2n}, \omega, g_\omega)$ and minimal Lagrangian submanifolds in $(M^{2n}, \omega, g_\omega)$. We recall that the contactization $Cont(M^{2n}, \omega)$ is a S^1-principal bundle over (M^{2n}, ω) with the S^1-connection form α whose curvature $d\alpha$ is $\pi^*(\omega)$. Here π also denotes the projection $Cont(M^{2n}, \omega) \to (M^{2n}, \omega)$. This connection form α is the canonical contact form α on $Cont(M^{2n}, \omega)$.

For example the standard contact manifold (S^{2n+1}, α) is the compact contactization of the Kähler symplectic manifold $(\mathbf{C}P^n, \omega)$.

Now we shall consider the unique compatible Riemannian metric g_α on the contactization $Cont(M^{2n}, \omega, g_\omega)$ such that π is a Riemannian submersion. We shall call this metric g_α **the canonical metric** on $Cont(M^{2n}, \omega, g_\omega)$.

3.1. Lemma. *The projection of the mean curvature field of a Legendrian submanifold Leg in the contactization $(Cont(M^{2n}), \alpha, g_\alpha)$ coincides with the mean curvature of the projected (immersed) Lagrangian submanifold in $(M^{2n}, \omega, g_\omega)$. In particular the projection of a minimal Legendrian submanifold in $(Cont(M^{2n}), \alpha, g_\alpha)$ is an immersed minimal Lagrangian submanifold in $(M^{2n}, \omega, g_\omega)$.*

Proof. The fact that the projection of a Legendrian submanifold Leg in $(Cont(M^{2n}), \alpha)$ is a Lagrangian submanifold in (M^{2n}, ω) follows from the definition. Now let us compare the mean curvature of the projected manifold $\pi(Leg)$ with the one of Leg. The second fundamental form of a Riemannian submanifold is defined by

$$\Phi_{II}(X, Y) = (\nabla_X Y)^N,$$

where Z^N denotes the normal component of the vector Z. Let X, Y be tangent vectors in $T_x Leg$. We extend these vectors a neighborhood of x to commutative vector fields, which we also denote by X, Y. Clearly their projections $\pi_*(X), \pi_*(Y)$ are also commutative.

Applying the formula

$$2 < \nabla_X Y, Z >_g = X < Y, Z >_g + Y < X, Z >_g - Z < X, Y >_g$$

$$+ < [X, Y], Z >_g + < [X, Z], Y >_g + < [Z, Y], X >_g \qquad (3.2)$$

we see immediately, that $< \Phi_{II}(X, Y), R >= 0$, since the Reeb field R is a Killing vector field on $(Cont(M^{2n}), \alpha, g_\alpha)$.

Now applying (3.2) to X, Y, Z (and g_α) and their projection $\pi_*(X), \pi_*(Y), \pi_*(Z)$ (and g_ω) we get immediately that the mean curvature $H((Leg)) = Tr(\Phi_{II})$ is equal to the projection $\pi(H(Leg))$ of the mean curvature $H(Leg)$. □

It is known that the mean curvature field $H(Lag)$ of a Lagrangian submanifold Lag in a Calabi-Yau manifold X is symplectically dual to the angle form $d\theta$ as follows ([H-L 1982]

$$H = J(d\theta^{\#}) \tag{3.3}$$

where $^{\#} : \Omega^1(Lag) \to Vect(Lag)$ is defined by

$$X(Y) =< X^{\#}, Y >$$

and θ is the real part of the complex value of $vol_{hol}(T_xM)$. Here vol_{hol} denotes the holomorphic complex volume form on X.

We shall prove formula (3.8.1) for the mean curvature of a Legendrian submanifold, which is analog to (3.3).

Let us recall that a contact manifold M with a compatible matric g_α is called **Sasaki**, if the cone $C(M)$ equipped with following extended metric $\bar{g}$

$$(C(M), \bar{g}) = (\mathbf{R}_+ \times M, dr^2 + r^2 g_\alpha) \tag{3.4}$$

is Kähler w.r.t the following **canonical almost complex structure** J on $TC(M) = \mathbf{R} \oplus < R > \oplus \ker \alpha$:

$$J(r\partial r) = -R, \; J(R) = r\partial r,$$

and the restriction of J to $\ker \alpha$ coincides with the compatible almost complex structure J, defined on M. For example, the standard contact sphere S^{2n-1} is a Sasaki contact manifold with the standard Riemannian metric.

If a Kähler manifold $(M^{2n}, \omega, g_\omega)$ is also Einstein, then its Ricci form ρ satisfies the following equation

$$\rho = \lambda \cdot \omega, \tag{3.5}$$

where λ is a constant (which is also called **Einstein constant**).

