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by

Peter B. Gilkey, Klaus Kirsten, Dmitri Vassilevich, and Andrei Zelnikov

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Duality symmetry of the p-form effective action and super trace of the twisted de Rham complex

P. Gilkey [a], K. Kirsten [a], D. Vassilevich [a,b], A. Zelnikov [c,d]

[a] *Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22-26, 04103 Leipzig, Germany*

[b] *V. Fock Institute of Physics, St.Petersburg University, Russia*

[c] *Theoretical Physics Institute, University of Alberta, Edmonton Alberta T6G 2J1, Canada*

[d] *Lebedev Physics Institute, Moscow, Russia*

Abstract

We consider quantum p-form fields interacting with a background dilaton. We calculate the variation with respect to the dilaton of a difference of the effective actions in the models related by a duality transformation. We show that this variation is defined essentially by the supertrace of the twisted de Rham complex. The supertrace is then evaluated on a manifold of an arbitrary dimension, with or without boundary.

Key words: Dualities, Dilaton, Euler-Poincaré characteristic, Heat trace asymptotics, Pfaffian
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1 Introduction

Let M be a smooth compact Riemannian manifold of dimension m. Let A_p be a p-form field with the field strength $F = dA_p$. Let ϕ be a scalar field (dilaton).

Email addresses: `gilkey@darkwing.uoregon.edu` (P. Gilkey),
`klaus.kirsten@mis.mpg.de` (K. Kirsten),
`vassil@itp.uni-leipzig.de` (D. Vassilevich),
`zelnikov@phys.ualberta.ca` (A. Zelnikov).

The classical action for this system is then given by

$$S = \int_M e^{-2\phi} F \wedge \star F\,, \qquad F = dA_p\,. \tag{1}$$

Here $\star$ is the Hodge operator, $\star^2 = (-1)^{p(m-p)}$. Later we shall also use a normalized Hodge operator $\tilde{\star}$. Interactions of this type appear in many physical applications. Among them are various reductions from higher dimensions, extended supergravities [9], and bosonic M theory [23]. Instead of the dilaton, also a tachyon coupling may appear [24].

We consider the case when the field A_p is quantized while the dilaton ϕ and the metric are kept as a classical background. The complete information about quantum properties of the system is contained in an effective action $W_p(\phi)$, which is a non-local functional of background fields. All quantum mean values of physical observables and their correlators can be derived by variation of the effective action with respect to ϕ and the background metric.

The main aim of this paper is to study the symmetry properties of $W_p(\phi)$ under the duality transformation

$$p \to m - p - 2\,, \qquad \phi \to -\phi\,. \tag{2}$$

We shall demonstrate that the difference $W_p(\phi) - W_{m-p-2}(-\phi)$ is described by the supertrace of the twisted de Rham complex and calculate the supertrace in any dimension m. We are motivated by higher dimensional supergravity theories where (2) is a part of the S-duality transformation.

Probably the first calculation of this type was performed when $m = 2$ and $p = 0$ by Schwarz and Tseytlin [32] who used the methods of an earlier paper [31]. A similar problem appears in two-dimensional dilaton gravity [20] where some symmetry properties of the effective action for a scalar field coupled to the dilaton were found in [21] by direct calculations. Also for $m = 2$, an exact expression for a class of fermionic determinants was obtained [25] using the supertrace of the Dirac complex. A more systematic approach has been suggested in [34] where the duality symmetry has been studied in dimension 2 for a "non-abelian" (matrix-valued) dilaton field and for the dilaton–Maxwell theory in dimension 4.

As a historical side-remark we note that the twisted de Rham complex has been also used in supersymmetric quantum mechanics [1,36] and in Morse theory [36].

Here is a brief outline to this paper. Section 2 contains some basic definitions. In Section 3, we study the variation of the effective action with respect to the dilaton. In particular, we demonstrate that a variation of two effective actions in the models related by the duality transformation (2) is essentially defined by a specific combination of localized heat trace coefficients (Theorem 3.1) which we identify with the supertrace. An explicit expression for the supertrace of the twisted de Rham complex is given by Theorem 4.2; the proof of this result uses the associated functorial properties and techniques of invariance theory. In Section 5, we generalize these results to the category of manifolds with boundary. In Section 6 we give some concluding remarks and calculate the difference of two dual effective actions for $M = \mathbb{R}^m$.

2 The twisted de Rham complex, heat trace, and zeta function

Let (M, g) be a compact Riemannian manifold without boundary. Let ϕ be an auxiliary smooth function on M which we use to twist the exterior derivative operator d by setting:

$$d_\phi := e^{-\phi} d e^{\phi} \quad \text{on} \quad C^\infty(\Lambda M)\,.$$

Let $\delta_{\phi,g}$ and $\Delta_{\phi,g}$ be the associated twisted coderivative and twisted Laplacian:

$$\delta_{\phi,g} := e^{\phi}\delta_g e^{-\phi} \qquad \text{and} \qquad \Delta_{\phi,g} := (d_\phi + \delta_{\phi,g})^2 = d_\phi\delta_{\phi,g} + \delta_{\phi,g}d_\phi\,.$$

