Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Local stress regularity in scalar non-convex variational problems

by

Carsten Carstensen and Stefan Müller

Preprint no.: 9 2002

LOCAL STRESS REGULARITY IN SCALAR NON-CONVEX VARIATIONAL PROBLEMS

CARSTEN CARSTENSEN AND STEFAN MÜLLER

Abstract. Motivated by relaxation in the calculus of variations, this paper addresses convex but not necessarily strictly convex minimization problems. A class of energy functionals is described for which any stress field σ in $L^q(\Omega)$ with $\operatorname{div}\sigma$ in $W^{1,p'}(\Omega)$ (from Euler Lagrange equations and smooth lower order terms) belongs to $W^{1,q}_{loc}(\Omega)$. Applications include the scalar double-well potential, an optimal design problem, a vectorial double-well problem in a compatible case, and Hencky elastoplasticity with hardening. If the energy density depends only on the modulus of the gradient we also show regularity up to the boundary.

1. Introduction

Given a volume term $f \in L^q_{loc}(\Omega)$ and Dirichlet data $u_0 \in W^{1,p}(\Omega)$ let the admissible displacements $\mathcal{A}$ be a nonvoid closed convex subset of $W^{1,p}(\Omega)$ with $u_0 + W^{1,p}_0(\Omega) \subseteq \mathcal{A} \subseteq W^{1,p}(\Omega)$. The task to

$$\text{(1.1)} \quad \text{minimize } E(u) := \int_\Omega W(Du)\,dx - \int_\Omega f\,u\,dx \text{ amongst } u \in \mathcal{A}$$

may fail to have a solution in $\mathcal{A}$. Typically, infimizing sequences exist and are bounded in the seminorm of $W^{1,p}(\Omega)$ and weakly convergent towards some u in $\mathcal{A}$; but, u may fail to minimize the energy E as the functional $E : \mathcal{A} \to \mathbb{R}$ is not (sequentially) weakly lower semicontinous owing to its non-convexity.

Nevertheless, u describes the macroscopic, space-averaged state and so is of interest. Relaxation results in the calculus of variations show that u can be computed as a solution of the relaxed problem,

$$\text{(1.2)} \quad \text{minimize } RE(u) := \int_\Omega \varphi(Du)\,dx - \int_\Omega f\,u\,dx \text{ amongst } u \in \mathcal{A}.$$

In the general case, φ is the quasiconvexification of W [Dac89, Rou97]; the arguments of this paper are essentially restricted to the situation

Date: September 11, 2001.

Key words and phrases. non-convex minimization, regularization, relaxed problem, stress regularity.

where φ is the convex envelope of W and so is easier to compute or to approximate.

It was observed in [Fri94, CP97a] for scalar problems and recently in [BKK00] in the general case, that the stress fields $\sigma_j := DW(Du_j)$ of an infimizing sequence u_j converge in a weak sense. The limit σ is given as the stress of a relaxed functional φ of W, i.e., $\sigma = D\varphi(Du)$. Hence, the stress field associated with (1.1) can be computed from (1.2); for the regularity of σ, is suffices to study (1.2).

This paper establishes local regularity of the stress variable σ under minimal conditions on u. We consider a class of convex (but not necessarily strictly convex) C^1 functions φ with

$$\begin{aligned} |D\varphi(A) - D\varphi(B)|^2 &\leq c(1 + |A|^s + |B|^s) \\ &\quad \times (D\varphi(A) - D\varphi(B)) : (A - B) \end{aligned} \tag{1.3}$$

for all $A, B \in \mathbb{M}^{m\times n}$ ($\mathbb{M}^{m\times n}$ denotes the real $m \times n$ matrices) and a multiplicative constant c. Our interpretation of (1.3) is as follows: As a function of the two variables (A, B), the left-hand side has zeros where the right-hand side has, but, off the diagonal in $\mathbb{M}^{m\times n} \times \mathbb{M}^{m\times n}$, they are of higher order. This local bound plus sufficient growth conditions for a proper choice of $s \geq 0$ yield (1.3). Note carefully that (1.3) implies convexity of φ but not strict convexity.

Theorem 2.1 of Section 2 asserts that the monotonicity condition (1.3) and $\operatorname{div} D\varphi(Du) = f$ in $W^{1,q}(\Omega)$ for some solution u of (1.2) yield $\sigma = D\varphi(Du)$ in $W^{1,q}_{loc}(\Omega)$. Examples follow for the scalar two-well potential in Section 3 and for a relaxed energy density of an optimal design problem in Section 4.

A symetric variant of (1.1)-(1.2) where Du (for $n = m$) is replaced by the symmetric part $\varepsilon(u) := \operatorname{sym} Du$,

$$\text{minimize } RE(u) := \int_\Omega \varphi(\varepsilon(u))\, dx - \int_\Omega f\, u\, dx \text{ amongst } u \in \mathcal{A}, \tag{1.4}$$

is discussed in Section 5. Emphasis is put on robustness of the stress in the Lamé constant $\lambda \to \infty$ involved in the elastic contribution of the models. Applications to Hencky elastoplasticity and a vector two-well example in Section 6 and 7 conclude this paper.

Throughout this paper, $\mathbb{M}^{m\times n}$ denotes the real $m \times n$ matrices endowed with the Euclidean scalar product :, $A : B := \sum_{j=1}^m \sum_{k=1}^n A_{jk}B_{jk}$ and induced (Frobenius matrix) norm $|\cdot|$, $|A| := (A : A)^{1/2}$. We use standard notation for Sobolev and Lebesgue spaces and norms resp. seminorms.

2. Abstract Stress Regularity Result

Let Ω be an open set in $\mathbb{R}^n$ and let $\varphi : \mathbb{M}^{m\times n} \to \mathbb{R}$ be C^1 and let $D\varphi$ be its derivative. Suppose that there exist constants $1 < p < \infty$, $1 < r < \infty$, $0 \le s < \infty$ and $0 < c_1$, such that, for all $A, B \in \mathbb{M}^{m\times n}$,

$$|D\varphi(A) - D\varphi(B)|^r \le c_1\,(1 + |A|^s + |B|^s) \times (D\varphi(A) - D\varphi(B)) : (A - B). \tag{2.1}$$

Theorem 2.1. *Assume furthermore that*

$$u \in W^{1,p}(\Omega;\mathbb{R}^m) \quad \textit{and} \quad \sigma := D\varphi(Du))$$

satisfy, for $p' := p/(p-1)$ *and* $q := r/(1+s/p)$,

$$\sigma \in L^q_{loc}(\Omega;\mathbb{M}^{m\times n}) \quad \textit{and} \quad \operatorname{div}\sigma \in W^{1,p'}_{loc}(\Omega;\mathbb{R}^m).$$

Suppose $p' \le q$ *and* $r \le 2$. *Then*

$$\sigma \in W^{1,q}_{loc}(\Omega;\mathbb{M}^{m\times n}).$$

Remarks 2.1. (a) The point is that (2.1) implies that φ is convex; but φ need not to be strictly convex since the lower bound is in terms of stress differences but not in terms of $|A - B|$.
(b) The assumptions on u can be localized to $u \in W^{1,p}_{loc}(\Omega;\ \mathbb{R}^n)$ by choosing another Ω.