We shall now prove

3.6. Lemma. *The canonical metric g_α on $Cont(M^{2n}, \omega, g_\omega)$ is Sasaki, if $(M^{2n}, \omega, g_\omega)$ is Kähler, and g_α is Einstein, if moreover g_ω is Kähler-Einstein.*

Proof. To show that the metric g_α is Sasaki, it suffices to show that the complex structure J on the cone $C(Cont(M^{2n}, \omega))$ is integrable. Clearly we can consider $C(Cont(M^{2n}))$ as the $\mathbf{C}^*$-bundle over (M^{2n}, J, ω) associated with the principal S^1 contact bundle $Cont(M^{2n}, \omega)$. It is easy to see that the complex structure on the cone $C(Cont(M^{2n}))$ is induced from that one on the holomorphic line bundle associated with the curvature form ω. Thus $C(Cont(M^{2n}))$ is Kähler. The fact that the canonical metric on $Cont(M^{2n})$ is Einstein, if g_ω is Kähler-Einstein was proved by S. Kobayashi [Ko1963] (see also [Be1987, Theorem 9.76]). $\square$

3.7. Remark. A computation using the following formula for the Ricci curvature on the cone $C(M)$ in [L-W 2001b, Appendix] (see also [Be1987, 9.106]) tells us that this cone possesses a Calabi-Yau metric, if M is the contactization of a symplectic Kähler-Einstein manifold with positive scalar curvature. This Calabi-Yau metric can be obtained by formula (3.4) with g_α being some multiple of the canonical Einstein metric on M. For the convenience of the reader we write down this formula here.

Proposition. [L-W 2001 Appendix] *The Ricci curvature Ric of the cone CM^{2n+1} satisfies*

$$Ric(\partial r, \partial r) = -\frac{2n+1}{r}\frac{\partial^2(r)}{(\partial r)^2} = 0,$$

$$Ric(\partial r, X) = 0, \text{ if } X \in T(M \times \{r\}),$$

$$Ric(X,Y) = Ric_M(\pi_* X, \pi_* Y) + < X, Y > [(\frac{\partial^2(r)}{(\partial r)^2 r} - (2n)\frac{1}{r^2}) \circ \pi]$$

$$= Ric_M(\pi_* X, \pi_* Y) - (2n) < \pi_* X, \pi_* Y >, \text{ if } X, Y \in T(M \times \{r\}).$$

If Leg is a Legendrian submanifold in a Sasaki contact manifold (M, α, J), we also have an analog of (3.4). More presicely we denote by $\det(M)$ the determinant bundle of the contact plan bundle $\ker \alpha$ over M and by $\mathcal{L}eg(M)$ the bundle of oriented Legendrian plan in $\ker \alpha$. We also denote by det the following bundle map

$$\det : \mathcal{L}eg(M) \to \det(M) : \ w \mapsto w \wedge J(w).$$

3.8. Lemma. *Let us denote by $\tilde{\alpha}$ the canonical connection form on the determinant bundle* $\det(M)$ *over a contact Sasaki manifold* (M, α, g, J). *Then the mean curvature $H(Leg)$ of an oriented Legendrian submanifold $Leg \subset M$ is symplecticcally dual to*

$$h_{Leg} = (\det \circ \rho)^*(\tilde{\alpha}), \tag{3.8.1}$$

i.e. $h_{Leg} = J(H(Leg)^{\#})$. *Here ρ denotes the Gauss map $Leg \to \mathcal{L}eg(M)$ which sends each point $x \in Leg$ to the plan $T_x Leg$.*

Proof. The canonical Kähler metric on the cone $\mathbf{R}_+ \times M$ is compatible with the symplectic form $\omega = d(r^2\alpha)$ on the symplectization $\mathbf{R} \times M$ (see e.g. [L-W 2001b]). In other words the symplectic form ω is Kähler with respect to $\bar{g}$. Next we note that at any point x in a Legendrian submanifold $Leg \subset M$ the mean curvature $H(Leg)(x)$ coincides with the mean curvature of the cone $CLeg$ at $(1, x)$:

$$H(Leg)(x) = H(CLeg)(1, x). \tag{3.9}$$

We also define the Maslov form on $CLeg$ by

$$h_{CLeg}(t, x) = J(H(CLeg)^{\sharp}). \tag{3.10}$$

On the other hand it is known that [Le1990] there exists a Maslov one form μ on the Lagrangian Grassmanian bundle $\mathcal{L}ag(CM, \omega)$ over CM [2] such that

$$h_{CLeg} = \rho_1^*(\mu), \tag{3.11}$$

where ρ_1 denotes the Gauss map $CLeg \to \mathcal{L}ag(CM)$. We recall that [Le1990] the pull back of the Maslov form μ to the unitary bundle $U(CM)$ is the trace of the canonical connection form on $U(CM)$. Thus μ is the pull-back of the canonical connection form α on the determinant bundle $\det M$. Hence we can rewrite (3.11) as follows

$$h_{CLeg} = (\det \circ \rho_1)^*\alpha, \tag{3.12}$$

here the map $det : Lag(M) \to \det(M)$ is defined as follows: $\det(w) = w \wedge Jw$. Now we put

$$\bar{h}_{Leg} := (\det \circ \rho_1 \circ e)^*\alpha, \tag{3.13}$$

[2] actually we have a statment for an almost Hermitian manifold CM

where e denotes the embedding $Leg \to CLeg : \ x \mapsto (1, x)$. Then we get from (3.9), (3.10) and (3.12)

$$\bar{h}_{Leg} = J(H(Leg)^{\#}). \tag{3.14}$$

We consider the natural embedding i from the Grassmannian $\mathcal{L}eg(M)$ into $\mathcal{L}ag(CM)$:

$$i(l) = \partial r \wedge l,$$

which satisfies

$$i \circ \rho(L) = \rho_1 \circ e(L). \tag{3.15}$$

On the other hand we have the following commutative diagramm

$$\begin{tikzcd}
Leg \arrow[r, "\rho"] \arrow[d, "e"] & \mathcal{L}eg(M) \arrow[r, "\det"] \arrow[d, "i"] & \det(M) \arrow[d, "i_1"] \\
CLeg \arrow[r, "\rho_1"] & \mathcal{L}ag(CM) \arrow[r, "\det"] & \det(CM)
\end{tikzcd}$$