Since $d_\phi^2 = 0$, we have an elliptic complex $d_\phi^p : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^{p+1}M)$ and we use the $\mathbb{Z}$ grading of this complex to decompose:

$$\Delta_{\phi,g} = \oplus_p \Delta^p_{\phi,g} \quad \text{where} \quad \Delta^p_{\phi,g} : C^\infty(\Lambda^p M) \to C^\infty(\Lambda^p M)$$

is a non-negative operator of Laplace type. If we set $\phi = 0$, then we recover the ordinary untwisted de Rham complex. Let

$$\chi(M) := \sum_p (-1)^p \dim H^p(M;\mathbb{R})$$

be the Euler-Poincaré characteristic of the manifold M. Since the index of an elliptic complex is invariant under perturbations, we may use the Hodge-de Rham isomorphism which identifies $\ker(\Delta^p_{0,g}) = H^p(M;\mathbb{R})$ to see that:

$$\operatorname{index}(d_\phi) = \operatorname{index}(d_0) = \sum_p (-1)^p \dim\ker(\Delta^p_{0,g}) = \chi(M). \tag{3}$$

The Hodge decomposition theorem extends to the twisted setting. Thus there is an orthogonal direct sum decomposition

$$C^\infty(\Lambda^p M) = d_\phi C^\infty(\Lambda^{p-1}M) \oplus \delta_{\phi,g} C^\infty(\Lambda^{p+1}M) \oplus \ker \Delta^p_{\phi,g}$$

which decomposes any form as the sum of a twisted exact, twisted co-exact, and twisted harmonic form:

$$A_p = d_\phi A_{p-1} + \delta_{\phi,g} A_{p+1} + \gamma_p \quad \text{for} \quad \gamma_p \in \ker \Delta^p_{\phi,g}.$$

The associated projections on the spaces of twisted exact and co-exact forms will be denoted by the subscripts $||$ and $\perp$ respectively; note that the space of twisted harmonic forms $\ker \Delta^p_{\phi,g}$ is finite dimensional.

Let D be an operator of Laplace type on M. The fundamental solution of the heat equation, e^{-tD}, is an infinitely smoothing operator. Let $f \in C^\infty(M)$ be an auxiliary smearing function. As $t \downarrow 0$, there is a complete asymptotic expansion:

$$\mathrm{Tr}_{L^2}(fe^{-tD}) \sim \sum_{n=0}^{\infty} t^{(n-m)/2} a_{n,m}(f, D) \tag{4}$$

where the *heat trace coefficients* $a_{n,m}(f, D)$ are locally computable; this means that there are local invariants $a_{n,m}(D)(x)$ so that:

$$a_{n,m}(f, D) = \int_M f(x) a_{n,m}(D)(x) dx$$

where dx is the Riemannian element of volume. The function f localizes the question and permits us to recover the divergence terms which would otherwise not be detected. The invariants $a_{n,m}$ vanish identically if n is odd.

Suppose additionally that D is non-negative. Let Π_D be orthogonal projection on the zero eigenspace and let $\widehat{D}$ be D acting on $\ker(D)^\perp$ (i.e. project out the zero eigenspace). Define the smeared zeta function by setting:

$$\zeta_D(f, s) := \mathrm{Tr}_{L^2}(f\widehat{D}^{-s}) \,.$$

The zeta function is related to the heat trace by means of the Mellin transform

$$\zeta_D(f, s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}_{L^2}\{f(e^{-tD} - \Pi_D)\} dt \,. \tag{5}$$

Since $\mathrm{Tr}_{L^2}\{f(e^{-tD} - \Pi_D)\}$ decays exponentially as $t \to \infty$, eqn. (4) and eqn. (5) imply that the zeta function $\zeta_D(f, s)$ is regular at $s = 0$ and that

$$\zeta_D(f, 0) = a_{m,m}(f, D) - \mathrm{Tr}_{L^2}\{f\Pi_D\} \,.$$

Let prime denote differentiation with respect to the parameter s. We use the zeta function to define the determinant of the operator D by setting [10,22,29]:

$$\ln \det \widehat{D} := -\zeta'_D(1, 0) \,. \tag{6}$$

3 The effective action

We consider quantum effects in a system described by the action given in eqn. (1). It is natural to include the dilaton field ϕ as well in the definition of the inner product by setting:

$$\langle A_p, B_p \rangle = \int_M e^{-2\phi} A_p \wedge \star B_p \,. \tag{7}$$

This equation defines also the path integral measure. For a different measure, the effective action will receive a contribution from the scale anomaly. This contribution is relatively easy to control. For the rest of this paper we adopt the measure defined by (7). The fields $\tilde{A}_p := e^{-\phi} A_p$ have a standard Gaussian measure and are to be considered as fundamental fields in the path integral. The action given in eqn. (1) is invariant under the gauge transformation which sends $\tilde{A}_p$ to $\tilde{A}_p + d_\phi \tilde{A}_{p-1}$. This means that the p-forms which are d_ϕ exact have to be excluded from the path integral, but that a Jacobian factor corresponding to the ghost fields $\tilde{A}_{p-1}$ has to be included in the path integral measure. Next we note that d_ϕ-exact $(p-1)$-forms do not generate a non-trivial transformation of $\tilde{A}_p$. Hence, such fields must be excluded from the ghost sector. Then we have to include "ghosts for ghosts". This goes on until the zero forms have been reached. By giving these arguments an exact meaning, one arrives at the Faddeev–Popov approach to quantization of the p-form actions [4,11,27,33]. We note that the procedure of [4,27] is valid also in the presence of a dilaton interaction if one simply replaces the ordinary derivatives by the twisted ones. As a result, we have the following expression for the effective action:

$$W_p(\phi) := \frac{1}{2} \sum_{k=0}^{p} (-1)^{p+k} \ln \det(\Delta^k_{\phi,g}|_\perp) + W_p^{\mathrm{top}} \tag{8}$$

where all determinants are restricted to the spaces of twisted co-exact forms. Let $b_j := \dim H^j(M;\mathbb{R})$ be the j^{th} Betti number. According to [4] the "topological" part of the effective action is given by [1] :

$$W_p^{\mathrm{top}} := \ln \sigma \sum_{j=0}^{p} (-1)^{p+j} b_j .$$

The coupling constant σ (an overall factor in front of the action (1)) has not been written explicitly. The topological effective action W_p^{top} will play no role in the subsequent calculations since it does not depend on ϕ.

