Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

The Principle of the Fermionic Projector III, Normalization of the Fermionic States

by

Felix Finster

Preprint no.: 99 2002

The Principle of the Fermionic Projector III, Normalization of the Fermionic States

Felix Finster
Fakultät für Mathematik
Universität Regensburg, Germany

October 2002

Abstract

The normalization of the states of the fermionic projector is analyzed. By considering the system in finite 4-volume and taking the infinite volume limit, it is made precise what "idempotence" of the fermionic projector means. It is shown that for each fermionic state, the probability integral has a well-defined infinite volume limit.

When working out the continuum limit [4], we were concerned with the microscopic structure of the fermionic projector on the Planck scale. But we did not pay attention to a problem on the large scale: the fermionic states are in general not normalizable in infinite volume. We disregarded this problem using a formalism which involved a δ-normalization in the mass parameter. In the present paper, we will analyze the normalization of the fermionic states more carefully by considering the system in finite volume and taking the infinite volume limit. This will give a justification for the formalism used in [4]. Furthermore, we show that for the states of the fermionic projector, the probability integral

$$(\Psi \mid \Psi) \;=\; \int (\overline{\Psi} \gamma^0 \Psi)(t, \vec{x})\, d\vec{x}$$

has a well-defined infinite volume limit. This will give a simple quantitative connection between the normalization used for the states of the fermionic projector and the usual normalization condition $(\Psi|\Psi) = 1$.

1 Normalization of Massive Fermions

We postpone the complications related to the chiral fermions to Chapter 2 and thus assume here that the chiral asymmetry matrix $X = \mathbb{1}$. As in [5, Chapter 1], we consider a system of fermions of masses $m_{a\alpha}$ with the family index $a = 1, \ldots, N$ and the generation index $\alpha = 1, 2, 3$. We make the physically reasonable assumption that the *masses* are *non-degenerate in the generations*, meaning that

$$m_{a\alpha} \neq m_{a\beta} \qquad \text{for all } a \text{ and } \alpha \neq \beta. \tag{1.1}$$

We introduce the matrix Y via the relation

$$m\, Y^{(a\alpha)}_{(b\beta)} \;=\; \delta^a_b\, \delta^\alpha_\beta\, m_{a\alpha}\;, \tag{1.2}$$

where $m > 0$ is a mass parameter which we keep fixed throughout. In the formalism of causal perturbation theory [2], the masses of the fermions are varied. To this end, we introduce a variable parameter $\mu > 0$ which shifts all masses by the same amount. Thus we describe the non-interacting system by the Dirac operator

$$i\partial\!\!\!/ - mY - \mu \mathbb{1} \; . \tag{1.3}$$

As in [2], we denote the spectral projectors corresponding to the free Dirac operator $(i\partial\!\!\!/ - m)\Psi = 0$ by p_m and k_m, and the Green's function by s_m. The operator $t_m \equiv \frac{1}{2}(p_m - k_m)$ describes a Dirac sea of mass m. The spectral projectors $p_{+\mu}$ corresponding to (1.3) are obtained by taking direct sums of the operators p_m,

$$p_{+\mu} \;=\; \bigoplus_{a,\alpha} p_{m_{a\alpha}+\mu} \; .$$

Similarly, we define $k_{+\mu}$ and $s_{+\mu}$. The normalization of the spectral projectors involves a δ-distribution in the mass parameter,

$$p_{+\mu}\, p_{+\mu'} \;=\; k_{+\mu}\, k_{+\mu'} \;=\; \delta(\mu - \mu')\, p_{+\mu} \tag{1.4}$$
$$p_{+\mu}\, k_{+\mu'} \;=\; k_{+\mu}\, p_{+\mu'} \;=\; \delta(\mu - \mu')\, k_{+\mu} \; . \tag{1.5}$$

The interaction is described by inserting a perturbation operator $\mathcal{B}$ into the Dirac operator,

$$i\partial\!\!\!/ + \mathcal{B} - mY - \mu \mathbb{1} \; .$$

Exactly as in [3], we assume that $\mathcal{B}$ is a smooth *multiplication operator*, which decays so fast at infinity that

$$x^j x^k\, \mathcal{B}(x) \;\in\; L^1(\mathbb{R}^4) \; . \tag{1.6}$$

Then the causal perturbation expansion gives us a unique definition of the spectral projectors with interaction $\tilde{p}_{+\mu}$ and $\tilde{k}_{+\mu}$. The auxiliary fermionic projector $P_{+\mu}$ is defined by

$$P_{+\mu} \;=\; \frac{1}{2}\,(p_{+\mu} - k_{+\mu}) \; .$$

It is normalized according to

$$P_{+\mu}\, P_{+\mu'} \;=\; \delta(\mu - \mu')\, P_{+\mu} \; . \tag{1.7}$$

Finally, the fermionic projector is introduced by taking the partial trace,

$$(P_{+\mu})^a_b \;=\; \sum_{\alpha,\beta=1}^{3} (P_{+\mu})^{(a\alpha)}_{(b\beta)} \; . \tag{1.8}$$

1.1 Infrared Regularization in the Vacuum

Our goal is to give a rigorous justification for the δ-normalization of the fermionic projector and to describe the normalization of its individual states. In this section we consider the normalization in the vacuum, i.e. for the Dirac operator (1.3). Since this operator is diagonal on the sectors, it suffices to consider a single Dirac sea.

In order to ensure that all normalization integrals are finite, we need to introduce an infrared regularization. To this end, we first replace space by the three-dimensional box

$$T^3 \;=\; [-l_1, l_1] \times [-l_2, l_2] \times [-l_3, l_3] \qquad \text{with } 0 < l_i < \infty \tag{1.9}$$

and set $V = |T^3| = 8\, l_1 l_2 l_3$. We impose periodic boundary conditions; this means that we restrict the momenta $\vec{k}$ to the lattice L^3 given by

$$L^3 \;=\; \frac{\pi\,\mathbb{Z}}{l_1} \times \frac{\pi\,\mathbb{Z}}{l_2} \times \frac{\pi\,\mathbb{Z}}{l_3} \;\subset\; \mathbb{R}^3\;.$$

The free spectral projectors p_m, k_m, and the Green's function s_m (which appear in the perturbation expansion for the fermionic projector, see [2]) can be adapted to the periodic boundary conditions as follows. We leave the distributions in momentum space unchanged and in the transformation to position space replace the Fourier integral over 3-momentum by a Fourier series according to

$$\int \frac{d\vec{k}}{(2\pi)^3} \;\longrightarrow\; \frac{1}{V} \sum_{\vec{k} \in L^3}\;. \tag{1.10}$$

When taking products of the resulting operators, we must take into account that the spatial integral is now finite. For example, we obtain that

$$\begin{aligned}
(p_m\, p_{m'})(x,y) \;&=\; \int_{\mathbb{R}\times T^3} p_m(x,z)\, p_{m'}(z,y)\; d^4z \\
&=\; \int_{\mathbb{R}\times T^3} d^4z \left(\int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}\in L^3} p_m(k)\, e^{-ik(x-z)} \right) \left(\int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{l}\in L^3} p_{m'}(l)\, e^{-il(z-y)} \right) \\
&=\; \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}\in L^3} p_m(k)\, e^{-ikx} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{l}\in L^3} p_{m'}(l)\, e^{-ily}\; 2\pi\, \delta(k^0 - l^0)\; V\, \delta_{\vec{k},\vec{l}} \\
&=\; \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}\in L^3} p_m(k)\, p_{m'}(k)\, e^{-ik(x-y)} \;=\; \delta(m-m')\, p_m(x,y)\;,
\end{aligned} \tag{1.11}$$

where $p_m(k) = (k\!\!\!/ + m)\, \delta(k^2 - m^2)$. More generally, all calculation rules for products of the operators k_m, p_m, and s_m (as derived in [2]) remain valid in finite 3-volume.

In (1.11) we are still using a δ-normalization in the mass parameter. In order to go beyond this formalism and to get into the position where we can multiply operators whose mass parameters concide, we "average" the mass over a small interval $[m, m+\delta]$. More precisely, we set

$$\bar{p}_m \;=\; \frac{1}{\delta} \int_m^{m+\delta} p_\mu\, d\mu \qquad \text{and} \qquad \bar{k}_m \;=\; \frac{1}{\delta} \int_m^{m+\delta} k_\mu\, d\mu\;. \tag{1.12}$$

Then

$$\begin{aligned}
\bar{p}_m\, \bar{p}_m \;&=\; \frac{1}{\delta^2} \int_m^{m+\delta} d\mu \int_m^{m+\delta} d\mu'\; p_\mu\, p_{\mu'} \\
&=\; \frac{1}{\delta^2} \int_m^{m+\delta} d\mu \int_m^{m+\delta} d\mu'\; \delta(\mu - \mu')\, p_\mu \\
&=\; \frac{1}{\delta^2} \int_m^{m+\delta} p_\mu\, d\mu \;=\; \frac{1}{\delta}\, \bar{p}_m\;,
\end{aligned}$$

and thus, apart from the additional factor δ^{-1}, $\bar{p}_m$ is idempotent. Similarly, we have the relations

$$\bar{k}_m\, \bar{k}_m \;=\; \frac{1}{\delta}\, \bar{p}_m \qquad \text{and} \qquad \bar{k}_m\, \bar{p}_m \;=\; \bar{p}_m\, \bar{k}_m \;=\; \frac{1}{\delta}\, \bar{k}_m\;.$$

Thus introducing the *infrared regularized fermionic projector* corresponding to a Dirac sea of mass m by

$$P \;=\; \frac{\delta}{2}\,(\bar{p}_m - \bar{k}_m)\;, \tag{1.13}$$

this operator is indeed a projector,

$$P^* \;=\; P \qquad \text{and} \qquad P^2 \;=\; P\;. \tag{1.14}$$

The *infinite volume limit* corresponds to taking the limits $l_1, l_2, l_3 \to \infty$ and $\delta \searrow 0$.

We conclude this section by discussing the above construction. Clearly, our regularization method relies on special assumptions (3-dimensional box with periodic boundary conditions, averaging of the mass parameter). This restriction is partly a matter of convenience, but partly also a necessity, because much more general regularizations would lead to unsurmountable technical difficulties. Generally speaking, infrared regularizations change the system only on the macroscopic scale near spatial infinity and possibly for large times. In this regime, the system is well-described by the continuum limit and furthermore, due to our decay assumptions on the bosonic potentials (1.6), the system is only weakly interacting. This should make infrared regularizations insensitive to the details of the regularization procedure, and it is reasonable to expect (although it is strictly speaking not proven) that if the infinite volume limit exists, it should be independent of which regularization method is used. Here we simply take this assumption for granted and thus restrict attention to a special regularization scheme. At least, we will see that the infinite volume limit is independent of how the limits $l_i \to \infty$ and $\delta \searrow 0$ are taken.

Let us be more specific and compare our infrared regularization with other potential regularization methods. First of all, we point out that a simple regularization in a 4-dimensional box does not work. Namely, if instead of averaging the mass parameter we restrict the time integral to the finite interval $t \in [-T, T]$, we obtain

$$\begin{aligned}
(p_m\, p_m)(x,y) \;&=\; \int_{-T}^{T} dt \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}, \vec{l} \in L^3,\, \vec{k}=\vec{l}} e^{-i(k^0-l^0)t}\, p_m(k)\, p_m(l)\, e^{ikx-ily} \\
&=\; \int_{-T}^{T} dt \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}, \vec{l} \in L^3,\, \vec{k}=\vec{l}} e^{-i(k^0-l^0)t} \\
&\qquad\qquad \times (\not{k}+m)\, (\not{l}+m)\, \delta((k^0)^2-(l^0)^2)\, \delta(k^2-m^2)\, e^{ikx-ily} \\
&=\; \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k} \in L^3} \frac{mT}{|k^0|}\, p_m(k)\, e^{-ik(x-y)} \;+\; \mathcal{O}(T^0)\;,
\end{aligned}$$

and due to the factor $|k^0|^{-1}$ in the last line, this is not a multiple of $p_m(x,y)$. This problematic factor $|k^0|^{-1}$ also appears under more general circumstances (e.g. when we introduce boundary conditions at $t = \pm T$ and/or take averages of the mass parameter), and thus it seems impossible to arrange that the fermionic projector is idempotent. We conclude that the 4-dimensional box is a too simple regularization.

The mass averaging in (1.12) leads to the bizarre effect that for fixed $\vec{k}$, a whole continuum of states of the fermionic projector, namely all states with

$$k^0 \in [-\sqrt{|\vec{k}|^2 + (m+\delta)^2},\; -\sqrt{|\vec{k}|^2+m^2}]\;, \tag{1.15}$$

are occupied. If one prefers to occupy for every $\vec{k}$ only a finite number of states, one can achieve this by taking the mass averages for the bra- and ket-states separately. For example, we could define the fermionic projector instead of (1.13) by

$$P(x,y) \;=\; \delta \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k},\vec{l} \in L^3,\; \vec{k}=\vec{l}} \bar{t}_m(k)\, \bar{t}_m(l)\, e^{-ikx+ily} \tag{1.16}$$

with $\bar{t}_m = \frac{1}{2}(\bar{p}_m - \bar{k}_m)$. This fermionic projector is for every $\vec{k}$ composed of a finite number of states. Furthermore, it is a projector (1.14). In contrast to (1.13), (1.16) is not homogeneous in time, but decays on the scale $t \sim \delta^{-1}$. However, if we restrict attention to a fixed region of space-time for which $t \ll \delta^{-1}$, then (1.13) and (1.16) differ only by terms of higher order in δ, and therefore we can expect that (1.12) and (1.14) yield the same infinite volume limit. The definition (1.13) has the advantage that it is easier to introduce the interaction.

1.2 Idempotence of the Fermionic Projector with Interaction

We now turn attention to systems of interacting Dirac seas. For the infrared regularization, we replace space by the three-dimensional box T^3. More precisely, for the operators $p_{+\mu}$, $k_{+\mu}$, and $s_{+\mu}$ we introduce periodic boundary conditions according to (1.10), whereas the perturbation is defined simply by restricting $\mathcal{B}$ to $\mathbb{R} \times T^3$. Then the operators $p_{+\mu}$, $k_{+\mu}$, and $s_{+\mu}$ satisfy the canonical multiplication rules as given in [2], and thus the causal perturbation expansion allows us to introduce the corresponding operators with interaction $\tilde{p}_{+\mu}$ and $\tilde{k}_{+\mu}$. These operators satisfy similar to (1.4) and (1.5) the normalization conditions

$$\tilde{p}_{+\mu}\, \tilde{p}_{+\mu'} \;=\; \tilde{k}_{+\mu}\, \tilde{k}_{+\mu'} \;=\; \delta(\mu-\mu')\, \tilde{p}_{+\mu} \tag{1.17}$$

$$\tilde{p}_{+\mu}\, \tilde{k}_{+\mu'} \;=\; \tilde{k}_{+\mu}\, \tilde{p}_{+\mu'} \;=\; \delta(\mu-\mu')\, \tilde{k}_{+\mu}\,. \tag{1.18}$$

In analogy to (1.12) and (1.13) we define the auxiliary fermionic projector by

$$P \;=\; \frac{1}{2} \int_0^\delta (\tilde{p}_{+\mu} - \tilde{k}_{+\mu})\, d\mu\,, \tag{1.19}$$

and the fermionic projector is again obtained by taking the partial trace,

$$P^a_b \;=\; \sum_{\alpha,\beta=1}^{3} P^{(a\alpha)}_{(b\beta)}\,.$$

The next theorem shows that the fermionic projector is idempotent in the infinite volume limit, independent of how the limits $l_i \to \infty$ and $\delta \searrow 0$ are taken.

Theorem 1.1 *Consider a system of massive fermions with non-degenerate masses (1.1), which interact via a multiplication operator $\mathcal{B}$ which decays at infinity (1.6). Then the fermionic projector defined by (1.19) and (1.8) satisfies the relations*

$$\int_{\mathbb{R}\times T^3} d^4z \sum_{b=1}^{N} P^a_b(x,z)\, P^b_c(z,y) \;=\; P^a_c(x,y) \;+\; \delta^2\, Q^a_c(x,y)\,,$$

where Q has an expansion as a sum of operators which all have a well-defined infinite volume limit.

