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by

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AND THE PLURICOMPLEX GREEN FUNCTION

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1. Introduction

Diederich and Ohsawa [12] have shown that if Ω is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ then the following lower bound for the Bergman distance in Ω holds: for a fixed $w_0 \in \Omega$ and $w \in \Omega$ close to the boundary one has

$$\operatorname{dist}_\Omega(w, w_0) \geq \frac{1}{C} \log\log \frac{1}{\delta_\Omega(w)}, \tag{1.1}$$

where $\delta_\Omega(w)$ denotes the euclidean distance of w to $\partial\Omega$ and C is a constant depending only on Ω. They also asked if (1.1) could be improved to

$$\operatorname{dist}_\Omega(w, w_0) \geq \frac{1}{C} \log \frac{1}{\delta_\Omega(w)} \tag{1.2}$$

which is known to hold for example in strongly pseudoconvex domains.

The main goal of this paper is to show that one can improve (1.1) to

$$\operatorname{dist}_\Omega(w, w_0) \geq \frac{\log \frac{1}{\delta_\Omega(w)}}{C \log\log \frac{1}{\delta_\Omega(w)}} \tag{1.3}$$

for C^2 smooth bounded pseudoconvex Ω in $\mathbb{C}^n$. Our main tool will be the pluricomplex Green function. We recall that for a bounded domain Ω in $\mathbb{C}^n$ and a pole $w \in \Omega$ it is defined by

$$g_{\Omega,w} := \sup\{u \in PSH(\Omega) : u < 0, \limsup_{z \to w}(u(z) - \log|z - w|) < \infty\}.$$

We refer to [18] for basic properties of g_Ω. The direct relation between the Bergman metric and the Green function has been explored quite extensively in recent years (see for example [5], [6], [11], [14]). In [12] a certain technical function similar but different from g_Ω was used. Here however, unlike in [12], we are able to apply the Green function directly. The main relation for us with the Bergman metric will be the following quite general result (it is special case of Theorem 3.6 below).

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Theorem 1.1. *There exists a positive constant c_n, depending only on n, such that if Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and $w, \widetilde{w} \in \Omega$ are such that $\{g_{\Omega,w} < -1\} \cap \{g_{\Omega,\widetilde{w}} < -1\} = \emptyset$, then*

$$\operatorname{dist}_\Omega(w, \widetilde{w}) \geq c_n. \tag{1.4}$$

The main ingredients of the proof of Theorem 1.1 is the Kobayashi estimate for the Bergman distance [19] and an L^2-estimate for the $\overline{\partial}$ operator essentially due to Donnelly and Fefferman [13]. As shown by Berndtsson [1]-[3], it is in fact a simple consequence of the original Hörmander theory [16]. It should be pointed out that in many papers (see for example [6], [7], [12]) much more complicated L^2-estimates were used for the $\overline{\partial}$ operator.

Therefore, in order to get bounds for the Bergman distance from below it is enough to estimate the pluricomplex Green function from below in order to study the behavior of its sublevel sets. We do it in section 4 following two basic ideas from [15]. The first is to use an inequality for the complex Monge-Ampère operator from [4] to estimate $|g_{\Omega,w}(\widetilde{\zeta})|$ from above in terms of $|g_{\Omega,\zeta}(w)|$ for some $\widetilde{\zeta}$ close to ζ. Then one estimates the modulus of continuity of $g_{\Omega,w}$ which is known to be continuous precisely when Ω is hyperconvex, that is it admits a bounded plurisubharmonic exhaustion function (see [9]). As a result, we improve some estimates from [15] and [11], by the way simplifying the part of Herbort's argument involving the estimate for the modulus of continuity of $g_{\Omega,w}$ ([15, Main Lemma]). In particular, we get the following result (see Theorem 4.2 below).

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with diameter R for which there exists $v \in PSH(\Omega)$ and positive constants A and a such that in Ω we have*

$$\frac{1}{A}\delta_\Omega^a \leq |v| \leq A\delta_\Omega^a. \tag{1.5}$$

Then there exist positive constants C_1, C_2 depending only on n, A, a and R such that if $w \in \Omega$ is such that $r := \delta_\Omega(w) \leq e^{-2}$, then

$$\{g_{\Omega,w} \leq -1\} \subset \{C_1^{-1} r(\log 1/r)^{-1/a} \leq \delta_\Omega \leq C_2 r(\log 1/r)^{n/a}\}.$$