The twisted exterior derivative d_ϕ intertwines $\Delta^p_{\phi,g}|_\parallel$ and $\Delta^{p-1}_{\phi,g}|_\perp$. This shows for the total zeta function in terms of the coexact ones,

$$\zeta_{\Delta^p_{\phi,g}}(1,s) = \zeta_{\Delta^p_{\phi,g}}|_\perp(1,s) + \zeta_{\Delta^{p-1}_{\phi,g}}|_\perp(1,s),$$

[1] Slightly different quantization schemes yield different W_p^{top} (cf. [4,11]). Some choices of W_p^{top} may lead to non-local contributions to the scale anomaly [12].

the inverse of which is

$$\zeta_{\Delta^p_{\phi,g}}|_\perp(1,s) = \sum_{k=0}^{p}(-1)^{p+k}\zeta_{\Delta^k_{\phi,g}}(1,s).$$

Each individual term in (8) can be calculated via (6) and we have

$$W_p(\phi) = -\frac{1}{2}\sum_{k=0}^{p}(-1)^{p+k}\zeta'_{\Delta^k_{\phi,g}}|_\perp(1,0) + W_p^{\mathrm{top}}\,. \tag{9}$$

In general, dealing with the zeta functions, we use the method of [3,30]. We first assume that s is sufficiently large to keep us away from the singularities, then use the Mellin transformation described in eqn. (5), perform the variation, then perform the Mellin transformation backwards, and then continue the results to $s=0$. This is a perfectly standard procedure which allows us to work with variations of positive integer powers of D; see [13] for further details.

We consider the variation of $\zeta_{\Delta^p_{\phi,g}|_\perp}(1,s)$ under an infinitesimal variation of ϕ and remark that the symbol 'δ' has two different meanings in the following equation:

$$\begin{aligned}\delta\zeta_{\Delta^p_{\phi,g}|_\perp}(1,s) &= \delta\mathrm{Tr}_{L^2}((\delta_{\phi,g}d_\phi|_{\Lambda^p_\perp})^{-s}) = \delta\mathrm{Tr}_{L^2}((e^\phi\delta_g e^{-2\phi}de^\phi|_{\Lambda^p_\perp})^{-s})\\ &= -2s\mathrm{Tr}_{L^2}\left((\delta\phi)\delta_{\phi,g}d_\phi(\Delta^p_{\phi,g})^{-s-1}|_{\Lambda^p_\perp}\right)\\ &\quad +2s\mathrm{Tr}_{L^2}\left((\delta\phi)d_\phi\delta_{\phi,g}(\Delta^p_{\phi,g})^{-s-1}|_{\Lambda^{p+1}_\parallel}\right)\\ &= -2s\left(\zeta_{\Delta^p_{\phi,g}|_\perp}(\delta\phi,s) - \zeta_{\Delta^{p+1}_{\phi,g}|_\parallel}(\delta\phi,s)\right)\,.\end{aligned} \tag{10}$$

This formal derivation can be justified using the procedure outlined above. We remark that the localized zeta function $\zeta_{\Delta^p_{\phi,g}|_\parallel}(\delta\phi,s)$ may have a pole at $s=0$ (for an explicit example see [34]) while $\zeta_{\Delta^p_{\phi,g}|_\parallel}(1,s)$ and $\zeta_{\Delta^p_{\phi,g}}(\delta\phi,s)$ are regular at this point.

Our aim is to study symmetry properties of $W_p(\phi)$ with respect to the reflection $\phi\to-\phi$. To proceed further we need several identities between spectral functions of the Laplacians with ϕ and $-\phi$.

Let $\tilde\star_g$ be the normalized Hodge operator defined by a local orientation of M. Then, the normalizations having taken into account the sign conventions, one has the following relationships:

$$\begin{aligned}&\tilde\star_g^2 = \mathrm{id}, &&\tilde\star_g\, d_\phi\, \tilde\star_g = \delta_{\phi,g},\\ &\tilde\star_g\, \delta_{\phi,g}\, \tilde\star_g = d_\phi, &&\tilde\star_g\, \Delta^p_{\phi,g}\, \tilde\star_g = \Delta^{m-p}_{-\phi,g},\\ &\tilde\star_g\, \Delta^p_{\phi,g}|_\perp\, \tilde\star_g = \Delta^{m-p}_{-\phi,g}|_\parallel, &&\zeta_{\Delta^p_{\phi,g}|_\perp}(\delta\phi,s) = \zeta_{\Delta^{m-p}_{-\phi,g}|_\parallel}(\delta\phi,s)\,.\end{aligned} \tag{11}$$

Theorem 3.1 *We have that*

$$\delta\left(W_p(\phi) - W_{m-p-2}(-\phi)\right) = \sum_{k=0}^{m}(-1)^{p+k}\left[a_{m,m}(\delta\phi, \Delta^k_{\phi,g}) - \mathrm{Tr}_{L^2}\{(\delta\phi)\Pi_{\Delta^k_{\phi,g}}\}\right].$$

