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Nilpotent Szabo Osserman and Ivanov-Petrova Pseudo-Riemannian Manifolds

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NILPOTENT SZABÓ, OSSERMAN AND IVANOV-PETROVA PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We exhibit pseudo Riemannian manifolds which are Szabó nilpotent of arbitrary order, or which are Osserman nilpotent of arbitrary order, or which are Ivanov-Petrova nilpotent of order 3.

1. Introduction

Let R be the Riemann curvature tensor of a pseudo-Riemannian manifold (M, g) of signature (p, q). The *Szabó operator* $\mathcal{S}$ is the self-adjoint linear map which is characterized by the identity:

$$g(\mathcal{S}(x)y, z) = \nabla R(y, x, x, z; x).$$

One says that (M, g) is *Szabó* if the eigenvalues of $\mathcal{S}(x)$ are constant on the pseudo-spheres of unit timelike and spacelike vectors:

$$S^{\pm}(M, g) := \{x \in TM : g(x, x) = \pm 1\}.$$

Szabó [20] used techniques from algebraic topology to show in the Riemannian setting $(p = 0)$ that any such metric is locally symmetric. He used this observation to give a simple proof that any 2 point homogeneous space is either flat or is a rank 1 symmetric space. Subsequently Gilkey and Stavrov [14] extended his results to show that any Szabó Lorentzian $(p = 1)$ manifold has constant sectional curvature. By replacing g by $-g$, one can interchange the roles of p and of q, thus these results apply to the cases $q = 0$ and $q = 1$ as well.

The eigenvalue zero is distinguished. One says that (M, g) is *Szabó nilpotent of order* n if $\mathcal{S}(x)^n = 0$ for every $x \in TM$ and if there exists a point $P_0 \in M$ and a tangent vector $x_0 \in T_{P_0}M$ so that $\mathcal{S}(x_0)^{n-1} \neq 0$. One says that (M, g) is *Szabó nilpotent* if (M, g) is Szabó nilpotent of order n for some n. Note that (M, g) is Szabó nilpotent if and only if 0 is the only eigenvalue of $\mathcal{S}$; consequently any Szabó nilpotent manifold is Szabó. There is some evidence [11, 19] to suggest, conversely, that any Szabó manifold is Szabó nilpotent.

If (M, g) is Szabó nilpotent of order 1, then $\mathcal{S}(x) = 0$ for all $x \in TM$. This implies [14] that $\nabla R = 0$ so (M, g) is a local symmetric space; this is to be regarded, therefore, as a trivial case. Gilkey, Ivanova, and Zhang [12] have constructed pseudo-Riemannian manifolds of any signature (p, q) with $p \geq 2$ and $q \geq 2$ which are Szabó nilpotent of order 2; these were the only previously known examples of Szabó manifolds which were not local symmetric spaces. In this brief note, we shall construct pseudo-Riemannian metrics g_n on $\mathbb{R}^{n+2}$ which are Szabó nilpotent of order $n \geq 2$; the metric will be *balanced* (i.e. $p = q$) if n is even and *almost balanced* (i.e. $p = q \pm 1$) if n is odd. By taking an isometric product with a suitable flat manifold, the signature can be increased without changing the order of nilpotency.

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thm1.1 **Theorem 1.1.** *Let $n \geq 2$. There exists a pseudo-Riemannian metric g_n on $\mathbb{R}^{n+2}$ which is Szabó nilpotent of order n. If $n = 2p$, then g_n has signature $(p+1, p+1)$; if $n = 2p+1$, then g_n has signature $(p+1, p+2)$.*

The *Jacobi operator* is defined analogously; it is characterized by the identity:

$$g(J(x)y, z) = R(y, x, x, z).$$

One says that (M, g) is *Osserman* if the eigenvalues of J are constant on $S^{\pm}(M)$. In the Riemannian setting, Osserman wondered [17] if this implied (M, g) was a 2 point homogeneous space. This question has been answered in the affirmative in the Riemannian setting [4, 16] for dimensions $\neq 8, 16$, and in all dimensions in the Lorentzian setting [1, 5].

We shall say that (M, g) is *Osserman nilpotent of order n* if $J(x)^n = 0$ for every $x \in TM$ and if there exists a point $P_0 \in M$ and a tangent vector $x_0 \in T_{P_0}M$ so that $J(x_0)^{n-1} \neq 0$, i.e. 0 is the only eigenvalue of J. Such manifolds are necessarily Osserman. Osserman nilpotent manifolds of orders 2 and 3 have been constructed previously [2, 7, 6, 8]. These manifolds need not be homogeneous, thus the question Osserman raised has a negative answer in the higher signature setting. A byproduct of our investigation of Szabó manifolds yields new examples of Osserman manifolds; again, the signature can be increased by taking isometric products with flat factors.

thm1.2 **Theorem 1.2.** *Let $n \geq 2$. There exists a pseudo-Riemannian metric $\tilde{g}_n$ on $\mathbb{R}^{n+2}$ which is Osserman nilpotent of order n. If $n = 2p$, $\tilde{g}_n$ has signature $(p+1, p+1)$; if $n = 2p+1$, $\tilde{g}_n$ has signature $(p+1, p+2)$.*