Proof. It follows immediately from (1.17), (1.18), and (1.19) that the auxiliary fermionic projector is idempotent,

$$\sum_{b,\beta} P^{(a\alpha)}_{(b\beta)} \, P^{(b\beta)}_{(c\gamma)} \;=\; P^{(a\alpha)}_{(c\gamma)} \;.$$

Thus it remains to show that

$$\sum_b \sum_{\alpha,\gamma} \sum_{\beta,\beta' \text{ with } \beta \neq \beta'} P^{(a\alpha)}_{(b\beta)} \, P^{(b\beta)}_{(c\gamma)} \;=\; \delta^2 \, Q^a_c(x,y) \;. \tag{1.20}$$

According to the non-degeneracy assumption (1.1), there are constants $c, \delta > 0$ such that for all sufficiently small δ,

$$|(m_{b\beta} + \mu) - (m_{b\beta'} + \mu')| \;\geq\; c \qquad \text{for all } b,\ \beta \neq \beta', \text{ and } 0 < \mu, \mu' < \delta \;. \tag{1.21}$$

On the left side of (1.20) we substitute in (1.19) and the operator product expansion [2]. Using (1.21), the resulting operator products are all finite and can be estimated using the relations

$$\int_0^\delta d\mu \int_0^\delta d\mu' \, (\cdots A_{+\mu})^{(a\alpha)}_{(b\beta)} \, (A_{+\mu'} \cdots)^{(b\beta')}_{(c\gamma)} \;=\; c^{-1} \, \mathcal{O}(\delta^2) \;,$$

where each of the factors A stands for p, k, or s. This gives (1.20). ∎

1.3 The Probability Integral

In finite volume and without interaction, a Dirac sea is composed of a discrete number of fermionic states. More precisely,

$$\begin{aligned} t_m(x,y) &= \int \frac{dk^0}{2\pi} \, \frac{1}{V} \sum_{\vec{k} \in L^3} (\not{k} + m) \, \delta(k^2 - m^2) \, \Theta(-k^0) \, e^{-ik(x-y)} \\ &= \frac{1}{2\pi V} \sum_{\vec{k} \in L^3} \frac{1}{2\,|k^0|} \, (\not{k} + m) \, e^{-ik(x-y)} \Big|_{k^0 = -\sqrt{|\vec{k}|^2 + m^2}} \;. \end{aligned} \tag{1.22}$$

Here the image of $(\not{k}+m)$ is two-dimensional; it is spanned by the two plane-wave solutions of the Dirac equation of momentum k with spin up and down, respectively. Thus we can write t_m as

$$t_m(x,y) \;=\; \sum_{\vec{k} \in L^3} \sum_{s = \pm 1} -|\Psi_{\vec{k}s}(x) \succ \prec \Psi_{\vec{k}s}(y)| \;, \tag{1.23}$$

where $\Psi_{\vec{k}s}$ are the suitably normalized negative-energy plane-wave solutions of the Dirac equation, and s denotes the two spin orientations. The minus sign in (1.23) takes into account that $\prec \Psi_{\vec{k}s} | \Psi_{\vec{k}s} \succ \, < 0$, whereas the matrix $(\not{k} + m)$ in (1.22) is positive. If an interaction is present, it is still possible to decompose the fermionic projector similar to (1.23) into individual states. But clearly, each of these states is perturbed by $\mathcal{B}$; we denote these perturbed states by a tilde. The next theorem shows that the probability integral for these states is independent of the interaction and of the size of T^3.

Theorem 1.2 *Under the assumptions of Theorem 1.1, every state* $\tilde{\Psi}$ *of* $\tilde{t}_{+\mu} = \frac{1}{2}(\tilde{p}_\mu - \tilde{k}_\mu)$ *is normalized according to*

$$\int_{T^3} \prec \tilde{\Psi} | \gamma^0 | \tilde{\Psi} \succ (t, \vec{x}) \, d\vec{x} \;=\; \frac{1}{2\pi} \,. \tag{1.24}$$

Proof. Since $\tilde{\Psi}$ is a solution of the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - mY - \mu \mathbb{1})\tilde{\Psi} = 0$, it follows from the current conservation that the probability integral (1.24) is time independent. Thus it suffices to compute it in the limits $t \to \pm\infty$, when according to our decay assumptions on $\mathcal{B}$ the system is not interacting. Since in the vacuum, $t_{+\mu}$ splits into a direct sum of Dirac seas, we may restrict attention to a single Dirac sea (1.23). Using that the probability integral is the same for both spin orientations,

$$\int_{T^3} \prec \Psi_{\vec{k}s} | \gamma^0 \mid \Psi_{\vec{k}s} \succ (t, \vec{x}) \, d\vec{x} \;=\; \int_{T^3} \frac{1}{2} \sum_{s=\pm} \mathrm{Tr} \left(\gamma^0 \, |\Psi_{\vec{k}s} \succ \prec \Psi_{\vec{k}s}| \right) d\vec{x} \,,$$

and comparing with (1.22) gives

$$\begin{aligned} \int_{T^3} \prec \Psi_{\vec{k}s} | \gamma^0 | \Psi_{\vec{k}s} \succ (t, \vec{x}) \, d\vec{x} \;&=\; \frac{1}{4\pi \, V} \int_{T^3} \frac{1}{2k^0} \, \mathrm{Tr}(\gamma^0 \, (k\!\!\!/ + m)) \big|_{k^0 = -\sqrt{|\vec{k}|^2 + m^2}} \, d\vec{x} \\ &=\; \frac{1}{4\pi \, V} \int_{T^3} \frac{4k^0}{2k^0} \, d\vec{x} \;=\; \frac{1}{2\pi} \,. \end{aligned}$$

∎

Let us consider what this result means for the states of the fermionic projector (1.19). As pointed out at the end of Subection 1.1, the fermionic projector of the vacuum for each $\vec{k} \in L^3$ is composed of a continuum of states (1.15). However, if we choose the space-time points in the fixed time interval $-T < t < T$ and let $\delta \searrow 0$, we need not distinguish between the frequencies in (1.15) and obtain that only the discrete states with $\vec{k} \in L^3$, $k^0 = -\sqrt{|\vec{k}|^2 + m^2}$ are occupied. In the causal perturbation expansion, each of these states is perturbed, and thus also the interacting fermionic projector for small δ can be regarded as being composed of discrete states. We write in analogy to (1.23),

$$P(x, y) \;=\; \sum_a -|\tilde{\Psi}_a \succ \prec \tilde{\Psi}_a| \,,$$

where a runs over all the quantum number of the fermions. According to (1.19) and Theorem 1.2, the probability integral is

$$\int_{T^3} \prec \tilde{\Psi}_a \mid \gamma^0 \mid \tilde{\Psi}_a \succ (t, \vec{x}) \, d\vec{x} \;=\; \frac{\delta}{2\pi} \,. \tag{1.25}$$

By substituting the formulas of the light-cone expansion [3] into (1.19), one sees that the contributions of the light-cone expansion to the fermionic projector all involve at least one factor of δ. Thus after rescaling P by a factor δ^{-1}, the probability integral (1.22) as well as the formulas of the light-cone expansion have a well-defined and non-trivial continuum limit.

We finally remark that Theorem 1.2 can be generalized in a straightforward way to include a gravitational field, if (1.24) is replaced by

$$\int_{\mathcal{H}} \prec \tilde{\Psi} \mid \gamma^j \, \nu_j \mid \tilde{\Psi} \succ \, d\mu_{\mathcal{H}} \;=\; \frac{1}{2\pi} \,, \tag{1.26}$$

where $\mathcal{H}$ is a space-like hypersurface with future-directed normal ν. However, we need to assume that the gravitational field decays at infinity. More precisely, space-time must be asymptotically flat and for $t \to \pm\infty$ must go over asymptotically to Minkowski space. In particular, realistic cosmological models like the Friedman-Robertson-Walker space-times are excluded. We do not expect that the large-scale structure of space-time should have an influence on the normalization constant in (1.26), but this is an open problem which still needs to be investigated.

2 Normalization of Chiral Fermions

We shall now develop a method to treat the fermionic projector with chiral asymmetry. The main difficulty is that for a proper normalization one needs to give the chiral fermions a small rest mass; this will be discusssed in Section 2.1 for a single Dirac sea in Minkowski space. In Section 2.2 we develop a method for analyzing the normalization of chiral fermions with a small generalized "mass," whereas Section 2.2 gives the general construction including the infrared regularization and the interaction.

2.1 Massive Chiral Fermions – Preparatory Discussion

Before introducing the infrared regularization, we need to understand how a chiral Dirac sea can be normalized in infinite volume using some kind of "δ-normalization." To this end, we consider a non-interacting left-handed Dirac sea in Minkowski space,

$$P(x,y) \;=\; \chi_L\, t_m(x,y)|_{m=0} \;. \tag{2.1}$$

Naively, products of this kernel vanish due to chiral cancellations,

$$\begin{aligned} P^2(x,y) \;&=\; \int d^4z\, P(x,z)\, P(z,y) \;=\; \int d^4z\, \chi_L\, t_0(x,z)\, \chi_L\, t_0(z,y) \\ &=\; \int d^4z\, \chi_L\, \chi_R\, t_0(x,z)\, t_0(z,y) \;\overset{\text{formally}}{=}\; 0\;. \end{aligned} \tag{2.2}$$

However, this formal calculation has no meaning in the formalism of causal perturbation theory [2], because in this formalim we are not allowed to multiply Dirac seas of the same fixed mass. Instead, we must treat the masses as variable parameters. Thus before we can give products of chiral Dirac seas a mathematical meaning, we must extend the definition of a chiral Dirac sea to non-zero rest mass.

Giving chiral Dirac particles a mass is a delicate issue which often leads to confusion and misunderstandings. Therefore, we discuss the situation in the example (2.1) in detail. In momentum space, the distribution t_m, $m \geq 0$, takes the form

$$t_m(k) \;=\; (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\;.$$

The range of the (4×4)-matrix $\not{k} + m$ is 2-dimensional; this corresponds to a twofold degeneracy of the eigenspaces of the Dirac operator $(\not{k} - m)$ for any fixed k. If $m = 0$, the Dirac equation is invariant on the left- and right-handed subspaces, and this makes it possible to project out half of the eigenvectors simply by multiplying by χ_L,

$$P(k) \;=\; \chi_L\, \not{k}\, \delta(k^2)\, \Theta(-k^0)\;. \tag{2.3}$$

If $m > 0$, this method cannot be applied because the left- and right-handed subspaces are no longer invariant. In particular, the product $\chi_L t_m$ for $m > 0$ is not Hermitian and is no solution of the Dirac equation. Nevertheless, we can project out one of the degenerate eigenvectors as follows. We choose (for given k on the lower mass cone) a vector q with

$$kq \;=\; 0 \qquad \text{and} \qquad q^2 \;=\; -1 \;. \tag{2.4}$$

A short calculation shows that

$$[t_m(k),\, \rho \not{q}] \;=\; 0 \qquad \text{and} \qquad (\rho \not{q})^2 \;=\; 1$$

(where $\rho \equiv \gamma^5$ is the pseudoscalar matrix). This means that the matrix $\rho \not{q}$ has eigenvalues ± 1, and that the Dirac equation is invariant on the corresponding eigenspaces. Projecting for example onto the eigenspace corresponding to the eigenvalue -1 gives

$$P_m(k) \;:=\; \frac{1}{2}\,(\mathbb{1} - \rho \not{q})\,(\not{k} + m)\,\delta(k^2 - m^2)\,\Theta(-k^0)\;. \tag{2.5}$$

Thus similar to the procedure in the massless case (2.3), P_m is obtained from t_m by projecting out half of the Dirac eigenstates on the lower mass shell. But in contrast to (2.3), the construction of P_m depends on the vector field q, which apart from the conditions (2.4) can be chosen arbitrarily. A short calculation shows that P_m is idempotent in the sense that

$$P_m\, P_{m'} \;=\; \delta(m - m')\, P_m \;. \tag{2.6}$$

The distribution (2.5) can be regarded as a generalization of the chiral Dirac sea (2.3) to the massive case. In order to make this connection clearer, we show that (2.5) reduces to (2.3) in the limit $m \searrow 0$: For fixed $\vec{k} \neq 0$ and variable $m > 0$, we let k be on the lower mass shell, $k(m) = (-\sqrt{|\vec{k}|^2 + m^2}, \vec{k})$, and choose $q(m)$ such that (2.4) is satisfied. A simple example for q is

$$q(m) \;=\; \frac{1}{m} \left(-|\vec{k}|,\; \sqrt{|\vec{k}|^2 + m^2}\; \frac{\vec{k}}{|\vec{k}|} \right) . \tag{2.7}$$

In this example, k and mq coincide as $m \searrow 0$; more precisely,

$$k - mq = \mathcal{O}(m^2)\;.$$

This relation holds for a large class of functions $q(m)$. Thus we concentrate on the situation where

$$k - mq \;=\; m^2\, v \qquad \text{with} \qquad v(m) = \mathcal{O}(m^0). \tag{2.8}$$

Solving this relation for q and substituting into (2.5) gives

$$P_m(k) \;=\; \frac{1}{2} \left(\mathbb{1} - \rho\, \frac{\not{k}}{m} + m \rho \not{v} \right) (\not{k} + m)\,\delta(k^2 - m^2)\,\Theta(-k^0)\;. \tag{2.9}$$

Using that on the mass shell $\not{k}(\not{k} + m) = m(\not{k} + m)$, we get

$$P_m(k) \;=\; \frac{1}{2}\,(\mathbb{1} - \rho + m \rho \not{v})\,(\not{k} + m)\,\delta(k^2 - m^2)\,\Theta(-k^0)\;. \tag{2.10}$$

If now we take the limit $m \searrow 0$, we obtain precisely (2.3), i.e.

$$\lim_{m \searrow 0} P_m \;=\; P \tag{2.11}$$

with convergence as a distribution. This calculation shows that (2.5) indeed includes (2.3) as a limiting case and that the dependence on q drops out as $m \searrow 0$.

The distribution (2.5) gives a possible definition of a massive chiral Dirac sea. However, it would be too restrictive to found our constructions only on (2.5), because there are other common ways to give chiral Dirac particles a rest mass. These alternatives are more general than (2.5) in that the wave functions are no longer solutions of the Dirac equation. To give a simple example, one could describe a massive left-handed Dirac sea for $m > 0$ by

$$P_m(k) \;=\; \left(\chi_L \not{k} + \frac{m}{4}\right) \delta\Big(k^2 - \frac{m^2}{4}\Big)\; \Theta(-k^0)\;. \tag{2.12}$$

This distribution has the advantage over (2.5) that it is Lorentz invariant, but it is clearly not a solution of the Dirac equation. As $m \searrow 0$, we again recover the massless chiral Dirac sea (2.1). We compute the operator product $P_m P_{m'}$ in momentum space,

$$\begin{aligned}
(P_m\, P_{m'})(k) \;&=\; \left(\chi_L \not{k} + \frac{m}{4}\right)\left(\chi_L \not{k} + \frac{m'}{4}\right) \delta\Big(k^2 - \frac{m^2}{4}\Big)\; \delta\Big(k^2 - \frac{m'^2}{4}\Big)\; \Theta(-k^0) \\
&=\; \delta\Big(\frac{m^2}{4} - \frac{m'^2}{4}\Big) \left(\frac{m+m'}{4}\, \chi_L \not{k} + \frac{mm'}{16}\right) \delta\Big(k^2 - \frac{m^2}{4}\Big)\, \Theta(-k^0) \\
&=\; \delta(m-m')\left(\chi_L \not{k} + \frac{m}{8}\right) \delta\Big(k^2 - \frac{m^2}{4}\Big)\, \Theta(-k^0)\;,
\end{aligned}$$

where in the last step we used that $m, m' > 0$. Note that in the last line the summand $m/8$ appears (instead of the summand $m/4$ in (2.12)), and therefore P_m is not idempotent in the sense (2.6). On the other hand, one can argue that (2.6) is a too strong normalization condition, because we are interested in the situation when the masses of the chiral particles are arbitrarily small, and thus it seems sufficient that (2.6) should hold in the limit $m, m' \searrow 0$. In this limit, the problematic summands $m/4$ and $m/8$ both drop out, and thus we can state the idempotence of P_m as follows,

$$\lim_{m,m' \searrow 0} \big(P_m\, P_{m'} - \delta(m-m')\, P_m\big) \;=\; 0\;. \tag{2.13}$$

The above example shows that, in order to have more flexibility to give the chiral Dirac particles a mass, it is preferable to work instead of (2.6) with the weaker normalization condition (2.13). Comparing with the naive calculation (2.2), one sees that introducing the mass changes the behavior of the operator products completely, even if the masses are arbitrarily small. Therefore, we refer to the limit $m, m' \searrow 0$ in (2.13) as the *singular mass limit*.