Theorems 1.1 and 1.2 imply that (1.3) is satisfied in C^2 smooth pseudoconvex domains in $\mathbb{C}^n$. Namely, by [10] such domains satisfy the assumption of Theorem 1.1. Therefore (1.4) holds provided that

$$\delta_\Omega(\widetilde{w}) \geq \delta_\Omega(w)(\log 1/\delta_\Omega(w))^C. \tag{1.6}$$

Also note that if $\gamma(\rho) := \rho(\log 1/\rho)^C$, $C > 1$, $\rho < e^{-C}$, then for $r < e^{-C}$ and positive integer k such that

$$r(\log 1/r)^{(k-1)C} \leq e^{-C}$$

we have

$$\gamma^k(r) \geq r(\log 1/r)^{kC}.$$

In [12] the jump in the Bergman distance was obtained for $w, \widetilde{w}$ with

$$\delta_\Omega(\widetilde{w}) \geq \delta_\Omega(w)^{1/C}$$

and then (1.1) immediately followed. On the other hand, a slightly weaker condition than (1.5) was assumed in [12]. To obtain (1.2) one would need to improve (1.6) to

$$\delta_\Omega(\widetilde{w}) \geq C\delta_\Omega(w)$$

This we are able to prove in arbitrary bounded convex domains with the constant $C = (e+1)^2/(e-1)^2$ (Theorem 4.4 below). Thus (using Proposition 3.7 below) we obtain the following.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ such that for every $z_0 \in \partial\Omega$ there exists an open neighborhood U of z_0, open V in $\mathbb{C}^n$ and a biholomorphism $F : U \to V$ such that $F(\Omega \cap U)$ is convex. Then there exists a positive constant C depending only on Ω such that (1.2) holds for every $w \in \{\delta_\Omega \le e^{-2}\}$.*

Note that again no assumption is made on the regularity of Ω (of course in the latter case the boundary must always be Lipschitz continuous).

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2. The Kobayashi construction

In this section we will briefly sketch the construction of Kobayashi [19] and discuss some of its consequences. We assume that Ω is a bounded domain in $\mathbb{C}^n$. By $H^2(\Omega)$ we denote the Hilbert space of square integrable holomorphic functions in Ω and $K_\Omega(z, w)$ is the Bergman kernel of Ω (holomorphic in z, antiholomorphic in w). We define the immersion of Ω into the (infitely dimensional) projective space $\mathbb{P}(H^2(\Omega))$ as follows

$$\tau : \Omega \ni w \longmapsto [K_\Omega(\cdot, w)] \in \mathbb{P}(H^2(\Omega)).$$

One can show that the Bergman metric in Ω is precisely the pull-back of the Fubini-Study metric in $\mathbb{P}(H^2(\Omega))$. Therefore

$$\mathrm{dist}_\Omega(w, \widetilde{w}) \ge \mathrm{dist}_{\mathbb{P}(H^2(\Omega))}(\tau(w), \tau(\widetilde{w})), \quad w, \widetilde{w} \in \Omega.$$

Moreover, $\mathbb{P}(H^2(\Omega))$ (with the Fubini-Study metric) is complete and

$$\mathrm{dist}_{\mathbb{P}(H^2(\Omega))}([f], [g]) = \arccos \frac{|\langle f, g\rangle|}{||f||\,||g||} \quad f, g \in H^2(\Omega) \setminus \{0\}.$$

We can now easily deduce the following two results.

Proposition 2.1. *For a bounded domain Ω in $\mathbb{C}^n$ we have*

$$\mathrm{dist}_\Omega(w, \widetilde{w}) \ge \arccos \frac{|K_\Omega(w, \widetilde{w})|}{\sqrt{K_\Omega(w, w) K_\Omega(\widetilde{w}, \widetilde{w})}}, \quad w, \widetilde{w} \in \Omega. \quad \square$$

Proposition 2.2. *If a bounded domain Ω in $\mathbb{C}^n$ satisfies*

$$\limsup_{w \to \partial\Omega} \frac{|f(w)|}{\sqrt{K_\Omega(w, w)}} < ||f||_{L^2(\Omega)}, \quad f \in H^2(\Omega) \setminus \{0\}, \tag{2.1}$$

then it is Bergman complete.