If $\{f_1, f_2\}$ is an oriented orthonormal basis for a non-degenerate oriented 2 plane π, we define the skew-symmetric curvature operator by setting $\mathcal{R}(\pi) := R(f_1, f_2)$. We say (M, g) is Ivanov-Petrova nilpotent of order n if $\mathcal{R}(\pi)^n = 0$ for any non-degenerate oriented 2 plane π and if there exists π so $\mathcal{R}(\pi)^{n-1} \neq 0$. We refer to [8] for further details concerning Ivanov-Petrova manifolds. Another byproduct of our investigation yields new examples of these manifolds:

thm1.3 **Theorem 1.3.** *There exist Ivanova-Petrova pseudo-Riemannian manifolds which are nilpotent of order 2 and of order 3.*

Here is a brief outline to the paper. In Section 2, we give a general procedure for constructing pseudo-Riemannian manifolds with certain kinds of curvature and covariant derivative curvature tensors. We apply this procedure in Section 3 to complete the proof of Theorem 1.1. Lemma 3.1 deals with the cases $n = 2$ and $n = 3$, Lemma 3.2 deals with the case $n = 2\ell + 1 \geq 5$, and Lemma 3.3 deals with the case $n = 2\ell + 2 \geq 4$. In Section 4, we prove Theorem 1.2 and in Section 5, we prove Theorem 1.3.

One can also work with the Jordan normal form; one says (M, g) is Jordan Szabo (resp. Jordan Osserman or Jordan IP) if the Jordan normal form of $\mathcal{S}$ (resp. J or $\mathcal{R}$) is constant on the appropriate domains of definition. The examples constructed in this paper do **not** fall into this framework; in particular, there are no known Jordan Szabo pseudo-Riemannian manifolds which are not locally symmetric.

2. A family of pseudo-Riemannian manifolds

Sect2 We introduce the following notational conventions. Let $(x, u_1, ..., u_\nu, y)$ be coordinates on $\mathbb{R}^{\nu+2}$. We shall use several different notations for the coordinate frame:

$$\mathcal{B} = \{e_0, e_1, ..., e_{\nu+1}\} = \{X, U_1, ..., U_\nu, Y\} := \{\partial_x, \partial_{u_1}, ..., \partial_{u_\nu}, \partial_y\}.$$

Let indices $i, j, ...$ range from 0 through $\nu+1$ and index the full coordinate frame. Let indices a, b range from 1 through ν and index the tangent vectors $\{U_1, ..., U_\nu\}$. In the interests of brevity, we shall give non-zero entries in a metric g, curvature tensor R, and covariant derivative curvature tensor ∇R up to the obvious $\mathbb{Z}_2$ symmetries.

lem2.1 **Lemma 2.1.** *Let $f = f(u)$ be a smooth function on $\mathbb{R}^\nu$ and let Ξ be a constant invertible symmetric $\nu \times \nu$ matrix. Define a metric g_f on $\mathbb{R}^{\nu+2}$ by setting:*

$$g_f(X, X) = f(u), \quad g_f(X, Y) = 1, \quad \textit{and} \quad g_f(U_a, U_b) = \Xi_{ab}.$$

All other scalar products equals zero.

1. *Then the non-zero entries in R_{g_f} are $R_{g_f}(X, U_a, U_b, X) = -\frac{1}{2} U_a U_b(f)$.*
2. *The non-zero entries in ∇R_{g_f} are $\nabla R_{g_f}(X, U_a, U_b, X; U_c) = -\frac{1}{2} U_a U_b U_c(f)$.*

Proof. Since $d\Xi = 0$, the non-zero Christoffel symbols of the first kind are:

eqn2.a
$$\Gamma_{a00} = \Gamma_{0a0} = -\Gamma_{00a} = \tfrac{1}{2} U_a(f). \tag{2.a}$$

Let Ξ^{ab} be the inverse matrix. We adopt the Einstein convention and sum over repeated indices to compute:

$$\begin{aligned}
&\Gamma_{ijb} = g(\nabla_{e_i} e_j, e_b) = g(\Gamma_{ij}{}^k e_k, e_b) = \Gamma_{ij}{}^a \Xi_{ab} && \text{so } \Gamma_{ij}{}^a = \Xi^{ab} \Gamma_{ijb},\\
&\Gamma_{ij\nu+1} = g(\nabla_{e_i} e_j, e_{\nu+1}) = g(\Gamma_{ij}{}^k e_k, e_{\nu+1}) = \Gamma_{ij}{}^0 && \text{so } \Gamma_{ij}{}^0 = 0,\\
&\Gamma_{ij0} = g(\nabla_{e_i} e_j, e_0) = g(\Gamma_{ij}{}^k e_k, e_0) = f \Gamma_{ij}{}^0 + \Gamma_{ij}{}^{\nu+1} && \text{so } \Gamma_{ij}{}^{\nu+1} = \Gamma_{ij0}.
\end{aligned}$$

Thus the non-zero Christoffel symbols of the second kind are:

eqn2.b
$$\Gamma_{a0}{}^{\nu+1} = \Gamma_{0a}{}^{\nu+1} = \tfrac{1}{2} U_a(f) \quad \text{and} \quad \Gamma_{00}{}^a = -\tfrac{1}{2} \textstyle\sum_b \Xi^{ab} U_b(f). \tag{2.b}$$

The components of the curvature tensor relative to the coordinate frame are:

eqn2.c
$$R_{ijkl} = e_i \Gamma_{jkl} - e_j \Gamma_{ikl} + \textstyle\sum_n \{\Gamma_{inl} \Gamma_{jk}{}^n - \Gamma_{jnl} \Gamma_{ik}{}^n\}. \tag{2.c}$$

By equation (2.b), $\Gamma_{ik}{}^0 = \Gamma_{jk}{}^0 = 0$. By equation (2.a), $\Gamma_{i,\nu+1,k} = \Gamma_{j,\nu+1,k} = 0$. Thus the index n in equation (2.c) is neither 0 nor $\nu+1$. Thus by equation (2.a) and equation (2.b), $i = j = k = l = 0$. This shows that the terms which are quadratic in Γ play no role in equation (2.c). Assertion (1) then follows from equation (2.a).