For the correct understanding of the singular mass limit, it is important to observe that, in contrast to operator products as considered in (2.13), the formalism of the continuum limit is well-behaved as $m \searrow 0$. Namely, in the continuum limit we consider an expansion in powers of m. The different orders in m have a different singular behavior on the light cone. In particular, to every order on the light cone only a finite number of orders in m contribute. Thus to every order on the light cone, the m-dependence is polynomial and therefore smooth. Expressed in terms of the kernel, the limit $m \searrow 0$ is singular when we form the product $P(x,z)P(z,y)$ and integrate over z (as in (2.2)). But if we take the closed chain $P(x,y)\,P(y,x)$ and consider the singularities on the light cone, the limit $m \searrow 0$ is regular and well-behaved. This justifies why in [5], it was unnecessary to give the neutrinos a mass and take the limit $m \searrow 0$ afterwards. We could treat the neutrino sector simply as being composed of massless chiral particles. In particular, the chiral cancellations in the formalism of the continuum limit are consistent with the singular mass limit.

Our next goal is to develop the mathematical framework for analyzing the singular mass limit for a fermionic projector with interaction. Clearly, this framework should be general enough to include the examples (2.5) and (2.12). Thus we first return to (2.5). After writing P_m in the form (2.10), it seems natural to interpret the leading factor as a generalization of the chiral asymmetry matrix X. This is indeed convenient in the vacuum, because introducing the operator X_m by

$$X_m(k) \;=\; \frac{1}{2}\,(\mathbb{1} - \rho - m\rho \slashed{v}(k))\,, \tag{2.14}$$

we obtain in analogy to the corresponding formulas for massless chiral particles that

$$P_m \;=\; X_m\, t_m \;=\; t_m\, X_m^* \;.$$

Unfortunately, the operator X_m does not seem to be useful in the case with interaction. The reason is that X_m depends on the momentum k, and this leads to the following serious difficulties. First, the k-dependence of X_m makes it very difficult to satisfy the analogue of the causality compatibility condition

$$X_m^*\,(i\slashed{\partial} + \mathcal{B} - m) \;=\; (i\slashed{\partial} + \mathcal{B} - m)\, X_m \;.$$

As a consequence, it is in general not possible to commute the chiral asymmetry matrix through the operator products of the causal perturbation expansion; in particular $X_m\,\tilde{t}_m$ and $\tilde{t}_m\,X_m$ do in general not coincide (where $\tilde{t}_m$ is the interacting Dirac sea as defined via the causal perturbation expansion). Even if we assume that there is a canonical definition of the fermionic projector P_m obtained by suitably inserting factors of X_m and X_m^* into the operator product expansion for $\tilde{t}_m$, we cannot expect that the correspondence to the massless Dirac sea is respected, i.e. (2.11) will in the case with interaction in general be violated. In order to explain how this comes about, we point out that our argument leading to (2.8) was based on the assumption that k converges to the mass cone as $m \searrow 0$. More precisely, if $\lim_{m \searrow 0} k(m)$ is not on the mass cone, the function v will diverge like $v(m) \sim m^{-2}$, so that $X_m(k)$ will not converge to X as $m \searrow 0$. Thus $\lim_{m \searrow 0} X_m = X$ only if in this limit all the momenta are on the mass cone. But in the causal perturbation expansion also off-shell momenta appear (note that the Green's functions are non-zero away from the mass cone). This means that in the limit $m \searrow 0$, the momenta are in general not on the lower mass cone, and so X_m will not converge to X. From these problems we conclude that it is not admissible to first perform the perturbation expansion for t_m and to multiply by X_m afterwards, but the k-dependence of X_m must be taken into account in the perturbation expansion.

At this point it is very helpful that we stated the normalization condition for a chiral Dirac sea in the form (2.13). The key observation is that, if we substitute (2.10) into (2.13), compute the operator product and take the limit $m, m' \searrow 0$, all contributions to (2.10) which are at least quadratic in m drop out. More precisely, if we expand P_m in the form

$$P_m \;=\; \Big(\chi_L\, \slashed{k} \;+\; \frac{m}{2}\,(\mathbb{1} - \rho) \;+\; \frac{m}{2}\,\rho \slashed{v}\slashed{k} + \mathcal{O}(m^2)\Big)\; \delta(k^2 - m^2)\,\Theta(-k^0)\;, \tag{2.15}$$

the error term is of no relevance for the normalization condition (2.13). Taking the inner product of (2.8) with k and using the first part of (2.4) as well as that $k^2 = m^2$, one sees that $vk = 1$. We use this identity in (2.15) to obtain

$$P_m \;=\; \left(\chi_L\, \not{k} \;+\; \frac{m}{2} \;+\; \frac{m}{4}\, \rho\, [\not{v}, \not{k}]\right)\, \delta(k^2 - m^2)\, \Theta(-k^0)\;. \tag{2.16}$$

Writing P_m in this form has the advantage that we can pull out the chiral projectors by setting

$$P_m \;=\; \frac{1}{2}\, (X\, \tilde{t}_m + \tilde{t}_m\, X^*) \tag{2.17}$$

with $X = \chi_L$ and

$$\tilde{t}_m \;=\; \left(\not{k} + m + \frac{m}{2}\, \rho\, [\not{v}, \not{k}]\right)\, \delta(k^2 - m^2)\, \Theta(-k^0)\;.$$

Again neglecting terms quadratic in m, $\tilde{t}_m$ is a solution of the Dirac equation,

$$(i\not{\partial} + \mathcal{B}_0 - m)\, \tilde{t}_m \;=\; 0\;, \tag{2.18}$$

where

$$\mathcal{B}_0(k) \;=\; -\frac{m}{2}\, \rho\, [\not{v}, \not{k}]\;. \tag{2.19}$$

The formulation of the vacuum (2.17) and (2.18),(2.19) has the advantage that the interaction can easily be introduced. Namely, in order to describe the interaction we simply insert the bosonic potentials into the Dirac equation (2.18). In this way, the problems mentioned after (2.14) have been resolved. Namely, instead of working with a k-dependent chiral asymmetry matrix X_m, the k-dependent vector field v in (2.10) is now taken into account by a perturbation $\mathcal{B}_0$ of the Dirac equation, making it possible to apply perturbative methods in the spirit of [2].

An apparent technical problem of this approach is that the perturbation operator $\mathcal{B}_0$, (2.19), is not of a form previously considered in that it is not causality compatible, is nonlocal, and does not decay at infinity. This problem will be analyzed in detail in Section 2.2. What makes the problem tractible is that $\mathcal{B}_0$ tends to zero as $m \searrow 0$ and is *homogeneous*, meaning that its kernel $\mathcal{B}_0(x, y)$ depends only on the difference $x - y$.

Let us verify in which generality the above method (2.17),(2.18) applies. In the example (2.12), we can write the chiral Dirac sea in the form (2.17) with

$$\tilde{t}_m \;=\; \left(\not{k} + \frac{m}{2}\right)\, \delta\!\left(k^2 - \frac{m^2}{4}\right)\, \Theta(-k^0)\;, \tag{2.20}$$

and $\tilde{t}_m$ is a solution of the Dirac equation (2.18) with $\mathcal{B}_0 = m/2$. Thus in this case, $\mathcal{B}_0$ is a homogeneous local operator. More generally, the method of pulling out the chiral asymmetry (2.17) applies to any distribution P_m of the form

$$P_m(k) \;=\; \big(\chi_L\, (\text{odd}) + (\text{even}) + \mathcal{O}(m^2)\big)\; \delta(k^2 - c\, m^2)\, \Theta(-k^0)\;,$$

where "(odd)" and "(even)" refer to a product of an odd and even number of Dirac matrices, respectively (and c is a constant). Namely, the corresponding $\tilde{t}_m$ is

$$\tilde{t}_m(k) \;=\; \big((\text{odd}) + 2\, (\text{even}) + \mathcal{O}(m^2)\big)\; \delta(k^2 - c\, m^2)\, \Theta(-k^0)\;.$$

Hence the only restriction of the method (2.17),(2.18) is that the *right-handed odd contribution to* P_m should be *of the order* $\mathcal{O}(m^2)$. For example, our method does not apply to

$$P_m(k) \;=\; \big(\chi_L \,\slashed{k} + m\,\chi_R\, f\, \slashed{k} + m + \mathcal{O}(m^2)\big)\;\delta(k^2-m^2)\,\Theta(-k^0)$$

with a scalar function $f(k)$, although in this case the normalization condition (2.13) is satisfied. Dropping this restriction would make it necessary to give up (2.17) and thus to treat the trace compatibility on a level which goes far beyond what we can accomplish here. It is our view that assuming that the right-handed odd contribution to P_m is of the order $\mathcal{O}(m^2)$ is a reasonable technical simplification.

We close our discussion with a comment on the example (2.12). We saw above that P_m can be written in the form (2.17) with $\tilde{t}_m$ according to (2.20), and that $\tilde{t}_m$ is a solution of the Dirac equation (2.18) with the perturbation $\mathcal{B}_0 = m/2$. An alternative point of view is that $\tilde{t}_m$ is a solution of the free Dirac equation of half the mass,

$$(i\slashed{\partial} - M)\,\tilde{t}_m \;=\; 0 \qquad \text{with} \qquad M = \frac{m}{2}\,. \tag{2.21}$$

We refer to the method of considering a Dirac equation in which the mass parameter is multiplied by a constant as the *modified mass scaling*. The modified mass scaling has the advantage that one can satisfy the normalization conditions for chiral Dirac seas (2.13) with P_m according to (2.17) and $\tilde{t}_m$ a solution of the free Dirac equation.

2.2 The Homogeneous Perturbation Expansion

In the above examples, we saw that there are different methods for giving a chiral Dirac sea a rest mass, which all correspond to inserting a suitable homogeneous operator $\mathcal{B}_0$ into the Dirac equation. Furthermore, we found that the terms quadratic in the mass were irrelevant for the normalization of the Dirac sea, and this suggests that it should be possible to treat $\mathcal{B}_0$ perturbatively. This is indeed possible, as we shall now show for general $\mathcal{B}_0$.

For simplicity, we again consider a single Dirac sea. We let $\mathcal{B}_0$ be a *homogeneous operator*, whose further properties will be specified below. In order to keep track of the different orders in perturbation theory, we multiply $\mathcal{B}_0$ by a small parameter $\varepsilon > 0$. Similar to (1.3), we also introduce a parameter μ into the Dirac equation, which then reads

$$(i\slashed{\partial} + \varepsilon\,\mathcal{B}_0 - \mu\,\mathbb{1})\,\Psi \;=\; 0\,.$$

Here the Dirac operator is homogeneous and is therefore diagonal in momentum space. Thus for given momentum k, it reduces to the 4×4 matrix equation

$$(\slashed{k} + \varepsilon\,\mathcal{B}_0(k) - \mu)\,\Psi(k) \;=\; 0\,. \tag{2.22}$$

Our aim is to introduce and analyze the spectral projectors and Green's functions of the Dirac operator $i\slashed{\partial}+\varepsilon\mathcal{B}_0$, where we regard μ as the eigenvalue. In preparation, we shall now analyze the matrix equation (2.22) for fixed k in a perturbation expansion to first order in ε. If $k^2 \neq 0$, the matrix $\slashed{k}$ is diagonalizable with eigenvalues and spectral projectors

$$\mu_\pm \;=\; \pm\mu_k\,, \qquad E_\pm \;=\; \frac{1}{2}\left(\mathbb{1} \pm \frac{\slashed{k}}{\mu_k}\right)\,, \tag{2.23}$$

where we set $\mu_k = \sqrt{k^2}$ (if $k^2 < 0$, our sign convention is such that μ_k lies in the upper complex half plane). The eigenspaces $\text{Im}\,E_\pm$ are two-dimensional. The spectral projectors

$E_\pm$ become singular as $k^2 \to 0$. The reason is that on the mass cone $\mathcal{C} = \{k \,|\, k^2 = 0\}$, the matrix $\not{k}$ is not diagonalizable. We will address this problem later and for the moment simply assume that $k^2 \neq 0$. We next consider the Dirac operator $\not{k} + \varepsilon \mathcal{B}_0$ for small ε. Perturbing the eigenspaces $\text{Im}\, E_\pm$ gives rise to two-dimensional invariant subspaces, and a standard calculation shows that the projectors $E^\varepsilon_\pm$ onto these subspaces are given by

$$E^\varepsilon_s \;=\; E_s \;+\; s\, \frac{\varepsilon}{2\mu_k}\, (E_s\, \mathcal{B}_0\, E_{\bar{s}} + E_{\bar{s}}\, \mathcal{B}_0\, E_s) \;+\; \mathcal{O}(\varepsilon^2) \tag{2.24}$$

with $s = \pm$ and $\bar{s} = -s = \mp$. It remains to diagonalize the operator $\not{k} + \varepsilon \mathcal{B}_0$ on the invariant subspaces $\text{Im}\, E^\varepsilon_s$. This is carried out in the next lemma. We choose three (possibly complex) Lorentz vectors $(q_i)_{i=1,2,3}$ such that

$$<q_i, k> = 0 \qquad \text{and} \qquad <q_i, q_j> = -\delta_{ij}\;. \tag{2.25}$$

More precisely, if k is time-like, we choose the (q_i) as a real orthonormal basis of the space-like hypersurface $<k>^\perp$. If on the other hand q is space-like, we choose q_1 and q_3 real and space-like, whereas q_3 is time-like and imaginary. We use the vector notation $\vec{q} = (q_1, q_2, q_3)$ and introduce the matrices $\Sigma_{1,2,3}$ by

$$\vec{\Sigma} \;=\; \rho\, \vec{\not{q}}\;. \tag{2.26}$$

Lemma 2.1 *Suppose that $k^2 \neq 0$ and that for small ε, the matrix $\not{k} + \varepsilon \mathcal{B}_0$ is diagonalizable. Then its eigenvalues $(\mu^a_s)_{s=\pm, a=1/2}$ are given by*

$$\mu^{1/2}_+ \;=\; \mu_k \;+\; \varepsilon\, (\nu_+ \pm \tau_+) \;+\; \mathcal{O}(\varepsilon^2) \tag{2.27}$$

$$\mu^{1/2}_- \;=\; -\mu_k \;+\; \varepsilon\, (\nu_- \mp \tau_-) \;+\; \mathcal{O}(\varepsilon^2)\;, \tag{2.28}$$

where

$$\nu_s \;=\; \frac{1}{2}\, Tr\, (E_s\, \mathcal{B}_0) \tag{2.29}$$

$$\vec{\tau}_s \;=\; \frac{1}{2}\, Tr(\vec{\Sigma}\, E_s\, \mathcal{B}_0) \tag{2.30}$$

$$\tau_s \;=\; \sqrt{(\tau^1_s)^2 + (\tau^2_s)^2 + (\tau^3_s)^2}\;. \tag{2.31}$$

The corresponding spectral projectors can be written as

$$E^a_s \;=\; \Pi^a\, E_s \;+\; s\, \frac{\varepsilon}{2\mu_k}\, (\Pi^a\, E_s\, \mathcal{B}_0\, E_{\bar{s}} + E_{\bar{s}}\, \mathcal{B}_0\, \Pi^a\, E_s) \;+\; \mathcal{O}(\varepsilon^2) \tag{2.32}$$

with

$$\Pi^{1/2} \;=\; \frac{1}{2} \left(\mathbb{1} \pm \frac{1}{\tau_s}\, \vec{\tau}_s \vec{\Sigma} \right)\;. \tag{2.33}$$