Proof. Let w_j be a Cauchy sequence with respect to dist_Ω, then $\tau(w_j)$ a Cauchy sequence with respect to $\mathrm{dist}_{\mathbb{P}(H^2(\Omega))}$. Since $\mathbb{P}(H^2(\Omega))$ is complete, we can find $f \in H^2(\Omega) \setminus \{0\}$ such that $\tau(w_j) = [K_\Omega(\cdot, w_j)] \to [f]$. In particular,

$$\frac{|f(w_j)|}{||f|| \sqrt{K_\Omega(w_j, w_j)}} = |\langle \frac{f}{||f||}, \frac{K_\Omega(\cdot, w_j)}{||K_\Omega(\cdot, w_j)||} \rangle| \to 1,$$

which by the assumption means that w_j has no accumulation point on $\partial\Omega$. But this of course means that w_j is also a Cauchy sequence with respect to the euclidean metric. $\square$

Zwonek [21] constructed a bounded, Bergman complete domain in $\mathbb{C}$ not satisfying

$$\limsup_{w\to\partial\Omega} \frac{|f(w)|}{\sqrt{K_\Omega(w,w)}} = 0, \quad f \in H^2(\Omega),$$

which was the criterion for Bergman completeness formulated in [19]. It remains an open problem to construct a Bergman complete domain Ω in $\mathbb{C}^n$ such that the (possibly) weaker condition (2.1) does not hold.

Proposition 2.1 shows that in order to estimate $\operatorname{dist}_\Omega(w,\widetilde{w})$ from below we need to estimate $|K_\Omega(w,\widetilde{w})|/\sqrt{K_\Omega(w,w)K_\Omega(\widetilde{w},\widetilde{w})}$ from above. Similarly as in [12], we will see that it is enough to construct a right function from $H^2(\Omega)$.

Proposition 2.3. *Let Ω be a bounded domain in $\mathbb{C}^n$, $w,\widetilde{w} \in \Omega$. Suppose that $f \in H^2(\Omega)$ is such that $f(w) = K_\Omega(w,\widetilde{w})/\sqrt{K_\Omega(w,w)K_\Omega(\widetilde{w},\widetilde{w})}$ and $f(\widetilde{w}) = 0$. Then*

$$\frac{|K_\Omega(w,\widetilde{w})|}{\sqrt{K_\Omega(w,w)K_\Omega(\widetilde{w},\widetilde{w})}} \le \frac{||f||_{L^2(\Omega)}}{\sqrt{1+||f||^2_{L^2(\Omega)}}}$$

and

$$\operatorname{dist}_\Omega(w,\widetilde{w}) \ge \frac{\pi}{2} - \arctan ||f||_{L^2(\Omega)}.$$

Proof. We first note that the second estimate is a direct consequence of the first one and Proposition 2.1. We may assume that $f \ne 0$. Set $h := K_\Omega(\cdot,\widetilde{w})/\sqrt{K_\Omega(\widetilde{w},\widetilde{w})}$. Then $\langle f,h\rangle = f(\widetilde{w})/\sqrt{K_\Omega(\widetilde{w},\widetilde{w})} = 0$ and therefore we can find an orthonormal basis $\{\varphi_0,\varphi_1,\dots\}$ of $H^2(\Omega)$ such that $\varphi_0 = h$ and $\varphi_1 = f/||f||$. Then

$$K_\Omega(z,z) = \sum_{j=0}^{\infty} |\varphi_j(z)|^2 \ge |h(z)|^2 + \frac{|f(z)|^2}{||f||^2}, \quad z \in \Omega.$$

Applying it for $z = w$ we get the desired estimate. $\square$

3. The Hörmander-Donnelly-Fefferman-Berndtsson L^2-estimate for the $\overline{\partial}$ operator and applications

Our main tool in constructing square integrable holomorphic functions will be the following estimate for the $\overline{\partial}$ operator, essentially due to Donnelly and Fefferman [13].

Theorem 3.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and φ,ψ plurisubharmonic functions in Ω such that $-e^{-\psi}$ is also plurisubharmonic. Assume that $\alpha \in L^2_{loc,(0,1)}(\Omega)$ is such that $\overline{\partial}\alpha = 0$ and that $i\alpha\wedge\overline{\alpha} \le H\, i\partial\overline{\partial}\psi$ for some nonnegative, locally integrable function H in Ω. Then there exists $u \in L^2_{loc}(\Omega)$ with $\overline{\partial}u = \alpha$ and such that*

$$\int_\Omega |u|^2 e^{-\varphi} \le 27 \int_\Omega H e^{-\varphi}.$$

Proof. If ψ is C^2 and strongly plurisubharmonic then $H \geq \sum_{j,k} \psi^{j\bar{k}} \alpha_j \overline{\alpha_k}$, where $(\psi^{j\bar{k}})$ denotes the inverse transposed matrix of $(\partial^2\psi/\partial z_j \partial \overline{z}_k)$ and α_j are given by $\alpha = \sum_j \alpha_j d\overline{z}_j$. Then Theorem 3.1 is a special case of [1, Theorem 3.1] with $\delta = 1/3$ and φ replaced by $\varphi + \psi/3$. The general case now follows using the standard approximation procedure (as for example in the proof of [16, Theorem 4.4.2]). □

Remark. As shown in [2] and [3], Theorem 3.2 (with a constant perhaps greater than 27, but still the numerical one) is essentially a formal consequence of the proof of [16, Lemma 4.4.1].