Proof. Given any direction $M \in \mathbb{M}^{m\times n}$, $|M| = 1$, and some $\eta \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\eta \subset \omega \subset \overline{\omega} \subset \Omega$ for some bounded open set ω which lies compactly in Ω. Set $\alpha := 1/(r-1)$, and $\beta := r/(r-1)$. Let $0 < h < h_0 := \operatorname{dist}(\operatorname{supp}\eta;\partial\omega)$ and define, for almost all $x \in \omega$,

$$\begin{aligned} \tau(x) &:= (\sigma(x + hM) - \sigma(x))/h, \\ e(x) &:= (u(x + hM) - u(x))/h, \\ \delta(x) &:= De(x). \end{aligned}$$

A standard argument in the approximation of weak derivatives by difference quotients shows

$$\|e\|_p := \|e\|_{L^p(\omega)} \le c_2\,\|u\|_{W^{1,p}(\Omega)} \tag{2.2}$$

with an h-independent constant c_2 (that depends on ω and Ω). Here and throughout the proof, $\|\cdot\|_t := \|\cdot\|_{L^t(\omega)}$ denotes the $L^t(\omega)$-norm with respect to the subdomain ω of Ω.

Owing to $u \in W^{1,p}(\Omega)$, $\|e\|_{L^p(\omega)}$ is bounded h-independently. A careful estimation using Hölder's inequality and

$$\operatorname{div}\sigma \in W^{1,p'}(\omega;\mathbb{M}^{m\times n}) \text{ resp. } \sigma \in L^{p'}(\omega;\mathbb{M}^{m\times n})$$

as well as $q' \le p$ yields, in analogy with (2.2), the h-independent bound

$$\| \varrho^{q/r} \|_{1+p/s}^{r/q} + \|e\|_p + \|e\|_{q'} + \|\eta^\beta \operatorname{div} \tau\|_{p'} + \| \eta \|_{W^{1,\infty}(\Omega)} \le c_3, \tag{2.3}$$

where $\varrho(x) := 1 + |Du(x)|^s + |Du(x+hM)|^s$ and η is fixed.

To verify the assertion, we have to bound $|\tau|_{L^q(K)}$ by h-independent quantities for each compact $K \subset \Omega$ (below K is a compact subset of the interior of $\operatorname{supp} \eta$).

Owing to the estimate (2.1) (with $A := Du(x+hM)$ and $B := Du(x)$) we have,

$$|\tau|^r \le c_1 \, h^{2-r} \, \varrho \, \tau : \delta \quad \text{a.e. in } \omega. \tag{2.4}$$

Raising (2.4) to the power q/r, then multiplying with $\eta^{\alpha q}$, and finally integrating the result over Ω, we infer

$$\|\eta^\alpha \tau\|_q^q \le c_1^{q/r} \, h^{q(2-r)/r} \int_\Omega \eta^{\alpha q} \, \varrho^{q/r} \, (\tau : \delta)^{q/r} dx. \tag{2.5}$$

Applying Hölder's inequality (for r/q resp. $(r/q)' = 1+p/s$) and raising the result to the power r/q we obtain (since $0 \le \tau : \delta$ and $\alpha r = \beta$)

$$\|\eta^\alpha \tau\|_q^r \le c_1 \, h^{2-r} \, \| \varrho^{q/r} \|_{1+p/s}^{r/q} \int_\Omega \eta^\beta \, \tau : \delta \, dx. \tag{2.6}$$

Since $\delta = De$ on ω, an integration by parts proves (recall $h \le h_0$)

$$\begin{aligned} \int_\Omega \eta^\beta \, \tau : \delta \, dx &= - \int_\Omega e \cdot \operatorname{div}(\eta^\beta \tau) dx \\ &\le \beta \| \eta \|_{1,\infty} \| \eta^{\beta-1} \, \tau \, e \|_1 + \| e \|_p \| \eta^\beta \operatorname{div} \tau \|_{p'}. \end{aligned} \tag{2.7}$$

Since $\beta - 1 = \alpha$, Hölder's inequality leads to

$$\| \eta^{\beta-1} \, \tau \, e \|_1 \le \| e \|_{q'} \| \eta^\alpha \, \tau \|_q. \tag{2.8}$$

The combination of (2.6)-(2.8) with (2.3) and $r \le 2$ proves

$$\|\eta^\alpha \tau\|_q^r \le c_1 c_3^3 h_0^{2-r} (1 + \| \eta^\alpha \tau \|_q). \tag{2.9}$$

With Young's inequality ($ab \le (ac)^r/r + (b/c)^{r'}/r'$ for positive a, b, c) we observe from (2.9), $r \le 2$, and $q > 1$ that $\|\eta^\alpha \tau\|_q$ is bounded h-independently; hence,

$$\limsup_{h \to 0} \|\eta^\alpha \tau\|_{L^q(\Omega)} < \infty \quad \text{for all } \eta \in \mathcal{D}(\Omega).$$

The proof is finished. □

This section is concluded with the simple example of a p-Laplace equation to illustrate the growth condition in (2.1).

Example 2.1. Given $2 \le p < \infty$, let $\varphi(F) := |F|^p/p$ for $F \in \mathbb{R}^n$, $m = 1$. Then, $D\varphi(F) = |F|^{p-2}\,F$ and so, for fixed $B \in \mathbb{R}^n$ and $A \in \mathbb{R}^n$ with $|A| \to \infty$,

$$\frac{|D\varphi(A) - D\varphi(B)|^2}{(D\varphi(A) - D\varphi(B) \cdot (A - B)} \approx |D\varphi(A)|/|A| = |A|^{p-2}.$$

Indeed, it is known (e.g., by a combination of Lemmas 2.1 to 2.3 in [CK01]) that, for any $A, B \in \mathbb{R}^n$,

$$\frac{|D\varphi(A) - D\varphi(B)|^2}{(D\varphi(A) - D\varphi(B) \cdot (A - B)} \le (1 + \max\{1, p-2\}^2)(|A|^{p-2} + |B|^{p-2}).$$

As a corollary of Theorem 2.1, we therefore obtain local regularity of the stress field, i.e., $\sigma := D\varphi(Du) \in W^{1,p'}_{loc}(\Omega; \mathbb{R}^n)$, for a minimizer $u \in W^{1,p}(\Omega)$ of (1.2) with $f \in W^{1,p'}_{loc}(\Omega)$.

3. An application to the scalar 2-well problem

This section concerns the scalar double-well problem where

$$W : \mathbb{R}^n \to \mathbb{R}, \quad F \mapsto |F - F_1|^2\,|F - F_2|^2 \tag{3.1}$$

for the given two distinct wells $F_1, F_2 \in \mathbb{R}^n$, $F_1 \ne F_2$. The scalar problem (1.1) with (3.1) (for $m = 1$) can be deduced from the Ericksen-James energy density in an anti-plane shear model; the version for $n = 1$, due to O. Bolza [Bol06]) (cf. also [You69]), is the model example in non-convex minimization.

Proposition 3.1 ([CP97a])**.** *Let $a := (F_2 - F_1)/2$ and $b := (F_1 + F_2)/2$. The convex envelope φ of* (3.1) *is*

$$\varphi(F) := \max\{|F - b|^2 - |a|^2, 0\}^2 + 4(|a|^2\,|F - b|^2 - [a \cdot (F - b)]^2)$$

and satisfies (2.1) *with $r = 2$, $s = 2$, and $c_1 = 4\max\{2, |F_1 - F_2|^2\}$.*

Corollary 3.2. *Adopt notation of Proposition 3.1 and let u be a minimizer of* (1.2)*. Then $\sigma := D\varphi(Du)$ belongs to $W^{1,4/3}_{loc}(\Omega; \mathbb{R}^n)$.*

Proof. The assertion follows from Theorem 2.1 and Proposition 3.1 since the Euler Lagrange equations of the minimization problem (1.2) provide $-\operatorname{div}\sigma = f \in W^{1,4/3}(\Omega)$. □

Remark 3.1. Further estimates in [CP97a] allow one to control other quantities. In particular,

$$\max\{0, |B - Du|^2 - |A|^2\} \in H^1_{loc}(\Omega) \text{ and } M \cdot Du \in H^1_{loc}(\Omega)$$

for all directions M perpendicular to A.