If $\vec{\tau}_s = 0$, the invariant subspace $Im\, E^\varepsilon_s$ is to first order in ε an eigenspace, i.e.

$$(\not{k} + \varepsilon \mathcal{B}_0)|_{Im\, E^\varepsilon_s} \;=\; (s\, \mu_k + \varepsilon\, \nu_s)\, \mathbb{1}|_{Im\, E^\varepsilon_s} \;+\; \mathcal{O}(\varepsilon^2)\;.$$

Proof. We restrict attention to the invariant subspace $\text{Im}\, E^\varepsilon_+$; for E^ε_- the proof is analogous. A short calculation using (2.26) and (2.23),(2.25) shows that

$$[\Sigma_i,\, E_+] \;=\; 0\;, \qquad \Sigma_i^2 \;=\; \mathbb{1}\;, \qquad \text{Tr}(\Sigma_i\, \Sigma_j\, E_+) \;=\; 2\, \delta_{ij}\;.$$

This means that the matrices Σ_i are invariant on $\text{Im}\, E_+$, have on this subspace the eigenvalues ± 1 and are orthogonal. Thus by choosing a suitable basis (and possibly after changing the orientation of $\vec{\Sigma}$ by exchanging Σ_1 with Σ_2), we can arrange that the matrices $\vec{\Sigma}|_{\text{Im}\, E_+}$ coincide with the Pauli matrices $\vec{\sigma}$. To first order in ε, the eigenvalues are obtained by diagonalizing $\mathcal{B}_0$ on the unperturbed invariant subspace Im E_+. According to a well-known formula, the 2×2 matrix $\nu \mathbb{1} + \vec{\tau}\vec{\sigma}$ has the eigenvalues $\nu \pm \tau$ and corresponding spectral projectors $\Pi_{1/2} = \frac{1}{2}(\mathbb{1} + \frac{1}{\tau}\, \vec{\tau}\vec{\sigma})$ with $\tau = \sqrt{(\tau_1)^2 + (\tau_2)^2 + (\tau_3)^2}$. This explains (2.27) and (2.28). Finally, (2.32) follows from standard perturbation theory without degeneracies. ∎

To avoid confusion, we point out that in general $\tau_s \neq |\vec{\tau}_s|$ because (2.31) involves ordinary squares instead of absolute squares. In particular, it is possible that $\tau_s = 0$ although $\vec{\tau}_s \neq 0$. However, in this case the 2×2 matrix $\varepsilon \mathcal{B}_0|_{\text{Im}\, E_s}$ is not diagonalizable, and thus the above lemma does not apply.

Remark 2.2 In the proof of the previous lemma, we used that the three matrices $\Sigma_i|_{\text{Im}\, E_+}$ can be represented as the Pauli matrices σ_i. It is instructive to verify explicitly that these matrices satisfy the correct commutation relations, e.g.

$$\frac{i}{2}\, [\Sigma_1,\, \Sigma_2]|_{\text{Im}\, E_+} \;=\; \Sigma_3|_{\text{Im}\, E_+}\;.$$

We now give this calculation in detail. By a choice of coordinates, we can arrange that $k = (\omega, \vec{p})$ and $q_{1/2} = (0, \vec{q}_{1/2})$. The standard identity between the Dirac matrices $i\sigma_{jk} = \frac{\rho}{2}\, \epsilon_{jklm}\, \sigma^{lm}$ yields that (possibly after changing the orientation of $\vec{\Sigma}$),

$$i \not{q}_1 \not{q}_2 \;=\; \frac{\rho}{2|\vec{p}|}\, [\not{k}, \gamma^0]\;. \tag{2.34}$$

From the definition of $\vec{\Sigma}$, (2.26), one sees that $[\Sigma_1, \Sigma_2] = -2\not{q}_1 \not{q}_2$, and using (2.34) as well as the identity $[\mu_k, \gamma^0] = 0$, we conclude that

$$\frac{i}{2}\, [\Sigma_1,\, \Sigma_2] \;=\; -\frac{\rho}{2|\vec{p}|}\, [\not{k} - \mu_k,\, \gamma^0]\;. \tag{2.35}$$

In order to simplify the rhs of (2.35) on Im E_+, we use that E_+ satisfies the Dirac equation

$$(\not{k} - \mu_k)\, E_+ \;=\; 0\;. \tag{2.36}$$

Namely, this identity allows us to replace the commutator with $\not{k} - \mu_k$ by an anti-commutator,

$$[\not{k} - \mu_k,\, \gamma^0]\, E_+ \;=\; \{\not{k} - \mu_k,\, \gamma^0\}\, E_+ \;=\; (2\omega - 2\mu_k \gamma^0)\, E_+\;. \tag{2.37}$$

Multiplying (2.36) by $2\omega/\mu_k$ and adding (2.37) gives

$$[\not{k} - \mu_k,\, \gamma^0]\, E_+ \;=\; \frac{2\omega}{\mu_k} \left(\not{k} - \frac{\mu^2}{\omega}\, \gamma^0 \right) E_+\;.$$

Using this identitiy in (2.35) gives

$$\frac{i}{2}\, [\Sigma_1,\, \Sigma_2]|_{\text{Im}\, E_+} \;=\; \rho \not{q}_3|_{\text{Im}\, E_+}$$

with

$$q_3 \;=\; -\frac{\omega}{\mu_k\, |\vec{p}|} \left(k\!\!\!/ - \frac{\mu_k^2}{\omega}\, \gamma^0 \right) ,$$

and a short calculation shows that this vector q_3 has indeed all the properties listed after (2.25).

We shall now define the spectral projectors and Green's functions corresponding to the Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$. We denote the spectrum of the matrix in (2.22) by $\sigma^\varepsilon(k)$,

$$\sigma^\varepsilon(k) \;=\; \sigma(k\!\!\!/ + \varepsilon \mathcal{B}_0(k)) \;.$$

It is natural to define the spectrum σ^ε of the Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$ as the union of the $\sigma^\varepsilon(k)$s,

$$\sigma^\varepsilon \;=\; \bigcup_{k \in \mathbb{R}^4} \sigma^\varepsilon(k) \;.$$

As we saw above, the matrix $k\!\!\!/ + \varepsilon \mathcal{B}_0(k)$ in general is not diagonalizable, and thus we cannot introduce the spectral projectors for all k pointwise. But since the diagonalizable matrices are dense in $\mathrm{Gl}(\mathbb{C}^4)$, it is reasonable to assume that the matrix $k\!\!\!/ + \varepsilon \mathcal{B}_0(k)$ is *diagonalizable for almost all (a.a.) k*. Our formalism will involve momentum integrals where sets of measure zero are irrelevant. Therefore we may in what follows restrict attention to those k for which the matrix $k\!\!\!/ + \varepsilon \mathcal{B}_0(k)$ is diagonalizable. Moreover, we shall assume that $\mathcal{B}_0$ is *smooth* and *bounded*. According to (2.23), the spectrum of the unperturbed Dirac operator is $\sigma^{\varepsilon=0} = \mathbb{R} \cup i\mathbb{R}$. The next lemma shows that the real part of the spectrum is stable under perturbations.

Lemma 2.3 *Suppose that $k^2 > 0$. Then for ε sufficiently small, $\sigma^\varepsilon(k) \subset \mathbb{R}$.*

Proof. Choosing coordinates such that $k = (\omega, \vec{0})$, it is obvious that the eigenspaces of $k\!\!\!/$ are definite, i.e.

$$\prec\!\Psi \,|\, \Psi\!\succ \;\neq 0 \qquad \text{for all eigenvectors } \Psi.$$

By continuity, the eigenspaces of $k\!\!\!/ + \varepsilon \mathcal{B}_0(k)$ will also be definite for sufficiently small ε. As a consequence, the corresponding eigenvalues are real, because

$$\lambda \prec\!\Psi \,|\, \Psi\!\succ \;=\; \prec\!\Psi \,|\, (k\!\!\!/ + \varepsilon \mathcal{B}_0)\, \Psi\!\succ \;=\; \prec\!(k\!\!\!/ + \varepsilon \mathcal{B}_0)\, \Psi \,|\, \Psi\!\succ \;=\; \overline{\lambda} \prec\!\Psi \,|\, \Psi\!\succ \;.$$

∎

However, a-priori we have no control of how the imaginary part of the spectrum changes with ε. For this reason, it is most convenient to introduce the spectral projectors for all $\mu \in \mathbb{C}$, such that they vanish identically for $\mu \notin \sigma^\varepsilon$. For the normalization, we work with δ-distributions supported at one point in the complex plane. More precisely, we set

$$\begin{aligned} \delta^2(z) \;&=\; \delta(\mathrm{Re}\, z)\, \delta(\mathrm{Im}\, z) \\ \int_{\mathbf{C}} d^2z \,\cdots \;&=\; \int_{\mathbb{R}^2} d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z)\, \cdots \;. \end{aligned}$$

Def. 2.4 *For $\mu \in \mathbb{C}$ and $k \in \mathbb{R}^4$ we set*

$$p^\varepsilon_\mu(k) \;=\; \sum_{s=\pm,\, a=1/2} E^a_s(k)\, \delta^2(\mu - \mu^a_s(k)) \qquad (2.38)$$

$$k^\varepsilon_\mu(k) \;=\; \epsilon(k^0)\, p^\varepsilon_\mu(k) \qquad (2.39)$$

$$s^\varepsilon_\mu(k) \;=\; \int_{\mathbf{C}} d^2\nu\, \frac{PP}{\mu-\nu}\, p^\varepsilon_\nu(k)\;. \qquad (2.40)$$

We also consider p^ε_μ, k^ε_μ, and s^ε_μ as multiplication operators in momentum space.

In formal calculations, the operators p^ε_μ and k^ε_μ are solutions of the Dirac equation,

$$(i\partial\!\!\!/ + \varepsilon\mathcal{B}_0 - \mu)\, p^\varepsilon_\mu \;=\; 0 \;=\; (i\partial\!\!\!/ + \varepsilon\mathcal{B}_0 - \mu)\, k^\varepsilon_\mu\;,$$

and satisfy in analogy to (1.4) and (1.5) the multiplication rules

$$p^\varepsilon_\mu\, p^\varepsilon_{\mu'} \;=\; k^\varepsilon_\mu\, k^\varepsilon_{\mu'} \;=\; \delta^2(\mu-\mu')\, p^\varepsilon_\mu \qquad (2.41)$$

$$p^\varepsilon_\mu\, k^\varepsilon_{\mu'} \;=\; k^\varepsilon_\mu\, p^\varepsilon_{\mu'} \;=\; \delta^2(\mu-\mu')\, k^\varepsilon_\mu \qquad (2.42)$$

as well as the "completeness relation"

$$\int_{\mathbf{C}} p^\varepsilon_\mu\, d^2\mu \;=\; \mathbb{1}\;.$$

Using these identities in (2.40) yields that

$$(i\partial\!\!\!/ + \varepsilon\mathcal{B}_0 - \mu)\, s^\varepsilon_\mu \;=\; \mathbb{1}\;.$$

Thus on a formal level, the operators p^ε_μ, k^ε_μ, and s^ε_μ are the spectral projectors and Green's functions of the Dirac operator, respectively. In order to give these operators a mathematical meaning, we can proceed as follows. Let k be such that the matrix $k\!\!\!/ + \varepsilon\mathcal{B}_0(k)$ can be diagonalized. Then the functional calculus for finite matrices (as defined e.g. via the approximation by polynomials) allows us to introduce for $f \in C^1(\mathbb{C})$ the matrix $f(k\!\!\!/ + \varepsilon\mathcal{B}_0(k))$. Formally, we can write the functional calculus with the spectral projectors,

$$\int_{\mathbf{C}} f(\mu)\, p^\varepsilon_\mu(k)\, d^2\mu \;=\; f(k\!\!\!/ + \varepsilon\mathcal{B}_0(k))\;. \qquad (2.43)$$

We can use this relation to give the integral in (2.43) a rigorous sense for a.a. k. The same argument applies to k^ε_μ. For s^ε_μ, we can similarly use the formal identity

$$\int_{\mathbf{C}} f(\mu)\, s^\varepsilon_\mu(k)\, d^2\mu \;\overset{(2.39)}{=}\; \int_{\mathbf{C}} g(\mu)\, p^\varepsilon_\nu(k)\, d^2\mu \qquad (2.44)$$

with

$$g(\nu) \;=\; \int_{\mathbf{C}} \frac{\mathrm{PP}}{\mu-\nu}\, f(\mu)\, d^2\mu\;.$$

In this way, one sees that the operators p^ε_μ, k^ε_μ, and s^ε_μ are well-defined when evaluated weakly in μ and k.

Under additional assumptions, we can make sense of the operators in Def. 2.4 even for fixed real μ. We first justify the δ-distribution and the principal part.

Lemma 2.5 *Suppose that for a given interval $I \subset \mathbb{R}$, the spectral projectors E^a_s in (2.38) are bounded uniformly in $\mu \in I$. Then for a.a. $\mu \in I$, the operators p^ε_μ, k^ε_μ, and s^ε_μ are well-defined distributions in momentum space.*

Proof. We write the Dirac equation $(\not{k} + \varepsilon \mathcal{B}_0(k))\Psi = 0$ in the Hamiltonian form

$$\omega \, \Psi \;=\; H(\omega, \mu) \, \Psi \qquad \text{with} \qquad H(\omega,\mu) \;=\; -\gamma^0 \, (\vec{\not{k}} + \varepsilon \mathcal{B}_0(\omega, \vec{k}) - \mu \mathbb{1})$$

and $k = (\omega, \vec{k})$. In what follows, we keep $\vec{k}$ fixed and consider this equation for variable parameters $\omega, \mu \in \mathbb{R}$. The matrix $H(\omega, \mu)$ is Hermitian w.r.to the positive scalar product $(.|.) = \prec .|\gamma^0|. \succ$. Thus it can be diagonalized; we denote its eigenvalues (counting multiplicities) by $\Omega_1 \leq \cdots \leq \Omega_4$. The min-max principle (see [6]) allows us to write Ω_n as

$$\Omega_n \;=\; \min_{U,\, \dim U = n} \;\; \max_{u \in U,\, \|u\|=1} \|Hu\| \;,$$

where $\|.\|$ is the norm induced by $(.|.)$ and U denotes a subspace of $\mathbb{C}^4$. It follows from this representation that the Ω_n depend Lipschitz-continuously on ω and μ. Namely,

$$\begin{aligned}
\Omega_n(\omega) \;&=\; \min_{U,\, \dim U = n} \;\; \max_{u \in U,\, \|u\|=1} \|H(\omega)\, u\| \\
&=\; \min_{U,\, \dim U = n} \;\; \max_{u \in U,\, \|u\|=1} \|H(\omega')\, u + (H(\omega) - H(\omega'))\, u\| \\
&\leq\; \min_{U,\, \dim U = n} \;\; \max_{u \in U,\, \|u\|=1} \big(\|H(\omega')\, u\| + \|H(\omega) - H(\omega')\| \, \|u\| \big) \\
&=\; \Omega_n(\omega') + \|H(\omega) - H(\omega')\| \;.
\end{aligned}$$