Clearly we have

$$\tilde{\alpha} = i_1^*(\alpha). \tag{3.16}$$

From (3.10), (3.15) and (3.16) we get

$$\bar{h}_{Leg} = (\det \circ \rho_1 \circ e)^*(\alpha) = \det \circ \rho(\tilde{\alpha}) = h_{Leg}. \tag{3.17}$$

Now Lemma (3.8) follows from (3.14) and (3.17). $\square$

If M is Kähler-Einstein (or more generally if the Ricci form ρ is non degenerated) we can also consider $(\det M, \tilde{\alpha})$ as the contactization of (M, ρ). We get immediately from Lemma 3.8 the following

3.18. Theorem. *For any Lagrangian submanifold Lag in a Kähler-Einstein manifold N there exists a canonical section* $h : Lag \to \det(N)_L$*. The section* $h(Lag)$ *is a Legendrian submanifold in* $(\det(N)_L, \tilde{\alpha})$*, if and only if Lag is a minimal submanifold.*

Indeed the canonical section h in (3.18) is defined as follows: $h := \det \circ \rho$, where ρ is the Gauss map from Lag to $\mathcal{L}ag(M)$.

3.19. Corollary. *Assume that M is a Kähler-Einstein manifold with a rational Kähler constant p/q. Then any minimal Lagrangian submanifold Lag can be lifted to a minimal Legendrian submanifold in $Cont(M)$. The holonomy group of the flat bundle $Cont(M)_{|Lag}$ is a subgroup of $\mathbf{Z}_q$.*

This Corollary 3.19 was first proved by Oh in [Oh1994] for $M = \mathbf{C}P^n$.

3.20 Corollary. (see also [Br1987]) *The mean curvature form h_L of a Lagrangian submanifold (Legendrian submanifold resp.) in a Kähler-Einstein manifold (Sasaki-Einstein manifold resp.) is closed.*

This corollary will be further studied in our Appendix.

3.21. Corollary. *If a Legendrian submanifold Leg in a Sasaki contact manifold M is contact isotopic to a minimal Legendrian submanifold, then the restriction of the Chern class of the contact plan bundle over N to Leg are trivial. If moreover M is Einstein Sasakian then its Maslov class (i.e. the cohomology of the curvature form h_{Leg}) is trivial.*

Now let us consider the Kähler-Einstein manifold $\mathbf{C}P^n$. The following results are obtained in a discussion with Kaoru Ono.

3.22. Proposition. *Let Lag be an oriented Lagrangian submanifold in a symplectic manifold $(\mathbf{C}P^n, \omega)$ (i.e. ω need not to be a Kähler symplectic form). Then the Euler class of Lag is vanished.*

Proof. Since Lag is an embedded oriented Langrangian submanifold, its Euler number equals the self -intersection number $[Lag]\cdot[Lag]$ in $\mathbf{C}P^n$. Since the cohomology ring $H^*(\mathbf{C}P^n, \mathbf{R})$ is generated by the single symplectic class $[\omega]$ then the interection number $[Lag] \cdot [Lag]$ is zero. □

In particular only torus T^2 admits a Lagrangian embedding into $\mathbf{C}P^2$ among orientable connected surfaces. There are some more known obstructions to a Lagrangian embedding into $\mathbf{C}P^n$ based on Floer homology, see e.g. [Sei2000], [B-C2001].

We can also get an analogous neccesary condition for a Lagrangian embedding of an unorientable submanifold Lag in $\mathbf{C}P^n$, but in this case an addition information on the image of the fundamental class $[Lag]$ in $H_*(\mathbf{C}P^n, \mathbf{Z}_2)$ is required. [3]

[3] recently Nimerovski proved that there is no embedded Langrangian Klein bottle in $\mathbf{C}P^2$.

4 Gluing Legendrian surfaces.

The following results are obtained in a discussion with Kaoru Ono.

In this section we shall show a glueing construction of two orientable embedded connected Legendrian submanifolds in a contact manifold N such that the result is also a connected orientable embedded Legendrian submanifold in N. The technique we use here is an analog of the Lagrangian surgery by Polterovich [Po1991], actually we can simplify some of his arguments.

4.1. Lemma. *Suppose that two Legendrian submanifolds Leg_1 and Leg_2 have only finite number of isolated transversal intersection points. Then we can perturb Leg_2 in a small neighborhood of the intersection points such that the perturbed Legendrian submanifold Leg_2' has no intersection point with Leg_1.*

Proof. We use a version of the Weinstein-Darboux theorem on a neighborhood of a Legendrian submanifold (see e.g. Lemma. 2.1) to represent a small neighborhood $U_2 \subset Leg_2$ at an isolated intersection point $x \in Leg_2 \cap Leg_1$ as a graph $(f, df) \subset J^1(U_1)$ of some function f over a small convex neighborhood $U_1 \subset L_1$. Here $J^1(U)$ denotes the 1-jet-bundle over U_1 with the natural contact structure $\alpha = df - p_i dx_i$, where x_i are local coordinates on U_i, and p_i are the fiber coordinates. There are two cases we have to consider. Case 1: $f \geq 0$ or $f \leq 0$, so the intersection point x is the minimum or maximum of f, since $f(x) = 0$. Case 2: the intersection point x is not the maximum or the minimum point of f.