4. An application to an optimal design problem

The relaxed model for an optimal design problem derived in [GKR86] has the form (1.2) where $\varphi(F) = \psi(|F|)$. Given positive parameters $0 < t_1 < t_2$ and $0 < \mu_2 < \mu_1$ with $t_1\mu_1 = t_2\mu_2$, the C^1 function $\psi : [0,\infty) \to [0,\infty)$ is defined by $\psi(0) = 0$ and

$$\psi'(t) := \begin{cases} \mu_1\, t & \text{if } 0 \le t \le t_1, \\ t_1\mu_1 = t_2\mu_2 & \text{if } t_1 \le t \le t_2, \\ \mu_2\, t & \text{if } t_2 \le t. \end{cases}$$

Proposition 4.1 ([CP97a]). *The function $\varphi(F) = \psi(|F|)$ satisfies* (2.1) *with $r = 2$, $s = 0$, and $c_1 = 1/\mu_1$.*

Therefore, Theorem 2.1 yields local stress regularity for minimizers of (1.2) when $f \in H^1_{loc}(\Omega)$.

Corollary 4.2. *Adopt notation of Proposition 4.1 and let u be a minimizer of* (1.2) *in $\mathcal{A} := H^1_0(\Omega)$. Then $\sigma := D\varphi(Du)$ belongs to $W^{1,2}_{loc}(\Omega;\mathbb{R}^n)$.*

The rest of this section is devoted to establish regularity up to the boundary.

Theorem 4.3. *Suppose that $f \in W^{1,2}_0(\Omega)$ and that Ω is a $C^{2,1}$ domain. If u is a minimizer of* (1.2)*, then $\sigma := D\varphi(Du)$ belongs to $W^{1,2}(\Omega;\mathbb{R}^n)$.*

The remaining part of this section is devoted to a proof of Theorem 4.3 via a local reflection argument. Owing to the local regularity of Corollary 4.2, it remains to prove $\sigma \in W^{1,2}(\Omega \cap B(x_0,\delta);\mathbb{R}^n)$ for each point x_0 on the boundary $\partial\Omega$ and some small $\delta > 0$. Without loss of generality, we suppose $x_0 = 0$ and that the Cartesian coordinate system at hand directly allows a $C^{2,1}$ parameterization.

Definition 4.1. *Let $\chi : B'_0 \to \mathbb{R}$ be a (scalar) $C^{2+\alpha}$ function where $B_0 := B(0,\delta_0) \subset \mathbb{R}^n$ and $B'_0 := \{x' \in \mathbb{R}^{n-1} : |x'| < \delta_0\} \subset \mathbb{R}^{n-1}$ denotes the δ_0-ball around $x_0 = 0$ in n and $(n-1)$ dimensions, respectively. Suppose that χ parameterizes the boundary $\Gamma := \partial\Omega$ near $x_0 = 0$, i.e.,*

$$\begin{aligned} \Gamma \cap B_0 &= \{(x',\chi(x')) \in B_0 : x' \in B'_0\}, \\ \Omega \cap B_0 &= \{(x',x_n) \in B_0 : x' \in B', x_n > \chi(x')\}, \\ B_0 \backslash \overline{\Omega} &= \{(x',x_n) \in B_0 : x' \in B', x_n < \chi(x')\}. \end{aligned}$$

Let ν be the unit normal vector on Γ and set

$$\Psi(x) := (x',\chi(x')) - x_n\, \nu(x',\chi(x')) \qquad \text{for all } x =: (x',x_n) \in B_0 \subset B'_0 \times \mathbb{R}.$$

Lemma 4.4. *The pull-back metric* $g := D\Psi^T D\Psi : B_0 \to \mathbb{M}^{n\times n}_{sym}$ *is* $C^{1+\alpha}$ *and, with* $E := I_{n-1} + e_n \otimes D\chi(c') - x_n D_{x'}\nu(x', \chi(x')) \in \mathbb{M}^{n\times(n-1)}$ *for the* $n \times (n-1)$ *unit matrix* I_{n-1}*, given by*

$$g(x) = \begin{pmatrix} E^T E & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{M}^{n\times n}_{sym} \text{ for } (x', x_n) \in B.$$

Proof. The C^1 property follows from direct calculations; the derivative $D_{x'}\nu(x', \chi(x'))$ replaces the $n \times (n-1)$ matrix of $\nu_j(x', \chi(x'))$ differentiated by $x_1, \dots, x_{n-1}$. Since $\partial\chi/\partial x_k$ is tangential on Γ for $k = 1, 2, \dots, n-1$ there holds $\nu \cdot \partial\chi/\partial x_k = 0$; whence $\nu^T D\chi = 0$. From $|\nu|^2 = 1$, we deduce $\nu^T D\nu = 0$. Those orthogonalities yield the block structure asserted for $g(x)$. □

Definition 4.2. *Suppose that* δ *is small enough,* $0 < \delta < \delta_0$*, such that* $\Psi(B_+) =: \omega \subset \Omega$*,* $B_\pm := \{(x', x_n) \in B : \pm x_n > 0\}$ *where* $B := B(0, \delta)$ *and* $B' := \{x' \in \mathbb{R}^{n-1} : |x'| < \delta\}$ *denote the* δ*-ball around* $x_0 = 0$ *in* n *and* $(n-1)$ *dimensions, respectively. For any* $x = (x', x_n) \in B_+$ *set* $Sx := (x', -x_n) \in B_-$ *and*

$$\begin{array}{rclcl} \tilde{u}(x) & = & -\tilde{u}(Sx) & := & u(\Psi(x)), \\ \tilde{\sigma}(x) & = & -\tilde{\sigma}(Sx)S & := & \sigma(\Psi(x)) \operatorname{cof} D\Psi, \\ \tilde{f}(x) & = & -\tilde{f}(Sx) & := & (\det g(x))^{1/2} f(\Psi(x)), \\ \tilde{g}(x) & = & \tilde{g}(Sx) & := & g(x). \end{array}$$

Lemma 4.5. *There holds* $\tilde{u} \in W^{1,2}(B)$*,* $\tilde{f} \in W^{1,2}(B)$*,* $\tilde{\sigma} \in H(\operatorname{div}, B)$*,*

$$\tilde{\sigma} = D\varphi(\nabla\tilde{u}\, \tilde{g}^{-1/2}) \operatorname{cof} \tilde{g}^{1/2} \text{ in } B,$$

and

$$\operatorname{div} \tilde{\sigma} = \tilde{f} \text{ in } \mathcal{D}'(B).$$

Proof. A polar decomposition $QU = D\Psi(x)$ shows $g(x) = U^2$, $g(x)^{1/2} = U$. Since $Q = D\Psi(x) g(x)^{-1/2}$ is orthonormal,

$$|\nabla u(\Psi(x))| = |\nabla u(\Psi(x))\, D\Psi(x)\, g(x)^{-1/2}| = |\nabla\tilde{u}(x)\, g(x)^{-1/2}|.$$

Since $\varphi(\cdot) = \psi(|\cdot|)$ solely depends on the modulus, this shows, at $\xi := \Psi(x)$, $x \in B_+$,

$$\begin{aligned} \sigma(\xi) &= D\varphi(\nabla u(\xi)) \\ &= \psi'(|\nabla u(\xi)|) \operatorname{sign} \nabla u(\xi) \\ &= \psi'(|\nabla\tilde{u}(x)\, g(x)^{-1/2}|) \operatorname{sign}(\nabla\tilde{u}(x)\, D\Psi^{-1}(x)) \\ &= D\varphi(\nabla\tilde{u}(x)\, g^{-1/2}(x))\, Q^T. \end{aligned}$$

Since $\operatorname{adj} D\psi(x) = \operatorname{cof} g^{1/2}(x)\, Q^T$, this proves the asserted identity for $\tilde{\sigma}$ in B_+.