The covariant derivative of the curvature tensor is given by:

eqn2.d
$$R_{ijkl;n} = e_n R_{ijkl} - \textstyle\sum_p \{\Gamma_{ni}{}^p R_{pjkl} + \Gamma_{nj}{}^p R_{ipkl} + \Gamma_{nk}{}^p R_{ijpl} + \Gamma_{nl}{}^p R_{ijkp}\}. \tag{2.d}$$

By equation (2.b) $\Gamma_{**}{}^0 = 0$. Thus we may assume $p \neq 0$ in equation (2.d). Furthermore, by assertion (1), $R_{\nu+1***} = R_{*\nu+1**} = R_{**\nu+1*} = R_{***\nu+1} = 0$ so we may also assume $p \neq \nu + 1$ in equation (2.d). Thus $\Gamma_{ni}{}^p R_{pjkl} = 0$ unless $i = j = 0$ and similarly $\Gamma_{nj}{}^p R_{ipkl} = 0$ unless $i = j = 0$. Thus these two terms cancel. Similarly $\Gamma_{nk}{}^p R_{ijpl}$ cancels $\Gamma_{nl}{}^p R_{ijkp}$. Thus $R_{ijkl;n} = e_n R_{ijkl}$ and assertion (2) follows. □

Remark 2.2. Let ρ be the associated Ricci tensor; $\rho(\xi, \xi) = \text{Trace}(J(\xi))$. We have $\rho(e_i, e_j) = \sum_{kl} g^{kl} R(e_i, e_k, e_l, e_j)$. Since R vanishes on $e_{\nu+1}$, we may sum over $k, l \leq \nu$. Since $g^{0k} = g^{k0} = 0$ for $k \leq \nu$, $\rho(e_i, e_j) = \sum_{ab} g^{ab} R(e_i, e_a, e_b, e_j)$. Thus $\rho(e_i, e_j) = 0$ for $(i, j) \neq (0, 0)$ and the only non-zero entry of the Ricci tensor is $\rho(e_0, e_0) = -\frac{1}{2} \sum_{ab} \Xi^{ab} \partial_a \partial_b f$. The associated Jacobi operator will be nilpotent if and only if this sum vanishes. Raising indices yields a Ricci operator $\hat{\rho}$ with the property that $\hat{\rho}(e_0) = -\frac{1}{2} \sum_{ab} \Xi^{ab} \partial_a \partial_b f$ and $\hat{\rho}(e_i) = 0$ for $i > 0$. Thus $\hat{\rho}^2 = 0$ so the Ricci operator is nilpotent of order 2 and non-trivial if and only if g is not Osserman.

If f is quadratic, then R is constant on the coordinate frame; if f is cubic, then ∇R is constant on the coordinate frame. However, these tensors are not curvature homogeneous in the sense of Kowalski, Tricerri, and Vanhecke [15] since the metric relative to the coordinate frames is not constant.

The tensors of Lemma 2.1 are related to hypersurface theory. Let M be a non-degenerate hypersurface in $\mathbb{R}^{(a,b)}$; we assume M is spacelike but similar remarks hold in the timelike setting. Let L be the associated second fundamental form and let $S = \nabla L$ be the covariant derivative of L; L is a totally symmetric 2 form and S is a totally symmetric 3 form. We may then, see for example [8], express:

eqn2.e

$$\begin{aligned} R_L(x_1, x_2, x_3, x_4) &= L(x_1, x_4)L(x_2, x_3) - L(x_1, x_3)L(x_2, x_4), \\ \nabla R_{L,S}(x_1, x_2, x_3, x_4; x_5) &= S(x_1, x_4, x_5)L(x_2, x_3) + L(x_1, x_4)S(x_2, x_3, x_5) \\ &- S(x_1, x_3, x_5)L(x_2, x_4) - L(x_1, x_3)S(x_2, x_4, x_5). \end{aligned} \tag{2.e}$$

If L is an arbitrary symmetric 2 tensor and if S is an arbitrary totally symmetric 3 tensor, then we may use equation (2.e) to define tensors we continue to denote by R_L and $\nabla R_{L,S}$. We refer to [9] for the proof of assertion (1) and to [10] for the proof of assertion (2) in the following result:

thm2.1 **Theorem 2.3.**

1. *The tensors R_L which are defined by a symmetric 2 form L generate the space of all algebraic curvature tensors.*
2. *The tensors $\nabla R_{L,S}$ which are defined by a symmetric 2 form L and by a totally symmetric 3 form S generate the space of all algebraic covariant derivative curvature tensors.*

The tensors of Lemma 2.1 (2) are of this form. Let

$$f(u) := -\tfrac{1}{3}\textstyle\sum_{a,b,c} c_{a,b,c} u_a u_b u_c$$

be a cubic polynomial in the u variables which is independent of x and y. Then:

$$\nabla R = \nabla R_{L,S} \quad \text{for} \quad L(\partial_i, \partial_j) := \delta_{0,i}\delta_{0,j} \quad \text{and} \quad S(\partial_i, \partial_j, \partial_k) := -\tfrac{1}{2}\partial_i\partial_j\partial_k f.$$

3. Nilpotent Szabó manifolds

Sect3 In this section we will use Lemma 2.1 to prove Theorem 1.1 by choosing Ξ and f appropriately. We shall consider metrics of the form:

$$g(X, X) = f(t, u, v),\ g(X, Y) = 1,\ g(T, T) = 1,\ g(U_a, V_b) = \delta_{ab};$$

the spacelike vector T will not be present in some cases. The vectors $\{U_a, V_a\}$ are a hyperbolic pair.

