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The eta invariant and the real
connective K-theory of the classifying
space for quaternion groups

by

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THE ETA INVARIANT AND THE REAL CONNECTIVE
K-THEORY OF THE CLASSIFYING SPACE FOR QUATERNION
GROUPS

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Abstract. We express the real connective $K$ theory groups $\tilde{k}o_{4k-1}(BQ)$ of the quaternion group $Q_\ell$ of order $\ell = 2^j \geq 8$ in terms of the representation theory of $Q_\ell$ by showing $\tilde{k}o_{4k-1}(BQ)$ = $KSp(S^{4k+3}/\tau Q)$ where $\tau$ is any fixed point free representation of $Q_\ell$ in $U(2k+2)$.

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1. Introduction

A compact Riemannian manifold $(M,g)$ is said to be a spherical space form if $(M,g)$ has constant sectional curvature $+1$. A finite group $G$ is said to be a spherical space form group if there exists a representation $\tau : G \rightarrow U(k)$ for $k \geq 2$ which is fixed point free - i.e. $\det(I - \tau(\xi)) \neq 0 \forall \xi \in G - \{1\}$. Let $M^{2k-1}(G,\tau) := S^{2k-1}/\tau(G)$ be the associated spherical space form; $G$ is then the fundamental group of the manifold $M^{2k-1}(G,\tau)$. Every odd dimensional spherical space form arises in this manner; the only even dimensional spherical space forms are the sphere $S^{2k}$ and real projective space $RP^{2k}$. The spherical space form groups all have periodic cohomology; conversely, any group with periodic cohomology acts without fixed points on some sphere, although not necessarily orthogonally. We refer to [18] for further details concerning spherical space form groups.

Any cyclic group is a spherical space form group since the group of $\ell$th roots of unity acts without fixed points by complex multiplication on the unit sphere $S^{2k-1}$ in $\mathbb{C}^k$. Let $H = \text{span}_\mathbb{R}\{1, I, J, K\}$ be the quaternions, let $\ell = 2^j \geq 8$, and let $\xi := e^{4\pi I/\ell} \in H$ be a primitive $(\frac{\ell}{2})$th root of unity. The quaternion group $Q_\ell$ is the subgroup of $H$ of order $\ell$ generated by $\xi$ and $J$:

$$Q_\ell := \{1, \xi, ..., \xi^{\ell/2-1}, J, \xi J, ..., \xi^{\ell/2-1} J\}.$$  \hspace{1cm} (1.1)

Let $BG$ be the classifying space of a finite group and let $ko_*(BG)$ be the associated real connective $K$ theory groups; we refer to [2, 3, 7, 9, 14] for a further discussion of connective $K$ theory and related matters.

The $p$ Sylow subgroup of a spherical space form group $G$ is cyclic if $p$ is odd and either cyclic or a quaternion group $Q_\ell$ for $\ell = 2^j \geq 8$ if $p = 2$. This focuses attention on these two groups. We showed previously in [4] that:

**Theorem 1.1.** Let $Z_\ell$ be the cyclic group of order $\ell = 2^j > 1$. Let $k \geq 1$. Let $\tau : Z \rightarrow U(2k+2)$ be a fixed point free representation. Then

$$\tilde{k}o_{4k-1}(BZ_\ell) = KSp(M^{4k+3}(Z_\ell, \tau)).$$

In this paper, we generalize Theorem 1.1 to the quaternion group:

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Theorem 1.2. Let $Q_\ell$ be the quaternion group of order $\ell = 2^j \geq 3$. Let $k \geq 1$. Let $\tau : Q_\ell \rightarrow U(2k+2)$ be a fixed point free representation. Then
\[ \tilde{\text{ko}}_{4k-1}(BQ_\ell) = \tilde{K}Sp(M^{4k+3}(Q_\ell, \tau)). \]

The quaternion (symplectic) $K$ theory groups $\tilde{K}Sp(M^{4k+3}(Q_\ell, \tau))$ are expressible in terms of the representation theory - see Theorem 4.1. Thus Theorem 1.2 expresses $\tilde{\text{ko}}_{4k-1}(BQ_\ell)$ in terms of representation theory. If $\ell = 8$, then these groups were determined previously [3, 5].

Here is a brief outline to this paper. In Section 2, we review some facts concerning the representation theory of $Q_\ell$ which we shall need. In Section 3, we review some results concerning the eta invariant. In Section 4, we use the eta invariant to study $\tilde{K}Sp(M^{4k+3}(Q_\ell, \tau))$. In Section 5, we use the eta invariant to study $\tilde{\text{ko}}(BQ_\ell)$ and complete the proof of Theorem 1.2.

The proof of Theorem 1.2 is quite a bit different from the proof of Theorem 1.1 given previously; the extension is not straightforward. This arises from the fact that unlike the classifying space $BZ_\ell$, the 2 localization of $BQ_\ell$ is not irreducible. Let $SL_2(\mathbb{F}_q)$ be the group of $2 \times 2$ matrices of determinant 1 over the field $\mathbb{F}_q$ with $q$ elements where $q$ is odd. Then the 2-Sylow subgroup of $SL_2(\mathbb{F}_q)$ is $Q_\ell$ for $\ell = 2^j$ where $j$ is the power of 2 dividing $q^2 - 1$. There is a stable 2-local splitting of the classifying space $BQ_\ell$ in the form
\[ BQ_\ell = BSL_2(\mathbb{F}_q) \vee \Sigma^{-1} BS^3/BN \vee \Sigma^{-1} BS^3/BN \]
where $N$ is the normalizer of a maximal torus in $S^3$ [16, 15]. It is necessary to find a corresponding splitting of $\tilde{K}Sp(M^{4k+3}(Q_\ell, \tau))$ that mirrors this decomposition; see Remark 5.2.