The next theorem is a direct consequence of Theorem 3.1.

Theorem 3.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let φ, v be plurisubharmonic in Ω. Assume that $v < 0$ and that it is locally bounded near $\partial\Omega$. Let moreover $\chi \in C^{0,1}(\mathbb{R})$ and let f be a holomorphic function defined in a neighborhood of the union of the supports of $\chi(\log(-v))$ and $\chi'(\log(-v))$. Then one can find a holomorphic function F in Ω satisfying the following estimate*

$$\int_\Omega |F - f\chi(\log(-v))|^2 e^{-\varphi} \leq 27 \int_\Omega |f\chi'(\log(-v))|^2 e^{-\varphi}.$$

Proof. We use Theorem 3.1 with $\psi := -\log(-v)$ and $\alpha := -f\overline{\partial}(\chi(\log(-v))$. Note that by [8], $v \in W^{1,2}_{loc}(\Omega)$, and thus $\alpha \in L^2_{(0,1),loc}(\Omega)$. We can take $H = |f\chi'(\log(-v))|^2$ and we conclude that $u + f\chi(\log(-v))$ is almost everywhere equal to a holomorphic F and satisfies the right estimate. □

Theorem 3.1 seems to be quite a universal tool in obtaining various estimates related to the Bergman kernel. First, we get the following estimate due to Herbort [14] (with a different constant though, depending also on the diameter of Ω).

Theorem 3.3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $f \in H^2(\Omega)$. Then*

$$\frac{|f(w)|}{\sqrt{K_\Omega(w,w)}} \leq \frac{1+3\sqrt{3}}{\gamma(n)} ||f||_{L^2(\{g_{\Omega,w} \leq -1\})}, \quad w \in \Omega,$$

where

$$\gamma(y) := \int_y^\infty \frac{dx}{xe^x}.$$

Proof. We apply Theorem 3.2 with $\varphi := 2ng_{\Omega,w}$, $v := g_{\Omega,w}$ and

$$\chi(t) := \begin{cases} 0, & t \leq 0, \\ \int_0^t e^{-2ne^s} ds, & t > 0. \end{cases}$$

By Theorem 3.2 we can find a holomorphic F in Ω such that $F(w) = \chi(\infty) f(w) = \gamma(n) f(w)$ (this is because $e^{-\varphi}$ is not integrable near w) and

$$||F||_{L^2(\Omega)} \leq (1 + 3\sqrt{3}) ||f||_{L^2(\{g_{\Omega,w} \leq -1\})}.$$

The estimate now follows from the fact that

$$\sqrt{K_\Omega(w,w)} = \sup\{ \frac{|f(w)|}{||f||_{L^2(\Omega)}} : f \in H^2(\Omega) \setminus \{0\} \}. \tag{3.1}$$ □

Theorem 3.3 together implies in particular, thanks to Proposition 2.2, that if Ω is bounded pseudoconvex in $\mathbb{C}^n$ and

$$\lim_{w\to\partial\Omega}\mathrm{vol}(\{g_{\Omega,w}\le -1\})=0$$

then Ω must be Bergman complete.

Next, we generalize results from [7] to several variables.

Theorem 3.4. *Let Ω and U be bounded domains in $\mathbb{C}^n$ such that $\Omega\cup U$ is pseudoconvex with diameter R. Assume that $U\subset B(z_0,r)$. Then for every $f\in H^2(\Omega)$ there exists $F\in H^2(\Omega\cup U)$ such that for every $\lambda>1$ we have*

$$||F-f||_{L^2(\Omega)}\le\left(1+\frac{3\sqrt{3}}{\log\lambda}\right)||f||_{L^2(\Omega\cap\overline{B}(z_0,(r/R)^{1/\lambda})}.$$