We begin by discussing the cases $n = 2$ and $n = 3$.

lem3.1 **Lemma 3.1.**

1. *Let $\mathcal{B}_2 := \{X, U, V, Y\} = \{\partial_x, \partial_u, \partial_v, \partial_y\}$ be the coordinate frame on $\mathbb{R}^4$ relative to the coordinate system (x, u, v, y). Define a metric g_2 by:*

   $$g_2(X, X) = -\tfrac{1}{3}u^3,\ g_2(X, Y) = 1,\ g_2(U, V) = 1.$$

   Then g_2 has signature $(2, 2)$ on $\mathbb{R}^4$ and g_2 is Szabó nilpotent of order 2.
2. *Let $\mathcal{B}_3 := \{X, T, U, V, Y\} = \{\partial_x, \partial_t, \partial_u, \partial_v, \partial_y\}$ be the coordinate frame on $\mathbb{R}^5$ relative to the coordinate system (x, t, u, v, y). Define a metric g_3 by:*

   $$g_3(X, X) = -tu^2,\ g_3(T, T) = 1,\ g_3(U, V) = 1,\ g_3(X, Y) = 1.$$

   Then g_3 has signature $(2, 3)$ on $\mathbb{R}^5$ and g_3 is Szabó nilpotent of order 3.

Proof. Let $\mathcal{B}^* = \{e^0, \dots, e^{\nu+1}\}$ be the corresponding *dual basis* of $\mathcal{B}$; it is characterized by the relations $g(e_i, e^j) = \delta_i^j$. For example, we have

eqn3.x

$$\mathcal{B}_2^* = \{Y, V, U, X - fY\} \quad \text{and} \quad \mathcal{B}_3^* = \{Y, T, V, U, X - fY\} \tag{3.a}$$

By Lemma 2.1, the only non-zero component of ∇R_{g_2} is

$$\nabla R_{g_2}(X, U, U, X; U) = 1.$$

Let $\xi = \xi_0 X + \xi_1 U + \xi_2 V + \xi_3 Y$ be a tangent vector. We use equation (3.a) to raise indices and conclude:

$$\mathcal{S}_{g_2}(\xi)X = \xi_1^3 Y - \xi_0\xi_1^2 V, \quad \mathcal{S}_{g_2}(\xi)U = -\xi_0\xi_1^2 Y + \xi_0^2\xi_1 V, \quad \mathcal{S}_{g_2}(\xi)Y = \mathcal{S}(\xi)V = 0.$$

Thus $\mathcal{S}_{g_2}(\xi)^2=0$ for all ξ while $\mathcal{S}_{g_2}(\xi)\neq 0$ for generic ξ. Assertion (1) now follows.

Similarly, the only non-zero components of ∇R_{g_3} are

$$\nabla R_{g_3}(X,U,U,X;T)=\nabla R_{g_3}(X,U,T,X;U)=1.$$

We use equation (3.a) to raise indices and compute:

$$\begin{array}{l}\mathcal{S}_{g_3}(\xi)X=\star T+\star Y+\star V,\quad \mathcal{S}_{g_3}(\xi)Y=0,\\ \mathcal{S}_{g_3}(\xi)T=\star Y+\star V,\\ \mathcal{S}_{g_3}(\xi)U=\star T+\star Y+\star V,\quad \mathcal{S}_{g_3}(\xi)V=0\end{array}$$

where $\star=\star(\xi)$ denotes suitably chosen cubic polynomials in the coefficients of ξ that is generically non-zero; as the precise value of this coefficient is not important, we shall suppress it in the interests of notational simplicity. It is now clear that $\mathcal{S}_{g_3}(\xi)^3=0$ for all ξ while $\mathcal{S}_{g_3}(\xi)^2$ is generically non-zero. □

Next we consider the case $n=2\ell+1\geq 5$. Let $(x,t,u_2,...,u_{\ell+1},v_2,...,v_{\ell+1},y)$ be coordinates on $\mathbb{R}^{2\ell+3}$ which define the associated coordinate frame:

$$\mathcal{B}:=\{X,T,U_2,...,U_{\ell+1},V_2,...,V_{\ell+1},Y\}=\{\partial_x,\partial_t,\partial_{u_2},...,\partial_{u_{\ell+1}},\partial_{v_2},...,\partial_{v_{\ell+1}},\partial_y\}.$$

lem3.2 **Lemma 3.2.** *Let $\ell\geq 2$. Define a metric $g_{2\ell+1}$ on $\mathbb{R}^{2\ell+3}$ by setting:*

$$g_{2\ell+1}(X,X)=-tu_2^2-\textstyle\sum_{2\leq a\leq\ell}(u_a+v_a)u_{a+1}^2,$$

$$g_{2\ell+1}(X,Y)=1,\quad g_{2\ell+1}(T,T)=1,\quad g_{2\ell+1}(U_a,V_b)=\delta_{ab}.$$

Then $g_{2\ell+1}$ is a metric of signature $(\ell+1,\ell+2)$ and Szabó nilpotent of order $2\ell+1$.