2. The Representation Theory of $Q_\ell$

We say that $f : Q_\ell \rightarrow C$ is a class function if $f(xgx^{-1}) = f(g)$ for all $x, g \in Q_\ell$; let $\text{Class}(Q_\ell)$ be the Hilbert space of all class functions with the $L^2$ inner product
\[ \langle f_1, f_2 \rangle = \ell^{-1} \sum_{g \in Q_\ell} f_1(g) \overline{f_2}(g). \]
Let $\text{Irr}(Q_\ell)$ be a set of representatives for the equivalence classes of irreducible unitary representations of $Q_\ell$. The orthogonality relations show that $\{\text{Tr}(\sigma)\}_{\sigma \in \text{Irr}(Q_\ell)}$ is an orthonormal basis for $\text{Class}(Q_\ell)$, i.e. we may expand any class function:
\[ f = \sum_{\sigma \in \text{Irr}(Q_\ell)} \langle f, \text{Tr}(\sigma) \rangle \text{Tr}(\sigma). \]

The unitary group representation ring $RU(Q_\ell)$ and the augmentation ideal $RU_0(Q_\ell)$ are defined by:
\[ RU(Q_\ell) = \text{Span}_\mathbb{Z}\{\sigma\}_{\sigma \in \text{Irr}(Q_\ell)}, \quad \text{and} \]
\[ RU_0(Q_\ell) = \{\sigma \in RU(Q_\ell) : \text{dim} \sigma = 0\}. \]
We shall identify a representation with the class function defined by its trace henceforth; a class function $f$ has the form $f = \text{Tr}(\tau)$ for some $\tau \in RU(Q_\ell)$ if and only if $\langle f, \sigma \rangle \in \mathbb{Z}$ for all $\sigma \in \text{Irr}(Q_\ell)$.

Let $RSp(Q_\ell)$ and $RO(Q_\ell)$ be the $\mathbb{Z}$ vector spaces generated by equivalence classes of irreducible quaternion and real representations, respectively. Forgetting the symplectic structure and complexification of a real structure define natural inclusions $RSp(Q_\ell) \subset RU(Q_\ell)$ and $RO(Q_\ell) \subset RU(Q_\ell)$. We have:
\[ RO(Q_\ell) \cdot RO(Q_\ell) \subset RO(Q_\ell), \]
\[ RSp(Q_\ell) \cdot RSp(Q_\ell) \subset RO(Q_\ell), \]
\[ RO(Q_\ell) \cdot RSp(Q_\ell) \subset RSp(Q_\ell). \]
The $\frac{\ell}{4} + 3$ conjugacy classes of $Q_\ell$ have representatives:

$$\{1, \xi, \ldots, \xi^{\ell/4} = -1, \ J, \ \xi J\}.$$ 

There are $\frac{\ell}{4} + 3$ irreducible inequivalent complex representations of $Q_\ell$. Four of these representations are the 1 dimensional representations defined by:

$$\rho_0(\xi) = 1, \ \kappa_1(\xi) = -1, \ \kappa_2(\xi) = 1, \ \kappa_3(\xi) = -1,$$

$$\rho_0(J) = 1, \ \kappa_1(J) = 1, \ \kappa_2(J) = -1, \ \kappa_3(J) = -1.$$ 

We define representations $\gamma_u : Q_\ell \to U(2)$ by setting:

$$\gamma_u(\xi) = \begin{pmatrix} \xi^u & 0 \\ 0 & \xi^{-u} \end{pmatrix}, \ \gamma_u(J) = \begin{pmatrix} 1 & (-1)^u \\ 0 & 1 \end{pmatrix}.$$ 

The representations $\gamma_u$, $\gamma_{-u}$, and $\gamma_u + \xi^u$ are all equivalent. The representations $\gamma_u$ are irreducible and inequivalent for $1 \leq u \leq \frac{\ell}{4} - 1$; $\gamma_0$ is equivalent to $\rho_0 + \kappa_2$ and $\gamma_4$ is equivalent to $\kappa_1 + \kappa_3$. We have:

$$\text{Irr} (Q_\ell) = \{\rho_0, \kappa_1, \kappa_2, \kappa_3, \gamma_1, \ldots, \gamma_{\frac{\ell}{4} - 1}\}.$$ 

If $\bar{s} = (s_1, \ldots, s_k)$ is a $k$ tuple of odd integers, then

$$\gamma_{\bar{s}} := \gamma_{s_1} \oplus \ldots \oplus \gamma_{s_k}$$

is a fixed point free representation from $Q_\ell$ to $U(2k)$; conversely, every fixed point free representation of $Q_\ell$ is conjugate to such a representation. The associated spherical space forms are the quaternion spherical space forms.

The representations $\{\rho_0, \kappa_1, \kappa_2, \kappa_3\}$ are real, the representations $\gamma_{2i}$ are real, and the representations $\gamma_{2i+1}$ are quaternion. We have:

$$\text{RO}(Q_\ell) = \text{span}_\mathbb{Z}\{\rho_0, \kappa_1, \kappa_2, \kappa_3, 2\gamma_{1}, \ gamma_{2}, \ldots, 2\gamma_{\ell/4 - 1}\},$$

$$\text{RSp}(Q_\ell) = \text{span}_\mathbb{Z}\{2\rho_0, 2\kappa_1, 2\kappa_2, 2\kappa_3, \ gamma_{1}, 2\gamma_{2}, \ldots, \gamma_{\ell/4 - 1}\}.$$ 

We define:

$$\Theta_1(g) := \begin{cases} \frac{\xi}{4} & \text{if } g = \pm I, \\ -2 & \text{if } g = \xi^{2i} J, \\ 0 & \text{otherwise}, \end{cases}$$

(2.2)

$$\Theta_2(g) := \begin{cases} \frac{\xi}{4} & \text{if } g = \pm I, \\ -2 & \text{if } g = \xi^{2i+1} J, \\ 0 & \text{otherwise}. \end{cases}$$

The two class functions $\Theta_i$ will be used to mirror in $RU(Q_\ell)$ the splitting of $BQ_\ell$ given in equation (1.2).

We identify virtual representations with the class functions they define henceforth. Let

$$\Delta := 2\rho_0 - \gamma_1; \ \ \text{Tr} (\Delta) = \det(I - \gamma_1).$$

Lemma 2.1.

(1) We have $\Theta_1 \in \text{RO}_0(Q_\ell)$ and $\Theta_2 \in \text{RO}_0(Q_\ell)$.

(2) Let $c_i := \ell^{-1} \sum_{g \in Q_\ell} \Delta(g)^i$. We have $c_0 = \frac{\ell - 1}{4}$. If $i > 0$, then $c_{2i} \in \mathbb{Z}$ and $c_{2i - 1} \in 2\mathbb{Z}$.

Proof: We use equation (2.2) to compute:

for any $\ell$ (i) $\langle \Theta_1, \rho_0 \rangle = 0$, (ii) $\langle \Theta_1, \gamma_{2i+1} \rangle = 0$, (iii) $\langle \Theta_1, \gamma_{2i} \rangle = (-1)^i$,

for $\ell = 8$ (i) $\langle \Theta_1, \kappa_1 \rangle = -1$, (ii) $\langle \Theta_1, \kappa_2 \rangle = 1$, (iii) $\langle \Theta_1, \kappa_3 \rangle = 0$,

for $\ell > 8$ (i) $\langle \Theta_1, \kappa_1 \rangle = 0$, (ii) $\langle \Theta_1, \kappa_2 \rangle = 1$, (iii) $\langle \Theta_1, \kappa_3 \rangle = 1$,

$$\langle \Theta_2, \kappa_1 \rangle = 1, \ \langle \Theta_2, \kappa_2 \rangle = 1, \ \langle \Theta_2, \kappa_3 \rangle = 0.$$
Lemma 3.2.