Using that $\mathcal{B}_0(k)$ is C^1 with bounded derivatives, we obtain the estimate

$$\|H(\omega) - H(\omega')\| \;\leq\; \|\varepsilon \gamma^0 \, (\mathcal{B}_0(\omega) - \mathcal{B}_0(\omega'))\| \;\leq\; \varepsilon c \, |\omega - \omega'|$$

and thus $\Omega_n(\omega) - \Omega_n(\omega') \leq \varepsilon c |\omega - \omega'|$. Exchanging the roles of ω and ω' gives the bound

$$|\Omega_n(\omega) - \Omega_n(\omega')| \;\leq\; \varepsilon c \, |\omega - \omega'| \;. \tag{2.45}$$

A similar calculation shows that

$$|\Omega_n(\mu) - \Omega_n(\mu')| \;\leq\; |\mu - \mu'| \;. \tag{2.46}$$

We next consider for given n the equation

$$\omega \;=\; \Omega_n(\omega, \mu) \;. \tag{2.47}$$

The following argument shows that for sufficiently small ε, this equation has a unique solution ω_n, which depends Lipschitz-continuously on μ. Let ϕ (for fixed μ and n) be the mapping

$$\phi \;:\; \mathbb{R} \to \mathbb{R} \;:\; \omega \mapsto \Omega_n(\omega, \mu) \;.$$

According to (2.45),

$$|\phi(\omega) - \phi(\omega')| \;=\; |\Omega_n(\omega) - \Omega_n(\omega')| \;\leq\; \varepsilon c \, |\omega - \omega'| \;.$$

Thus if we choose ε small enough, ϕ is a contraction. The Banach fixed point theorem yields a unique fixed point ω_n. The dependence on the parameter μ is controlled by (2.45) and (2.46). Namely,

$$\begin{aligned} |\omega_n(\mu) - \omega_n(\mu')| &= |\Omega_n(\omega_n(\mu), \mu) - \Omega_n(\omega_n(\mu'), \mu')| \\ &\leq \varepsilon c\, |\omega_n(\mu) - \omega_n(\mu')| \,+\, |\mu - \mu'| \end{aligned}$$

and thus

$$|\omega_n(\mu) - \omega_n(\mu')| \;\leq\; (1 - \varepsilon c)^{-1}\, |\mu - \mu'|\,. \tag{2.48}$$

If we regard the spectral projector (2.38) as a distribution in ω, it is supported at those ω for which the Dirac equation $(\not{k} + \varepsilon \mathcal{B}_0 - \mu)\Psi = 0$ has a non-trivial solution. These are precisely the solutions ω_n of the equation (2.47). Thus we can write p^ε_μ as

$$p^\varepsilon_\mu \;=\; \sum_{n=1}^4 E^a_s(\omega_n)\; \delta(\omega - \omega_n)\; \delta(\mathrm{Im}\, \mu) \left| \frac{\partial \omega(\mu)}{\partial \mu} \right| \,, \tag{2.49}$$

where the parameters $a = a(n)$ and $s = s(n)$ must be chosen such that $\mu^a_s(\omega_n) = \mu$. Since $\omega_n(\mu)$ is Lipschitz (2.48), the factor $|\partial_\mu \omega_n(\mu)|$ in (2.49) is well-defined for a.a. μ and is uniformly bounded. Thus $p^\varepsilon_\mu(\omega)$ is a well-defined distribution for a.a. μ. The same argument applies to k^ε_μ.

It remains to justify the Green's function s^ε_μ. We can write it in the Hamiltonian framework as

$$s^\varepsilon_\mu \;=\; \frac{\mathrm{PP}}{\not{k} + \varepsilon \mathcal{B}_0 - \mu\, \mathbb{1}} \;=\; \frac{\mathrm{PP}}{\omega - H(\omega, \mu)}\, \gamma^0\,.$$

Thus denoting the spectral projectors of H by $(F_n)_{n=1,\ldots,4}$, we have

$$s^\varepsilon_\mu(\omega) \;=\; \sum_{n=1}^4 \frac{\mathrm{PP}}{\omega - \Omega_n(\omega, \mu)}\, F_n(\omega, \mu)\, \gamma^0\,. \tag{2.50}$$

According to (2.45), $\Omega_n(\omega)$ is Lipschitz and thus differentiable almost everywhere with $|\partial_\omega \Omega_n| \leq \varepsilon c$. The spectral projectors $F_n(\omega)$ can also be chosen to be Lipschitz. As a consequence, the principal part in (2.50) is well-defined for a.a. μ. ■

This lemma involves the strong assumption that the spectral projectors E^a_s must be uniformly bounded. We shall now analyze this assumption in detail. As one sees from (2.23) in the limit $\mu \to 0$, the spectral projectors can have poles and thus in general are not uniformly bounded. Thus we need to impose an extra condition, which we will state using the following notion.

Def. 2.6 *Let A be a 4×4 matrix, which is Hermitian w.r.to $\prec .|. \succ$. A point $\mu \in \sigma(A)$ is called* **ϵ-definite** *if there is a subset $\sigma_+ \subset \sigma(A)$ such that*

(i) *The invariant subspace I_+ corresponding to σ_+ is definite.*

(ii) *$\mathrm{dist}(\sigma_+,\, \sigma(A) \setminus \sigma_+) > \varepsilon$.*

Lemma 2.7 *If $\mu \in \sigma(A)$ is ε-definite, the matrix A is diagonalizable on I_+, and its spectral projectors E_a are bounded by*

$$\|E_a\| \;\leq\; c \left(\frac{\|A\|}{\varepsilon} \right)^3 ,$$

where $\|.\|$ is a matrix norm and c is a constant which depends only on the choice of $\|.\|$.

Proof. It clearly suffices to consider a particular matrix norm. We introduce the positive scalar product $(.|.) = \prec .|\gamma^0|.\succ$, let $\|.\| = (.|.)^{\frac{1}{2}}$ be the corresponding norm and set

$$\|A\| \;=\; \sup_{\Psi \text{ with } \|\Psi\|=1} \|A\Psi\| \;.$$

We denote the projector onto I_+ by E. E can be constructed with a functional calculus: Let $\mathcal{P}(z)$ be a complex polynomial satisfying the conditions

$$\mathcal{P}|_{\sigma_+} \;=\; 1 \qquad \text{and} \qquad \mathcal{P}|_{\sigma_-} \;=\; 0 \;.$$

Since these are at most four conditions, $\mathcal{P}$ can be chosen of degree three,

$$\mathcal{P}(z) \;=\; \sum_{n=0}^{3} c_n\, z^n \;.$$

Furthermore, the fact that A is ε-definite can be used to bound the coefficients c_n by bounded by

$$|c_n| \;\leq\; \frac{C}{\varepsilon^n} \tag{2.51}$$

with a suitable algebraic constant C (this is easily seen from a scaling argument). The projector E is given by $E = \mathcal{P}(A)$, and (2.51) gives the estimate

$$\|E\| \;\leq\; \sum_{n=0}^{3} \frac{C}{\varepsilon^n}\, \|A\|^n \;\leq\; C \left(\frac{\|A\|}{\varepsilon} \right)^3 , \tag{2.52}$$

where we used in the last step that $\varepsilon < \|A\|$.

By definition, $\text{Im}\, E = I_+$ is a definite subspace. We can assume without loss of generality that it is positive, i.e.

$$\prec \Psi \,|\, E\, \Psi \succ \;\geq\; 0 \qquad \text{for all } \Psi.$$

The matrix $A|_{I_+}$ is Hermitian w.r.to the positive scalar product $\prec .|.\succ|_{I_+}$. Thus it has a spectral decomposition with eigenvalues μ_a and corresponding spectral projectors E_a, $a = 1, \ldots, N$,

$$A|_{I_+} \;=\; \sum_{a=1}^{n} \mu_a\, E_a|_{I_+} \;.$$

Extending the E_a by zero to the invariant subspace corresponding to $\sigma(A) \setminus \sigma_+$, the spectral projectors satisfy the relations

$$E_a^* = E_a = E_a^2 \;, \qquad \sum_{a=1}^{N} E_a = E \;, \qquad \prec \Psi \,|\, E_a\, \Psi \succ \;\geq 0 \;\text{ for all } \Psi,$$

where "*" denotes the adjoint w.r.to $\prec.|.\succ$.

We introduce the operators F and F_a by

$$F \;=\; \gamma^0\, E\;, \qquad\qquad F_a \;=\; \gamma^0\, E_a\;.$$

It is straightforward to check that these operators have the following properties,

$$F_a^+ \;=\; F_a\;, \qquad\qquad (\Psi \,|\, F_a\, \Psi) \;\geq\; 0 \tag{2.53}$$

$$\sum_a F_a \;=\; F\;, \tag{2.54}$$

where "+" denotes the adjoint w.r.to $(.|.)$. The relations (2.53) mean that the F_a are positive self-adjoint operators on a Hilbert space. This makes it possible to estimate the norm of the spectral projectors as follows,

$$\|E_a\| \;=\; \|\gamma^0\, F_a\| \;\leq\; \|\gamma^0\|\, \|F_a\| \;\leq\; \|F_a\| \;=\; \sup_{\Psi \text{ with } \|\Psi\|=1} (\Psi \,|\, F_a\, \Psi)$$

$$\leq\; \sup_{\Psi \text{ with } \|\Psi\|=1} \sum_{b=1}^N (\Psi \,|\, F_b\, \Psi) \;=\; \sup_{\Psi \text{ with } \|\Psi\|=1} (\Psi \,|\, F\, \Psi) \;=\; \|F\| \;=\; \|\gamma^0\, E \;\leq\; \|E\|\;.$$

Now apply (2.52). ■

Def. 2.8 *The Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$ has an ε-***definite kernel** *if for all $\mu \in (-\varepsilon, \varepsilon)$ and all k with $\mu \in \sigma^\varepsilon(k)$, μ is in the ε-definite spectrum of the matrix $k\!\!\!/ + \varepsilon \mathcal{B}_0(k)$.*

Combining Lemma 2.5 and Lemma 2.7 gives the following result.

Theorem 2.9 *If the Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$ has an ε-definite kernel, then its spectral projectors and Green's functions as given in Def. 2.4 are for a.a. $\mu \in (-\varepsilon, \varepsilon)$ well-defined distributions in momentum space.*

It remains to specify under which assumptions on $\mathcal{B}_0$ the Dirac operator has an ε-definite kernel. We decompose $\mathcal{B}_0$ as

$$\mathcal{B}_0(k) \;=\; \alpha\, \mathbb{1} + i\beta\, \rho + v\!\!\!/ + \rho\, a\!\!\!/ + \frac{i\rho}{2}\, w_{ij}\, \sigma^{ij}\;. \tag{2.55}$$

Here α, β, v, a, and w are real potentials (Namely the scalar, pseudoscalar, vector, axial, and bilinear potentials, respectively. Clearly, we assume w to be anti-symmetric). We introduce the function $\Delta(k)$ as the following combination of the axial and bilinear potentials,

$$\Delta^2 \;=\; -k^2 <a,a> + <a,k>^2 - w_{ij} k^j\, w^{il} k_l\;. \tag{2.56}$$

The first two summands can also be written as

$$-k^2 <a,a> + <a,k>^2 \;=\; -k^2 \left(a - \frac{1}{k^2} <a,k>\, k \right)^2\;. \tag{2.57}$$

For timelike k, the vector inside the round brackets is spacelike, and thus (2.57) ≥ 0. Similarly, the vector $w_{ij} k^j$ is spacelike for k timelike. We conclude that

$$\Delta(k) \;\geq\; 0 \qquad\qquad \text{if } k^2 > 0. \tag{2.58}$$

Furthermore, $\Delta(q)$ vanishes on the mass cone $\mathcal{C} = \{q^2 = 0\}$ if and only if q is collinear to the vector a and is an eigenvector of w,

$$a \;=\; \nu q \qquad \text{and} \qquad w_{ij} q^j \;=\; \lambda\, q_i \qquad (\nu, \lambda \in \mathbb{R},\ q \in \mathcal{C}). \tag{2.59}$$

Expanding (2.56), one sees that in this case, Δ is finite to the next order on the light cone, i.e.

$$\Delta(q) = 0 \quad \Longrightarrow \quad l \equiv \lim_{k \to q} \frac{1}{k^2}\, \Delta(k) \text{ exists.} \tag{2.60}$$

Qualitatively speaking, the next theorem states that the Dirac operator has an ε-definite kernel if and only if the scalar potential is non-zero and dominates the axial and bilinear potentials.

Theorem 2.10 *Suppose that for all $q \in \mathcal{C}$,*

$$|\alpha(q)| \;>\; \frac{3}{2} + \left\{ \begin{array}{cl} \left| \dfrac{w_{ij}(q) a^i q^j}{\Delta(q)} \right| & \text{if } \Delta(q) \neq 0 \\ \left(1 + \Theta(1 - 2\sqrt{|l(q)|}\right)\, \sqrt{|l(q)|} & \text{if } \Delta(q) = 0. \end{array} \right. \tag{2.61}$$

Then for sufficiently small ε, the Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$ has an ε-definite kernel. If conversely there is $q \in \mathcal{C}$ for which the opposite inequality holds (i.e. (2.61) with ">" replaced by "<"), then the Dirac operator has no ε-definite kernel.

Proof. A short calculation using (2.29), (2.23), and (2.55) gives

$$\nu_\pm \;=\; \alpha \pm \frac{1}{\mu_k} <v, k> . \tag{2.62}$$

In the special case $k\!\!\!/ = \mu_k\, \gamma^0$ and $q\!\!\!/ = \vec{\gamma}$, we obtain furthermore from (2.30) that

$$(\tau_\pm)_r \;=\; a_r \pm w_{r0} \qquad (r = 1, 2, 3).$$

Thus, according to (2.31),

$$(\tau_\pm)^2 \;=\; \sum_{r=1}^{3} (a_r)^2 \pm 2\, a_r\, w_{r0} + (w_{r0})^2 ,$$

and this can be written covariantly as

$$(\tau_\pm)^2 \;=\; -<a, a> + \frac{1}{\mu_k^2} <a, k>^2 - \frac{1}{\mu_k^2}\, w_{ij} k^i\, w^{il} k_l \mp \frac{2}{\mu_k}\, w_{ij} a^i k^j . \tag{2.63}$$

This tensor equation is valid for any time-like k, and it is easy to check that it holds for spacelike k as well.