Proof. It is enough to apply Theorem 3.2 (in $\Omega\cup U$) with $v(z):=\log|z-z_0|/R$, $\varphi:=0$ and, for $\rho>r$,

$$\chi(t):=\begin{cases}1, & t\le\log\log R/\rho-\log\lambda,\\ \frac{\log\log R/\rho-t}{\log\lambda}, & \log\log R/\rho-\log\lambda<t\le\log\log R/\rho,\\ 0, & t>\log\log R/\rho.\end{cases}$$

We then get, since the support of $\chi(\log(-v))$ is contained in Ω,

$$||F-f||_{L^2(\Omega)}\le||f(1-\chi(\log(-v)))||_{L^2(\Omega)}+3\sqrt{3}||f\chi'(\log(-v))||_{L^2(\Omega)}$$

and the estimate will follow if we let ρ tend to r. □

Corollary 3.5. *Assume that Ω is a bounded domain in $\mathbb{C}^n$ satisfying the following property: for every $z_0\in\partial\Omega$ there exists a neighborhood basis U_j of z_0 such that $\Omega\cup U_j$ is pseudoconvex for every j. (Note that this is always true if $n=1$.) Then, if*

$$\lim_{w\to\partial\Omega}K_\Omega(w,w)=\infty,$$

it follows that Ω is Bergman complete.

Proof. Let $\Omega\ni w_k\to z_0\in\partial\Omega$ and $f\in H^2(\Omega)$. By Theorem 3.4 there exists a sequence $F_j\in H^2(\Omega\cup U_j)$ such that $||F_j-f||_{L^2(\Omega)}\to 0$. We have

$$\frac{|f(w_k)|}{\sqrt{K_\Omega(w_k,w_k)}}\le\frac{|F_j(w_k)|}{\sqrt{K_\Omega(w_k,w_k)}}+||F_j-f||_{L^2(\Omega)}.$$

For every fixed j, since F_j is holomorphic in a neighborhood of z_0, the sequence $|F_j(w_k)|$ is bounded. Thus, if we first let $k\to\infty$ and then $j\to\infty$, the corollary follows from Proposition 2.2. □

Theorem 3.6. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that $w, \widetilde{w} \in \Omega$ and $\alpha, \widetilde{\alpha} > 0$ are such that $\{g_{\Omega,w} < -\alpha\} \cap \{g_{\Omega,\widetilde{w}} < -\widetilde{\alpha}\} = \emptyset$. Then*

$$\operatorname{dist}_\Omega(w, \widetilde{w}) \geq \frac{\pi}{2} - \arctan\left(1 + 3\sqrt{3}\frac{e^{n\widetilde{\alpha}}}{\gamma(n\alpha)}\right),$$

where γ is as in Theorem 3.3.

Proof. Let $h := K_\Omega(\cdot, \widetilde{w})/\sqrt{K_\Omega(\widetilde{w}, \widetilde{w})} \in H^2(\Omega)$ so that $||h||_{L^2(\Omega)} = 1$. Set $\varphi := 2n(g_{\Omega,w} + g_{\Omega,\widetilde{w}})$, $v := g_{\Omega,w}$ and

$$\chi(t) := \begin{cases} 0, & t \leq \log\alpha, \\ \dfrac{1}{\gamma(2n)} \displaystyle\int_{\log\alpha}^{t} e^{-2ne^s} ds, & t > \log\alpha. \end{cases}$$

Then in particular $\chi(\log(-v)) = \chi(\infty) = 1$ at w and $\chi(\log(-v)) = 0$ at $\widetilde{w}$. By Theorem 3.2 there exists $f \in H^2(\Omega)$ such that $f(w) = h(w)$, $f(\widetilde{w}) = 0$ and

$$||f||_{L^2(\Omega)} \leq ||h\chi(\log(-v))||_{L^2(\Omega)} + ||f - h\chi(\log(-v))||_{L^2(\Omega)} \leq 1 + 3\sqrt{3}\frac{e^{n\widetilde{\alpha}}}{\gamma(n\alpha)}.$$

The theorem now follows from Proposition 2.3. □

The following simple localization principle for the Bergman metric shows that the way the Bergman distance grows near the boundary is a local property (see also [20, Proposition 3.1]).