By assumption, $\operatorname{div}\sigma = f$ in $\mathcal{D}'(\Omega)$, and elementary transformations show, for all $\eta \in \mathcal{D}(B_+)$ with the test function $\eta \circ \Psi^{-1}$,

$$\begin{aligned}\int_{B_+} \tilde{f}(x)\,\eta(x)\,dx &= \int_\omega f(\xi)\eta(\Psi^{-1}(\xi))\,d\xi \\ &= -\int_\omega \nabla\eta(\Psi^{-1}(\xi))\cdot D\Psi^{-1}(\xi)\sigma(\xi)\,d\xi.\end{aligned}$$

The substitution of $\tilde{\sigma}$ and a re-transformation give

$$\int_{B_+} \tilde{f}(x)\,\eta(x)\,dx = \int_{B_+} \tilde{\sigma}(x)\cdot\nabla\eta(x)\,dx.$$

This proves $\operatorname{div}\tilde{\sigma} = \tilde{f}$ in $\mathcal{D}'(B_+)$.

The block structure of g shows that g^α commutes with $S = \operatorname{diag}(1, \ldots, 1, -1)$, i.e., $Sg^\alpha = g^\alpha S$ for $\alpha \in \mathbb{R}$. Since $\varphi(\cdot)$ depends solely on the modulus, $D\varphi$ commutes with S as well, i.e., $D\varphi(-S\cdot) = -SD\varphi(\cdot)$. Then, for $x \in B_-$, $\xi \in B_+$, $x = S\xi$,

$$\begin{aligned}&\operatorname{cof}\tilde{g}^{1/2}(x)\,D\varphi(\tilde{g}^{-1/2}(x)\,\nabla\tilde{u}(x)) \\ &= \operatorname{cof}\tilde{g}^{1/2}(\xi)\,D\varphi(-\tilde{g}^{-1/2}(\xi)\,S\nabla\tilde{u}(\xi)) \\ &= -S\operatorname{cof}\tilde{g}^{1/2}(\xi)\,D\varphi(\tilde{g}^{-1/2}(\xi)\,\nabla\tilde{u}(\xi)) \\ &= -\tilde{\sigma}(\xi)S = \tilde{\sigma}(x).\end{aligned}$$

Thus, $\tilde{\sigma} = \operatorname{cof}(\tilde{g}^{1/2})\,D\varphi(\tilde{g}^{-1/2}\,\nabla\tilde{u})$ holds almost everywhere in B.

Owing to $\tilde{f} = 0 = \tilde{u}$ on $\overline{B}_+ \cap \overline{B}_- = B \cap (B' \times \{0\})$, $\tilde{u}$ and $\tilde{f}$ belong to $W^{1,2}(B)$. Notice that $g \in C(B)$. Clearly, $\tilde{\sigma} \in L^2(B)$ and $\tilde{\sigma}|_{B_\pm} \in H(\operatorname{div}, B_\pm)$. Hence it remains to prove $\operatorname{div}\tilde{\sigma} = \tilde{f}$ in $\mathcal{D}'(B)$. Given $\eta \in \mathcal{D}(B)$ set $\alpha := (\eta + \eta\circ S)/2$ and $\beta := (\eta - \eta\circ S)/2$. Since $\nabla\alpha(x) = (\nabla\eta(x) + \nabla\eta(Sx)S)/2 = \nabla\alpha(Sx)S$,

$$\int_B \tilde{\sigma}\cdot\nabla\alpha\,dx = \int_{B_+} (\tilde{\sigma}(x) + \tilde{\sigma}(Sx))\cdot\nabla\alpha(x)\,dx = 0.$$

Since $\beta = 0$ on $B' \times \{0\}$ and $\nabla\beta(Sx) = -\nabla\beta(x)S$, a transformation to B_+ and an integration by parts in B_+ lead to

$$\begin{aligned}&\textstyle\int_B \tilde{\sigma}\cdot\nabla\eta\,dx = \int_{B_+} \tilde{\sigma}(x)\cdot\nabla\beta(x)\,dx + \int_{B_+} \tilde{\sigma}(Sx)\cdot\nabla\beta(Sx)\,dx \\ &= 2\textstyle\int_{B_+} \tilde{\sigma}(x)\cdot\nabla\beta(x)\,dx = 2\int_{B_+} \tilde{f}(x)\beta(x)\,dx = \int_B \tilde{f}(x)\beta(x)\,dx.\end{aligned}$$

Hence $-\operatorname{div}\tilde{\sigma} = \tilde{f}$ in $\mathcal{D}'(B)$ and the proof of Lemma 4.5 is finished. □

The preceeding results on the reflection near x_0 at the boundary provides $\tilde{\sigma}$ perturbed by the metric $\tilde{g}^{-1/2}$. Hence, Theorem 2.1 does not directly lead to $\tilde{\sigma} \in W^{1,2}_{loc}(B;\mathbb{M}^{n\times n})$ which then shows $\sigma \in W^{1,2}(\Omega\cap$

$B(x_0, \delta/2); \mathbb{R}^n)$ and so concludes the proof of Theorem 4.3. Instead, we need to follow the proof of Theorem 2.1 for the perturbed situation.

Proof of Theorem 4.3. Given $x \in B$, $h > 0$, $M \in \mathbb{R}^n$, $|M| = 1$ set for brevity, $x_2 := x + hM$, $x_0 := x - h\,M$, and $x_1 := x$ and, for $j = 1, 2$,

$$F_j := \nabla \tilde{u}(x_j),\ \sigma_j := \tilde{\sigma}(x_j),\ U_j := \tilde{g}^{-1/2}(x_j),$$
$$V_j := \operatorname{cof} U_j^{-1},\ \Sigma_j := \sigma_j \det U_j,\ T_j := \Sigma_j U_j^{-1}.$$

Moreover, let $a \le C\,b$ be abbreviated as $a \lesssim b$ if C is a generic constant that is independent of (sufficient small) $\delta > 0$, $h > 0$. The constant $C > 0$, however, may depend on g, U_j, V_j, e.g., through $\|\tilde{g}\|_{W^{1,\infty}(B)}$, $\|\operatorname{cof} \tilde{g}\|\|_{W^{1,\infty}(B)}$, $\|\tilde{g}^{-1}\|_{W^{1,\infty}(B)}$, $\|\operatorname{cof} \tilde{g}^{-1}\|_{W^{1,\infty}(B)}$, or $\|\eta\|_{W^{1,\infty}(B)}$. Then,

$$\begin{aligned} |\sigma_2 - \sigma_1|^2 &= |D\varphi(F_2U_2)V_2 - D\varphi(F_1U_1)V_1)|^2 \\ &\lesssim |V_2 - V_1|^2 |D\varphi(F_1U_1)|^2 + |D\varphi(F_2U_2) - D\varphi(F_1U_1)|^2. \end{aligned}$$

With Proposition 4.1, the above notation, and the identity $T_jU_j = \Sigma_j$ we infer

$$\begin{aligned} & |D\varphi(F_2U_2) - D\varphi(F_1U_1)|^2 \\ \lesssim\ & (D\varphi(F_2U_2) - D\varphi(F_1U_1)) \cdot (F_2U_2 - F_1U_1) \\ =\ & (T_2 - T_1) \cdot (F_2U_2 - F_1U_1) \\ =\ & (\Sigma_2 - \Sigma_1) \cdot (F_2 - F_1) + (T_2 - T_1) \cdot (F_1 + F_2)(U_2 - U_1) \\ & + T_1 \cdot F_1(U_2 - U_1) - T_2 \cdot F_2(U_2 - U_1). \end{aligned}$$