Proof. We first use eqn. (10) and eqn. (11) to demonstrate that

$$\begin{aligned} & -\frac{1}{2}\delta\left[\sum_{k=0}^{p}(-1)^{k+p}\zeta_{\Delta^k_{\phi,g}|_\perp}(1,s) - \sum_{q=0}^{m-p-2}(-1)^{p+q}\zeta_{\Delta^q_{-\phi,g}|_\perp}(1,s)\right] \\ & = s\sum_{k=0}^{m}(-1)^{p+k}\zeta_{\Delta^k_{\phi,g}}(\delta\phi, s)\,. \end{aligned} \tag{12}$$

We differentiate eq. (12) with respect to s, take the limit $s \to 0$, and use eq. (9) together with properties of the zeta function described in Section 2 to complete the proof. The term with the harmonic forms takes care of the zero mode subtraction. □

4 Supertrace of the twisted de Rham complex

Motivated by the term appearing in Theorem 3.1, we express

$$\begin{aligned} \sum_p(-1)^p\mathrm{Tr}_{L^2}(fe^{-t\Delta^p_{\phi,g}}) \sim \sum_{n=0}^{\infty}t^{(n-m)/2}\int_M a^{d+\delta}_{n,m}(\phi,g)(x)f(x)dx \quad \text{where} \\ a^{d+\delta}_{n,m}(\phi,g) := \sum_{p=0}^{m}(-1)^p a_{n,m}(\Delta^p_{\phi,g})\,. \end{aligned} \tag{13}$$

The cancellation argument of McKean and Singer [26] together with eq. (3) then yields, setting the smearing function $f = 1$, that:

$$\int_M a^{d+\delta}_{n,m}(\phi,g)(x)dx = \begin{cases} \chi(M) & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{14}$$

From eq. (14) we immediately see that for constant $\delta\phi$ the variation in Theorem 3.1 is zero. The same statement can be also derived from the invariance of d_ϕ and $\delta_{\phi,g}$ under constant shifts of the dilaton.

Let R_{ijkl} be the components of the Riemann curvature tensor relative to a local orthonormal frame with the sign convention that $R_{1221} = +1$ on the standard sphere in $\mathbb{R}^3$. We define the anti-symmetric permutation tensor ε by setting:

$$\varepsilon^{i_1...i_m;j_1...j_m} := (e^{i_1} \wedge ... \wedge e^{i_m}, e^{j_1} \wedge ... \wedge e^{j_m}) .$$

We adopt the Einstein convention and sum over repeated indices. If $m = 2\bar{m}$ is even, then the *Pfaffian* or *Euler form* is defined by setting:

$$\mathcal{E}_m := (4\pi)^{-\bar{m}} \tfrac{1}{2^{\bar{m}}\bar{m}!} \varepsilon^{i_1...i_m;j_1...j_m} R_{i_1 i_2 j_2 j_1} ... R_{i_{m-1} i_m j_m j_{m-1}} . \tag{15}$$

We set $\mathcal{E}_m = 0$ if m is odd. For example, we have:

$$\mathcal{E}_2 = \tfrac{1}{4\pi} R_{ijji} \quad \text{and} \quad \mathcal{E}_4 = \tfrac{1}{32\pi^2}((R_{ijji})^2 - 4|R_{ijjk}|^2 + |R_{ijkl}|^2) .$$

The following theorem of Patodi [28] deals with the untwisted case:

Theorem 4.1 *If $\phi = 0$, then $a_{n,m}^{d+\delta} = 0$ if $n < m$ and $a_{m,m}^{d+\delta} = \mathcal{E}_m$.*

Combining Theorem 4.1 with eq. (14), yields a heat equation proof of the Chern-Gauss-Bonnet theorem [7] for manifolds without boundary:

$$\chi(M) = \int_M \mathcal{E}_m(g)(x) dx .$$

Shortly after Patodi's original proof, Atiyah, Bott, and Patodi [2] and Gilkey [15] also gave proofs of this result using very different methods. Subsequently many proofs of the index theorem using heat equation methods have been given; we refer to [5] for an excellent historical survey.

In the present note, we shall follow the development in [15] to extend Theorem 4.1 to the twisted setting:

Theorem 4.2 *For arbitrary ϕ, $a_{n,m}^{d+\delta} = 0$ if $n < m$ and $a_{m,m}^{d+\delta} = \mathcal{E}_m$.*

Theorem 4.2 shows that the crucial term where $n = m$ does **not** involve ϕ. We remark that the higher divergence terms where $n > m$ do involve ϕ [19].

The proof of Theorem 4.2 will rest upon a detailed analysis of the invariants $a_{n,m}^{d+\delta}$. We begin by introducing some spaces of invariants. Let $\mathcal{Q}_{n,m}$ be the set of all invariants which are homogeneous of weight n in the derivatives of the metric and the derivatives of an auxiliary function ϕ with coefficients which are smooth functions of the metric tensor and which are defined on underlying manifolds of dimension m. These spaces are trivial if n is odd. Let $\mathcal{P}_{n,m} \subset \mathcal{Q}_{n,m}$ be the subspace of invariants not involving the auxiliary function ϕ. For example, $\phi_{;ii} \in \mathcal{Q}_{2,m}$, $R_{ijji;kk} \in \mathcal{P}_{4,m}$, and $\mathcal{E}_m \in \mathcal{P}_{m,m}$.