Let $\Theta$ be a measure of the spectral asymmetry of $M$. We use equation (2.1) to complete the proof of assertion (1):

\[
\Theta_1 = \begin{cases} 
\operatorname{Tr} \{s_2 - s_1\} & \text{if } \ell = 8, \\
\operatorname{Tr} \{s_2 + s_3 + \sum_{1 \leq i < \ell / 8} (-1)^i \gamma_{2i}\} & \text{if } \ell \geq 16,
\end{cases}
\]

\[
\Theta_2 = \begin{cases} 
\operatorname{Tr} \{s_2 - s_3\} & \text{if } \ell = 8, \\
\operatorname{Tr} \{s_2 + s_1 + \sum_{1 \leq i < \ell / 8} (-1)^i \gamma_{2i}\} & \text{if } \ell \geq 16.
\end{cases}
\]

The first identity of assertion (2) is immediate. Let $r > 0$. As $\operatorname{Tr} (\Delta^r)(1) = 0$,

\[
c_r = \ell^{-1} \sum_{g \in Q_L} \operatorname{Tr} (\Delta^r)(g) = (\Delta^r, \rho_0) \in \mathbb{Z}.
\]

If $r$ is odd, then $\gamma^r_1$ is quaternion so $\langle \gamma^r_1, \rho_0 \rangle \in 2\mathbb{Z}$. Since $\Delta^r \equiv \gamma^r_1 \mod 2RU(Q_L)$, $\langle \Delta^r, \rho_0 \rangle \in 2\mathbb{Z}$ if $r$ is odd. \qed

3. The eta invariant, $K$ theory, and bordism

Let $V$ be a smooth complex vector bundle over a compact Riemannian manifold $M$. Let $V$ be equipped with a unitary (Hermitian) inner product. Let

\[
P : C^\infty (V) \rightarrow C^\infty (V)
\]

be a self-adjoint elliptic first order partial differential operator. Let $\{\lambda_i\}$ denote the eigenvalues of $P$ repeated according to multiplicity. Let

\[
\eta(s, P) := \sum_i \operatorname{sign}(\lambda_i) |\lambda_i|^{-s}.
\]

The series defining $\eta$ converges absolutely for $\Re(s) > 0$ to define a holomorphic function of $s$. This function has a meromorphic extension to the entire complex plane with isolated simple poles. The value $s = 0$ is regular and one defines

\[
\eta(P) := \frac{i}{2} (\eta(s, P) + \dim(\ker P))|_{s=0}
\]

as a measure of the spectral asymmetry of $P$; we refer to [11] for further details concerning this invariant which was first introduced by [1] and which plays an important role in the index theorem for manifolds with boundary.

We say that $P$ is quaternion if $V$ has a quaternion structure and if the action of $P$ commutes with this structure. We say that $P$ is real if $V$ is the complexification of an underlying real vector bundle and if $P$ is the complexification of an underlying real operator.

Lemma 3.1. Let $M$ be a spin manifold of dimension $m$.

1. If $m \equiv 3, 4 \mod 8$, then the Dirac operator is quaternion.
2. If $m \equiv 7, 8 \mod 8$, then the Dirac operator is real.

Proof: Let $\text{Clif} (m)$ be the real Clifford algebra on $\mathbb{R}^m$. We have:

\[
\text{Clif} (3) = \mathbb{H} \oplus \mathbb{H},
\]

\[
\text{Clif} (4) = M_2(\mathbb{H}),
\]

\[
\text{Clif} (7) = M_8(\mathbb{R}) \oplus M_8(\mathbb{R}),
\]

\[
\text{Clif} (8) = M_{16}(\mathbb{R}), \quad \text{and}
\]

\[
\text{Clif} (m + 8) = \text{Clif} (m) \otimes_{\mathbb{R}} M_{16}(\mathbb{R}).
\]

Therefore, the fundamental spinor representation of $\text{Clif} (m)$ is quaternion if we have $m \equiv 3, 4 \mod 8$ and real if we have $m \equiv 7, 8 \mod 8$. The Lemma now follows. \qed

The following deformation result will be crucial to our investigations:

Lemma 3.2. Let $P_u$ be a smooth 1 parameter family of self-adjoint first order elliptic partial differential operators on a compact manifold $M$.

1. The reduction mod $\mathbb{Z}$ of $\eta(P_u)$ is a smooth $\mathbb{R}/\mathbb{Z}$ valued function.
2. The variation $\partial_u \eta(P_u)$ is locally computable.
3. If the operators $P_u$ are quaternion, then the reduction mod $2\mathbb{Z}$ of $\eta(P_u)$ is a smooth $\mathbb{R}/2\mathbb{Z}$ valued function.
Theorem 3.4. Let $P$ be an elliptic self-adjoint first order partial differential operator. Let $\sigma \in RU_0(\pi_1(M))$. 

This invariant is a homotopy invariant.

Lemma 3.3. Let $P_u$ be a smooth 1 parameter family of elliptic first order self-adjoint partial differential operators over $M$.

1. If $\sigma \in RU_0(\pi_1(M))$, then the mod $\mathbb{Z}$ reduction of $\eta^\sigma(P_u)$ is independent of the parameter $u$.

2. If all the operators $P_u$ are quaternion and $\sigma \in RO_0(\pi_1(M))$ or if all the operators $P_u$ are real and $\sigma \in RSp_0(\pi_1(M))$, then the mod $2\mathbb{Z}$ reduction of $\eta^\sigma(P_u, \sigma)$ is independent of the parameter $u$.

Proof: If $\sigma$ is a representation of $\pi_1(M)$, then the mod $\mathbb{Z}$ reduction of $\eta^\sigma(P_u)$ is smooth a smooth function of $u$ by Lemma 3.2. Since $P_u^\sigma$ is locally isomorphic to $\dim \sigma$ copies of $P_u$ and since the variation is locally computable,

$$\partial_u \eta^\sigma(P_u) = \dim \sigma \cdot \partial_u \eta(P_u).$$

This formula continues to hold for virtual representations. In particular, if we have that $\sigma \in RU_0(\pi_1(M))$, then $\dim \sigma = 0$ so $\partial_u \eta^\sigma(P_u) = 0$; (1) follows.