Given $\eta \in \mathcal{D}(B)$, $0 \le \eta \le 1$, which equals one in a neighborhood of $x_0 = 0$ and provided $|h|$ sufficiently small, the combination of the last two estimates is multiplied by η^2/h^2 and integrated over $\operatorname{supp} \eta$. With the notation $\tilde{\tau}(x) := (\tilde{\sigma}(x_2) - \tilde{\sigma}(x_1))/h$ and $\tilde{e}(x) := (\tilde{u}(x_2) - \tilde{u}(x_1))/h$, we deduce,

$$\begin{aligned} \int \eta^2(x)|\tilde{\tau}(x)|^2\,dx \lesssim & \int \eta^2|\tilde{\sigma}(x)|^2\,dx \\ & + 1/h^2 \int \eta^2(\Sigma_2 - \Sigma_1) \cdot (F_2 - F_1)\,dx \\ & + 1/h \int \eta^2|T_2 - T_1|\,(|\nabla \tilde{u}(x)| + |\nabla \tilde{u}(x + h\,M)|)\,dx \\ & + 1/h^2 \int \eta^2 T_1(U_2 - U_1) \cdot F_1\,dx \\ & - 1/h^2 \int \eta^2 T_2(U_2 - U_1) \cdot F_2\,dx \\ & =: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} - \mathrm{V}\,. \end{aligned}$$

Term I is bounded since $\tilde\sigma \in L^2(B)$. Term II is recast into

$$\begin{aligned} II &= 1/h^2 \int \eta^2 (\Sigma_2 - \Sigma_1)\cdot(F_2 - F_1)\,dx \\ &= \int \eta^2(x)\,\det \tilde g^{-1/2}(x)\,\tilde\tau(x)\cdot\nabla\tilde e(x)\,dx \\ &\quad + \int \eta^2(x)\left(\det \tilde g^{-1/2}(x_2) - \det \tilde g^{-1/2}(x_1)\right)/h \\ &\qquad \times \tilde\sigma(x_2)\cdot\nabla\tilde e(x)\,dx. \end{aligned}$$

Since $\tilde e\eta^2 \det \tilde g^{-1/2} \in H^1(B)$ is a feasible test function we have

$$\begin{aligned} \int \eta^2 \det \tilde g^{-1/2}\tilde\tau\cdot\nabla\tilde e\,dx &= -\int \tilde e\tilde\tau\cdot\nabla(\eta^2\det\tilde g^{-1/2})\,dx \\ &+ \int \tilde e(x)\eta^2(x)\det g^{-1/2}(x)\,(f(x_2)-f(x_1))/h\,dx \\ &\lesssim \|\tilde u\|_{1,2}(\|f\|_{1,2} + \|\eta\tilde\tau\|_2) \end{aligned}$$

with the abbreviations $\|\cdot\|_p := \|\cdot\|_{L^p(B)}$ and $\|\cdot\|_{1,p} := \|\cdot\|_{W^{1,p}(B)}$. A shift in the variable x_2 in the term $\nabla\tilde e(x)$ yields

$$\begin{aligned} &\int \eta^2(x)\big(\det \tilde g^{-1/2}(x_2) - \det \tilde g^{-1/2}(x_1)\big)/h\,\tilde\sigma(x_2)\cdot\nabla\tilde e(x)\,dx \\ &= -\int (\eta^2(x_1) - \eta^2(x_0))/h \\ &\qquad \times (\det \tilde g^{-1/2}(x_1) - \det \tilde g^{-1/2}(x_0))/h\,\tilde\sigma(x)\cdot\nabla\tilde u(x)\,dx \\ &\quad - \int \eta^2(x)(\det \tilde g^{-1/2}(x_2) - \det \tilde g^{-1/2}(x))/h\,\tilde\tau(x)\cdot\nabla\tilde u(x)\,dx \\ &\quad - \int \eta^2(x)\Big(\det \tilde g^{-1/2}(x_2) - 2\det \tilde g^{-1/2}(x_1) \\ &\qquad + \det \tilde g^{-1/2}(x_0)\Big)/h^2\,\tilde\sigma(x)\cdot\nabla\tilde u(x)\,dx \\ &\lesssim (\|\tilde\sigma\|_2 + \|\eta\tilde\tau\|_2)\cdot\|\tilde u\|_{1,2}. \end{aligned}$$

In the last step we employed $g \in C^{1,1}$ and so required that $\partial\Omega$ is $C^{2,1}$. Altogether,

$$\mathrm{II} \lesssim \|\tilde u\|_{1,2}(\|f\|_{1,2} + \|\tilde\sigma\|_{1,2} + \|\eta\tilde\tau\|_2).$$

Since $T_j = D\varphi(\nabla\tilde u(x_j)) = \sigma_j \operatorname{cof} U_j$ similar arguments lead to

$$\begin{aligned} \mathrm{III} &= 1/h \int \eta^2 |T_2 - T_1|(|\nabla\tilde u(x_1)| + |\nabla\tilde u(x_2)|)\,dx \\ &\lesssim (\|\eta\tilde\tau\|_2 + \|\tilde\sigma\|_2)\|u\|_{1,2}. \end{aligned}$$

A shift in the variable x_2 in term V and similar arguments result in

$$\begin{aligned}\text{IV - V} &= \int \eta^2 T_1(\tilde{g}^{-1/2}(x_2) - 2\tilde{g}^{-1/2}(x_1) + \tilde{g}^{-1/2}(x_0)/h^2 \cdot \nabla\tilde{u}(x)\,dx \\ &\quad + \int (\eta^2(x_1) - \eta^2(x_2)/h\, T_1\,(\tilde{g}^{-1/2}(x) - \tilde{g}^{-1/2}(x_0)/h \cdot \nabla\tilde{u}(x)\,dx \\ &\lesssim \|\tilde{\sigma}\|_2\,\|\tilde{u}\|_{1,2}.\end{aligned}$$

Absorbing $\|\eta\tau\|_2$ in II and III one concludes the proof. □

5. A symmetric variant for geometrically linear models

This section concerns Theorem 2.1 for symmetrized gradients. Some (geometrically) linear models in elasticity involve the symmetric Green strain

$$\varepsilon(u) := \operatorname{sym} Du := ((\partial u_j/\partial x_k + \partial u_k/\partial x_j) : \, j,k = 1,\dots,n)$$

for $m = n$. For the ease of this presentation we focus on $p = r = q = 2$ and $s = 0$ as in linear elasticity but emphasize robustness with respect to the incompressible limit $\lambda \to \infty$ (see below).

Let $\mathbb{M}^{n\times n}_{sym}$ denote the symmetric real $n\times n$ matrices. The fourth-order elasticity tensor $\mathbb{C} : \mathbb{M}^{n\times n}_{sym} \to \mathbb{M}^{n\times n}_{sym}$ is defined by

$$\mathbb{C}E := \lambda\, \operatorname{tr}(E)\,\mathbb{I} + 2\mu\, E \quad \text{for } E \in \mathbb{M}^{n\times n}_{sym}$$

for the positive Lamè constants λ, μ, the trace $\operatorname{tr}(E) := \sum_{j=1}^n E_{jj}$, and the $n \times n$ unit matrix $\mathbb{I}$. Since $\mathbb{C}$ is positive definite, there exist an inverse $\mathbb{C}^{-1}$ and their square roots $\mathbb{C}^{1/2}$ and $\mathbb{C}^{-1/2}$. The norm

$$|\,E\,|_{\mathbb{C}} := (E : \mathbb{C}E)^{1/2} = |\,\mathbb{C}^{1/2}E\,| \quad \text{for } E \in \mathbb{M}^{n\times n}_{sym}$$

is induced by the energy scalar product with respect to $\mathbb{C}$ in $\mathbb{M}^{n\times n}_{sym}$.