The following natural restriction map $r : \mathcal{Q}_{n,m} \to \mathcal{Q}_{n,m-1}$ will play a crucial role. If (N, g_N, ϕ_N) are structures in dimension $m-1$, then we can define corresponding structures in dimension m by setting

$$(M,\phi_M,g_M):=(N\times S^1,\phi_N,g_N+d\theta^2)\,.$$

If $x\in N$ is the point of evaluation, we take the corresponding point $(x,1)\in M$ for evaluation; which point on the circle chosen is, of course, irrelevant as S^1 has a rotational symmetry. The restriction map is then characterized dually by the formula:

$$r(Q)(\phi_N,g_N)(x)=Q(\phi_N,g_N+d\theta^2)(x,1)\,.$$

We can also describe the restriction map r in classical terms. One considers polynomials in the covariant derivatives of the function ϕ and of the curvature tensor R. The theorem of H. Weyl [35] on the invariants of the orthogonal group then shows that all invariants are built by contracting indices in pairs, where the indices range from 1 thru m. If P is given in terms of a Weyl spanning set, then $r(P)$ is given in terms of the same Weyl spanning set by simply restricting the range of summation to be from 1 thru $m-1$.

Lemma 4.3 *We have $a_{n,m}^{d+\delta}\in\mathcal{Q}_{n,m}\cap\ker(r)$.*

Proof. The heat trace invariants $a_{n,m}^{d+\delta}$ are homogeneous of degree n in the jets of the metric and of ϕ. Thus $a_{n,m}^{d+\delta}\in\mathcal{Q}_{n,m}$.

If D is any operator of Laplace type on a vector bundle V, then we have that $a_0(x,D)=(4\pi)^{-m/2}\dim V$. Consequently, $a_{0,m}^{d+\delta}=0$ for any dimension m as $\sum_p(-1)^p\dim(\Lambda^pM)=0$. On the circle, the metric is flat. If $\phi=0$, then the jets of ϕ play no role and thus $a_{n,1}^{d+\delta}=0$ for $n>0$. This shows

$$a_{n,1}^{d+\delta}(0,d\theta^2)=0\quad\text{for all}\quad n. \tag{16}$$

Suppose the structures decouple. Then we may decompose $\Lambda M=\Lambda M_1\otimes\Lambda M_2$, $d_\phi=d_1+d_2$ and $\delta_{\phi,g}=\delta_1+\delta_2$ where on $C^\infty(\Lambda^pM_1\otimes\Lambda^qM_2)$ we have:

$$\begin{aligned}&d_1:=d_{\phi_1}\otimes\mathrm{id},\ \ d_2:=(-1)^p\,\mathrm{id}\otimes d_{\phi_2},\\&\delta_1:=\delta_{\phi_1,g}\otimes\mathrm{id},\ \delta_2:=(-1)^p\,\mathrm{id}\otimes\delta_{\phi_2,g_2}.\end{aligned}$$

Consequently these operators satisfy the commutation relations:

$$d_1d_2+d_2d_1=0,\ d_1\delta_2+\delta_2d_1=0,\ \delta_1d_2+d_2\delta_1=0,\ \delta_1\delta_2+\delta_2\delta_1=0\,.$$

Thus we may express $\Delta_{\phi,g}=\Delta_{\phi_1,g_1}\otimes\mathrm{id}+\mathrm{id}\otimes\Delta_{\phi_2,g_2}$ so the fundamental solution of the heat equation is given by $e^{-t\Delta_{\phi,g}}=e^{-t\Delta_{\phi_1,g_1}}\otimes e^{-t\Delta_{\phi_2,g_2}}$. By taking the super trace and by equating coefficients of t in the resulting asymptotic expansions we see that:

$$a_{n,m}^{d+\delta}(\phi,g)(x_1,x_2)=\sum_{n_1+n_2=n}a_{n_1,m_1}^{d+\delta}(\phi_1,g_1)(x_1)\cdot a_{n_2,m_2}^{d+\delta}(\phi_2,g_2)(x_2). \tag{17}$$

We now set $(M_2, \phi_2, g_2) = (S^1, 0, d\theta^2)$ and apply eqn. (16) and eqn. (17) to see $r(a^{d+\delta}_{n,m}) = 0$. $\square$

The heat equation proof of the Chern-Gauss-Bonnet theorem given in [15] relied heavily on the following result:

Lemma 4.4 *Let $P \in \mathcal{P}_{n,m} \cap \ker(r)$. If $n < m$, $P = 0$. If $n = m$, $P = c_m \mathcal{E}_m$.*

We can extend this Lemma to the situation at hand by showing that ϕ plays no role for $n \leq m$:

Lemma 4.5 *Let $Q \in \mathcal{Q}_{n,m} \cap \ker(r)$. If $n < m$, $Q = 0$. If $n = m$, $Q = c_m \mathcal{E}_m$.*

Proof. Let $Q \in \mathcal{Q}_{n,m} \cap \ker(r)$. Fix a point $x_0 \in M$ and introduce a system of local coordinates $x = (x^1, ..., x^m)$ so that

$$g_{ij}(x_0) = \delta_{ij} \quad \text{and} \quad \partial_k g_{ij}(x_0) = 0 \quad \text{for} \quad 1 \leq i, j, k \leq m\,.$$

For example, geodesic polar coordinates centered at x_0 have this property. If $\alpha := (a_1, ..., a_m)$ and $\beta := (b_1, ..., b_m)$ are multi-indices, introduce variables

$$\phi_{/\alpha} := \partial^x_\alpha \phi \qquad \text{and} \qquad g_{ij/\beta} := \partial^x_\beta g_{ij}$$

for the ordinary partial (not covariant) derivatives of the function ϕ and of the metric g. We emphasize, we are **not** working invariantly. With the normalizations given above, we may then express:

$$Q = Q(\phi_{/\alpha}, g_{ij/\beta}) \qquad \text{for} \qquad |\alpha| \geq 1 \quad \text{and} \quad |\beta| \geq 2\,. \tag{18}$$