Proposition 3.7. *Let Ω, U, V be bounded domains in $\mathbb{C}^n$. Assume that Ω is pseudoconvex. Then there exist positive constants C_1, C_2 such that*

$$\frac{1}{C_1}\beta_{\Omega\cap U}(w, X) \leq \beta_\Omega(w, X) \leq C_2\beta_{\Omega\cap U}(w, X), \quad w \in \Omega \cap V,\ X \in \mathbb{C}^n.$$

Proof. It is well known that for any bounded domain G

$$\beta_G(w, X) = \frac{1}{\sqrt{K_G(w, w)}} \sup\{\frac{|D_X f(w)|}{||f||_{L^2(G)}} : f \in H^2(G) \setminus \{0\},\ f(w) = 0\}, \quad w \in G,\ X \in \mathbb{C}^n, \tag{3.2}$$

where $D_X f := \sum_j X_j\, \partial f/\partial z_j$. Let $\eta \in C^\infty(\mathbb{C}^n)$ be such that $0 \leq \eta \leq 1$ in $\mathbb{C}^n$, $\eta = 1$ in a neighborhood of $\overline{V}$ and $\operatorname{supp}\eta \subset U$. Fix $w \in \Omega \cap V$. We apply Theorem 3.1 with $\varphi(z) := (2n+2)\log|z - w|/R$, where R is the diameter of Ω, $\alpha := \overline{\partial}(\eta f) = f\overline{\eta}$, $\psi(z) := -\log(R^2 - |z - w|^2)$. We may then take $H := R^2|\partial\eta|^2|f|^2$ and Theorem 3.1 gives F holomorphic in Ω such that $F(w) = f(w) = 0$, $\nabla F(w) = \nabla f(w)$ and

$$||F||_{L^2(\Omega)} \leq C_1||f||_{L^2(\Omega\cap U)},$$

where

$$C_1 := \left(1 + 3\sqrt{3}\frac{R^{n+2}}{\operatorname{dist}(V, \operatorname{supp}\partial\eta)^{n+1}}\right)||\partial\eta||_{L^\infty}.$$

Together with (3.2) and the fact that $K_\Omega \le K_{\Omega\cap U}$ on $\Omega\cap U$ (by (3.1)), this proves the first inequality.

To show the second one we proceed similarly: note that

$$K_\Omega(w,w)\beta^2_\Omega(w,X) \le K_{\Omega\cap U}(w,w)\beta^2_{\Omega\cap U}(w,X), \quad w\in\Omega\cap U,\ X\in\mathbb{C}^n,$$

and therefore it is enough to show that

$$K_{\Omega\cap U}(w,w) \le C_2 K_\Omega(w,w), \quad w\in\Omega\cap V. \tag{3.3}$$

For $\widetilde{f}\in H^2(\Omega\cap U)$ we now apply Theorem 3.1 with $\widetilde{\varphi}(z)=2n\log|z-w|/R$ and η,α,ψ as before. We obtain $\widetilde{F}\in H^2(\Omega)$ such that $\widetilde{F}(w)=\widetilde{f}(w)$ and

$$||\widetilde{F}||_{L^2(\Omega)} \le C_2||\widetilde{f}||_{L^2(\Omega\cap U)},$$

where

$$C_2 := \left(1+3\sqrt{3}\frac{R^{n+1}}{\operatorname{dist}(V,\operatorname{supp}\partial\eta)^n}\right)||\partial\eta||_{L^\infty}.$$

This together with (3.1) gives (3.3). □

4. Estimates for the pluricomplex Green function

The following theorem will be the main step in estimating the Green function. The main idea of the proof comes from [15].

Theorem 4.1. *Assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ with the diameter R. Let $\zeta,w\in\Omega$ and $0<\varepsilon<\min\{r/2,|\zeta-w|/2\}$, where $r:=\delta_\Omega(w)$. Then*

$$|g_{\Omega,w}(\zeta)| \le \frac{\log R/\varepsilon}{\log r/(2\varepsilon)}\left(\sup_{\{\delta_\Omega=\varepsilon\}}|g_{\Omega,w}| + (n!)^{1/n}(\log R/\varepsilon)^{1-1/n}|g_{\Omega,\zeta}(w)|^{1/n}\right).$$

Proof. Let $\alpha:=\log R/\varepsilon$. By [4] and since $(dd^c g_{\Omega,w})^n=(2\pi)^n\delta_w$,

$$\int_\Omega |g_{\Omega,w}|^n(dd^c\max\{g_{\Omega,\zeta},-\alpha\})^n \le n!(2\pi)^n\alpha^{n-1}|g_{\Omega,\zeta}(w)|.$$