Suppose that $\varphi : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ is C^1 and that, for its derivative $D\varphi$, there exists a constant c_4 such that, for all $A, B \in \mathbb{M}^{n\times n}_{sym}$,

$$|D\varphi(A) - D\varphi(B)|^2_{\mathbb{C}^{-1}} \le c_4\,(D\varphi(A) - D\varphi(B)) : (A - B). \tag{5.1}$$

Theorem 5.1. *Assume furthermore that*

$$u \in H^1(\Omega;\mathbb{R}^n) \quad \textit{and} \quad \sigma := D\varphi(\varepsilon(u)))$$

satisfy

$$\sigma \in L^2_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}) \quad \textit{and} \quad \operatorname{div}\sigma \in H^1_{loc}(\Omega;\mathbb{R}^n).$$

Then

$$\sigma \in H^1_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}).$$

Moreover, if $\omega_0 \subset\subset \omega_1 \subset\subset \Omega$ for nonvoid open sets ω_0 and ω_1, there exists a λ-independent constant $c_5 > 0$ such that

$$\| \sigma \|_{H^1(\omega_0)} \le c_5 \, (\| u \|_{H^1(\omega_1)} + \| \sigma \|_{L^2(\omega_1)} + \| \operatorname{div} \sigma \|_{H^1(\omega_1)}). \tag{5.2}$$

Remarks 5.1. (a) Korn's inequality does not play an explicit role in the proof. It is used, however, in applications to guarantee $u \in H^1(\Omega)$ (and so the boundedness of e in $L^2_{loc}(\Omega)$ in the proof).
(b) The fourth order elasticity tensor could be more general; for the assertion $\sigma \in H^1_{loc}(\Omega; \mathbb{M}^{n\times n}_{sym})$ it is sufficient that $\mathbb{C}$ is a linear, continuous, and positive definite operator.
(c) The constant c_5 in (5.2) depends on c_4, μ, ω_0, and ω_1; but neither on σ nor on u.
(d) The functional $\varphi : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ may depend on $\mathbb{C}$ and λ; the constant c_5 depends on φ only through c_4 and stays λ-independent as long as c_4 does.

Proof of Theorem 5.1. The proof follows the arguments of the proof of Theorem 2.1 where the differential operator D is replaced by the symmetric variant ε, e.g., $\delta := \varepsilon(e)$. This results in

$$\begin{aligned} \| \eta \, \mathbb{C}^{-1/2} \tau \|_2^2 &\le c_6 \int_\omega \eta^2 \, \varepsilon(e) : \tau \, dx = -c_6 \int_\omega e \operatorname{div}(\tau \eta^2) \, dx \\ &\le c_6 \, \| e \|_2 \, \| \eta \|_{W^{1,\infty}(\Omega)} (2 \| \eta \, \tau \|_2 + \| \eta \operatorname{div} \tau \|_2) \\ &\le c_7 (\| u \|^2_{H^1(\omega)} + \| \operatorname{div} \sigma \|^2_{H^1(\omega)} + \| \eta \, \tau \|_2)^2 \end{aligned} \tag{5.3}$$

for some (h, λ, μ)-independent constant $c_7 > 0$. The first assertion follows (with a λ-depending constant) from this and

$$\| \eta \, \tau \|_2 \le (2\mu + \lambda)^{1/2} \| \eta \, \mathbb{C}^{-1/2} \tau \|_2.$$

In order to prove (5.2) we are given $\omega_0 \subset\subset \omega_1$ and suppose that ω is a bounded Lipschitz domain between ω_0 and ω_1, $\omega_0 \subset\subset \omega \subset\subset \omega_1$. Assume that $\eta \in \mathcal{D}(\omega)$ satisfies $0 \le \eta \le 1$ and equals $\eta = 1$ on ω_0. Then, we introduce the deviator $\operatorname{dev}(\tau) := \tau - \operatorname{tr}(\tau)/n \, \mathbb{I}$ and rewrite (5.3) as

$$\begin{aligned} \| \eta \, \mathbb{C}^{-1/2} \tau \|_2^2 &= \frac{\| \operatorname{dev}(\eta\tau) \|_2^2}{2\mu} + \frac{\| \operatorname{tr}(\eta\tau) \|_2^2}{n^2(2\mu/n + \lambda)} \\ &\le c_7 \Big(\| u \|^2_{H^1(\omega)} + \| \operatorname{div} \sigma \|^2_{H^1(\omega)} + \| \operatorname{dev}(\eta\tau) \|_2^2 + \| \operatorname{tr}(\eta\tau) \|_2^2 \Big). \end{aligned} \tag{5.4}$$

The λ-independent bound requires an extra argument using the Stokes problem [BF91, GR86] where integral means must be factored out. With the center ξ of mass of ω and $e_1 = (1, 0, \dots, 0)$ we define the

constant

$$\tau_0 := \int_\omega \operatorname{tr}(\eta\,\tau)dx/|\omega| \in \mathbb{R}$$

($|\omega|$ is the measure of ω) and the function $v_1 \in H^1(\omega;\mathbb{R}^n)$ by

$$v_1(x) := \tau_0((x-\xi)\cdot e_1)\,e_1 \quad \text{for } x \in \omega.$$

Then, $\operatorname{div} v_1 = \tau_0$ and $\int_\omega(\tau_0 - \operatorname{tr}(\eta\,\tau))dx = 0$. The solvability of the Stokes equations guarantees the existence of $v_2 \in H^1_0(\omega;\mathbb{R}^n)$ with $\operatorname{div} v_2 = \tau_0 - \operatorname{tr}(\eta\tau)$ and the bound

$$\| v_2 \|_{H^1(\omega)} \le c_8 \| \tau_0 - \operatorname{tr}(\eta\,\tau) \|_2 \le c_8 \| \operatorname{tr}(\eta\,\tau) \|_2.$$

Then, with $c_9 > 0$, $v := v_1 - v_2 \in H^1(\omega;\mathbb{R}^n))$ satisfies

$$\operatorname{div} v = \operatorname{tr}(\eta\,\tau) \quad \text{and} \quad \| v \|_{H^1(\omega)} \le c_9 \| \operatorname{tr}(\eta\,\tau) \|_2. \tag{5.5}$$

Recall $\operatorname{tr}(\eta\,\tau)/n\,\mathbb{I} := \eta\,\tau - \operatorname{dev}(\eta\,\tau)$ and deduce

$$\begin{aligned} \| \operatorname{tr}(\eta\,\tau) \|_2^2 &= \int_\omega \operatorname{tr}(\eta\,\tau)\operatorname{div} v\,dx = \int_\omega \operatorname{tr}(\eta\,\tau)\mathbb{I} : Dv\,dx \\ &= n\int_\omega (\eta\,\tau - \operatorname{dev}(\eta\,\tau)) : Dv\,dx. \end{aligned}$$

Cauchy inequalities and integration by parts result in

$$\begin{aligned} \frac{1}{n}\| \operatorname{tr}(\eta\,\tau) \|_2^2 &\le \| Dv \|_2 \| \operatorname{dev}(\eta\,\tau) \|_2 - \int_\Omega v\cdot(\tau\nabla\eta + \eta\operatorname{div}\tau)dx \\ &\le \| v \|_{H^1(\omega)}\Big(\| \operatorname{dev}(\eta\,\tau) \|_2 + \| \eta\ \operatorname{div}\tau \|_2\Big) \\ &\quad - \int_\Omega v\cdot\tau\nabla\eta\,dx. \end{aligned} \tag{5.6}$$