Suppose that $Q \neq 0$ and that $r(Q) = 0$. Then Q vanishes on product manifolds $N \times S^1$ where the structures are flat in the S^1 direction. Restricting to such product structures simply imposes the additional relations

$$\phi_{/\alpha} = 0 \quad \text{and} \quad g_{ij/\beta} = 0 \qquad \text{if} \qquad a_m > 0 \quad \text{and} \quad \delta_{i,m} + \delta_{j,m} + b_m > 0\,.$$

We let $\deg_\ell$ be the number of times the index ℓ appears in a given variable:

$$\deg_\ell(\phi_{/\alpha}) := a_\ell \qquad \text{and} \qquad \deg_\ell(g_{ij/\beta}) := \delta_{i\ell} + \delta_{j\ell} + b_\ell.$$

The condition $r(Q) = 0$ is then equivalent to the condition that $\deg_m(A) > 0$ for every monomial A of Q and, by symmetry, that $\deg_i(A) > 0$ for every monomial A of Q and every index i. Let

$$A = \phi_{/\alpha_1} \cdots \phi_{/\alpha_u} g_{i_1 j_1/\beta_1} \cdots g_{i_v j_v/\beta_v}$$

be a monomial of Q. Since Q is invariant under the change of coordinates $x_m \to -x_m$ and $x_i \to x_i$ for $i < m$, the index m (and hence every index) must appear an even number of times in A. Consequently as every index appears in A, every index appears at least twice. Using eqn. (18), we count indices to estimate:

$$\begin{aligned} &2m \le \textstyle\sum_i \deg_i(A) = \sum_\mu |\alpha_\mu| + \sum_\nu (2 + |\beta_\nu|) = n + 2v \\ &\le n + \textstyle\sum_\nu |\beta_\nu| = 2n - \sum_\mu |\alpha_\mu| \le 2n - u \le 2n. \end{aligned} \tag{19}$$

This is impossible if $n < m$ so $Q = 0$ if $n < m$. On the other hand, if $n = m$, all of the inequalities in eqn. (19) must have been equalities. Thus, $u = 0$ so Q does not involve ϕ at all. Consequently $Q \in \mathcal{P}_{m,m} \cap \ker(r)$ so by Lemma 4.4, $Q = c_m \mathcal{E}_m$. □

Proof of Theorem 4.1. We apply Lemmas 4.3 and 4.5; assertion (1) is now immediate. If m is odd, then $a^{d+\delta}_{m,m} = 0$ so there is nothing to prove. If m is even, then $a^{d+\delta}_{m,m} = c_m \mathcal{E}_m$ for some universal constant c_m. Evaluating the integral on $S^2 \times ... \times S^2$ with $f = 1$ where S^2 is given the standard metric and using eq. (14) then yields

$$c_m \int_{S^2 \times ... \times S^2} \mathcal{E}_m dx = \chi(S^2 \times ... \times S^2) = 2^{\bar{m}}.$$

Taking into effect the multiplicative nature of the constants and the factor of $\frac{1}{\bar{m}!}$ we see that

$$c_m \int_{S^2 \times ... \times S^2} \mathcal{E}_m dx = c_m \left\{ \int_{S^2} \mathcal{E}_2 dx_2 \right\}^{\bar{m}} = c_m \left\{ \frac{1}{8\pi} \int_{S^2} 4R_{1221} dx \right\}^{\bar{m}} = c_m 2^{\bar{m}}.$$

We combine these two equations to see $c_m = 1$. □

5 Manifolds with boundary

We now suppose that M has a non-empty, smooth boundary ∂M. Let $D_\mathcal{B}$ be the realization of an operator D of Laplace type with respect to suitable boundary conditions defined by a local boundary operator $\mathcal{B}$ as discussed, for example, in [6]. If $f \in C^\infty(M)$, let $\nabla^k_m(f)$ denote the k^{th} normal covariant derivative of f on the boundary with respect to the inward unit normal. There is a complete asymptotic expansion

$$\mathrm{Tr}_{L^2}(f e^{-tD_\mathcal{B}}) \sim \textstyle\sum_{n \ge 0} t^{(n-m)/2} a_{n,m}(f, D, \mathcal{B})$$

where the heat trace asymptotics $a_{n,m}$ are locally computable:

$$a_{n,m}(f, D, \mathcal{B}) = \int_M f(x) a_{n,m}(D)(x) dx + \sum_k \int_{\partial M} \nabla^k_m f(y) \cdot a^{\partial M}_{n,m,k}(D, \mathcal{B})(y) dy$$

where dy is the Riemannian volume element on ∂M. The new feature here is the presence of the normal derivatives of the smearing function f; the heat kernel behaves asymptotically like a distribution near the boundary and these additional terms reflect this behaviour. Furthermore, note that $a_{n,m}(f, D, \mathcal{B})$ can be non-zero for n odd. We refer to [6] for a further discussion of these matters.

Absolute and relative boundary conditions, which are motivated by index theory, may be defined as follows; we refer to [17] for further details. Near the boundary, we normalize the choice of coordinates $x = (y, x^m)$ so that ∂_m is the inward geodesic normal and so that $y = (y^1, ..., y^{m-1})$ are coordinates on the boundary. Let $I = \{1 \le a_1 < ... < a_p \le m-1\}$ be a multi-index and let $dy^I := dy^{a_1} \wedge ... \wedge dy^{a_p}$ be the corresponding tangential differential form on ∂M. We decompose an arbitrary differential form ω into tangential and normal components:

$$\omega := \theta_I dy^I + \psi_J dy^J \wedge dx^m$$

and define the absolute boundary operator by setting:

$$\mathcal{B}_a \omega := \{(\partial_m \theta_I) dy^I|_{\partial M}\} \oplus \{\psi_J dy^J|_{\partial M}\}\,.$$