The measure $(dd^c\max\{g_{\Omega,\zeta},-\alpha\})^n$ is supported on the set $\{g_{\Omega,\zeta}=-\alpha\}\subset\overline{B}(\zeta,\varepsilon)$ and its total mass is equal to $(2\pi)^n$. Therefore, there exists $\widetilde{\zeta}\in\overline{B}(\zeta,\varepsilon)$ such that

$$|g_{\Omega,w}(\widetilde{\zeta})|^n \le n!(\log R/\varepsilon)^{n-1}|g_{\Omega,\zeta}(w)|. \tag{4.1}$$

By u denote the relative extremal function of the ball $\overline{B}(w,\varepsilon)$, that is

$$u=\sup\{v\in PSH(\Omega): v|_\Omega<0, v|_{\overline{B}(w,\varepsilon)}\le -1\}.$$

One can easily check that

$$\log R/\varepsilon\, u \le g_{\Omega,w} \le \log r/\varepsilon\, u, \quad \text{on } \Omega\setminus B(w,\varepsilon). \tag{4.2}$$

In particular

$$|u(z)| \le \frac{\sup_{\{\delta_\Omega=\varepsilon\}} |g_{\Omega,w}|}{\log r/\varepsilon} =: \delta, \quad \text{if } \delta_\Omega(z) \le \varepsilon.$$

Set $\widetilde{\Omega} := \{z \in \Omega : z + \widetilde{\zeta} - \zeta \in \Omega\}$ and

$$h(z) := \begin{cases} \max\{u(z), u(z+\widetilde{\zeta}-\zeta)-\delta\}, & z \in \widetilde{\Omega}, \\ u(z), & z \in \Omega \setminus \widetilde{\Omega}. \end{cases}$$

It can be shown without difficulty that h is a negative plurisubharmonic function in Ω. Since

$$u(z) \le \frac{\log|z-w|/r}{\log r/\varepsilon}, \quad z \in \Omega \setminus B(w,\varepsilon),$$

we get

$$h \le \max\{-1, -\beta-\delta\} \le -\beta \quad \text{on } \overline{B}(w,\varepsilon),$$

where

$$\beta := \frac{\log r/(2\varepsilon)}{\log r/\varepsilon}.$$

Thus $h(z) \le \beta u(z)$. For $z = \zeta$ we get

$$u(\widetilde{\zeta}) - \delta \le h(\zeta) \le \beta u(\zeta),$$

which together with (4.1) and (4.2) gives the required inequality. □

Remark. For $n = 1$ the theorem recovers the symmetry of g_Ω: it is enough to let $\varepsilon \to 0$.

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ where we can find $v \in PSH(\Omega)$ and positive constants A, B, a, b such that in Ω the following estimate holds*

$$\frac{1}{A}\delta_\Omega^a \le |v| \le B\delta_\Omega^b. \tag{4.3}$$

Then there exist positive constants $C, \widetilde{C}$ depending only on n, A, B, a, b and R, the diameter of Ω such that for $\zeta, w \in \Omega$ with $r := \delta_\Omega(w) \le e^{-2}$ and $\rho := \delta_\Omega(\zeta) \le e^{-2}$ we have

$$|g_{\Omega,w}(\zeta)| \le \begin{cases} C\dfrac{\rho^b}{r^a}\log 1/r, & \text{if } \rho \le r/2, \\ \widetilde{C}\dfrac{r^{b/n}}{\rho^{a/n}}(\log 1/r)^{1-1/n}(\log 1/\rho)^{1/n}, & \text{if } \rho \ge 2r. \end{cases} \tag{4.4}$$

In particular,

$$\{g_{\Omega,w} \le -1\} \subset \{C^{-1/b} r^{a/b}(\log 1/r)^{-1/b} \le \delta_\Omega \le \widetilde{C}^{n/a} r^{b/a}(\log 1/r)^{n/a}\}.$$

Proof. Assume first that $\rho \le r/2$. If $0 < \delta \le r - \rho$ then the inequality

$$g_{\Omega,w}(z) \ge \log R/\delta \frac{v(z)}{\inf_{\overline{B}(\zeta,\delta)}|v|}$$

holds for $z \in \partial B(\zeta,\delta)$ and thus also for all $z \in \Omega \setminus B(\zeta,\delta)$. For $\delta := r/2$ and $z := \zeta$ (4.3) now gives

$$|g_{\Omega,w}(\zeta)| \le 2^a AB \frac{\rho^b}{r^a} \log(2R)/r$$

which implies the first inequality in (4.4).