In the case 1 we can perturb Leg_2 by pertubing f inside the neighborhood $U_1 \subset Leg_1$ so that the perturbed manifold Leg_2' has no intersection point with U_1. This perturbation can be done by adding to f (resp. substracting from f) a positive function which vanishes nearby the boundary ∂U_1. The graph $(\tilde{f}, d\tilde{f})$ of the perturbed function represent the required perturbation of U_2, since $\tilde{f}$ is stricly positive (negative resp.) on U_1. In other words we get rid of one intersection point of Leg_1 and Leg_2.

In the case 2 we also perturb Leg_2 inside U_2 by perturbing the function f in a neighborhood $U_1 \subset Leg_1$ such that the only critical point of the perturbed function $\tilde{f}$ is the maximum (resp. minimum) point. This perturbation of f can be done as follows. First we can assume that U_1 is a disk and therefore it can be considered as a cone with vertex in the maximum (resp. minimum) point x_0 of f on U_1. Now we choose such a function $\tilde{f}$ such that $\tilde{f}$ has a negative (resp. positive) derivative along the radius vector on the domain outside the maximum (resp. minimum) point x_0, moreover $\tilde{f}(x_0) = f(x_0)$ and $\tilde{f}$ coincides with f in a small neighborhood of of the boundary ∂U_1. Clearly the only critical point of $\tilde{f}$ is the maximum (minimum resp.) point x_0. Since $\tilde{f}(x_0) \neq 0$, the graph $(\tilde{f}, d\tilde{f})$ has no intersection with the (zero section of) U_1. Thus by using $\tilde{f}$ we can perturb Leg_2 inside U_2 so that the perturbed submanifold Leg_2 has no intersection with U_1. Thus we also get rid of one intersection point. $\square$

Next we note that if two Legendrian submanifolds Leg_1 and Leg_2 in a contact manifold N has no intersection point, then we can perturb them in a small domain which is a connected sum of two tubular neighborhoods of Leg_1 and Leg_2 so that the perturbed Legendrian submanifold Leg_1' intersects transversally to Leg_2' at exactly one point. If the contact manifold N is the standard contact sphere S^{2n-1}, then we can alternatively use the group $U(n)$ to move Leg_1 to Leg_1' so that Leg_1' intersects Leg_2 at a given point $x \in Leg_2$.

4.2 Proposition. *Suppose that Leg_1 and Leg_2 are two orientable connected Legendrian submanifolds which intersects transversally at exactly one point x in a contact manifold N^{2n+1}. Then for any small neighborhood $U \ni x$ in N there exists an orientable connected Legendrian submanifold $L_1 \# L_2$ such that $(L_1 \# L_2) \setminus (L_1 \cup L_2)$ is a submanifold in U.*

Proof. We can assume that U is so small such that $U_2 = L_2 \cap U$ is a graph (f, df) of a domain $U_1 = L_1 \cap U$ for some function f on U_1. Furthermore, using a contactomorphism we can assume that U_1 and U_2 are two domains in Legendrian subspaces $\mathbf{R}^n(x_i)$ and $\mathbf{R}^n(y_i)$ in the contact space $(\mathbf{R}^{2n+1}, dz - x_i dy_i)$, so the intersection point x is identified with the origin $0 \in \mathbf{R}^{2n+1}$. Now we consider a preliminary Legendrian handle L_0 which is the graph of the function $f(r, \theta) = \ln r + ||\theta||^2/2$ over $U_1 \setminus \{0\}$. We also choose U_1 to be a small disk $D^n \subset \mathbf{R}^n(x_i)$. We observe that the ends of L_0 project injectively to $\mathbf{R}^n(x_i)$ and to $\mathbf{R}^n(y_i)$ respectively. Hence we can perturb L_0 by first changing f in a small neighboorhood of the sphere ∂U_1 so that the new perturbed function $\tilde{f}$ equals zero on a smaller neighboorhood of ∂U_1. The new Legendrian handle L_0' is then glued smoothly to $L_1 \setminus U_1$. In the similar way we can perturb L_0' at the other its end, so that the perturbed Legendrian handle L_0'' is smoothly glued to $L_2 \setminus U_2$. Clearly the connected sum

$$L_1 \# L_2 := (L_1 \setminus U_1) \cup (L_2 \setminus U_2) \cup L_0''$$

satisifies the required condition in Proposition 4.2. An expert will recognize that our technique is an analog of the Poltervich construction [Po1991] of a Lagrangian surgery, and in fact here we have simplified some of his arguments. □

In view of Corollary 3.21 it is important to know the Maslov class of a glued Legendrian submanifold in a Sasakian Einstein contact manifold.