Let $\Delta_{\phi,g,\mathcal{B}_a}$ be the realization of the twisted Laplacian on the domain $\ker \mathcal{B}_a$. As before, let $\tilde{\star}_g$ be the normalized Hodge operator. Relative boundary conditions are given by setting $\mathcal{B}_r := \mathcal{B}_a\, \tilde{\star}_g$ and the associated realization of the Laplacian is denoted by $\Delta_{\phi,g,\mathcal{B}_r}$. We may decompose

$$\Delta_{\phi,g,\mathcal{B}_a} = \oplus_p \Delta^p_{\phi,g,\mathcal{B}_a} \quad \text{and} \quad \Delta_{\phi,g,\mathcal{B}_r} = \oplus_p \Delta^p_{\phi,g,\mathcal{B}_r}\,.$$

We define the super-trace for absolute ($\mathcal{B} = \mathcal{B}_a$) or relative ($\mathcal{B} = \mathcal{B}_r$) boundary conditions by setting:

$$a^{d+\delta}_{n,m}(f, \phi, g, \mathcal{B}) := \sum_p (-1)^p a_{n,m}(f, \Delta^p_{\phi,g}, \mathcal{B}).$$

This can then be expressed in terms of a local formula

$$a^{d+\delta}_{n,m}(f, \phi, g, \mathcal{B}) = \int_M a^{d+\delta}_{n,m}(\phi, g)(x) f(x) dx + \sum_k \int_{\partial M} \nabla^k_m f(y) \cdot a^{d+\delta,\mathcal{B}}_{n,m,k}(\phi, g)(y) dy$$

where the interior invariants $a^{d+\delta}_{n,m}$ are given by eqn. (13) and the boundary invariants are defined by

$$a^{d+\delta,\mathcal{B}}_{n,m,k}(\phi, g)(y) := \sum_{p=0}^{m} (-1)^p a_{n,m,k}(\Delta^p_{\phi,g}, \mathcal{B})(y).$$

If $\phi = 0$, we have:

$$\begin{aligned} \ker(\Delta^p_{g,a}) &= H^p(M; \mathbb{R}), & \operatorname{index}(d_{\mathcal{B}_a}) &= \chi(M) \\ \ker(\Delta^p_{g,r}) &= H^p(M, \partial M; \mathbb{R}), & \operatorname{index}(\delta_{\mathcal{B}_r}) &= \chi(M, \partial M). \end{aligned}$$

If we set the smearing function $f = 1$ and if ϕ satisfies Neumann boundary conditions, then we recover the index:

$$a_{n,m}^{d+\delta,\mathcal{B}_a}(1,\phi,g) = \begin{cases} \chi(M) & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$
$$a_{n,m}^{d+\delta,\mathcal{B}_r}(1,\phi,g) = \begin{cases} \chi(M,\partial M) & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

This motivates the study of absolute and relative boundary conditions.

We return to the general setting and do not assume ϕ satisfies Neumann boundary conditions. The normalized Hodge operator $\tilde{\star}_g$ intertwines relative and absolute boundary conditions and intertwines $\Delta_{\phi,g}^p$ and $\Delta_{-\phi,g}^{m-p}$ so:

$$a_{n,m}^{d+\delta,\mathcal{B}_a}(f,\phi,g) = (-1)^m a_{n,m}^{d+\delta,\mathcal{B}_r}(f,-\phi,g).$$

Consequently, we shall restrict to absolute boundary conditions henceforth.

If $\phi = 0$, then these invariants were analyzed in [16] where a proof of the Chern-Gauss-Bonnet theorem [8] for manifolds with boundary was given using heat equation methods. To describe those results, we must introduce some additional invariants. Let L be the second fundamental form of $\partial M \subset M$. Let $m = 2\bar{m}$ or $m = 2\bar{m}+1$. Let $\{e_1, ..., e_m\}$ be a local orthonormal frame near the boundary where e_m is the inward geodesic normal. Let indices a, b range from 1 thru $m-1$. For $2k \leq m-1$, define:

$$\begin{aligned}\mathcal{E}_{k,m} := (8\pi)^{-k} & \tfrac{1}{k!(m-1-2k)!} \tfrac{1}{\mathrm{vol}(S^{m-1-2k})} \varepsilon^{a_1,\ldots,a_{m-1};b_1,\ldots,b_{m-1}} \\ & \cdot R_{a_1a_2b_2b_1} \ldots R_{a_{2k-1}a_{2k}b_{2k}b_{2k-1}} L_{a_{2k+1}b_{2k+1}} \ldots L_{a_{m-1}b_{m-1}}.\end{aligned}$$

In this definition, there are no L terms if $2k = m-1$; there are no R terms if $k = 0$.

We refer to [16] for a proof of the following result:

Theorem 5.1 *Let $\phi = 0$. If $n < m-1$, then $a_{n,m,0}^{d+\delta,\mathcal{B}_a} = 0$. Furthermore, $a_{m,m,0}^{d+\delta,\mathcal{B}_a} = \sum_k \mathcal{E}_{k,m}$.*

Combining this result with the observations made above then gives a heat equation proof of the Chern-Gauss-Bonnet theorem for manifolds with boundary.