If $P_u$ is quaternion and $\sigma$ is real or if $P_u$ is real and if $\sigma$ is quaternion, then $P_u^\sigma$ is quaternion and $\eta^\sigma(P_u)$ is a smooth $\mathbb{R}/2\mathbb{Z}$ valued function of $u$. The same argument shows that $\partial_u \eta^\sigma(P_u) = 0$. □

We can use the eta invariant to construct invariants of $K$ theory. Let $P : C^{\infty}(V) \to C^{\infty}(V)$ be a first order self-adjoint elliptic partial differential operator with leading symbol $p$. Let $W$ be a unitary vector bundle over $M$. We use a partition of unity to construct a self-adjoint elliptic first order operator $P^W$ on $C^{\infty}(V \otimes W)$ with leading symbol $p \otimes \text{id}$; this operator is not, of course, canonically defined.

We can extend the invariant $\eta^\sigma$ to the the reduced unitary unitary and quaternion (symplectic) $K$ theory groups $\tilde{K}U$ and $\tilde{K}Sp$.

Theorem 3.4. Let $P$ be an elliptic self-adjoint first order partial differential operator. Let $\sigma \in RU_0(\pi_1(M))$. 

Proof: We sketch the proof briefly and refer to [11] Theorem 1.13.2 for further details. Since $\text{sign}(u)$ has an integer jump when $u = 0$, $\eta(P_u)$ can have integer valued jumps at values of $u$ where $\dim(\ker(P_u)) > 0$. However, in $\mathbb{R}/\mathbb{Z}$, the jump disappears so the mod $\mathbb{Z}$ reduction of $\eta(P_u)$ is a smooth $\mathbb{R}/\mathbb{Z}$ valued function of $u$; one uses the pseudo-differential calculus to construct an approximate resolvent and to show that the variation $\partial_u \eta(P_u)$ is locally computable. Assertions (1) and (2) then follow. If $P_u$ is quaternion, then the eigenvalues of $P_u$ inherit quaternion structures. Thus $\dim(\ker(P_u))$ is even so $\eta(P_u)$ has twice integer jumps as eigenvalues cross the origin. Consequently the reduction mod $2\mathbb{Z}$ of $\eta(P_u)$ is smooth and assertion (3) follows. □ 

Let $\tilde{M}$ be the universal cover of a connected manifold $M$ and let $\sigma$ be a representation of $\pi_1(M)$ in $U(k)$. The associated vector bundle is defined by:

$$V^{\sigma} : = \tilde{M} \times \mathbb{C}^k / \sim$$

where we identify

$$(\tilde{x}, z) \sim (g \cdot \tilde{x}, \sigma(g) \cdot z)$$

for $g \in \pi_1(M)$, $\tilde{x} \in \tilde{M}$, and $z \in \mathbb{C}^k$.

The trivial connection on $\tilde{M} \times \mathbb{C}^k$ descends to define a flat connection on $V^{\sigma}$. The transition functions of $V^{\sigma}$ are locally constant; they are given by the representation $\sigma$. Thus the bundle $V^{\sigma}$ is said to be locally flat. Let $P : C^{\infty}(V) \to C^{\infty}(V)$ be a self-adjoint elliptic first order operator on $M$;

$$P^{\sigma} : C^{\infty}(V \otimes V^{\sigma}) \to C^{\infty}(V \otimes V^{\sigma})$$

is a well defined operator which is locally isomorphic to $k$ copies of $P$. Define $\eta^\sigma(P) : = \eta(P^{\sigma})$; we extend by linearity to $\sigma \in RU(\pi_1(M))$. 

This invariant is a homotopy invariant.
(1) The map $W \to \eta^\sigma(p^W)$ extends to a map $\eta^\sigma_p : KU(M) \to \mathbb{R}/\mathbb{Z}$.

(2) Suppose that $P$ and $\sigma$ are both real or that $P$ and $\sigma$ are both quaternion. The map $W \to \eta^\sigma(p^W)$ extends to a map

$$\eta^\sigma_p : KSp(M) \to \mathbb{R}/2\mathbb{Z}.$$  

**Proof:** Let $p^W$ and $\tilde{p}^W$ be two first order self-adjoint partial differential operators on $C^\infty(V \otimes W)$ with leading symbol $p \oplus \text{id}$. Set:

$$P_u := uP^W + (1 - u)\tilde{p}^W.$$  

This is a smooth 1 parameter family of first order self-adjoint partial differential operators. As the leading symbol of $P_u$ is $p \oplus \text{id}$, the operators $P_u$ are elliptic. By Lemma 3.3, $\eta^\sigma(p_u) \in \mathbb{R}/\mathbb{Z}$ is independent of $u$. Consequently $\eta^\sigma_p(W) := \eta^\sigma(P^W) \in \mathbb{R}/\mathbb{Z}$ only depends on the isomorphism class of the bundle $W$. As the eta invariant is additive with respect to direct sums, we may extend $\eta^\sigma_p$ to $KU(M)$ as an $\mathbb{R}/\mathbb{Z}$ valued invariant. Let $W$ be quaternion. By Lemma 3.3, $\eta^\sigma(P_u) \in \mathbb{R}/2\mathbb{Z}$ is independent of $u$ if both $P$ and $\sigma$ are real or if both $P$ and $\sigma$ are quaternion and thus $\eta^\sigma$ extends to $KSp$ as an $\mathbb{R}/2\mathbb{Z}$ valued invariant in this instance. \(\square\)

We can use the Atiyah-Patodi-Singer index theorem [1] to see that the eta invariant also defines bordism invariants. Let $G$ be a finite group. A $G$ structure $f$ on a connected manifold $M$ is a representation $f$ from $\pi_1(M) \to G$. Equivalently, $f$ can also be regarded as a map from $M$ to the classifying space $BG$. We consider tuples $(M, g, s, f)$ where $(M, g)$ is a compact Riemannian manifold with a spin structure $s$ and a $G$ structure $f$. We introduce the bordism relation $[(M, g, s, f)] = 0$ if there exists a compact manifold $N$ with boundary $M$ so that the structures $(g, s, f)$ extend over $N$; this induces an equivalence relation and the equivariant bordism groups $MSpin_m(BG)$ consists of bordism classes of these triples. Disjoint union defines the group structure.

Let $MSpin_* := MSpin_*(B\{1\})$ be defined by the trivial group. Cartesian product makes $MSpin_*(BG)$ into an $MSpin_*$ module. Let $\mathcal{F}$ be the forgetful homomorphism which forgets the $G$ structure $f$. The reduced bordism groups are then defined by:

$$\tilde{MSpin}_*(BG) := \ker(\mathcal{F}) : MSpin_*(BG) \to MSpin_*.$$  

Since the eta invariant vanishes on $MSpin_*$, we restrict henceforth to the reduced groups.