To recast the last term with a summation by parts, let $\otimes$ denote the dyadic product and set

$$V_h(x) := \frac{1}{h}\Big((v\otimes\nabla\eta)(x) - (v\otimes\nabla\eta)(x-hM)\Big) \in \mathbb{M}^{n\times n} \quad \text{for a.e } x \in \omega.$$

Since $(v\otimes\nabla\eta)_{jk} = v_j\,\partial\eta/\partial x_k$ belongs to $H^1(\omega)$ we have

$$\lim_{h\to 0}\| V_h \|_2 \le | v\otimes\nabla\eta |_{H^1(\omega_1)} \le \| v \|_{H^1(\omega)}\| \eta \|_{W^{2,\infty}(\omega)}. \tag{5.7}$$

Since $\eta \in \mathcal{D}(\omega)$ is fixed, we infer (for sufficiently small h) with (5.7)

$$\begin{aligned} -\int_\Omega v\cdot\tau\nabla\eta\,dx &= \int_\Omega V_h : \sigma\,dx \\ &\le \| \sigma \|_{L^2(\omega_1)} \| V_h \|_{L^2(\omega_1)} \le c_{10} \| \sigma \|_{L^2(\omega_1)} \| v \|_{H^1(\omega)}. \end{aligned}$$

Using this in (5.6) and the estimate (5.5) to bound $\| v \|_{H^1(\omega)}$, we deduce

$$c_{11} \| \operatorname{tr}(\eta\,\tau) \|_2 \le \| \operatorname{dev}(\eta\,\tau) \|_2 + \| \operatorname{div} \tau \|_2 + \| \sigma \|_{L^2(\omega_1)}. \tag{5.8}$$

We return to (5.4) and substitute $\| \operatorname{tr}(\eta\,\tau) \|_2$ with the bound (5.8) on the right-hand side of (5.4). The resulting estimate reads

$$\begin{aligned} &\frac{\| \operatorname{dev}(\eta\tau) \|_2^2}{2\mu} + \frac{\| \operatorname{tr}(\eta\tau) \|_2^2}{n^2(2\mu/n+\lambda)} \\ &\qquad \le c_{12}\Big(\| u \|_{H^1(\omega)}^2 + \| \operatorname{div} \sigma \|_{H^1(\omega)}^2 + \| \sigma \|_{L^2(\omega_1)}^2 + \| \operatorname{dev}(\eta\tau) \|_2^2 \Big) \end{aligned}$$

and allows us to absorb $\| \operatorname{dev}(\eta\tau) \|_2$ with Young's inequality. Hence

$$c_{13} \| \eta\, \mathbb{C}^{-1/2} \tau \|_2 \le \| u \|_{H^1(\omega)} + \| \operatorname{div} \sigma \|_{H^1(\omega)} + \| \sigma \|_{L^2(\omega_1)}.$$

Another application of (5.8) yields finally

$$c_{14} \| \eta\, \tau \|_2 \le \| u \|_{H^1(\omega)} + \| \operatorname{div} \sigma \|_{H^1(\omega)} + \| \sigma \|_{L^2(\omega_1)}.$$

The proof is then concluded as in Theorem 2.1. □

6. An application to Hencky elastoplasticity with hardening

One time step within an elastoplastic evolution problem leads to Hencky's model. For various hardening laws and von-Mises yield condition, the minimization problem takes the form (1.4). After an elimination of internal variables [ACZ99] the problem reads, in the notation of the previous section,

$$\varphi(E) := \frac{1}{2}\, E : \mathbb{C}E - \frac{1}{4\mu} \max\{0, |\operatorname{dev} \mathbb{C}E| - \sigma_y\}^2/(1+\eta) \tag{6.1}$$

for $E \in \mathbb{M}^{n\times n}_{sym}$; $\mathbb{C}$ is the fourth-order elasticity tensor, $\sigma_y > 0$ is the yield stress and $\eta > 0$ is the modulus of hardening. The model of perfect plasticity corresponds to $\eta = 0$ [Tem83].

Proposition 6.1. *We have, for all $A, B \in \mathbb{M}^{n\times n}_{sym}$,*

$$|D\varphi(A) - D\varphi(B)|^2_{\mathbb{C}^{-1}} \le (D\varphi(A) - D\varphi(B)) : (A - B). \tag{6.2}$$

Proof. Set $\xi(x) := 1 - \max\{0, 1 - \sigma_y/(2\mu\, x)\}/(1+\eta)$ to define the continuous and monotonously decreasing function $\xi : [0,\infty) \to (0,1]$ with $\xi(0) = 1 \ge \xi(x) > \eta/(1+\eta) > 0$ for $0 < x < \infty$. Then,

$$D\varphi(E) = (\lambda + 2\mu/n)\, \operatorname{tr}(E)\, \mathbb{I} + 2\mu\, \xi(|\operatorname{dev} E|)\, \operatorname{dev} E \text{ for all } E \in \mathbb{M}^{n\times n}_{sym}.$$

Without loss of generality, we suppose $a := |\operatorname{dev} A| \le b := |\operatorname{dev} B|$ and abbreviate $\alpha := \xi(a)$ and $\beta := \xi(b)$. First we calculate

$$2\mu\, \delta := |D\varphi(A) - D\varphi(B)|^2_{\mathbb{C}^{-1}} - (D\varphi(A) - D\varphi(B)) : (A - B)$$

and then have to show that

$$\delta = |\operatorname{dev}(\xi(a)A - \xi(b)B)|^2 - \operatorname{dev}(\xi(a)A - \xi(b)B) : \operatorname{dev}(A - B)$$

is non-positive. To see $\delta \leq 0$, observe that $0 \leq (1-\alpha)\beta + \alpha(1-\beta)$. Expanding the squares and collecting terms we infer in combination with Cauchy's inequality

$$\begin{aligned}
\delta &= (\xi(a)a - \xi(b)b)^2 - (\xi(a)a - \xi(b)b)(a-b) \\
&\quad +(\operatorname{dev}(A) : \operatorname{dev}(B) - ab)\Big((1-\alpha)\beta + \alpha(1-\beta)\Big) \\
&\leq (\xi(a)a - \xi(b)b)^2 - (\xi(a)a - \xi(b)b)(a-b) \\
&= (\xi(a)a - \xi(b)b)\Big((\alpha-1)a - (\beta-1)b\Big).
\end{aligned}$$

An elementary analysis shows that $x\xi(x) \geq 0$ is monotonously increasing in $0 \leq x < \infty$ while $x(\xi(x)-1) \leq 0$ is monotonously decreasing. As a consequence, $a \leq b$ implies $\xi(a)a \leq \xi(b)b$ and $(\xi(a)-1)a \geq (\xi(b)-1)b$. Taking this in the last estimate of δ into account we conclude $\delta \leq 0$. □

We therefore have the following consequence of Theorem 5.1.

Corollary 6.2. *If u is a minimizer of* (1.2) *in $\mathcal{A} \subseteq H^1(\Omega)$ and $f \in H^1_{loc}(\Omega)$ then $\sigma := D\varphi(\varepsilon(u))$ belongs to $W^{1,2}_{loc}(\Omega;\mathbb{R}^n)$.* □

Remarks 6.1. (a) The corollary is essentially due to Seregin [Ser93].
(b) The case $\eta = 0$ corresponds to perfect plasticity [Tem83] and is excluded from our analysis. Then, u only belongs to $BD(\Omega)$, the space of bounded deformations.