Let $q \in \mathcal{C}$. We first consider the case $\Delta(q) \neq 0$. By continuity, $\Delta \neq 0$ in a neighborhood U of q, and according to (2.58), Δ is positive in U. We substitute (2.62) and (2.63) into (2.27) and (2.28). In order to remove the singularities at $\mu_k = 0$, we write the eigenvalues μ_s^a in the form

$$\left. \begin{array}{rcl} \mu_+^{1/2} &=& \sqrt{k^2 + 2\varepsilon \delta_{1/2}} + \varepsilon\, (\alpha \pm \kappa_+) + \mathcal{O}(\varepsilon^2) \\ \mu_-^{1/2} &=& -\sqrt{k^2 + 2\varepsilon \delta_{1/2}} + \varepsilon\, (\alpha \mp \kappa_-) + \mathcal{O}(\varepsilon^2) \end{array} \right\} , \tag{2.64}$$

where we set

$$\delta_{1\!/\!2} \;=\; <v,k> \pm \Delta \;, \qquad\qquad \kappa_\pm \;=\; \tau_\pm - \frac{1}{\mu_k}\,\Delta \;.$$

The functions $\kappa_\pm$ have the following expansion,

$$\kappa_\pm \;=\; \frac{1}{\mu_k} \left(\sqrt{\Delta^2 \mp 2\mu_k\, w_{ij} a^i k^j} - \Delta \right) \;=\; \mp \frac{w_{ij} a^i k^j}{\Delta} \;+\; \mathcal{O}(\mu_k) \;. \tag{2.65}$$

In particular, one sees that these functions are bounded locally uniformly in μ_k. Let us verify under which conditions the Dirac operator restricted to U has an ε-definite kernel. Suppose that $\mu^1_+ \in (-\varepsilon, \varepsilon)$. Then, due to the square root in (2.64),

$$k^2 + 2\varepsilon\delta_1 \;=\; \mathcal{O}(\varepsilon^2) \;.$$

It follows from (2.64) that

$$\begin{aligned} \mu^2_\pm \;&=\; \sqrt{k^2 + 2\varepsilon\delta_2} + \mathcal{O}(\varepsilon) \;=\; \sqrt{\mathcal{O}(\varepsilon^2) + 2\varepsilon(\delta_2 - \delta_1)} + \mathcal{O}(\varepsilon) \\ &=\; \sqrt{-4\varepsilon\,\Delta} + \mathcal{O}(\varepsilon) \;\sim\; \sqrt{\varepsilon} \end{aligned}$$

and therefore

$$|\mu^1_+ - \mu^2_+| \;\sim\; \sqrt{\varepsilon} \;\gg\; \varepsilon \;.$$

Moreover, we obtain from (2.64) and (2.65) that

$$\begin{aligned} \mu^1_+ - \mu^1_- \;&=\; 2\mu^1_+ + 2\varepsilon\alpha - \varepsilon(\kappa_+ - \kappa_-) + \mathcal{O}(\varepsilon^2) \\ &=\; 2\mu^1_+ + 2\varepsilon \left(\alpha + \frac{w_{ij}\, a^i k^j}{\Delta} + \mathcal{O}(\mu_k) \right) + \mathcal{O}(\varepsilon^2) \;. \end{aligned}$$

Thus the condition $|\mu^1_+ - \mu^1_-| > \varepsilon$ is satisfied if

$$\left| \alpha + \frac{w_{ij} a^i k^j}{\Delta} \right| \;>\; \frac{3}{2} \;.$$

As is proven in Lemma 2.11 below, the eigenspace corresponding to μ^1_+ is definite. We conclude that μ^1_+ is an ε-definite eigenvalue of A. Repeating the above argument in the three other cases $\mu^2_-, \mu^2_\pm \in (-\varepsilon, \varepsilon)$, one obtains that for sufficiently small ε, the kernel of the Dirac operator is ε-definite in U. If conversely (2.61) holds with "$>$" replaced by "$<$", it is straightforward to check that the Dirac operator for small ε has no ε-definite kernel.

It remains to consider the case $\Delta(q) = 0$. We write the eigenvalues μ^a_s as

$$\left.\begin{aligned} \mu^{1\!/\!2}_+ \;&=\; \sqrt{k^2 + 2\varepsilon <v,k>} + \varepsilon\,(\alpha \pm \tau_+) \\ \mu^{1\!/\!2}_- \;&=\; -\sqrt{k^2 + 2\varepsilon <v,k>} + \varepsilon\,(\alpha \mp \tau_-) \;. \end{aligned}\right\} \tag{2.66}$$

According to (2.60), the first three summands in (2.63) have a finite limit at q. Furthermore, (2.59) yields that

$$\frac{2}{\mu_k}\, w_{ij} a^i k^j \;=\; \mathcal{O}(\mu_k) \;.$$

We conclude that the functions $\tau_\pm$ in a neighborhood of q have the expansion

$$\tau_\pm \;=\; \sqrt{|l|} + \mathcal{O}(\sqrt{|\mu_k|}) \;. \tag{2.67}$$

For small ε, $\mu_k \sim \sqrt{\varepsilon}$, and so the term $\mathcal{O}(\sqrt{\mu_k})$ is of higher order in ε and can be omitted. Furthermore, the following continuity argument varying l shows that the eigenvalues μ^a_+ and μ^a_- correspond to positive and negative eigenvectors, respectively: If $l = 0$, only the scalar and vector potentials enter the perturbation calculation to first order in ε (see (2.66) and (2.67)). If only scalar and vector potentials are present, the spectral decomposition of the matrix $\not{k} + \varepsilon \mathcal{B}_0$ is easily obtained from the identity

$$[(\not{k} + \varepsilon \not{v} + \varepsilon \alpha) - \varepsilon \alpha]^2 \;=\; (k + \varepsilon v)^2 \, \mathbb{1} \; .$$

One sees that the eigenvalues are twofold degenerate, $\sigma^\varepsilon(k) = \{\mu_+, \mu_-\}$, and that if they are real, the corresponding eigenspaces are definite. The parameter l removes the degeneracy of these eigenspaces, but the resulting invariant subspaces remain definite.

Suppose that $\mu^1_+ \in (-\varepsilon, \varepsilon)$. We consider the two subcases $2\sqrt{|l|} > \varepsilon$ and $2\sqrt{|l|} < \varepsilon$ separately. In the first case, $|\mu^1_s - \mu^2_s| > \varepsilon$, and thus we must arrange that

$$|\mu^1_+ - \mu^2_\mp| > \varepsilon \qquad (2\sqrt{l} > \varepsilon). \tag{2.68}$$

In the second case, $|\mu^1_s - \mu^2_s| < \varepsilon$. Thus we must combine the eigenvalues to pairs and consider the definite eigenspaces corresponding to the sets $\sigma_s = \{\mu^1_s, \mu^2_s\}$, $s = \pm$, and must satisfy the condition

$$\text{dist}\,(\sigma_+, \sigma_-) \;>\; \varepsilon \qquad (2\sqrt{l} < \varepsilon). \tag{2.69}$$

Evaluating (2.68) and (2.69) using (2.66),(2.67) and analyzing similarly the three other cases $\mu^1_-, \mu^2_\pm \in (-\varepsilon, \varepsilon)$ gives the condition (2.61). ■

Lemma 2.11 *Let A be a Hermitian matrix (w.r.to $\prec .|. \succ$). If $\mu \in \sigma(A)$ is real and the corresponding invariant eigenspace I is one-dimensional, then I is a definite eigenspace.*

Proof. Since each invariant subspace contains at least one eigenvector, I is clearly an eigenspace. We must show that I is definite. Assume to the contrary that $I = <\Psi>$ is null, i.e.

$$A\Psi = \lambda \Psi \qquad \text{with} \qquad \lambda \in \mathbb{R} \text{ and } \prec \Psi | \Psi \succ = 0.$$

We denote the invariant subspaces of A by $(I_\mu)_{\mu \in \sigma(A)}$. Since $I_\lambda = <\Psi>$ is one-dimensional and null, there must be an invariant subspace I_μ, $\mu \neq \lambda$, which is not orthogonal to Ψ,

$$I_\mu \cap <\Psi>^\perp \;\neq\; \emptyset \; .$$

We choose on I_μ a basis $(e_1, \ldots, e_n)$ such that A is in the Jordan form, i.e.

$$A|_{I_\mu} \;=\; \begin{pmatrix} \mu & 1 & \cdots & 0 \\ 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \mu \end{pmatrix} .$$

Let $k \in \{1, \ldots, n\}$ be the smallest index such that $\prec e_k | \Psi \succ \neq 0$. Then

$$\begin{aligned} \lambda \prec e_k | \Psi \succ \;&=\; \prec e_k \,|\, A\Psi \succ \;=\; \prec A e_k \,|\, \Psi \succ \\ &=\; \mu \prec e_k \,|\, \Psi \succ \;+\; \prec e_{k-1} \,|\, \Psi \succ \;=\; \mu \prec e_k \,|\, \Psi \succ \; . \end{aligned}$$

This is a contradiction. ■

Suppose that the homogeneous operator $\mathcal{B}_0$ satisfies the condition (2.61) in Theorem 2.10. Then the Dirac operator has an ε-definite kernel. As a consequence, the distributions $t^\varepsilon_\mu = \frac{1}{2}(p^\varepsilon_\mu - k^\varepsilon_\mu)$ are well-defined (see Def. 2.4 and Theorem 2.9). Following (2.17), we introduce the fermionic projector by

$$P^\varepsilon_\mu \;=\; \frac{1}{2} \left(X\, t^\varepsilon_\mu + t^\varepsilon_\mu\, X^* \right) \tag{2.70}$$

with $X = \chi_L$. In order to analyze the normalization of P^ε_μ, we consider the product

$$P^\varepsilon_\mu\, P^\varepsilon_{\mu'} \;=\; \frac{1}{4} \left(X\, t^\varepsilon_\mu\, t^\varepsilon_{\mu'}\, X^* + X\, t^\varepsilon_\mu\, X\, t^\varepsilon_{\mu'} + t^\varepsilon_\mu\, X^*\, t^\varepsilon_{\mu'}\, X^* \right) . \tag{2.71}$$

According to (2.41) and (2.42),

$$t^\varepsilon_\mu\, t^\varepsilon_{\mu'} \;=\; \delta^2(\mu - \mu')\, t^\varepsilon_\mu \;. \tag{2.72}$$

Thus the only problem is to compute the products $t^\varepsilon_\mu X t^\varepsilon_{\mu'}$ and $t^\varepsilon_\mu X^* t^\varepsilon_{\mu'}$. Using the relations $\chi_{L\,/\,R} = \frac{1}{2}(\mathbb{1} \mp \rho)$ together with (2.72), this problem reduces to making mathematical sense of the operator product

$$t^\varepsilon_\mu\, \rho\, t^\varepsilon_{\mu'} \;.$$

It seems impossible to give this expression a meaning without making additional assumptions on $\mathcal{B}_0$. For simplicity, we shall impose a quite strong condition, which is motivated as follows. The spectral projectors p_μ corresponding to the unperturbed Dirac operator $i\partial\!\!\!/ - \mu$ satisfy the relations $\rho\, p_\mu\, \rho = p_{-\mu}$ and thus $p_\mu\, \rho\, p_\mu = 0$ $(\mu > 0)$. It is natural to demand that the last identity should also hold in the presence of the homogeneous perturbation for small ε.

Def. 2.12 *The kernel of the homogeneous Dirac operator $i\partial\!\!\!/ + \mathcal{B}(\varepsilon, k)$ is* ε**-orthogonal to** *ρ if for all $\mu, \mu' \in \sigma^\varepsilon(k) \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, the corresponding spectral projectors $E_\mu(k)$ and $E_{\mu'}(k)$ satisfy the condition*

$$E_\mu\, \rho\, E_{\mu'} \;=\; 0 \;. \tag{2.73}$$

If the kernel of the Dirac operator is ε-definite and ε-orthogonal to ρ, it follows immediately that for all $\mu, \mu' \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$,

$$t^\varepsilon_\mu\, \rho\, t^\varepsilon_{\mu'} \;=\; 0 \;. \tag{2.74}$$

Using (2.72) and (2.74) in (2.71), one sees that

$$P^\varepsilon_\mu\, P^\varepsilon_{\mu'} \;=\; \delta^2(\mu - \mu')\, \frac{1}{8} \left(X\, P^\varepsilon_\mu + P^\varepsilon_\mu\, X^* + 2\, X\, P^\varepsilon_\mu\, X^* \right) \;.$$

Now we can take the limits $\varepsilon, \mu \searrow 0$ to obtain

$$\lim_{\varepsilon \searrow 0}\; \lim_{\mu, \mu' \searrow 0} \left(P^\varepsilon_\mu\, P^\varepsilon_{\mu'} - \frac{1}{2}\, \delta^2(\mu - \mu')\, P^\varepsilon_\mu \right) \;=\; 0 \;. \tag{2.75}$$

In analogy to (2.13), this relation states that the fermionic projector is idempotent (apart from the factor $\frac{1}{2}$ which will be treated in Section 2.3 using the modified mass scaling).

In the remainder of this section, we analyze under which assumptions on $\mathcal{B}_0$ the kernel of the Dirac operator is ε-orthogonal to ρ. We begin with a simple calculation in first order perturbation theory.

Lemma 2.13 *Suppose that the Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$ has an ε-definite kernel and that the homogeneous potentials in (2.55) satisfy for all $k \in \mathbb{R}^4$ the relations*

$$\beta(k) \;=\; 0 \qquad \textit{and} \qquad \epsilon_{ijlm}\, w^{ij}(k)\, k^l \;=\; 0\,. \tag{2.76}$$

Then for all k and $\mu, \mu' \in \sigma^\varepsilon(k) \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$,

$$E_\mu(k)\, \rho\, E_\mu(k) \;=\; \mathcal{O}(\varepsilon^2)\,. \tag{2.77}$$

Proof. Choose k and $\mu, \mu' \in \sigma^\varepsilon(k) \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Since the Dirac operator has an ε-definite kernel, the invariant subspace I corresponding to the set $\{\mu, \mu'\} \subset \sigma^\varepsilon(k)$ is definite (notice that $\mu \in (-\varepsilon, \varepsilon)$ and $|\mu - \mu'| < \varepsilon$). We saw in the proof of Theorem 2.10 that the invariant subspaces Im E^ε_+ and Im E^ε_+ (with $E^\varepsilon_\pm$ according to (2.24)) are definite. Thus $I \subset$ Im E^ε_+ or $I \subset$ Im E^ε_-. Therefore, it suffices to show that for all $s = \pm$,

$$E^\varepsilon_s\, \rho\, E^\varepsilon_s \;=\; \mathcal{O}(\varepsilon^2)\,. \tag{2.78}$$

Substituting (2.24) and using the relations $\rho E_\pm \rho = E_\mp$, we obtain the equivalent condition

$$E_s\, \{\mathcal{B}_0, \rho\}\, E_s \;=\; 0\,. \tag{2.79}$$

This equation means that the matrix $\{\mathcal{B}_0, \rho\}$ must vanish on the two-dimensional subspace Im E_s. Since on this subspace, the matrices $\vec{\Sigma}$, (2.26), have a representation as the Pauli matrices, we can restate (2.79) as the four conditions

$$\mathrm{Tr}\,(E_s\, \{\mathcal{B}_0, \rho\}) \;=\; 0 \;=\; \mathrm{Tr}\left(\vec{\Sigma}\, E_s\, \{\mathcal{B}_0, \rho\}\right)\,.$$

Evaluating these relations using (2.23), (2.26), and (2.55) gives (2.76). ■

This lemma is not satisfactory because it gives no information on how the error term in (2.77) depends on k. More specifically, the error term may have poles on the mass cone (and explicit calculations show that such poles $\sim k^{-2n}$ indeed occur for $n = 1$ and $n = 2$). Since in the limit $\varepsilon \searrow 0$ the kernel of the Dirac operator is the mass cone, it is far from obvious how to control the error term in this limit. In other words, (2.77) cannot be interpreted as "the kernel of the Dirac operator is ε-orthogonal to ρ up to a small error term."

In order to resolve this difficulty, we must proceed non-perturbatively. In generalization of our previous ansatz $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$, we shall consider the Dirac operator $i\partial\!\!\!/ + \mathcal{B}^\varepsilon$, where we assume that $\mathcal{B}^\varepsilon(k)$ is a homogeneous potential which is smooth in both arguments and has the power expansion

$$\mathcal{B}^\varepsilon(k) \;=\; \varepsilon\, \mathcal{B}_0(k) + \varepsilon^2\, \mathcal{B}_1(k) + \varepsilon^3\, \mathcal{B}_2(k) + \cdots\,. \tag{2.80}$$

The higher order potentials $\mathcal{B}_1, \mathcal{B}_2, \ldots$ are irrelevant for Def. 2.8 because they are negligible for small ε. In particular, the statement of Theorem 2.10 remains valid without changes. Furthermore, the potentials $\mathcal{B}_1, \mathcal{B}_2, \ldots$ should be irrelevant for the statement of idempotence (2.75) because (2.75) involves a limit $\varepsilon \searrow 0$. Therefore, it seems unnecessary to enter a detailed study of these potentials. The only point of interest is under which assumptions on $\mathcal{B}_0$ there exist smooth potentials $\mathcal{B}_1$, $\mathcal{B}_2$,... such that the spectral projectors corresponding to the Dirac operator $i\partial\!\!\!/ + \mathcal{B}^\varepsilon$ satisfy the conditions (2.73) exactly.