We can generalize Theorem 5.1 to this setting by removing the hypothesis that $\phi = 0$ and by showing the normal jets f do not into the crucial term when $n = m$:

Theorem 5.2 *If $n+k \le m-1$, then $a_{n,m,k}^{d+\delta,\mathcal{B}_a} = 0$. Furthermore, $a_{m,m,0}^{d+\delta,\mathcal{B}_a} = \sum_k \mathcal{E}_{k,m}$.*

We illustrate Theorem 5.2 in dimensions $m = 2, 3, 4$:

$$\begin{aligned}
a_{2,2}^{d+\delta,\mathcal{B}_a}(f,\phi,g) &= \tfrac{1}{4\pi}\textstyle\int_{M^2} fR_{ijji}dx + \tfrac{1}{2\pi}\int_{\partial M^2} fL_{aa}dy,\\
a_{3,3}^{d+\delta,\mathcal{B}_a}(f,\phi,g) &= \tfrac{1}{8\pi}\textstyle\int_{\partial M^3} f\{R_{a_1a_2a_2a_1} + L_{a_1a_1}L_{a_2a_2} - L_{a_1a_2}L_{a_1a_2}\}dy\\
a_{4,4}^{d+\delta,\mathcal{B}_a}(f,\phi,g) &= \tfrac{1}{32\pi^2}\textstyle\int_{M^4} f((R_{ijji})^2 - 4|R_{ijjk}|^2 + |R_{ijkl}|^2)dx\\
&\quad +\tfrac{1}{24\pi^2}\textstyle\int_{\partial M^4} f\{3R_{abba}L_{cc} + 6R_{acbc}L_{ab} + 2L_{aa}L_{bb}L_{cc}\\
&\quad -6L_{ab}L_{ab}L_{cc} + 4L_{ab}L_{bc}L_{ac}\}dy.
\end{aligned}$$

Let $\mathcal{Q}_{n,m}^{\partial M}$ be the set of all boundary invariants which are homogeneous of weight n in the derivatives of the metric and the derivatives of an auxiliary function ϕ with coefficients which are smooth functions of the metric tensor and which are defined on underlying manifolds of dimension m and let $\mathcal{P}_{n,m}^{\partial M} \subset \mathcal{Q}_{n,m}^{\partial M}$ be the subspace of boundary invariants not involving the auxiliary function ϕ. These spaces are non trivial for all (n,m).

As before, taking a product with the circle where the structures are trivial on the circle defines a dual restriction $r : \mathcal{Q}_{n,m}^{\partial M} \to \mathcal{Q}_{n,m-1}^{\partial M}$. The same argument as that given for manifolds without boundary then shows $a_{n,m,k}^{d+\delta,\mathcal{B}_a} \in \mathcal{Q}_{n-k-1,m}^{\partial M}$. Thus Theorem 5.2 will follow from the following result in invariance theory and from Theorem 5.1.

Lemma 5.3 *Let $Q \in \mathcal{Q}_{n,m}^{\partial M} \cap \ker(r)$. If $n < m-1$, then $Q = 0$. If $n = m-1$, then $Q \in \mathcal{P}_{m-1,m}^{\partial M}$.*

Proof. We normalize the coordinates on the boundary so that

$$g_{mm} = 1, \quad g_{am} = 0, \quad \text{and} \quad g_{ab/c}(x_0) = 0\,.$$

Thus Q is a polynomial in the variables

$$\{\phi_{/\alpha}, g_{ij/\beta}, g_{ab/m}\} \quad \text{for} \quad |\alpha| \ge 1 \quad \text{and} \quad |\beta| \ge 2\,.$$

We consider a monomial $A = \phi_{/\alpha_1}\cdots\phi_{/\alpha_u}g_{i_1j_1/\beta_1}\cdots g_{i_vj_v/\beta_v}g_{a_1b_1/m}\cdots g_{a_wb_w/m}$. We argue as before to see that $\deg_a A$ is even and non-zero for $1 \le a \le m-1$. Consequently, we may estimate:

$$\begin{aligned}
2(m-1) &\le \sum_a \deg_a(A) \le \sum_\mu |\alpha_\mu| + \sum_\nu (2 + |\beta_\mu|) + 2w = 2v + w + n\\
&\le \sum_\nu |\beta_\mu| + w + n = 2n - \sum_\mu |\alpha_\mu| \le 2n - u \le 2n.
\end{aligned}$$

This implies that $Q = 0$ if $n < m - 1$, while if $n = m - 1$, all the inequalities must have been equalities. Thus $u = 0$ so ϕ does not enter. $\square$

6 Conclusions

In this paper we have analysed the variation with respect to the dilaton of a difference of two effective actions in the models related by a duality transformation. This variation is reduced to a contribution from twisted harmonic forms and to a combination of the heat trace coefficients (supertrace of the twisted de Rham complex). Direct evaluation of these heat trace coefficients is possible in low dimensions only. By using functorial properties of the supertrace we are able nevertheless to obtain an explicit formula in any dimension, also on manifolds with boundaries.

Theorem 3.1 defines the variation of the effective actions with respect to the dilaton. It can be integrated to give a relation between the dual actions. Since we do not have a closed expression for the twisted harmonic forms we suppose $M = \mathbb{R}^m$ and impose a fall-off condition on the dilaton field. We also assume that the metric is asymptotically flat so that the Laplace operators $\Delta^p_{\phi,g}$ have no normalizable zero modes (non-normalizable zero modes never appear in the path integral and, therefore, must not be subtracted). Also in this case $W_p(0) = W_{m-p-2}(0)$ [4] [2]. Let $\mathcal{E}_m$ be given by (15). We have:

$$W_p(\phi) - W_{m-p-2}(-\phi) = (-1)^p \int_M \phi \mathcal{E}_m \, dx \,.$$

One can extend the results presented in this paper also to domain wall and brane-world geometries. The heat trace asymptotics for these cases can be found in [18].

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[2] In $m = 4$ this has been demonstrated by Fradkin and Tseytlin [14].

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