Proof. Let $2\leq a\leq\ell$. By Lemma 2.1, the non-zero components of ∇R are:

$$\begin{array}{lllll}\nabla R(X,U_2,U_2,X;T)&=&\nabla R(X,T,U_2,X;U_2)&=&1,\\ \nabla R(X,U_{a+1},U_{a+1},X;U_a)&=&\nabla R(X,U_{a+1},U_a,X;U_{a+1})&=&1,\\ \nabla R(X,U_{a+1},U_{a+1},X;V_a)&=&\nabla R(X,U_{a+1},V_a,X;U_{a+1})&=&1.\end{array}$$

The dual basis is $\mathcal{B}^*=\{Y,T,V_2,...,V_{\ell+1},U_2,...,U_{\ell+1},X-fY\}$. Let ξ be an arbitrary tangent vector. We raise indices and compute:

$$\begin{array}{lll}\mathcal{S}(\xi)X&\in&\mathrm{Span}\{Y,T,U_2,...,U_\ell,V_2,...,V_{\ell+1}\},\\ \mathcal{S}(\xi)Y&=&0,\\ \mathcal{S}(\xi)T&=&\star Y+\star V_2,\\ \mathcal{S}(\xi)U_2&=&\star T+\star Y+\star V_2+\star V_3,\\ \mathcal{S}(\xi)U_a&=&\star U_{a-1}+\star Y+\star V_{a-1}+\star V_a+\star V_{a+1}\quad\text{for}\quad 3\leq a\leq\ell,\\ \mathcal{S}(\xi)U_{\ell+1}&=&\star U_\ell+\star Y+\star V_\ell+\star V_{\ell+1}\\ \mathcal{S}(\xi)V_a&=&\star Y+\star V_{a+1}\quad\text{for}\quad 2\leq a\leq\ell,\\ \mathcal{S}(\xi)V_{\ell+1}&=&0\end{array}$$

where $\star$ is a coefficient that is non-zero for generic ξ. If $\mathcal{E}$ is a subspace, let $\alpha=\beta+\mathcal{E}$ mean that $\alpha-\beta\in\mathcal{E}$. We compute:

$$\begin{array}{llll}\mathcal{S}(\xi)^\mu U_{\ell+1}&=&\star U_{\ell+1-\mu}+\mathrm{Span}\{V_2,...,V_{\ell+1},Y\},&1\leq\mu\leq\ell-1\\ \mathcal{S}(\xi)^\ell U_{\ell+1}&=&\star T+\mathrm{Span}\{V_2,...,V_{\ell+1},Y\},&\\ \mathcal{S}(\xi)^\mu U_{\ell+1}&=&\star V_{\mu+1-\ell}+\mathrm{Span}\{V_{\mu+2-\ell},...,V_{\ell+1},Y\},&\ell+1\leq\mu\leq 2\ell-1\\ \mathcal{S}(\xi)^{2\ell}U_{\ell+1}&=&\star V_{\ell+1}+\mathrm{Span}\{Y\}.&\end{array}$$

Thus $\mathcal{S}(\xi)^{2\ell}\neq 0$ for generic ξ. One shows similarly $\mathcal{S}(\xi)^{2\ell+1}=0$ for every ξ by:

$$\begin{array}{llll}\mathcal{S}(\xi)^\mu\mathcal{B}&\subseteq&\mathrm{Span}\{T,U_2,...,U_{\ell+1-\mu},V_2,...,V_{\ell+1},Y\},&1\leq\mu\leq\ell-1\\ \mathcal{S}(\xi)^\ell\mathcal{B}&\subseteq&\mathrm{Span}\{T,V_2,...,V_{\ell+1},Y\},&\\ \mathcal{S}(\xi)^\mu\mathcal{B}&\subseteq&\mathrm{Span}\{V_{\mu+1-\ell},...,V_{\ell+1},Y\},&\ell+1\leq\mu\leq 2\ell\end{array}$$

and $\mathcal{S}(\xi)^{2\ell+1}\mathcal{B}=\{0\}$. □

We complete the proof of Theorem 1.1 by considering the case $n = 2\ell + 2$ for $\ell \geq 1$. Let $(x, u_1, u_2, ..., u_{\ell+1}, v_1, ..., v_{\ell+1}, y)$ be coordinates on $\mathbb{R}^{2\ell+4}$ which define the associated coordinate frame:

$$\mathcal{B} := \{X, U_1, ..., U_{\ell+1}, V_1, ..., V_{\ell+1}, Y\} = (\partial_x, \partial_{u_1}, ..., \partial_{u_{\ell+1}}, \partial_{v_1}, ..., \partial_{v_{\ell+1}}, \partial_y\}.$$

lem3.3 **Lemma 3.3.** *Let $\ell \geq 1$. Define a metric $g_{2\ell+2}$ on $\mathbb{R}^{2\ell+4}$ by setting:*

$$\begin{aligned} &g_{2\ell+2}(X, X) = -\textstyle\sum_{1\leq a\leq \ell}(u_a + v_a)u_{a+1}^2 - \frac{1}{3}u_1^3, \\ &g_{2\ell+2}(X, Y) = 1, \quad g_{2\ell+2}(U_a, V_b) = \delta_{ab}. \end{aligned}$$

Then $g_{2\ell+2}$ is a metric of signature $(\ell+2, \ell+2)$ and Szabó nilpotent of order $2\ell+2$.