Theorem 2.14 *Suppose that the Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$ has an ε-definite kernel and that the homogeneous potentials in (2.55) satisfy for all k the relations (2.76). Then there is $\varepsilon > 0$ and a smooth potential $\mathcal{B}^\varepsilon(k)$ having the expansion (2.80) such that the kernel of the Dirac operator $i\partial\!\!\!/ + \mathcal{B}^\varepsilon$ is ε-orthogonal to ρ.*

Proof. Choose k and $\mu, \mu' \in \sigma^\varepsilon(k) \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Similar as described before (2.78), we know from the proof of Theorem 2.10 that the matrix $A \equiv k\!\!\!/ + \mathcal{B}^\varepsilon(k)$ has a positive and a negative definite invariant subspace, one of which contains $\mathrm{Im}\, E_\mu \cup \mathrm{Im}\, E_{\mu'}$. Again denoting the projectors onto these subspaces by E^ε_+ and E^ε_-, respectively, it thus suffices to show that for $s = \pm$,

$$E^\varepsilon_s \,\rho\, E^\varepsilon_s \;=\; 0\;. \tag{2.81}$$

We first evaluate these conditions in a special spinor basis. Namely, we let e_1 and e_2 be an orthonormal basis of $\mathrm{Im}\, E^\varepsilon_+$ and set $e_3 = \rho e_1$, $e_4 = \rho e_2$. The conditions (2.81) imply that e_3 and e_4 span $\mathrm{Im}\, E^\varepsilon_-$. Using the relation $\rho^2 = \mathbb{1}$ as well as that the subspaces $<\!\{e_1, e_2\}\!>$ and $<\!\{e_3, e_4\}\!>$ are invariant under A, we conclude that the matrices ρ and A are of the form

$$\rho \;=\; \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad A \;=\; \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \tag{2.82}$$

where we used a block matrix notation corresponding to the splitting $\mathbb{C}^4 = <\!\{e_1, e_2\}\!> \oplus <\!\{e_3, e_4\}\!>$, and "$*$" denotes an arbitrary block matrix entry. Furthermore, the relation $\rho^* = -\mathbb{1}$ yields that

$$\prec\! e_3 \,|\, e_3 \!\succ \;=\; -1 \;=\; \prec\! e_4 \,|\, e_4 \!\succ\;,$$

and thus the basis (e_α) is pseudo-orthonormal,

$$\prec\! \Psi \,|\, \Phi \!\succ \;=\; \sum_{\alpha=1}^4 s_\alpha\, \overline{\Psi^\alpha}\, \Phi^\alpha \qquad \text{with} \qquad s_1 = s_2 = 1,\; s_3 = s_4 = -1. \tag{2.83}$$

We see that the matrix ρ and the spin scalar product are in the usual Dirac representation. In this representation, the fact that A is block diagonal (2.82) can be expressed by saying that A must be a real linear combination of the 8 matrices

$$\mathbb{1},\quad \gamma^0,\quad \rho\vec{\gamma},\quad \rho\gamma^0\vec{\gamma}\;. \tag{2.84}$$

We next express this result in a general basis, but again in the Dirac representation. Since the representations of the matrix ρ, (2.82), and of the scalar product, (2.83), are fixed, the freedom in choosing the basis is described by even $U(2,2)$ transformations. This group, which we denote by $U(2,2)^{\text{even}}$, contains the normal Abelian subgroup $U = \{\exp(\vartheta\rho/2) : \vartheta \in \mathbb{R}\}$. Acting by U on (2.84) gives the matrices

$$\mathbb{1},\quad ((\cosh\vartheta + \rho\,\sinh\vartheta)\,\gamma^0,\quad ((\cosh\vartheta + \rho\,\sinh\vartheta)\,\rho\vec{\gamma},\quad \rho\gamma^0\vec{\gamma}\;. \tag{2.85}$$

When the factor group $U(2,2)^{\text{even}}/U$ acts on (2.85), the resulting transformations correspond precisely to Lorentz transformations of the tensor indices (for details see [1], where this correspondence is worked out for $U(2,2)$-transformations in position space). Thus the conditions (2.81) are satisfied if and only if A is of the form

$$A \;=\; \alpha\,\mathbb{1} + ((\cosh\vartheta + \rho\,\sinh\vartheta)\,v\!\!\!/ + ((\rho\,\cosh\vartheta + \sinh\vartheta)\,a\!\!\!/ + \rho v\!\!\!/ b\!\!\!/ \tag{2.86}$$

with a time-like vector field u and two vector fields a and b, which are orthogonal to u,

$$<u, a> \;=\; 0 \;=\; <u, b> \;. \qquad (2.87)$$

We substitute the identity $A = \not{k} + \mathcal{B}^\varepsilon(k)$ into (2.86) and solve for $\mathcal{B}^\varepsilon(k)$. Expanding in powers of ε gives the result. ■

2.3 The General Construction, Proof of Idempotence

In this section we shall make precise what "idempotence" means for a fermionic projector with chiral asymmetry in the presence of a general interaction. We proceed in several steps. We begin with a straightforward extension of the results of Section 2.2 to systems of Dirac seas. Then we introduce the interaction and perform the causal perturbation expansion. After putting in an infrared regularization, we can define the fermionic projector. Finally, idempotence is established as a singular mass limit.

We begin with a system of Dirac seas in the vacuum, described by the mass matrix Y, (1.2), and the chiral asymmetry matrix X (see [5, Chapter 1] for details). In order to give the chiral fermions a "small generalized mass," we introduce a homogeneous operator $\mathcal{B}_0$ and consider for $\varepsilon > 0$ the Dirac operator $i\not{\partial} + \varepsilon \mathcal{B}_0 - mY$. For simplicity, we assume that $\mathcal{B}_0$ is diagonal on the sectors and is non-trivial only in the chiral blocks, i.e.

$$(\mathcal{B}_0)^{(a\alpha)}_{(b\beta)} \;=\; \delta^a_b\, \delta^\alpha_\beta\, \mathcal{B}_0^{(a\alpha)} \qquad \text{with} \qquad \mathcal{B}_0^{(a\alpha)} = 0 \text{ if } X_a = \mathbb{1}.$$

Then on each sector the methods of Section 2.2 apply; let us collect the assumptions on $\mathcal{B}_0$ and the main results: For every index $(a\alpha)$ with $X^{(a\alpha)} \neq \mathbb{1}$ we assume that

(1) $\mathcal{B}_0^{(a\alpha)}(k)$ depends smoothly on $k \in \mathbb{R}^4$ and grows at most polynomially at infinity.

(2) The (4×4)-matrix $\not{k} + \varepsilon \mathcal{B}_0^{(a\alpha)}(k)$ is diagonalizable for a.a. k.

(3) $\mathcal{B}_0^{(a\alpha)}$ has the decomposition into scalar, vector, axial, and bilinear potentials,

$$\mathcal{B}_0^{(a\alpha)}(k) \;=\; \alpha\, \mathbb{1} + \not{v} + \rho\, \not{a} + \frac{i\rho}{2}\, w_{ij}\, \sigma^{ij} \;,$$

such that for all $k \in \mathbb{R}^4$ and $q \in \mathcal{C}$ the following conditions are satisfied,

$$\begin{aligned}
\epsilon_{ijlm}\, w^{ij}(k)\, k^l \;&=\; 0 \\
|\alpha(q)| \;&>\; \frac{3}{2} + \begin{cases} \left| \dfrac{w_{ij}(q) a^i q^j}{\Delta(q)} \right| & \text{if } \Delta(q) \neq 0 \\[2mm] \left(1 + \Theta(1 - 2\sqrt{|l(q)|}\right) \sqrt{|l(q)|} & \text{if } \Delta(q) = 0. \end{cases}
\end{aligned}$$

(with Δ and l defined by (2.56) and (2.60), $\mathcal{C} = \{k \mid k^2 = 0\}$ is the mass cone).

Then for sufficiently small ε, the Dirac operator $i\not{\partial} + \varepsilon \mathcal{B}_0^{(a\alpha)}$ has an ε-definite kernel (see Def. 2.8 and Theorem 2.10). Thus for a.a. $\mu \in (-\varepsilon, \varepsilon)$, the spectral projectors $p_\mu^{\varepsilon,(a\alpha)}$, $k_\mu^{\varepsilon,(a\alpha)}$ and the Green's functions $s_\mu^{\varepsilon,(a\alpha)}$ are well-defined distributions in momentum space (see Def. 2.4 and Theorem 2.9). Furthermore, the kernel of the Dirac operator is ε-orthogonal to ρ (see Def. 2.12 and Theorem 2.14; for simplicity we here omit the higher order potentials $\mathcal{B}_1, \mathcal{B}_2, \ldots$ in (2.80), this is justified because these potentials obviously drop out in the singular mass limit), and this can be stated in the form (cf. (2.38)

$$p_\mu^{\varepsilon,(a\alpha)}\, \rho\, p_{\mu'}^{\varepsilon,(a\alpha)} \;=\; 0 \qquad \text{for all } \mu, \mu' \in (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}) \;.$$

We build up the spectral projectors $p^\varepsilon_{+\mu}$, $k^\varepsilon_{+\mu}$ and the Green's function $s^\varepsilon_{+\mu}$ of the whole system by taking direct sums; namely,

$$A^\varepsilon_{+\mu} \;=\; \bigoplus_{a,\alpha} \left\{ \begin{array}{ll} A_{m_{a\alpha}+\mu} & \text{if } X_a = \mathbb{1} \\ A^{\varepsilon,(a\alpha)}_{\frac{\mu}{2}} & \text{if } X_a \neq \mathbb{1}, \end{array} \right. \tag{2.88}$$

where A stands for p, k, or s. Note that in the chiral blocks the mass parameter $\frac{\mu}{2}$ (and not μ) is used. The purpose of this *modified mass scaling* is to get rid of the factor $\frac{1}{2}$ in the normalization of a chiral Dirac sea (2.75) (also see the paragraph after (2.21)). The corresponding Dirac operator is

$$i\partial\!\!\!/ + \varepsilon \mathcal{B}_0 - mY - \mu Z \;,$$

where the matrix $Z \equiv \frac{1}{2}(X + X^*)$ takes into account the modified mass scaling. The spectral projectors satisfy the multiplication rules

$$\left. \begin{array}{rcl} p^\varepsilon_{+\mu}\, p^\varepsilon_{+\mu'} &=& k^\varepsilon_{+\mu}\, k^\varepsilon_{+\mu'} \;=\; \delta^2(\mu-\mu')\, Z^{-1}\, p^\varepsilon_{+\mu} \\ p^\varepsilon_{+\mu}\, k^\varepsilon_{+\mu'} &=& k^\varepsilon_{+\mu}\, p^\varepsilon_{+\mu'} \;=\; \delta^2(\mu-\mu')\, Z^{-1}\, k^\varepsilon_{+\mu} \end{array} \right\} \tag{2.89}$$

$$C^\varepsilon_{+\mu}\, \rho\, C^\varepsilon_{+\mu'} \;=\; 0 \qquad \text{for } \mu, \mu' \in (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}) \;, \tag{2.90}$$

where C stands for k or p. The Green's functions satisfy the relations

$$\left. \begin{array}{rcl} C^\varepsilon_{+\mu}\, s^\varepsilon_{+\mu'} &=& s^\varepsilon_{+\mu}\, C^\varepsilon_{+\mu'} \;=\; \dfrac{\mathrm{PP}}{\mu-\mu'}\, Z^{-1}\, C^\varepsilon_{+\mu} \\ s^\varepsilon_{+\mu}\, s^\varepsilon_{+\mu'} &=& \dfrac{\mathrm{PP}}{\mu-\mu'}\, Z^{-1}\, (s^\varepsilon_{+\mu} - s^\varepsilon_{+\mu'}) \end{array} \right\} \tag{2.91}$$

These multiplication rules differ from those in [2] only by the additional factor Z^{-1}.

To describe the interaction, we insert a potential $\mathcal{B}$ into the Dirac operator, which then reads

$$i\partial\!\!\!/ + \mathcal{B} + \varepsilon \mathcal{B}_0 - mY - \mu Z \;. \tag{2.92}$$

We assume that Y and $\mathcal{B}$ have the following properties:

(a) Only the chiral particles are massless, i.e.

$$Y^{(a\alpha)} > 0 \qquad \text{if } X_a = \mathbb{1}.$$

(b) $\mathcal{B}$ is the operator of multiplication with the Schwartz function $\mathcal{B}(x)$.

(c) Y and $\mathcal{B}$ are causality compatible, i.e.

$$X^*\, (i\partial\!\!\!/ + \mathcal{B} - mY) \;=\; (i\partial\!\!\!/ + \mathcal{B} - mY)\, X \;. \tag{2.93}$$

In order to introduce the spectral projectors with interaction $\tilde{p}^\varepsilon_{+\mu}$ and $\tilde{k}^\varepsilon_{+\mu}$, we take the operator expansion of causal perturbation theory [2, Section 4] and replace the operators according to $A \to A^\varepsilon_{+\mu}$ (with $A = p$, k, or s). All the operator products of the resulting expansion are well-defined for a.a. μ (note that $\tilde{\mathcal{B}}(k)$ has rapid decay and $A^\varepsilon_{+\mu}(k)$ grows at most polynomially at infinity).

For the infrared regularization, we proceed exactly as in Section 1.1 and replace space by the three-dimensional torus (1.9). Furthermore, we "average" the mass parameter μ. More precisely, combining (1.19) with (2.70), the auxiliary fermionic projector is defined by

$$P^{\varepsilon,\delta} \;=\; \frac{1}{2} \int_{(0,\delta)\times(-\delta,\delta)} (X\, \tilde{t}^\varepsilon_{+\mu} + \tilde{t}^\varepsilon_{+\mu}\, X^*)\, d^2\mu \;, \tag{2.94}$$

where as usual $\tilde{t}^\varepsilon_{+\mu} = \frac{1}{2}(\tilde{p}^\varepsilon_{+\mu} - \tilde{k}^\varepsilon_{+\mu})$. Finally, the *regularized fermionic projector* is obtained by taking the partial trace,

$$(P^{\varepsilon,\delta})^a_b \;=\; \sum_{\alpha,\beta=1}^{3} (P^{\varepsilon,\delta})^{(a\alpha)}_{(b\beta)} \;. \tag{2.95}$$

Before we can prove idempotence, we need to impose the following extension of the non-degeneracy assumption (1.1). We set

$$\sigma^\varepsilon_{(a\alpha)} \;=\; \sigma(\not{k} + \varepsilon \mathcal{B}^{(a\alpha)}_0) \;.$$

Def. 2.15 *The Dirac operator $i\not{\partial} + \varepsilon\mathcal{B}_0 - mY$ has* **ε-non-degenerate masses** *if for all a and $\beta \neq \gamma$,*

$$\sigma^\varepsilon_{(b\beta)} \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \neq \emptyset \quad\Longrightarrow\quad \sigma^\varepsilon_{(b\gamma)} \cap (-\varepsilon, \varepsilon) = \emptyset\,. \tag{2.96}$$

Roughly speaking, the next theorem states that the masses are ε-non-degenerate if they are non-degenerate in the massive sectors, and if the homogeneous potentials in the chiral sectors are sufficiently different from each other.

Theorem 2.16 *Suppose that Y and $\mathcal{B}_0$ have the following properties:*

(i) *In the massive blocks (i.e. $X_a = \mathbb{1}$), the masses are non-degenerate,*

$$Y^{(b\beta)} \neq Y^{(b\gamma)} \qquad \text{if } \beta \neq \gamma.$$

(ii) *In the chiral blocks (i.e. $X_a \neq \mathbb{1}$), for all $\beta \neq \gamma$ and all $q \in \mathcal{C}$ either*

$$<v^{(b\beta)}, q> + s\, \Delta^{(b\beta)}(q) \;\neq\; <v^{(b\gamma)}, q> + s'\, \Delta^{(b\gamma)}(q) \qquad \text{for all } s, s' \in \{\pm 1\} \tag{2.97}$$

or else

$$|\alpha^{(b\beta)}(q) - \alpha^{(b\gamma)}(q)| \;>\; 2 \;+\; 2\, |d^{(b\beta)}(q) + d^{(b\gamma)}(q)| \tag{2.98}$$

with

$$d(q) \;=\; \begin{cases} \left| \dfrac{w_{ij}(q) a^i q^j}{\Delta(q)} \right| & \text{if } \Delta(q) \neq 0 \\ \sqrt{|l(q)|} & \text{if } \Delta(q) = 0 \end{cases}$$

(and Δ, l according to (2.56) and (2.60)).