4.3. Proposition. *The glued Legendrian submanifold has the trivial Maslov class, if and only if each of the summands has the trivial Maslov class.*

Proof. If the dimension of a glued Legendrian submanifold is at least 3 then the Legendrian handle is simply connected. Thus the Maslov number of the glued Legendrian submanifold in this case is the least common divisor of the Maslov numbers of each summand. If the glued Legendrian submanifold L has dimension 2, then the only new generator of $H_1(L, \mathbf{Z})$ is the base of the Legendrian handle $S^1 \times [0, 1]$. We remember that in this case our Legendrian handle is constructed as a Legendrian deformation of the graph of a smooth funtion over the annulus $D^2 \setminus \{0\}$. Hence the evaluation of the Maslov class on this generator is trivial. □

5 Deformations towards minimal Legendrian submanifolds.

A straightforward computation shows that the mean curvature $H(Leg)$ of a Legendrian submanifold Leg in a contact manifold (M, α, g_α, J) is orthogonal to the Reeb field R. Hence (unlike the case of a Lagrangian submanifold in a Kähler-Einstein manifold) $H(Leg)$ cannot represent an infinitisemal Legendrian (or locally contact) transformation. There are two ways to deal with this problem if we wish to deform a Legendrian submanifold into another one of smaller volume.

The first natural idea is to use the gradient flow of the volume functional restricted to the space of Legendrian submanifolds to deform a Legendrian submanifold. [4]

[4]A question of the existence of such a flow was posed by Gang Tian to Guofang Wang. The evolution equation (5.4) was found in our discussion with Guofang Wang on this question.

The second one is to modify the mean curvature flow to get a Legendrian deformation, namely we want to find a normal contact vector field V over a Legendrian submanifold such that $< V, H_L > \geq 0$. For a contact Sasaki manifold which is a compact contactization of a Kähler-Einstein manifold there is a candidate for such a deformation, namely it is the lifting of the mean curvature flow from the base space [Smoc2002]. It can seen as generated by the projection of the mean curvature vector filed on the space of (Hamiltonian) contact deformations.

Let us now to consider the first idea. Here we shall use a special metric on the space of Legendrian deformations, so our gradient flow is not like the mean curvature flow. On the other hand, its form is quite simple.

An immersed Legendrian submanifold Leg is called $\mathcal{L}$-minimal if it is a critical point of the volume functional restricted to the space Λ_{Leg} of all Legendrian submanifolds. On Λ_{Leg} we define a L^2 metric as follows. Let $\mathcal{X}$, $\mathcal{Y} \in T_{Leg}\Lambda_{Leg}$. By (2.4) we have

$$\mathcal{X} = f_1 \cdot R - J(\nabla^{Leg} f_1) \text{ and } \mathcal{Y} = f_2 \cdot R - J(\nabla^{Leg} f_2),$$

for some functions f_1 and f_2 on L. We set

$$< \mathcal{X}, \mathcal{Y} > = \int_{Leg} f_1 f_2 \, dvol(Leg). \tag{5.1}$$

This L^2 metric does not coincide with the usual L^2-metric induced by g_α on the space of sections of the normal bundle of L in M, since we take into account only the Reeb field component.

Let us recall that the mean curvature form h_{Leg} of a Legendrian submanifold Leg in a metric contact manifold (M^{2n+1}, α, g, J) is defined as follows

$$h_{Leg} = (J(H(Leg))^{\#}.$$

5.2. Lemma. *An immersed Legendrian submanifold Leg is a critical point of the volume functional $\mathcal{A}$ on the space Λ_{Leg} if and only if, the curvature form h_L is co-closed, i.e.,*

$$d^* h_{Leg} = 0,$$

where $d^* : \Omega^1(Leg) \to \Omega^0(Leg)$ is the adjoint operator of the exterior differential d w.r.t the induced metric on Leg.

This Lemma is an immediate consequence of the following Proposition.

5.3. Proposition. *The L^2-gradient vector field $\nabla\mathcal{A}$ of the volume functional $\mathcal{A}$ on Λ_{Leg} is*

$$\nabla\mathcal{A} := f_{Leg}R - J(\nabla^{Leg} f_Leg), \tag{5.3.1}$$

where f_{Leg} is defined by

$$f_{Leg} = d^* h_{Leg}.$$

Proof of Proposition 5.3. For a given function f on Leg let us consider a family Leg_t of Legendrian submanifolds with

$$Leg_0 = Leg, \ \& \frac{\partial Leg_t}{\partial t}_{|t=0} = \mathcal{X} = fR - J(\nabla^{Leg} f).$$

The L^2 gradient vector field $\nabla\mathcal{A}$ of the volume functional $\mathcal{A}$, by definition, satisfies

$$\begin{aligned} < \nabla\mathcal{A}, \mathcal{X} > \ &:= \ \frac{\partial\mathcal{A}(Leg_t)}{\partial t}_{|t=0} = \int_{Leg} < H, fR - J(\nabla^{Leg} f) > \\ &= \ -\int_{Leg} < H, J(\nabla^{Leg} f) >= \int_{Leg} < JH, \nabla^{Leg} f > \\ &= \ \int_{Leg} f_{Leg} f =< f_{Leg}R - J(\nabla^{Leg} f_{Leg}), \mathcal{X} > . \end{aligned}$$

Hence $\nabla\mathcal{A}$ coincides with the RHS of (5.3.1). □

Now we introduce a **Legendrian submanifold diffusion flow** as follows

$$\frac{\partial}{\partial t} Leg_t = -\nabla\mathcal{A}(Leg_t). \tag{5.4}$$

5.5. Proposition. *The volume functional $\mathcal{A}$ is decreasing along the flow (5.4), except at the stationary points.*