If $s$ is a spin structure on $(M, g)$, let $P_{(M, g, s)}$ be the associated Dirac operator. If $\sigma \in RU_0(G)$, then $f^*\sigma \in RU_0(\pi_1(M))$ and we may define:

$$\eta^\sigma(M, g, s, f) := \eta^{f^*\sigma}(P_{(M, g, s)}).$$

**Theorem 3.5.** Let $G$ be a finite group. Assume either that $m \equiv 3 \text{ mod } 8$ and that $\sigma \in RO_0(G)$ or that $m \equiv 7 \text{ mod } 8$ and that $\sigma \in RS_0(G)$. Then the map $(M, g, s, f) \to \eta^\sigma(M, g, s, f)$ extends to a map

$$\eta^\sigma : \tilde{MSpin}_m(BG) \to \mathbb{R}/2\mathbb{Z}.$$  

**Proof:** We sketch the proof and refer to [6] for further details. Suppose that $m \equiv 3 \text{ mod } 4$ and that $[(M, g, s, f)] = 0$ in $MSpin_m(BG)$. Then $M = dN$ where the spin and $G$ structures on $M$ extend over $N$. We may also extend the given Riemannian metric on $M$ to a Riemannian metric on $N$ which is product near the boundary.

Let $\sigma \in RU_0(G)$. The Dirac operator $P_{(M, g, s)}$ on $M$ is the tangential operator of the spin complex $Q_{(N, g, s)}$ on $N$. We twist these operators by taking coefficients in the locally flat virtual bundle $V^{f^*\sigma}$.

Let $\tilde{A}(N, g, s)$ be the $A$-roof genus and let $ch(V^{f^*\sigma})$ be the Chern character. By the Atiyah-Patodi-Singer index theorem [1]:

$$\text{index}(Q^{f^*\sigma}_{(N, g, s)}) = \int_N \tilde{A}(N, g, s) \wedge ch(V^{f^*\sigma}) + \eta(P^\sigma_{(M, g, s)}).$$
Corollary 3.7. Let \( \sigma \in (V, f) \). Since \( \sigma \) is a virtual bundle of virtual dimension 0 which admits a flat connection, the Chern character of \( V^\sigma \) vanishes. Consequently:

\[
\eta^\sigma(M, g, s, f) = \eta(P(M, g, s)) = \text{index}(Q^\sigma(M, g, s)).
\]

The dimension of \( N \) is \( m + 1 \). We apply Lemma 3.1 to see that if \( m \equiv 3 \mod 8 \) and if \( \sigma \) is real or if \( m \equiv 7 \mod 8 \) and if \( \sigma \) is quaternion, then \( Q^\sigma(M, g, s, f) \) is quaternion. Thus \( \text{index}(Q^\sigma(M, g, s, f)) \in 2\mathbb{Z} \) so \( \eta^\sigma(M, g, s) \) vanishes as an \( \mathbb{R}/2\mathbb{Z} \) valued invariant if \( [(M, g, s, f)] = 0 \) in \( \text{MSpin}_m(BG) \). \( \square \)

There is a geometric description of the real connective \( K \) theory groups \( \tilde{k}_0m(BG) \) in terms of the spin bordism groups. Let \( \mathbb{H}P^2 \) be the quaternionic projective plane. Let \( \tilde{T}_m(BG) \) be the subgroup of \( \text{MSpin}_m(BG) \) consisting of bordism classes \( [(E, g, s, f)] \) where \( E \) is the total space of a geometrical \( \mathbb{H}P^2 \) spin fibration and where the \( G \) structure on \( E \) is induced from a corresponding \( G \) structure on the base. The following theorem is a special case of a more general result \( [17] \):

**Theorem 3.6.** Let \( G \) be a finite group. There is a 2 local isomorphism between \( k_0m(BG) \) and \( \text{MSpin}_m(BG)/\tilde{T}_m(BG) \).

We use Theorem 3.6 to draw the following consequence:

**Corollary 3.7.** Assume either that \( m \equiv 3 \mod 8 \) and \( \sigma \in \text{RO}_0(Q_t) \) or that \( m \equiv 7 \mod 8 \) and \( \sigma \in \text{RO}_0(Q_t) \). Then \( \eta^\sigma \) extends to a map from \( \tilde{k}_0m(BQ_t) \) to \( Q/2\mathbb{Z} \).

**Proof:** If \( [(E, s, f)] \in T_m(BQ_t) \), then \( \eta^\sigma(P(E, g, s, f)) = 0 \); see \( [6] \) Lemma 4.3 or \( [13] \) Lemma 2.7.10 for details. Thus by Theorems 3.5 and Theorem 3.6, the eta invariant extends to \( \tilde{k}_0m(BQ_t) \). By \( [6] \) Theorem 2.4, \( \tilde{k}_0m_{4k-1}(BQ_t) \) is a finite 2 group. Thus it is not necessary to localize at the prime 2 and the eta invariant takes values in \( Q/2\mathbb{Z} \). \( \square \)

The eta invariant is combinatorially computable for spherical space forms. The following theorem follows from \( [8] \).

**Theorem 3.8.** Let \( \tau : G \to SU(2k) \) be fixed point free, let \( P \) be the Dirac operator on \( M^{4k-1}(G, \tau) \), and let \( \sigma \in \text{RU}_0(G) \). Then

\[
\eta^\sigma(P) = t^{-1} \sum_{g \in G \setminus \{1\}} \text{Tr}(\sigma(g)) \det(I - \tau(g))^{-1}.
\]

**4. The Groups** \( \tilde{K}Sp(M^{4w-1}(Q_t, \nu \cdot \gamma_1)) \)

Let \( \Delta = \det(I - \gamma_1) \in \text{RO}_0(Q_t) \). By equation (2.1):

\[
\Delta^\nu \text{RO}_0(Q_t) \subset \text{RO}_0(Q_t) \quad \text{if } \nu \text{ is even},
\]

\[
\Delta^\nu \text{RO}_0(Q_t) \subset \text{RO}_0(Q_t) \quad \text{if } \nu \text{ is odd}.
\]

The following Theorem is well known - see, for example \( [10, 12] \):

**Theorem 4.1.** Let \( \tau : Q_t \to U(2\nu) \) be fixed point free. Then

\[
\tilde{K}Sp(M^{4w-1}(Q_t, \tau)) = \begin{cases} 
\text{RO}_0(Q_t)/\Delta^\nu \text{RO}_0(Q_t) & \text{if } \nu \text{ is even,} \\
\text{RO}_0(Q_t)/\Delta^\nu \text{RO}_0(Q_t) & \text{if } \nu \text{ is odd.}
\end{cases}
\]