Proof. Let $2 \leq a \leq \ell + 1$. The non-zero components of $\nabla R_{g_{2\ell+2}}$ are:

$$\begin{array}{lllll} \nabla R_{g_{2\ell+2}}(X, U_1, U_1, X; U_1) & = & 1, & & \\ \nabla R_{g_{2\ell+2}}(X, U_a, U_a, X; U_{a-1}) & = & \nabla R_{g_{2\ell+2}}(X, U_a, U_{a-1}, X; U_a) & = & 1, \\ \nabla R_{g_{2\ell+2}}(X, U_a, U_a, X; V_{a-1}) & = & \nabla R_{g_{2\ell+2}}(X, U_a, V_{a-1}, X; U_a) & = & 1. \end{array}$$

We compute:

$$\begin{array}{lll} \mathcal{S}(\xi)X & = & \star U_1 + ... + \star U_\ell + \star Y + \star V_1 + ... + \star V_{\ell+1}, \\ \mathcal{S}(\xi)U_1 & = & \star Y + \star V_1 + \star V_2, \\ \mathcal{S}(\xi)U_a & = & \star U_{a-1} + \star Y + \star V_{a-1} + \star V_a + \star V_{a+1} \quad \text{for} \quad 2 \leq a \leq \ell, \\ \mathcal{S}(\xi)U_{\ell+1} & = & \star U_\ell + \star Y + \star V_\ell + \star V_{\ell+1}, \\ \mathcal{S}(\xi)Y & = & 0, \\ \mathcal{S}(\xi)V_a & = & \star Y + \star V_{a+1} \quad \text{for} \quad 1 \leq a \leq \ell, \\ \mathcal{S}(\xi)V_{\ell+1} & = & 0. \end{array}$$

We may then show $\mathcal{S}(\xi)^{2\ell+1}$ is generically non-zero by computing:

$$\begin{array}{llll} S(\xi)^\mu U_{\ell+1} & = & \star U_{\ell+1-\mu} + \text{Span}\{Y, V_1, ..., V_{\ell+1}\}, & 1 \leq \mu \leq \ell \\ S(\xi)^\mu U_{\ell+1} & = & \star V_{\mu-\ell} + \text{Span}\{Y, V_{\mu+1-\ell}, ..., V_{\ell+1}\}, & \ell+1 \leq \mu \leq 2\ell \\ S(\xi)^{2\ell+1} U_{\ell+1} & = & \star V_{\ell+1} + \text{Span}\{Y\}. & \end{array}$$

A similar argument shows $\mathcal{S}(\xi)^{2\ell+2} = 0$ for all ξ. We can write

$$\begin{array}{llll} S(\xi)^\mu \mathcal{B} & \subseteq & \text{Span}\{U_1, ..., U_{\ell+1-\mu}, V_1, ..., V_{\ell+1}, Y\}, & 1 \leq \mu \leq \ell, \\ S(\xi)^\mu \mathcal{B} & \subseteq & \text{Span}\{V_{\mu-\ell}, ..., V_{\ell+1}, Y\}, & \ell+1 \leq \mu \leq 2\ell+1 \end{array}$$

and $S(\xi)^{2\ell+2}\mathcal{B} = \{0\}$. □

rmk3.4 **Remark 3.4.** One can also consider the purely pointwise question. We shall say that (M, g) is Szabó nilpotent of order n at $P \in M$ if $\mathcal{S}(x)^n = 0$ for all $x \in T_PM$ and if $\mathcal{S}(x_0)^{n-1} \neq 0$ for some $x_0 \in T_PM$. Throughout Section 3, we considered cubic functions to ensure that ∇R was constant on the coordinate frames; thus the point in question played no role. However, had we replaced u^3 by u^4, tu^2 by tu^3, $u_a u_{a+1}^2$ by $u_a u_{a+1}^3$, and $v_a u_{a+1}^2$ by $v_a u_{a+1}^3$, then we would have constructed metrics g_n which were Szabó nilpotent of order n on $T_P\mathbb{R}^{n+2}$ for generic points $P \in \mathbb{R}^{n+2}$, but where ∇R vanishes at the origin $0 \in \mathbb{R}^{n+2}$. Since the order of nilpotency would vary with the point of the manifold, these metrics clearly are not homogeneous.

4. Nilpotent Osserman manifolds

Sect4 In Section 3, we used cubic expressions to define our metrics to ensure the tensors $R_{ijkl;n}$ were constant on the coordinate frame. To discuss the Jacobi operator, we use the corresponding quadratic polynomials. We adopt the notation of Section 3 to define metrics:

$$\begin{aligned}
&\tilde g_2(X,X)=-u^2,\ \tilde g_2(X,Y)=1,\ \tilde g_2(U,V)=1,\\
&\tilde g_3(X,X)=-2tu-u^2,\ \tilde g_3(T,T)=1,\ \tilde g_3(U,V)=1,\ \tilde g_3(X,Y)=1,\\
&\tilde g_{2\ell+1}(X,X)=-2tu_2-u_2^2-\textstyle\sum_{2\le a\le\ell}\{2(u_a+v_a)u_{a+1}+u_{a+1}^2\},\\
&\quad \tilde g_{2\ell+1}(X,Y)=1,\ \tilde g_{2\ell+1}(T,T)=1,\ \tilde g_{2\ell+1}(U_a,V_b)=\delta_{uv},\quad (\ell\ge 2)\\
&\tilde g_{2\ell+2}(X,X)=-\textstyle\sum_{1\le a\le\ell}\{2(u_a+v_a)u_{a+1}+u_{a+1}^2\}-u_1^2,\\
&\quad \tilde g_{2\ell+2}(X,Y)=1,\quad \tilde g_{2\ell+2}(U_a,V_b)=\delta_{ab}\qquad (\ell\ge 1).
\end{aligned}$$