Proof. From the proof of Proposition 5.3, we have

$$\frac{\partial\mathcal{A}(Leg_t)}{\partial t}_{|t=0} = -\int_{Leg} < H, f_{Leg}R - J(\nabla^{Leg} f_{Leg}) >= -\int_{Leg} |f_{Leg}|^2.$$

Hence follows Proposition 5.5. □

According to Lemma 2.1 any Legendrian submanifold Leg_t in a contact manifold (M, α) corresponds to a function f_t on Leg_0, if Leg_t lies in a small (Weinstein-Darboux) neighborhood of Leg_0. More precisely let us denote by ϕ is a contactomorphism from an open neighboorhood $U \supset Leg_0$ in the standard contact manifold $J^1(Leg_0)$ to a small tubular neighborhood $N_\varepsilon(Leg_0) \subset M$. Then $Leg_t = \phi(f_t, df_t)$ for some function f_t on Leg_0. Using this notation we shall prove

5.6. Lemma. *The equation (5.4) locally (i.e. there exists $T > 0$ such that for all $t \in [0, T)$) is equivalent to the equation*

$$\frac{d\, f_t}{dt} = d_t^*(h_{\phi(f_t, df_t)}), \tag{5.6.1}$$

where d_t^* denotes the operator d^* on submanifold $\phi(f_t, df_t)$.

Proof. Clearly there exists a positive number T such that L_t belongs to this neighborhood, if $t \le T$.

First we show that the flow equation (5.4) implies the equation (5.6.1). We observe that the Reeb field R in the standard contact manifold $J^1(Leg)$ coincides with ∂z and the fiber $T^*_x Leg$ is always Legendrian. We also assume that the neighboorhood $U \subset J^1(Leg_0)$ has the induced metric $\phi^*(g_\alpha)$. Thus in these coordinates and using (5.3) we can rewrite (5.4) as

$$\frac{d}{dt}\phi(f_t, df_t) = \phi_*[d_t^* h_{(f_t, df_t)}\partial z - J\nabla^t(d_t^* h_{(f_t, df_t)})]. \tag{5.4.1}$$

The LHS of (5.4.1) the sum of the Reeb component $\phi_*((\frac{d}{dt}f_t)\partial z)$ and the "fiber" component $\phi_*(\frac{d}{dt}(df_t))$. The fiber component $\frac{d}{dt}df_t$ lies in the contact hyperplan in $J^1(Leg)$, so it is orthogonal to the Reeb field ∂z w.r.t to the compatible induced metric $\phi^*(g_\alpha)$. Hence, by comparing the Reeb component (in the orthogonal decomposition) of LHS and RHS of (5.4.1) we get immediately

$$\phi_*(\frac{df_t}{dt} f_t) = d_t^*(h_{\phi(f_t, df_t)}),$$

what is (5.6.1).

Now suppose that (5.6.1) holds. Then the Reeb component (in the orthogonal decomposition w.r.t. $\phi^*(g_\alpha)$) of the Legendrian deformation $\frac{d}{dt}(f_t, df_t)$ is $\frac{d}{dt}f_t$, as the fiber component $(d/dt)df_t$ lies in the contact hyperplan. According to Corollary (2.5) this component defines the infinitesimal Legendrian deformation uniquely. Hence (5.4) is a consequence of (5.6.1). □

Next we shall show that (5.6.1) is a quasilinear parabolic equation for a function f_t on Leg_0. Then applying the short time existence theorem, due to Huisken and Polden, for such scalar parabolic equation ([H-P1999, Thm 7.15]), we shall get the short time unique existence of a solution (5.6.1).

To see that (5.6.1) is a quasilinear equation we can either directly compute (5.6.1) in local coordinates, or we can use Lemma 3.8 in the following argument. We denote by Leg_t the image $\phi(f_t, df_t)$, and by P_t the Hodge operator $T^*Leg_t \to T^*Leg_t$ w.r.t. the induced metric on Leg_t. Clearly P_t depends linearly on the induced metric $g(t)$ on Leg_t which in turn depends on the first and second derivative (in x) of $f_t(x)$. Using Lemma 3.8, we write (5.6.1) as follows

$$\frac{d}{dt}f(t,x) = P_t \circ d \circ P_t(\rho_t^* \circ (\det)^*\tilde{\alpha}) \tag{5.14}$$

On the other hand in the local coordinates (x^i) on Leg_t we have

$$[\rho_t^*(\theta)(x)]^i = \frac{\partial \rho^j}{\partial x^i}\theta^j(\rho_t(x)), \tag{5.15}$$

$$\rho_t(x) = (f_t(x), df_t(x), \wedge_{i=1}^n(\partial x_i, \partial f_t/\partial x_i, \partial(df_t)/\partial x_i)). \tag{5.16}$$

Here for abbreviation we put $\theta = \det^*(\tilde{\alpha})$.

From (5.14), (5.15), (5.16) it follows that the RHS of (5.14) is a differential equation of 4th order and the coefficent of 4th order operator depends on derivative of maximal 3th order of f. Thus (5.6.1) is a quasilinear differential operator.