By Theorem 4.1, the particular representation \( \tau \) plays no role and we therefore set \( \tau = \nu \cdot \gamma_1 \). We use the eta invariant to study these groups. Let \( \eta^\nu_\nu(W) \) be the invariant described in Theorem 3.4 for the Dirac operator \( P \) on \( M^{4w-1}(Q_t, \nu \cdot \gamma_1) \). We define:

\[
\bar{\eta}^\nu(W) := \begin{cases} 
(\eta^0_0, \eta^1_0, \eta^2_0, \eta^2_\Delta, \eta^2_2, \ldots, \eta^\nu_\nu, \eta^\nu_0, \eta^\nu_\Delta, \eta^\nu_0^\nu, \eta^\nu_\Delta^\nu, \eta^\nu_0^\nu^\nu, \eta^\nu_\Delta^\nu^\nu)(W) & \text{if } \nu \text{ is even,} \\
(\eta^0_0, \eta^1_0, \eta^2_0, \eta^2_\Delta, \eta^2_2, \ldots, \eta^\nu_\nu, \eta^\nu_0, \eta^\nu_\Delta, \eta^\nu_0^\nu, \eta^\nu_\Delta^\nu, \eta^\nu_0^\nu^\nu)(W) & \text{if } \nu \text{ is odd.}
\end{cases}
\]

**Lemma 4.2.** Let \( M := M^{4w-1}(Q_t, \nu \cdot \gamma_1) \). Then

\[
\bar{\eta}^\nu : \tilde{K}Sp(M) \to (Q/2\mathbb{Z})^{
u+1}.
\]
Proof: We apply Lemma 3.1 and Theorem 3.4. We distinguish two cases:

(1) If $\nu$ is even, then $P$ is real. Thus $\eta'_\nu : \mathbb{K}Sp(M) \to \mathbb{Q}/2\mathbb{Z}$ for real $\sigma$ and the Lemma follows as we have used the real representations $\{\Theta_1, \Theta_2, 2\Delta, 2\Delta^2, \ldots, \Delta^{\nu-2}, 2\Delta^{\nu-1}\}$ to define $\eta'_\nu$.

(2) If $\nu$ is odd, then $P$ is quaternion. Thus $\eta'_\nu : \mathbb{K}Sp(M) \to \mathbb{Q}/2\mathbb{Z}$ if $\sigma$ is quaternion and the Lemma follows as we have used the quaternion representations $\{2\Theta_1, 2\Theta_2, \Delta, 2\Delta^2, \ldots, \Delta^{\nu-2}, 2\Delta^{\nu-1}\}$ to define $\eta'_\nu$. □

Let $\varepsilon_{2i} = 2$ and $\varepsilon_{2i-1} = 1$: $\{2\Theta_1, 2\Theta_2, \Delta, 2\Delta^2, \ldots, \varepsilon_{\nu-1}\Delta^{\nu-1}\}$ are quaternion. In Lemma 2.1, we defined constants

$$c_i := \ell^{-1} \sum_{g \in Q_{\ell}(1)} \det(I - \gamma_1(g))^i.$$ 

Since $\Delta(g) = \det(I - \gamma_1(g))$, we use Theorem 3.8 to compute:

$$\eta^{\Delta^s}_\nu = \ell^{-1} \sum_{g \in Q_{\ell}(1)} \Delta(g)^{r+s} \Delta^{\nu-r} = c_{r+s-\nu}. \tag{4.1}$$

Since $\Theta_1$ and $\Theta_2$ are supported on the elements of order 4 in $Q_{\ell}$ and since $\Delta(g) = 2$ for such an element, we may use Theorem 3.8 and equation (2.2) to see:

$$\eta^{\Delta^s}_\nu(\Theta_1) = \eta^{\Delta^s}_\nu(\Delta^s)' = \ell^{-1} \sum_{g \in Q_{\ell}(1)} 2^r \text{Tr}(\Theta_1(g)) \Delta^{\nu-r} = 0,$$

$$\eta^{\Delta^s}_\nu(\Theta_2) = \ell^{-1} \sum_{g \in Q_{\ell}(1)} \text{Tr}(\Theta_2(g)) \Delta^{\nu-r} = 2 \cdot \frac{\ell^2}{\pi} + 4 \cdot \frac{\ell^4}{4}, \tag{4.2}$$

We have $\ell = 2^i$. We use equation (4.1), equation (4.2), and Lemma 2.1 to see:

$$\tilde{\eta}'_{\nu} = \left( \begin{array}{c} 2\Theta_1 \\ 2\Theta_2 \\ \Delta \\ 2\Delta^2 \\ \vdots \\ \varepsilon_{\nu-1}\Delta^{\nu-1} \end{array} \right) = \left( \begin{array}{cc} A_\nu & 0 \\ 0 & B_\nu \end{array} \right) \in M_{\nu+1}(\mathbb{Q}/2\mathbb{Z})$$

where $A$ is the $2 \times 2$ matrix given by

$$A_\nu = 2^{1-\nu} \left( \begin{array}{cc} 2^{j-3} + 1 & 2^{j-3} \\ 2^{j-3} & 2^{j-3} \end{array} \right) \text{ if } \nu \text{ is even}$$

$$A_\nu = 2^{2-\nu} \left( \begin{array}{cc} 2^{j-3} + 1 & 2^{j-3} \\ 2^{j-3} & 2^{j-3} \end{array} \right) \text{ if } \nu \text{ is odd}$$

and where $B$ is the $\nu - 1 \times \nu - 1$ matrix given by:

$$B_\nu = \left( \begin{array}{cccc} 2c_{2-\nu} & c_{3-\nu} & 2c_{4-\nu} & \ldots & 2c_{2-\nu} \\ 4c_{3-\nu} & 2c_{4-\nu} & 4c_{5-\nu} & \ldots & 4c_{1-\nu} \\ 2c_{4-\nu} & c_{5-\nu} & 2c_{6-\nu} & \ldots & 2c_0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 2c_{2-\nu} & c_{1-\nu} & 2c_0 & \ldots & 0 \\ 4c_{1-\nu} & 2c_0 & \ldots & 0 & 0 \\ 2c_0 & 0 & \ldots & 0 & 0 \end{array} \right) \text{ if } \nu \text{ is even}$$
The Theorem now follows from equation (4.3).
Thus equations (4.4) and (4.7) show

\[
B_\nu = \begin{pmatrix}
c_2 c_3 c_4 & \cdots & c_2 c_3 c_0 \\
c_2 c_4 c_5 & \cdots & c_2 c_0 \\
c_4 c_5 c_6 & \cdots & c_0 \\
\vdots & \ddots & \vdots \\
c_{2\nu} c_{2\nu+1} & \cdots & c_{2\nu} c_{2\nu+1} \\
0 & \cdots & 0 \\
\end{pmatrix}
\]

if \( \nu \) is odd.