7. An application to a vector 2-well problem

Given two distinct wells E_1 and E_2 in $\mathbb{M}^{n\times n}_{sym}$ with minimal energies W^0_1 and W^0_2 in $\mathbb{R}$, we have a quadratic elastic energy

$$W_j(E) := \frac{1}{2}(E - E_j) : \mathbb{C}(E - E_j) + W^0_j \quad \text{for all } E \in \mathbb{M}^{n\times n}_{sym}. \tag{7.1}$$

Energy minimization balances the configuration of the two phases and so the strain energy density W is modeled by the minimum

$$W(E) = \min\{W_1(E), W_2(E)\} \quad \text{for all } E \in \mathbb{M}^{n\times n}_{sym}. \tag{7.2}$$

The two wells (transformation strains) are said to be compatible if the following condition holds

$$E_1 = E_2 + \frac{1}{2}(a \otimes b + b \otimes a) \quad \text{for some } a, b \in \mathbb{R}^n. \tag{7.3}$$

The constant γ is given by a certain projection onto the space of symmetric matrices and satisfies $0 < \gamma \leq \frac{1}{2}|E_2 - E_1|^2_{\mathbb{C}}$ and, in the compatible case (7.3), takes its upper bound $\gamma = \frac{1}{2}|E_2 - E_1|^2_{\mathbb{C}}$.

The quasiconvexification φ of W reads, owing to [Koh91],

$$\varphi(E) = \begin{cases} W_2(E) & \text{if } W_2(E) + \gamma \leq W_1(E), \\ \frac{1}{2}(W_2(E) + W_1(E)) - \frac{1}{4\gamma}(W_2(E) - W_1(E))^2 - \frac{\gamma}{4} & \\ & \text{if } |W_2(E) - W_1(E)| \leq \gamma, \\ W_1(E) & \text{if } W_1(E) + \gamma \leq W_2(E). \end{cases} \tag{7.4}$$

Lemma 7.1 ([CP97b]). *In the compatible case* (7.3)*, we have, for all* $A, B \in \mathbb{M}^{n\times n}_{sym}$,

$$|D\varphi(A) - D\varphi(B)|^2_{\mathbb{C}^{-1}} \leq \Big(D\varphi(A) - D\varphi(B)\Big) : (A - B). \tag{7.5}$$

We therefore have the following consequence of Theorem 5.1.

Corollary 7.2 ([Ser93]). *If u is a minimizer of* (1.4) *in* $\mathcal{A} \subseteq H^1(\Omega)$ *and* $f \in H^1_{loc}(\Omega)$ *then the stress* $\sigma := D\varphi(\varepsilon(u))$ *belongs to* $W^{1,2}_{loc}(\Omega; \mathbb{M}^{n\times n})$. □

Remarks 7.1. (a) The corollary is due to Seregin [Ser93, Theorem 2.2]; besides the local stress regularity, he shows that the strain tensor locally has bounded mean oscillation and investigates the pure phase area.
(b) In case of incompatible wells (i.e., if (7.3) fails), Lemma 7.1 fails (as it guarantees convexity of φ). Due to Seregin [Ser96], $\varphi(\operatorname{sym} F)$ can be rewritten as the sum of a convex function (which then satisfies an estimate of the form (7.5)) and a linear combination of second order minors of F. Then, up to cofactor matrices of the gradient F (stress free if pure Dirichlet boundary conditions are imposed), the stress belongs to $W^{1,2}_{loc}(\Omega; \mathbb{M}^{n\times n}_{sym})$. The interpretation of cof Du as a constant pressure may be formally correct (as the model is in material coordinates) but is doubtful from the physical point of view: A linearisation is behind (7.1) and so material and spatial coordinates coincide and incompressibility reads $\operatorname{div} u = 0$ and *not* $\det Du = 1$.
(c) A time-discretized model for hysteresis of [MTL] leads to a similar variational problem. From a stress estimate in [CP00], we obtain an analogue of Lemma 7.1 and can conclude $\sigma \in W^{1,2}_{loc}(\Omega; \mathbb{M}^{n\times n})$ as well.

Acknowledgments. The first author's research (CC) was supported by the Max Plank Institute in the Sciences of Leipzig, Germany, and the German Research Foundation through the DFG-Schwerpunktprogramm Multi-Scale Problems.

References

[ACZ99] J. Alberty, C. Carstensen, and D. Zarrabi. Adaptive numerical analysis in primal elastoplasticity with hardening. *Comp. Meth. Appl. Mech. Eng.*, 171:175–204, 1999.

[BF91] Franco Brezzi and Michel Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[BKK00] J.M. Ball, B. Kirchheim, and J. Kristensen. Regularity of quasiconvex envelopes. *Calc. Var.*, 11:333–359, 2000.

[Bol06] Oskar Bolza. A fifth necessary condition for a strong extremum of the integral $\int_{x_0}^{x_1} f(x, y, y')dx$. *Trans. Amer. Math. Soc.*, 7(2):314–324, 1906.

[CK01] C. Carstensen and R. Klose. Guaranteed a posteriori finite element error control for the p-laplace problem. *Preprint*, 2001.

[CP97a] C. Carstensen and P. Plecháč. Numerical solution of the scalar double-well problem allowing microstructure. *Math. Comp.*, 66:997–1026, 1997.

[CP97b] C. Carstensen and Petr Plecháč. Adaptive algorithms for scalar non-convex variational problems. *Appl. Num. Math.*, 26(1-2):203–216, 1997.

[CP00] C. Carstensen and Petr Plecháč. Numerical analysis of compatible phase transitions in elastic solids. *SIAM J. Numer. Anal.*, 37(6):2061–2081, 2000.

[Dac89] B. Dacarogna. *Direct methods in the calculus of variations.*, volume 78 of *Applied Mathematical Sciences*. Springer Verlag, Berlin, 1989.

[Fri94] G. Friesecke. A necessary and sufficient condition for non-attainment and formation of microstructure almost everywhere in scalar variational problems. *Proc. R. Soc. Edin.*, 124A:437–471, 1994.

[GKR86] Jonathan Goodman, Robert V. Kohn, and Luis Reyna. Numerical study of a relaxed variational problem from optimal design. *Comput. Methods Appl. Mech. Engrg.*, 57(1):107–127, 1986.

[GR86] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations*. Springer, Berlin, 1986.

[Koh91] R.V. Kohn. The relaxation of a double-well energy. *Continuum Mech. Thermodyn.*, 3:193–236, 1991.

[MTL] A. Mielke, F. Theil, and V. I. Levitas. A variational formulation of rate-independent phase transformations using an extremum principle. Preprint 2001.

[Rou97] T. Roubíček. *Relaxation in Optimization Theory and Variational Calculus.*, volume 4 of *Series in Nonlinear Analysis and Applications*. Walter de Gruyter, 1997.

[Ser93] G.A. Seregin. On the regularity of minimizers of some variational problems in the theory of plasticity. *St. Petersburg Math. J.*, 4(5):989–1020, 1993.

[Ser96] G.A. Seregin. The regularity properties of solutions of variational problems in the theory of phase transitions in elastic. *St. Petersburg Math. J.*, 7(6):979–1003, 1996.

[Tem83] R. Temam. *Problemes mathematiques en plasticite. (French) [Mathematical problems in plasticity] Mèthodes Mathematiques de l'Informatique [Mathematical Methods of Information Science]*. 12. Gauthier-Villars, Paris, 1983.

[You69] L. C. Young. *Lectures on the calculus of variations and optimal control theory.* W. B. Saunders Co., Philadelphia-London-Toronto, 1969.

Institute for Applied Mathematics and Numerical Analysis, Vienna University of Technology, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria

E-mail address: Carsten.Carstensen@tuwien.ac.at

Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany.

E-mail address: Stefan.Mueller@mis.mpg.de