5.17. Theorem. *Let $(M^{2n-1}, \alpha, g_\alpha)$ be an Einstein Sasakian contact manifold with a non-negative scalar curvature. Then the flow (5.4) is well-posed, i.e., for any smooth orientable Legendrian submanifold Leg_0, there exists a $T > 0$ and a unique family of Legendrian submanifolds Leg_t for $t \in [0,T)$ such that Leg_t satisfies (5.4) with the initial condition Leg_0.*

Proof. Clearly it suffices to prove the short time existence for the solution to (4.6.1). We shall use here the following theorem due to Huisken and Polden [H-P1999].

5.18 Theorem ([H-P1999, Theorem 7.15] *Suppose that for a smooth initial data u_0 the operators of $2p$ order*

$$A(u) = A^{i_1j_1\cdots i_pj_p}(x, u, \nabla u, \cdots, \nabla^{2p-1}u)D_{i_1j_1\cdots i_pj_p}u$$

is smooth and strongly elliptic in a neigborhood of u_0. Then the evolution equation

$$D_t u = A(u) + b \tag{5.18.a}$$

where $b = b(x, u, \nabla u, \cdots, \nabla^{2p-1}u)$ is smooth, has a unique smooth solution on some interval $[0, T)$.

Continuation of the proof of Theorem 5.17. Since the evolution equation (5.6.1) is a scalar quasilinear, in views of the Huisken -Polden theorem it suffices to show that (5.6.1) is parabolic. Namely it suffices to show that the symbol of the RHS of (5.6.1) is the square of a positive definite matrix. Since the differential operator in RHS of (5.6.1) depends only on local coordinates on $Leg = \phi(f_t, df_t)$, we shall compute this symbol in an open simply connected domain on Leg. For the simplicity we shall denote this domain also by Leg.

Recall from Lemma 3.8 that we have for a Legendrian submanifold Leg in a Sasakian manifold M

$$h_{Leg} = (\det \circ \rho)^*(\tilde{\alpha}). \tag{5.18}$$

Since M is weakly Einstein Sasakian, the form h_{Leg} is closed, therefore the restriction of $\det M$ to Leg is a flat S^1-bundle. Since Leg is simply connected we can choose a trivialization

$$\Pi : \det M_{|Leg} \to S^1$$

which is compatible with this connection, i.e. $i^*(\tilde{\alpha}) = \Pi^*(d\theta) = d\Pi$. Here i denotes the embedding of $\det M_{|Leg}$ into $\det M$, and $d\theta$ is the canonical 1-form in the circle S^1 with coordinate θ. Thus we can rewrite (5.18) as follows

$$h_{Leg} = d(\Pi \circ \det \circ \rho). \tag{5.19}$$

So we can rewrite the equation (5.6.1) as follows

$$\frac{df_t}{dt} = d_t^* \circ d(\Pi \circ \det \circ \rho_t). \tag{5.20}$$

As we have observed above, the operator d_t^* depends at most on the second derivative of f. Thus it suffices to use the following Lemma 5.21 to compute the symbol of the RHS of (5.20). [5]

5.21. Lemma. *The symbol of the linearization of* $\Pi \circ \det \circ \rho \circ \phi(f)$ *is a positive multiple of the identity matrix.*

Proof. The argument in Remark 3.7 tells us that the cone $C(M)$ has a natural Calabi-Yau metric g_{C-Y}, namely it is obtained by multiplying the Sasaki-Einstein metric on M with a positive constant σ. Clearly $\sigma \cdot g_\alpha$ remains a Sasakian Eistein metric, and our "new" flow equation (5.4) (and (5.6.1)) for a Legendrian submanifold Leg in the new metric $\sigma \cdot g$ is the rescaling by factor σ of the "old" flow equation in the metric g_α. Thus it suffices to prove Lemma 5.21 in the case that the associated metric $\bar{g}_\alpha$ is the Calabi-Yau metric g_{C-Y} .

Let us denote by $\Pi_0 : \det M \to S^1$ the canonical trivialization of $\det M^{2n-1}$ on the Calabi-Yau. Let us denotes by $\bar{\Pi}$ the trivialization $\det(CM)_{|CLeg}$ which is induced from $\Pi : \det M_{Leg} \to S^1$. Since two trivialization are compatible with the canonical connection form α on $\det CM$, so they are the same. Our computation in section 3 shows that the curvature forms on Leg and on $CLeg$ are related by

$$h_{Leg} = i^*(h_{CLeg}).$$

Therefore the linearization $D(\Pi \circ \det \circ \rho \circ \phi)$ is equal to the restriction of the linearization $D(\Pi_0 \circ i \circ \rho_1 \circ C(\phi))$ to homogeneous functions, i.e. the set of functions $f(r, x)$ on $CLeg$ with $f(r, x) = r^2 f(x)$.