**Theorem 4.3.** Let \( B_\nu \) be the subgroup of \((\mathbb{Q}/2\mathbb{Z})^{\nu-1}\) spanned by the rows of the matrix \( B_\nu \) defined above. Let \( M = M^{4\nu-1}(Q_\ell, \nu \cdot \gamma_1) \). Then

\[
\tilde{K}Sp(M) = \begin{cases} 
\mathbb{Z}_{2\nu} \oplus \mathbb{Z}_{2\nu} \oplus B_\nu & \text{if } \nu \text{ is even}, \\
\mathbb{Z}_{2\nu-1} \oplus \mathbb{Z}_{2\nu-1} \oplus B_\nu & \text{if } \nu \text{ is odd}.
\end{cases}
\]

**Proof:** Let \( \mathcal{K}_\nu \) be the subspace of \( \tilde{K}Sp(M) \) spanned by the virtual vector bundles defined by \( \{2\Theta_1, 2\Theta_2, \Delta, 2\Delta^2, \ldots, \nu-1\Delta^{\nu-1}\} \). It is then immediate from the form of the matrix \( B_\nu \) that

\[
\eta_\nu(\mathcal{K}_\nu) = \begin{cases} 
\mathbb{Z}_{2\nu} \oplus \mathbb{Z}_{2\nu} \oplus B_\nu & \text{if } \nu \text{ is even}, \\
\mathbb{Z}_{2\nu-1} \oplus \mathbb{Z}_{2\nu-1} \oplus B_\nu & \text{if } \nu \text{ is odd}.
\end{cases}
\]

We use Lemma 2.1 to see \( c_0 = \ell^{-1} \). Thus \( c_0 \) is an element of order \( \ell \) in \( \mathbb{Q}/2\mathbb{Z} \). We use the diagonal nature of matrix \( B_\nu \) to see that:

\[
|\eta_\nu(\mathcal{K}_\nu)| \geq \begin{cases} 
4^\nu \ell^{\nu-1} & \text{if } \nu \text{ is even}, \\
4^{\nu-1} \ell^{\nu-1} & \text{if } \nu \text{ is odd}.
\end{cases}
\]

The \( E_2 \) term in the Atiyah-Hirzebruch spectral sequence for the \( K \) theory groups \( \tilde{K}Sp^v(M) \) is

\[
\oplus_{u+v=w} \hat{H}^w(M; KSp^v(pt)).
\]

We take \( w = 0 \) and study the reduced groups to obtain the estimate:

\[
|\tilde{K}Sp(M)| \leq |\oplus_{u+v=0} \hat{H}^w(M; KSp^v(pt))|.
\]

We have that:

\[
KSp^v(pt) = \begin{cases} 
\mathbb{Z} & \text{if } v \equiv 0, 4 \text{ mod } 8, \\
\mathbb{Z}_2 & \text{if } v \equiv -5, -6 \text{ mod } 8,
\end{cases}
\]

\[
KSp^v(pt) = \begin{cases} 
0 & \text{otherwise},
\end{cases}
\]

\[
\hat{H}^w(M; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}_\ell & \text{if } u \equiv 0, 4 \text{ mod } 8, u < 4\nu - 1,
\hat{H}^w(M; \mathbb{Z}_2) = \begin{cases} 
\mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } u \equiv 1, 2, 5, 6 \text{ mod } 8, u \leq 4\nu - 1.
\end{cases}
\end{cases}
\]

Equations (4.5) and (4.6) then imply:

\[
|\tilde{K}Sp(M)| \leq \begin{cases} 
4^\nu \ell^{\nu-1} & \text{if } \nu \text{ is even}, \\
4^{\nu-1} \ell^{\nu-1} & \text{if } \nu \text{ is odd}.
\end{cases}
\]

Thus equations (4.4) and (4.7) show \( |\tilde{K}Sp(M)| \leq |\eta_\nu(\mathcal{K}_\nu)| \). As the opposite inequality is immediate, we have

\[
\eta_\nu(\mathcal{K}_\nu) = \mathcal{K}_\nu = \tilde{K}Sp(M).
\]

The Theorem now follows from equation (4.3). \( \square \)

5. The groups \( \tilde{k}o_{4k-1}(BQ_\ell) \)

Let \( x = (M, g, s, f) \) where \( s \) is a spin structure and \( f \) is a \( G \) structure on a compact Riemannian manifold \((M, g)\) of dimension \( 4k - 1 \). Let \( \eta^v(x) \) be the eta invariant of the associated Dirac operator with coefficients in \( f^*\sigma \). We reverse the parities of the invariant defined in the previous section to define:

\[
\tilde{\eta}_\ell(x) := \begin{cases} 
(\eta^{2\Theta_1}(x), \eta^{2\Theta_2}(x), \eta^{2\Delta}(x), \eta^{2\Delta^2}(x), \ldots, \eta^{2\Delta^{k}}(x)) & (k \text{ even}) \\
(\eta^{\Theta_1}(x), \eta^{\Theta_2}(x), \eta^{2\Delta}(x), \eta^{2\Delta^2}(x), \ldots, \eta^{2\Delta^{k}}(x)) & (k \text{ odd}).
\end{cases}
\]
We have used real representations if $k$ is odd and quaternion representations if $k$ is even. Therefore, by Corollary 3.7, $\bar{\eta}_k$ extends to:

$$\bar{\eta}_k : \tilde{k}_{4k-1}(BG) \to (\mathbb{Q}/2\mathbb{Z})^{k+2}.$$  

The group $Q_\ell$ has 3 non-conjugate elements of order 4: $\{I, J, \xi J\}$ which generate the 3 non-conjugate subgroups $\{(I), (J), (\xi J)\}$ of order 4. The representation $\gamma_1$ restricts to a fixed point free representation of any subgroup of $Q_\ell$. We define the following spherical space forms:

$$M_{Q_{4k-1}} := M_{4k-1}(Q_\ell, k\gamma_1), \quad M_{J_{4k-1}} := M_{4k-1}((I), k\gamma_1) \quad M_{\xi J_{4k-1}} := M_{4k-1}((J), k\gamma_1).$$