Now assume that $\rho \ge 2r$. By $C_1, C_2 \dots$ we will denote positive constants depending only on n, A, B, a, b and R. If $0 < \varepsilon < r/2$ then by the first inequality

$$\sup_{\{\delta_\Omega = \varepsilon\}} |g_{\Omega,w}| \le C_1 \frac{\varepsilon^b}{r^a} \log 1/r$$

and

$$|g_{\Omega,\zeta}(w)| \le C_1 \frac{r^b}{\rho^a} \log 1/\rho,$$

if r is sufficiently small. Therefore, by Theorem 4.1

$$|g_{\Omega,w}(\zeta)| \le C_2 \frac{\log 1/\varepsilon}{\log r/(2\varepsilon)} \left(\frac{\varepsilon^b}{r^a} \log 1/r + \frac{r^{b/n}}{\rho^{a/n}} (\log 1/\varepsilon)^{1-1/n} (\log 1/\rho)^{1/n} \right). \tag{4.5}$$

We set

$$\varepsilon := r^\alpha (\log 1/r)^{-\frac{n-1}{bn}},$$

where

$$\alpha := \frac{a(n-1)}{bn} + \frac{1}{n} + 1 \ge 2,$$

since $a \ge b$. Then

$$\begin{aligned} \frac{\varepsilon^b}{r^a} \log 1/r &\le r^{(b-a)/n} (\log 1/r)^{1/n} \\ &\le r^{(b-a)/n} (\log 1/\varepsilon)^{1-1/n} (\log 1/r)^{1/n} \\ &\le C_3 \frac{r^{b/n}}{\rho^{a/n}} (\log 1/\varepsilon)^{1-1/n} (\log 1/\rho)^{1/n} \end{aligned} \tag{4.6}$$

(recall that $\rho \ge 2r$). We also have

$$\frac{\log 1/\varepsilon}{\log r/(2\varepsilon)} \le C_4 \tag{4.7}$$

and

$$\log 1/\varepsilon \le C_5 \log 1/r. \tag{4.8}$$

Combining (4.5)-(4.8) we arrive at the second inequality in (4.4). □

Theorem 4.3 immediately gives the following result which slightly generalizes the main result from [15].

Corollary 4.3. *Let Ω be as in Theorem 4.2. Then for every compact subset K of Ω we have*

$$\lim_{w\to\partial\Omega}\sup_K |g_{\Omega,w}| = 0. \qquad \square$$

Remark. It remains an open problem if Corollary 4.3 holds for arbitrary bounded hyperconvex Ω. Note that then we know from [5] that

$$\lim_{w\to\partial\Omega} ||g_{\Omega,w}||_{L^p(\Omega)} = 0$$

for every $p < \infty$.

Theorem 4.2 can be improved and its proof simplified in case when Ω is convex.

Theorem 4.4. *Let Ω be a bounded convex domain in $\mathbb{C}^n$. For given $\zeta, w \in \Omega$ set $\rho := \delta_\Omega(\zeta)$, $r := \delta_\Omega(w)$. Then*

$$g_{\Omega,w}(\zeta) \geq \log\frac{|\rho - r|}{\rho + r}.$$

In particular,

$$\{g_{\Omega,w} \leq -1\} \subset \{\frac{e-1}{e+1} r \leq \delta_\Omega \leq \frac{e+1}{e-1} r\}.$$

Proof. By the Lempert theorem (see [17]) $g_{\Omega,w}(\zeta)$ is symmetric in w and ζ and thus we way assume that $\rho > r$. Let H be a real hyperplane in $\mathbb{C}^n$ with $H \cap \Omega = \emptyset$ and $\delta_\Omega(w) = \operatorname{dist}(w, H)$. After an orthonormal change of variables we may assume that $H = \{Re\, z_1 = 0\}$, $\Omega \subset \{Re\, z_1 > 0\}$, $w = (r, 0, \dots, 0)$ and $\widetilde{\rho} := \operatorname{dist}(\zeta, H) = Re\, \zeta_1 \geq \rho$. Then

$$g_{\Omega,w}(\zeta) \geq \log\frac{|\zeta_1 - r|}{|\zeta_1 + r|} \geq \log\frac{\widetilde{\rho} - r}{\widetilde{\rho} + r} \geq \log\frac{\rho - r}{\rho + r}. \qquad \square$$

Remark. Actually, one can avoid the use of the Lempert theorem in the proof of Theorem 4.4. Namely, for $\rho < r$ one has to repeat the same argument but with the hyperplane $\widetilde{H}$ such that $\widetilde{H} \cap \Omega = \emptyset$ and $\delta_\Omega(\zeta) = \operatorname{dist}(\zeta, \widetilde{H})$.

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