The same argument as in our proof of Proposition 2.16 yields

$$D(\Pi \circ i \circ \rho \circ C(\phi))(f, df) = 2n\, f(x) + \triangle f. \tag{5.22}$$

There is a second simple argument to get (5.22). For a giiven function f on Leg let us denote by $\tilde{f}$ the homogeneous function of the cone $CLeg$ such that $f(r, x) = r^2 f(x)$. Then $\tilde{f}$ is the infinitesimal deformation of the cone $CLeg$ in the Calabi-Yau manifold $(C(M), g_{C-Y})$ which coresponds to (the

[5] We can think that Lemma 5.21 is a an analog of Theorem 2.13 in [H-L1982, Chapter III], where Harvey and Lawson computed the linearization of the function $\sin \circ \det(f, \nabla f)$ at the minimal Lagrangian submanifolds $(f, \nabla f)$ in $\mathbf{R}^{2n}$ with the standard Kähler structure.

cone) of the deformation of Leg by f. Hence taking into account the known fact [McL1998]

$$D(\Pi_0 \circ i \circ \rho \circ C(\phi))(d\tilde{f}) = 2n\,\tilde{f}(x) + \triangle \tilde{f}. \tag{5.23}$$

We rewrite this expression (5.23) in coordinates on Leg we get the RHS of (5.22). □

Now Theorem 5.17 follows immediately from Lemma 5.21 and the fact that $d^*d = d^*d + dd^*$ is the standard Laplacian. □

6 Appendix. A characterization of Kähler-Einstein manifolds and Sasaki-Einstein manifolds.

We say that a Sasakian contact manifold is weakly Einstein, if the restriction of the Ricci form to each contact plan is proportional to the metric g.

Proposition A.1. *A Kähler manifold M is Einstein, if and only for each Lagrangian submanifold $Lag \subset M$, the 1-form h_{Lag} associated with the mean curvature of Lag is closed.*

Proposition A.2. *A contact Sasakian manifold N is weakly Einstein, if and only if for each Legendrian submanifold $Leg \subset N$, the 1-form h_{Leg} associated with the mean curvature of Leg is closed.*

The definition of h_{Lag} and h_{Leg} can be found e.g. in section 3.

Proof of Proposition A.2. First we remind that the 1-form h_{Lag} on a Lagrangian submanifold Lag in a Kähler manifold M is related to the Ricci form (or the Chern form) ρ of M as follows

$$dh_{Lag} = \rho_{|Lag}.$$

Applying this to our Langrangian cone we have

$$dh_{CLeg} = \rho^C_{|CLeg}$$

where ρ^C denotes the Ricci form on the Kähler cone CN. Clearly we have

$$d_{Leg}(h_{Leg}) = d_{CLeg}(h_{CLeg})_{|Leg}.$$

Thus we get

$$d_{Leg}(h_{Leg}) = \rho^C_{|Leg}.$$

Now Proposition in [L-W 2001, Appendix] yields that the Ricci form ρ_N is a linear combination of the Ricci form ρ^C on the cone and the symplectic form on CN. Here we consider N as a subspace $(1, x)$ in the cone $CN = (r, x)$. Since L is Legendrian we get

$$d_{Leg}(h_{Leg}) = (\rho_N)_{|Leg}.$$

Thus if N is a Sasakian weakly Einstein manifold, then h_L is closed.

Next we assume that that the form h_{Leg} is closed for all Legendrian submanifold $Leg \subset N$. This means that the Ricci form ρ^C vanishes on all Legendrian submanifolds in N. In particular for any $k \in \mathbf{R}$ we have the form

$$\rho_k = \rho^C + k\omega$$

also vanishes on any Legendrian plan in N. We choose k big enough such that ρ_k is nondegenerated in order to apply the following Lemma.

A.3. Lemma. *If on $\mathbf{C}^n$ there are two symplectic forms ω_1 and ω_2 which are compatible with J and has the the same Lagrangian Grassmannian then they are propotional.*

Proof . We prove by induction. Suppose that the Lemma is correct for $\mathbf{C}^n$ we shall now prove for the case $\mathbf{C}^{n+1}$. It suffices to show that there is constant λ such that for any $v \in \mathbf{R}^{2n+2} = \mathbf{C}^{n+1}$ we have

$$\omega_1(V, JV) = \lambda\omega_2(V, JV).$$

Clearly if $n \geq 3$, then the induction statement is trivial, because the restriction of ω_i to any complex hyperplan is a symplectic form. Now we proceed to consider the case $n = 2$. First we choose a Lagrangian plan $v_1 \wedge v_2$. We can assume that v_i is an orthonormal basis in this plan w.r.t. the metric g_1 which is compatible to ω_1. We claim that v_1 is also orthogonal to v_2 w.r.t. the second compatible metric g_2. To see it we notice that the set of al Lagrangian plan containing v_1 is generated by the second vector lying in the plan $v_2 \wedge Jv_1$.

To complete the proof of Lemma A.3 for $n = 2$ we normalize ω_2 such that $\omega_2(v_1, Jv_1) = 1$. It remains to show that v_i, Jv_i is a unitary basis w.r.t. the second metric g_2. We observe that

$$\omega_1(Jv_1 + v_2, v_1 + Jv_2) = 0.$$

Clearly

$$\omega_2(Jv_1 + v_2, v_1 + Jv_2) = 0,$$

if and only the norms of v_i w.r. to g_2 are equal. This completes the proof of our Lemma. □

Continuation of the the proof of Proposition A.2. Our Proposition follows immediately from the Lemma A.3 and the relation between the Ricci form and the Chern form on a metric contact manifold. □

In the same way we prove Proposition A.1. The only new argument here is that a Kähler weakly Einstein manifold is Kähler-Einstein. □

A.4. Remark. Using Proposition A.2 we can obtain many example of weakly Einstein-Sasaki manifolds but not Einstein. We begin with a contact Sasakian-Einstein manifold (M, α, g_α). Then using Proposition A.2 we can see that the contact manifold $(M, \sigma \cdot d\alpha)$ has a compatible metric which is Sasakian weakly Einstein but not Einstein.

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