Give the lens spaces $M_g$ the $Q_\ell$ structure induced by the natural inclusion $\langle g \rangle \subset Q_\ell$. We project into the reduced group $\tilde{M}_{\text{Spin}}_{4k-1}(Q_\ell)$; this does not affect the eta invariant as $\eta^\sigma(M_{\text{Spin}}_8(pt)) = 0$. Let $i > 0$. By Theorem 3.8:

$$\eta^{i^2}(M_{4k-1} \times Z^{i^2}) = \eta^\sigma(M_{4k-1})\tilde{A}(Z^{i^2}) = \begin{cases} 2\eta^\sigma(M_{4k-1}) & \text{if } j \text{ is odd}, \\ \eta^\sigma(M_{4k-1}) & \text{if } j \text{ is even}. \end{cases}$$

Let $K^4$ be a spin manifold with $\tilde{A}(K^4) = 2$ and let $B^8$ be a spin manifold with $\tilde{A}(B^8) = 1$. Let $Z^{8k-4} := K^4 \times B^{8k-8}$ and $Z^{8k} = (B^8)^{k}$. Standard product formulas [10] then show

$$\eta^\sigma(M_{4k-1} \times Z^{i^2}) = \eta^\sigma(M_{4k-1})\tilde{A}(Z^{i^2}) = \begin{cases} 2\eta^\sigma(M_{4k-1}) & \text{if } j \text{ is odd}, \\ \eta^\sigma(M_{4k-1}) & \text{if } j \text{ is even}. \end{cases}$$

Let $B_\nu$ and $B_{\nu'}$ be as defined in Section 4. There is a dimension shift involved as we must set $\nu = k + 1$. We use the same arguments as those given previously to see

$$\bar{\eta}_k \begin{pmatrix} M_{4k-1} - M_{J_{4k-1}} \\ M_{I_{4k-1}} - M_{\xi J_{4k-1}} \\ M_{Q_{4k-1}} - M_{I_{4k-1}} \times Z^4 \\ \vdots \\ M_{Q_{4k-1}} \times Z^{4k-4} \end{pmatrix} = \begin{pmatrix} C_k & 0 \\ 0 & B_{k+1} \end{pmatrix} \in M_{k+2}(\mathbb{Q}/2\mathbb{Z})$$

where $C_k$ is the $2 \times 2$ matrix given by

$$C_k = \begin{pmatrix} 2^{1-k} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } \ell = 8 \text{ and } k \text{ is even}, \\ 2^{1-k} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } \ell > 8 \text{ and } k \text{ is even}, \\ 2^{-k} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } \ell = 8 \text{ and } k \text{ is odd}, \\ 2^{-k} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} & \text{if } \ell > 8 \text{ and } k \text{ is odd}, \end{pmatrix}$$

Theorem 1.2 will follow from Theorem 4.3 and from the following:

**Theorem 5.1.** We have

$$\tilde{k}_{4k-1}(BQ_\ell) = \begin{cases} \mathbb{Z}_{2k} \oplus \mathbb{Z}_{2k} \oplus B_{k+1} & \text{if } k \text{ is even}, \\ \mathbb{Z}_{2k+1} \oplus \mathbb{Z}_{2k+1} \oplus B_{k+1} & \text{if } k \text{ is odd}. \end{cases}$$
Remark 5.2. Let 

We use the same argument used to prove Theorem 4.3. Let 

Proof: We use the same argument used to prove Theorem 4.3. Let 

\[ \mathcal{L}_k := \text{Span}_\mathbb{Z}[M^4_{k-1} - M^4_{k-1}, M^4_{k-1} - M^4_{k-1}, M^4_{k-1}, M^4_{k-1} \times Z^4, ..., M^3 \times Z^{4k-4}] \subset \tilde{ko}_{4k-1}(BQ \ell). \]

We then have that 

\[ \bar{\eta}(\mathcal{L}_k) = \begin{cases} Z_{2k} \oplus Z_{2k} \oplus B_{k+1} & \text{if } k \text{ is even}, \\ Z_{2k+1} \oplus Z_{2k+1} \oplus B_{k+1} & \text{if } k \text{ is odd}. \end{cases} \]

By Lemma 2.1 we have \( c_0 = \frac{2}{k} \) and thus \( 2c_0 \) is an element of order \( 2 \) in \( \mathbb{Q}/2\mathbb{Z} \). We use the diagonal nature of the matrix \( B_{k+1} \) to see that: 

\[ |\bar{\eta}(\mathcal{L}_k)| \geq \begin{cases} 4^{k+1}k & \text{if } k \text{ is even}, \\ 4^{k+1}k & \text{if } k \text{ is odd}. \end{cases} \]

We use [6] Theorem 2.4 see: 

\[ |\tilde{ko}_{4k-1}(BQ \ell)| \geq \begin{cases} 4^{k+1}k & \text{if } k \text{ is even}, \\ 4^{k+1}k & \text{if } k \text{ is odd}. \end{cases} \]

Remark 5.2. Let \( n \geq 0 \). One has [3] that: 

\[ \tilde{ko}_{8n+2}(\Sigma^{-1} BS^3/BN) = \begin{cases} \mathbb{Z}_2 & \text{if } \varepsilon = 1, 2, \\ \mathbb{Z}_{2^n+2} & \text{if } \varepsilon = 3, 7, \\ 0 & \text{if } \varepsilon = 4, 5, 6, 8, \end{cases} \]

We may use equation (1.2) to decompose: 

\[ \tilde{ko}_{\ast}(BQ \ell) = \tilde{ko}_{\ast}(\Sigma^{-1} BS^3/BN) \oplus \tilde{ko}_{\ast}(\Sigma^{-1} BS^3/BN) \]

\[ \oplus \tilde{ko}_{\ast}(B_{2}\mathcal{L}_2(\mathbb{F}_q)). \]

This is the decomposition given in Theorems 4.3 and 5.1: 

\[ \mathcal{A}_k = \tilde{ko}_{4k-1}(\Sigma^{-1} BS^3/BN) \oplus \tilde{ko}_{4k-1}(\Sigma^{-1} BS^3/BN) \]

\[ = \text{Span} \{[V^g],[V^h]\} \subset \bar{K}Sp(M^{4k+3}(Q \ell, \tau)) \]

\[ = \text{Span} \{[M^4_{k-1} - M^4_{k-1}],[M^4_{k-1} - M^4_{k-1}]\} \subset \tilde{ko}_{4k-1}(BQ \ell), \]

\[ B_k = \tilde{ko}_{4k-1}(B_{2}\mathcal{L}_2(\mathbb{F}_q)) \]

\[ = \text{Span} \{[V^g],[\Delta^1]\} \subset \bar{K}Sp(M^{4k+3}(Q \ell, \tau)) \]

\[ = \text{Span} \{[M^4_{k-1-4\mu} \times Z^{4\mu}]\} \subset \tilde{ko}_{4k-1}(BQ \ell). \]

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References


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