lem4.1 **Lemma 4.1.**

1. $\tilde g_2$ *has signature* $(2,2)$ *and is Osserman nilpotent of order* 2.
2. $\tilde g_3$ *has signature* $(2,3)$ *and is Osserman nilpotent of order* 3.
3. $\tilde g_{2\ell+1}$ *has signature* $(\ell+1,\ell+2)$ *and is Osserman nilpotent of order* $2\ell+1$.
4. $\tilde g_{2\ell+2}$ *has signature* $(\ell+2,\ell+2)$ *and is Osserman nilpotent of order* $2\ell+2$.

Proof. By Lemma 2.1, the non-zero components of $R_{\tilde g_2}$ are

eqn4.a
$$R_{\tilde g_2}(X,U,U,X)=1. \tag{4.a}$$

Assertion (1) now follows since:

$$J_{\tilde g_2}(\xi)X=\star Y+\star V,\quad J_{\tilde g_2}(\xi)U=\star Y+\star V,\quad J_{\tilde g_2}(\xi)Y=J(\xi)V=0$$

where $\star$ denotes suitably chosen quadratic polynomials in the components of ξ which are non-zero for generic ξ.

Similarly, the only non-zero component of $\nabla R_{\tilde g_3}$ are

eqn4.b
$$R_{\tilde g_3}(X,U,U,X)=1\quad\text{and}\quad R_{\tilde g_3}(X,U,T,X)=1. \tag{4.b}$$

Assertion (2) now follows since:

$$\begin{aligned}
&J_{\tilde g_3}(\xi)X=\star T+\star Y+\star V,\quad J_{\tilde g_3}(\xi)Y=0,\\
&J_{\tilde g_3}(\xi)T=\star Y+\star V,\\
&J_{\tilde g_3}(\xi)U=\star T+\star Y+\star V,\quad J_{\tilde g_3}(\xi)V=0.
\end{aligned}$$

We take $\ell\ge 2$ to prove assertion (3). Let $2\le a\le\ell$. The non-zero components of $R_{2\ell+1}$ are:

eqn4.c
$$\begin{aligned}
1&=R_{\tilde g_{2\ell+1}}(X,U_2,U_2,X)=R_{\tilde g_{2\ell+1}}(X,T,U_2,X)\\
&=R_{\tilde g_{2\ell+1}}(X,U_{a+1},U_{a+1},X)=R_{\tilde g_{2\ell+1}}(X,U_{a+1},U_a,X)\\
&=R_{\tilde g_{2\ell+1}}(X,U_{a+1},V_a,X).
\end{aligned} \tag{4.c}$$

Assertion (3) follows from the same argument as that used to prove Lemma 3.2 as:

$$\begin{aligned}
&J(\xi)X\in\operatorname{Span}\{Y,T,U_2,...,U_\ell,V_2,...,V_{\ell+1}\},\\
&J(\xi)Y=0,\\
&J(\xi)T=\star Y+\star V_2,\\
&J(\xi)U_2=\star T+\star Y+\star V_2+\star V_3,\\
&J(\xi)U_a=\star U_{a-1}+\star Y+\star V_{a-1}+\star V_a+\star V_{a+1}\quad\text{for}\quad 3\le a\le\ell,\\
&J(\xi)U_{\ell+1}=\star U_\ell+\star Y+\star V_\ell+\star V_{\ell+1},\\
&J(\xi)V_a=\star Y+\star V_{a+1}\quad\text{for}\quad 2\le a\le\ell,\\
&J(\xi)V_{\ell+1}=0
\end{aligned}$$

To prove assertion (4), we take $\ell\ge 1$. Let $2\le a\le\ell+1$. The non-zero components of $R_{\tilde g_{2\ell+2}}$ are:

eqn4.d
$$\begin{aligned}
1&=R_{\tilde g_{2\ell+2}}(X,U_1,U_1,X)=R_{\tilde g_{2\ell+2}}(X,U_a,U_a,X)\\
&=R_{\tilde g_{2\ell+2}}(X,U_a,U_{a-1},X)=R_{\tilde g_{2\ell+2}}(X,U_a,V_{a-1},X).
\end{aligned} \tag{4.d}$$

We may then compute:

$$\begin{array}{lll} J(\xi)X & = & \star U_1+...+\star U_\ell+\star Y+\star V_1+...+\star V_{\ell+1}, \\ J(\xi)U_1 & = & \star Y+\star V_1+\star V_2, \\ J(\xi)U_a & = & \star U_{a-1}+\star Y+\star V_{a-1}+\star V_a+\star V_{a+1} \quad \text{for} \quad 2\le a\le \ell, \\ J(\xi)U_{\ell+1} & = & \star U_\ell+\star Y+\star V_\ell+\star V_{\ell+1}, \\ J(\xi)Y & = & 0, \\ J(\xi)V_a & = & \star Y+\star V_{a+1} \quad \text{for} \quad 1\le a\le\ell, \\ J(\xi)V_{\ell+1} & = & 0. \end{array}$$