Then for sufficiently small ε, the Dirac operator has ε-non-degenerate masses.

Proof. The condition in **(i)** follows immediately from the fact that the eigenvalues μ_s^a in the two sectors differ precisely by $m(Y^{(b\beta)} - Y^{(b\gamma)})$. For part **(ii)** we consider the formulas for the eigenvalues (2.64) and (2.66). If (2.97) holds, the eigenvalues in the two sectors all differ by contributions of the order $\sqrt{\varepsilon}$, and so (2.96) is satisfied for small ε. If on the other hand (2.97) is violated, there are eigenvalues in two different sectors such that the square roots in (2.64) and/or (2.66) coincide. Thus these eigenvalues differ by $(\alpha+\sigma)^{(b\beta)} - (\alpha+\sigma)^{(b\gamma)}$, where each σ is an element of the set $\{\pm\kappa_+, \pm\kappa_-, \pm\tau_+, \pm\tau_-\}$. The condition (2.98) guarantees that this difference is greater than 2ε, and so (2.96) is again satisfied. ■

We can now state the main result of this chapter.

Theorem 2.17 (Idempotence) *Consider the Dirac operator (2.92) under the above assumptions* **(1)**–**(3)** *and* **(a)**–**(c)**. *Assume furthermore that the masses are ε-non-degenerate (see Def. 2.15 and Theorem 2.16). Then the corresponding fermionic projector (2.94), (2.95) satisfies the identity*

$$\lim_{\varepsilon\searrow 0}\lim_{\delta\searrow 0}\delta\left(\int_{\mathbb{R}\times T^3}\sum_{b=1}^N P^a_b(x,z)\,P^b_c(z,y)\,d^4z \;-\; P^a_c(x,y)\right) \;=\; 0 \tag{2.99}$$

with convergence as a distribution to every order in perturbation theory.

Proof. Similar to (2.40), the Green's function $s^\varepsilon_{+\mu}$ has a spectral representation in a mass parameter ν. We want to decompose $s^\varepsilon_{+\mu}$ into contributions $\dot{s}^\varepsilon_{+\mu}$ and $\breve{s}^\varepsilon_{+\mu}$ where $|\nu-\mu|$ is small and large, respectively. To this end, we introduce in each sector the operator

$$\left\{\begin{array}{lclr} \dot{s}^{\varepsilon,(a\alpha)}_{\mu} &=& \displaystyle\int_{B_{\varepsilon/4}(\mu)} \frac{\mathrm{PP}}{\nu-\mu}\,p^\varepsilon_\mu\,d^2\nu & \text{if } X_1\neq \mathbb{1} \\ \dot{s}_{m_{a\alpha}+\mu} &=& \displaystyle\int_{B_{\varepsilon/2}(\mu)} \frac{\mathrm{PP}}{\nu-\mu}\,p_{m_{a\alpha}+\nu}\,d^2\nu & \text{if } X_1 = \mathbb{1} \end{array}\right.$$

and define $\dot{s}^\varepsilon_{+\mu}$ by taking as in (2.88) the direct sum. Setting $\breve{s}^\varepsilon_{+\mu} = s^\varepsilon_{+\mu} - \dot{s}^\varepsilon_{+\mu}$, we obtain the decomposition

$$s^\varepsilon_{+\mu} \;=\; \dot{s}^\varepsilon_{+\mu} + \breve{s}^\varepsilon_{+\mu}\;. \tag{2.100}$$

Our first step is to show that for small μ, the matrix $\breve{s}^\varepsilon_{+\mu}(k)$ is bounded; more precisely, that

$$\|s^\varepsilon_{+\mu}(k)\| \;\leq\; \frac{C(k)}{\varepsilon^7} \qquad \text{for } \mu\in(-\frac{\varepsilon}{2},\frac{\varepsilon}{2}) \tag{2.101}$$

with $C(k)$ a smooth function with at most polynomial growth at infinity (the exponent 7 is probably not optimal, but (2.101) is sufficient for our purpose). It clearly suffices to prove (2.101) in a given sector $(a\alpha)$; for simplicity the sector index will be omitted (i.e. $\mathcal{B}_0 \equiv \mathcal{B}_0^{(a\alpha)}$). Furthermore, we only consider the case $X_a \neq \mathbb{1}$; the other case is analogous (and even simpler, because in the massive sectors no homogeneous potentials are present). We introduce the projector $E(k)$ by

$$E \;=\; \sum_{(a,s)\in\mathcal{S}} E^a_s \qquad\text{with}\qquad \mathcal{S} = \left\{(a,s)\text{ with } |\mu^a_s-\mu| < \frac{\varepsilon}{4}\right\}$$

According to Lemma 2.7,

$$\|E\| \;\leq\; \frac{C_1(k)}{\varepsilon^3} \tag{2.102}$$

with $C_1(k)$ smooth with at most polynomial growth at infinity. The matrix $\breve{s}^\varepsilon_{+\mu}$ has a simple spectral representation,

$$\breve{s}^\varepsilon_{+\mu} \;=\; \sum_{(a,s)\not\in\mathcal{S}} \frac{1}{\mu^a_s - \mu}\, E^a_s \;.$$

Unfortunately, this representation is not suitable for estimates, because we have no control of $\|E^a_s\|$ for $(a,s) \not\in \mathcal{S}$. To avoid this problem, we rewrite $\breve{s}^\varepsilon_{+\mu}$ as follows,

$$\begin{aligned}
\breve{s}^\varepsilon_{+\mu} \;&=\; \left(\sum_{(a,s)\not\in\mathcal{S}} \frac{1}{\mu^a_s - \mu}\, E^a_s \;+\; \sum_{(a,s)\in\mathcal{S}} \frac{1}{\mu^a_s - \mu + \varepsilon}\, E^a_s \right) (\mathbb{1} - E) \\
&=\; (\not{k} + \varepsilon \mathcal{B}_0 - \mu + \varepsilon E)^{-1}\, (\mathbb{1} - E) \;.
\end{aligned}$$

Introducing the "Hamiltonian" $H = -\gamma^0\, (\vec{\not{k}} + \varepsilon \mathcal{B}_0(\omega, \vec{k}) - \mu + \varepsilon E)$, we obtain

$$\breve{s}^\varepsilon_{+\mu} \;=\; (\omega - H)^{-1}\, \gamma^0\, (\mathbb{1} - E) \;. \tag{2.103}$$

The matrix $H(k)$ is Hermitian w.r.to the positive scalar product $(.|.) = \prec.|\gamma^0|.\succ$ and can thus be diagonalized, i.e.

$$H \;=\; \sum_{n=1}^4 \Omega_n\, F_n$$

with real eigenvalues Ω_n and spectral projectors F_n. Substituting into (2.103) gives

$$\breve{s}^\varepsilon_{+\mu} \;=\; \sum_{n=1}^4 \frac{1}{\omega - \Omega_n}\, F_n\, \gamma^0\, (\mathbb{1} - E) \;,$$

and (2.102) yields the bound

$$\|\breve{s}^\varepsilon_{+\mu}\| \;\leq\; 2 \max_n \frac{1}{|\omega - \Omega_n|}\, \|\mathbb{1} - E\| \;\leq\; \frac{C_2(k)}{\varepsilon^3}\, \max_n \frac{1}{|\omega - \Omega_n|} \;. \tag{2.104}$$

It remains to estimate the factors $|\omega - \Omega_n|$ from below. We use that the determinant is multiplicative to obtain

$$\begin{aligned}
\prod_{n=1}^4 (\omega - \Omega_n) \;&=\; \det(\omega - H) \;=\; \det(\gamma^0\, (\omega - H)) \\
&=\; \det(\not{k} + \varepsilon \mathcal{B}_0 - \mu + \varepsilon E) \;=\; \prod_{(a,c)\not\in\mathcal{S}} (\mu^a_c - \mu) \prod_{(a,c)\in\mathcal{S}} (\mu^a_c - \mu + \varepsilon) \;.
\end{aligned}$$

Taking the absolute value, the factors $|\mu^a_c - \mu|$ and $|\mu^a_c - \mu + \varepsilon|$ are all greater than $\frac{\varepsilon}{4}$, and thus

$$\prod_{n=1}^4 |\omega - \Omega_n| \;\geq\; \left(\frac{\varepsilon}{4}\right)^4 . \tag{2.105}$$

Since the eigenvalues of the Hermitian matrix $\omega - H$ can be estimated by the sup-norm of the matrix,

$$|\omega - \Omega_n| \;\leq\; \|\omega - H\| \;,$$

we can deduce from (2.105) that each factor $|\omega - \Omega_n|$ is bounded by

$$|\omega - \Omega_n| \;\geq\; \|\omega - H\|^{-3} \left(\frac{\varepsilon}{4}\right)^4 \;.$$

Substituting this inequality into (2.104) gives (2.101).

The causal perturbation expansion expresses $\tilde{t}^\varepsilon_{+\mu}$ as a sum of operator products of the form

$$\tilde{t}^\varepsilon_{+\mu} \;\asymp\; A^\varepsilon_{+\mu}\, \mathcal{B}_0\, A^\varepsilon_{+\mu} \,\cdots\, A^\varepsilon_{+\mu}\, \mathcal{B}_0\, A^\varepsilon_{+\mu} \;, \tag{2.106}$$

where each factor A stands for p, k, or s. Since $\tilde{\mathcal{B}}_0(k)$ has rapid decay and $A^\varepsilon_\mu(k)$ grows at most polynomially, these operator products are well-defined. According to (2.94) and (2.95), the first summand inside the brackets in (2.99) can be written as

$$\begin{aligned}\frac{1}{4} \int_0^\delta d\mu \int_0^\delta d\mu' \sum_b \sum_{\beta,\gamma} \Big(& X_a\, (t^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, (t^\varepsilon_{+\mu'})^{(b\gamma)}_{(c\delta)}\, X^*_c \;+\; X_a\, (t^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, X_b\, (t^\varepsilon_{+\mu'})^{(b\gamma)}_{(c\delta)} \\ & + (t^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, X^*_b\, (t^\varepsilon_{+\mu'})^{(b\gamma)}_{(c\delta)}\, X^*_c \;+\; (t^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, X^*_b\, X_b\, (t^\varepsilon_{+\mu'})^{(b\gamma)}_{(c\delta)} \Big) \,.\end{aligned} \tag{2.107}$$

When we substitute (2.106) into (2.107), the difficult point is to multiply the rightmost factor A of the first factor t to the leftmost factor A of the second factor t. More precisely, we must analyze the following operator products,

$$(\cdots\, A^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, (A^\varepsilon_{+\mu'}\, \cdots)^{(b\gamma)}_{(c\delta)} \tag{2.108}$$

$$(\cdots\, A^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, \rho\, (A^\varepsilon_{+\mu'}\, \cdots)^{(b\gamma)}_{(c\delta)} \tag{2.109}$$

with $A = p$, k, or s.

If one of the factors A in (2.108) or (2.109) is the Green's function, we substitute (2.100) and expand. Since $\breve{s}^\varepsilon_{+\mu}$ is bounded (2.101), the products involving $\breve{s}^\varepsilon_{+\mu}$ have a finite limit as $\delta \searrow 0$. Since the two integrals in (2.107) give a factor δ^2, these products all drop out when the limit $\delta \searrow 0$ is taken in (2.99). Thus it suffices to consider the case when the factors A in (2.108) and (2.109) stand for p, k, or $\dot{s}$.

Since the Dirac operator has ε-non-degenerate masses, the distributions $A^\varepsilon_{+\mu}(k)$ have disjoint supports in different sectors. More precisely, for all $\mu, \mu' \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$,

$$\text{supp}\, (A^{\varepsilon,(b\beta)}_\mu) \cap \text{supp}\, (A^{\varepsilon,(b\gamma)}_{\mu'}) \;=\; \emptyset \qquad \text{if } \beta \neq \gamma \text{ and } X_b \neq \mathbb{1},$$

where each factor A stands for p, k, or $\dot{s}$. A similar relation holds in the massive blocks. Therefore, (2.108) and (2.109) vanish if $\beta \neq \gamma$.

In the case $\beta = \gamma$, (2.109) is zero because the Dirac operator is ε-orthogonal to ρ (2.90). Thus, using a matrix notation in the sectors, we only need to take into account the operator products

$$(\cdots\, A^\varepsilon_{+\mu})(A^\varepsilon_{+\mu'}\, \cdots)$$

with $A = p$, k, or s (here we may again consider s instead of $\dot{s}$ because, as we saw above, all factors $\breve{s}$ drop out in the limit $\delta \searrow 0$). Now we can apply the multiplication rules (2.89) and (2.91). Applying (2.89) gives a factor $\delta^2(\mu - \mu')$, and we can carry out the μ'-integral.

After dividing by δ, we can take the limits $\delta \searrow 0$ and $\varepsilon \searrow 0$. Using that in this limit the Dirac operator is causality compatible (2.93), we can "commute X through" the resulting operator products (see [2, Section 4]). In this way, one recovers precisely the unregularized fermionic projector $P = \lim_{\varepsilon,\delta \searrow 0} P^{\varepsilon,\delta}$. If (2.91) is applied, the resulting principal part is bounded after the integrals over μ and μ' are carried out, and we can take the limits $\delta \searrow 0$ and $\varepsilon \searrow 0$. After commuting X through the resulting operator products we find that all terms cancel. ∎

For understanding better what the above results mean physically, it is instructive to consider a cosmological situation where the 4-volume of space-time is finite. In this case, the limits $\varepsilon, \delta \searrow 0$ in (2.99) are merely a mathematical idealization corresponding to the fact that the size of the universe is very large compared to the usual length scales on earth. We can extrapolate from (2.94),(2.95) to get some information on how the properly normalized physical fermionic projector should look like: The parameter δ is to be chosen of the order T^{-1} with T the lifetime of the universe (also see Section 1.1). Then due to the μ-integral in (2.94), the Dirac seas are built up from those fermionic states whose momentum lies in a thin strip around the mass cone. Naively, the modified mass scaling implies that for the neutrinos this strip must be thinner. However, this naive picture is misleading, because the detailed form of the chiral Dirac seas depends strongly on the homogeneous operator $\mathcal{B}_0$, which is unknown. We point out that in (2.99) the order of limits is essential: we must first take the infinite volume limit and then the limit $\varepsilon \searrow 0$. This means for our cosmology in finite 4-volume that the homogeneous perturbation $\varepsilon \mathcal{B}_0$ must be large compared to T^{-1}. One possibility to realize this is to give the neutrinos a small rest mass. But, as shown above, the same can be achieved by more general, possibly nonlocal potentials which do not decay at infinity.

Acknowledgments: I would like to thank the Max Planck Institute for Mathematics in the Sciences, Leipzig, for its support and hospitality.

References

[1] F. Finster, "Local $U(2,2)$ symmetry in relativistic quantum mechanics," *J. Math. Phys.* 39 (1998) 6276-6290

[2] F. Finster, "Definition of the Dirac sea in the presence of external fields," hep-th/9705006, *Adv. Theor. Math. Phys.* 2 (1998) 963-985

[3] F. Finster, "Light-cone expansion of the Dirac sea in the presence of chiral and scalar potentials," hep-th/9809019, *J. Math. Phys.* 41 (2000) 6689-6746

[4] F. Finster, "The principle of the fermionic projector I, introduction and continuum limit," hep-th/0001048

[5] F. Finster, "The principle of the fermionic projector II, derivation of the effective gauge group," hep-th/0202059

[6] M. Reed, B. Simon, "Methods of Modern Mathematical Physics," vol. 4, Academic Press (1997)

NWF I – Mathematik, Universität Regensburg, 93040 Regensburg, Germany,
Felix.Finster@mathematik.uni-regensburg.de