Assertion (4) now follows from the argument used to establish Lemma 3.3. □

rmk4.2 **Remark 4.2.** Again, one can consider pointwise questions. We shall say that (M,g) is Ossersman nilpotent of order n at $P\in M$ if $J(x)^n=0$ for all $x\in T_PM$ and if $J(x_0)^{n-1}\ne 0$ for some $x_0\in T_PM$. By replacing u^2 by u^3, tu by tu^2, u_au_{a+1} by $u_au_{a+1}^2$, and v_au_{a+1} by $v_au_{a+1}^2$, we could construct metrics $\tilde g_n$ on $\mathbb{R}^{n+2}$ which are Osserman of order n on $T_P\mathbb{R}^{n+2}$ for generic points $P\in\mathbb{R}^{n+2}$, but where R vanishes at the origin $0\in\mathbb{R}^{n+2}$. This gives rise to metrics where the order of nilpotency varies with the point of the manifold; such examples, clearly, are neither symmetric nor homogeneous.

rmk4.3 **Remark 4.3.** Stanilov and Videv [18] defined a higher order analogue of the Jacobi operator in the Riemannian setting which was subsequently extended to arbitrary signature. Let $\mathrm{Gr}_{r,s}(M,g)$ be the Grassmannian bundle of all non-degenerate subspaces of TM of signature (r,s). We assume $0\le r\le p$, $0\le s\le q$, and $0<r+s<p+q$ to ensure $\mathrm{Gr}_{r,s}(M,g)$ is non-empty and does not consist of a single point; such a pair (r,s) will be said to be *admissible*. Let $\mathcal{B}=\{e_1^+,...,e_r^+,e_1^-,...,e_s^-\}$ be an orthonormal basis for $\pi\in\mathrm{Gr}_{r,s}(M,g)$. Then

$$J(\pi):=J(e_1^+)+...+J(e_r^+)-J(e_1^-)-...-J(e_s^-)$$

is independent $\mathcal{B}$ and depends only on π. Following Stanilov, one says that (M,g) is Osserman of type (r,s) if the eigenvalues of $J(\pi)$ are constant on $\mathrm{Gr}_{r,s}(M,g)$. Let J_n be defined by the metric $\tilde g_n$ defined in Lemma 4.1. The discussion given above then implies $J_n(\pi)^n=0$ for all π and thus $(\mathbb{R}^{n+2},\tilde g_n)$ is Osserman of type (r,s) for all admissible (r,s). We refer to the discussion in [3, 13] for other examples of higher order Osserman manifolds.

5. Ivanov-Petrova manifolds

lem5.1 **Lemma 5.1.** *The pseudo-Riemannian manifold* $(\mathbb{R}^{n+2},\tilde g_n)$ *defined in* Lemma 4.1 *is nilpotent Ivanov-Petrova of order* 2 *if* $n=2$ *and nilpotent Ivanova-Petrova of order* 3 *if* $n\ge 3$.

Proof. Suppose first $n=2$. We use equation (4.a) to see:

$$\mathcal{R}_{\tilde g_2}(\pi)X=\star V,\ \mathcal{R}_{\tilde g_2}(\pi)U=\star Y,\ \mathcal{R}_{\tilde g_2}(\pi)V=\mathcal{R}_{\tilde g_2}(\pi)Y=0,$$

where $\star$ are suitably chosen quadratic polynomials in the components of the generating vectors of $\pi=\mathrm{Span}\{f_1,f_2\}$ which are non-zero for generic f_i. Thus $\mathcal{R}_{\tilde g_2}(\pi)\ne 0$ for generic π while $\mathcal{R}_{\tilde g_2}(\pi)^2=0$ for all π.

We use equations (4.b), (4.c), and (4.d) to compute $\mathcal{R}_{\tilde g_n}(\pi)Y=0$ and:

$$\begin{array}{ll} \mathcal{R}_{\tilde g_3}(\pi)X\in\mathrm{Span}\{V,T\}, & \mathcal{R}_{\tilde g_3}(\pi)T\in\mathrm{Span}\{Y\}, \\ \mathcal{R}_{\tilde g_3}(\pi)V=0, & \mathcal{R}_{\tilde g_3}(\pi)U\in\mathrm{Span}\{Y\}, \\ \mathcal{R}_{g_{2\ell+1}}(\pi)X\in\mathrm{Span}\{T,U_a,V_a\}, & \mathcal{R}_{g_{2\ell+1}}(\pi)T\in\mathrm{Span}\{Y\}, \\ \mathcal{R}_{g_{2\ell+1}}(\pi)U_a\in\mathrm{Span}\{Y\}, & \mathcal{R}_{g_{2\ell+1}}(\pi)V_a\in\mathrm{Span}\{Y\}, \\ \mathcal{R}_{\tilde g_{2\ell+2}}(\pi)X\in\mathrm{Span}\{U_a,V_a\}, & \mathcal{R}_{\tilde g_{2\ell+2}}(\pi)U_a\in\mathrm{Span}\{Y\}, \\ \mathcal{R}_{\tilde g_{2\ell+2}}(\pi)V_a\in\mathrm{Span}\{Y\}. & \end{array}$$

This shows $\mathcal{R}^3_{\tilde g_n}(\pi)=0\ \forall\ \pi$ and $\mathcal{R}^2_{\tilde g_n}(\pi)\ne 0$ for generic π. □

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