Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Heat flow for horizontal harmonic maps into a class of Carnot-Caratheodory spaces

by

Jürgen Jost and Yi-Hu Yang

Preprint no.: 104 2003

Heat flow for horizontal harmonic maps into a class of Carnot-Caratheodory spaces

Jürgen Jost and Yi-Hu Yang*

1 Introduction

Let X and B be two Riemannian manifolds with $\pi : X \to B$ being a Riemannian submersion. Let $\mathcal{H}$ be the corresponding horizontal distribution, which is perpendicular to the tangent bundle of the fibres of $\pi : X \to B$. Then X (just considered as a differentiable manifold), together with the distribution $\mathcal{H}$, forms a so-called *Carnot-Caratheodory space* [1], when the Riemannian metric of X is restricted to $\mathcal{H}$. On X, as a Carnot-Caratheodory space, can then be defined the notions of Carnot-Caratheodory distance (sometimes called sub-Riemannian distance), (minimizing) geodesic, completeness (under the Carnot-Caratheodory distance), etc; a geodesic is actually a horizontal curve which locally realizes the Carnot-Caratheodory distance. In this note, we always assume that X is complete, as both a Riemannian manifold and a Carnot-Caratheodory space, and the Riemannian submersion $\pi : X \to B$ together with its horizontal distribution $\mathcal{H}$ satisfies the following conditions

1) the *Chow condition*: the vector fields of $\mathcal{H}$ $X_1, X_2, \cdots$, and their iterated Lie brackets $[X_i, X_j], [[X_i, X_j], X_k], \cdots$ span the tangent space T_xX at every point of X;

2) the sectional curvature of X (as a Riemannian manifold) in the direction of $\mathcal{H}$ is non-positive.

Remark. 1) The Chow condition guarantees that one has the so-called Hopf-Rinow theorem (cf. [1]): if X is complete under the Carnot-Caratheodory metric, then any two points can be joined by a minimizing geodesic (under the Carnot-Caratheodory distance); moreover, in any given homotopic class of horizontal curves connecting two points, there exists a minimzing geodesic (under the Carnot-Caratheodory distance) connecting these two points. 2) The Riemannian length of a horizontal curve is just equal to the Carnot-Caratheodory length by the definitions.

*The second named author supported partially by NSF of China (No. 10171077)

Our interest in this note is to study *horizontal* maps from a compact Riemannian manifold M into X, i.e. the image of the derivative of such a map lies in $\mathcal{H}$. We wish to find some such maps which furthermore satisfy some differential equation, e.g. harmonic map equation, as X is considered as a Riemannian manifold. First of all, let us consider the space of smooth maps from M into X which are horizontal and can be connected horizontally to a fixed horizontal map g, denoted by $B^{\circ}_{g,\mathcal{H}}(M;X)$; it is easy to see that, under a certain suitable metric (defined by using some suitable Sobolev's norm), $B^{\circ}_{g,\mathcal{H}}(M;X)$ can be completed into a Banach manifold, denoted by $B_{g,\mathcal{H}}(M;X)$, which is obviously an infinite dimensional smooth manifold; clearly, its tangent vectors are just horizontal vector fields of X (if necessary,they can be considered as sections of a certain pull-back bundle). Similarly, considering the space of all maps from M into X, which are not necessarily horizontal, one can get another Banach manifold, denoted by $B(M;X)$, and $B_{g,\mathcal{H}}(M;X)$ can be considered as a submanifold of $B(M;X)$. It should be pointed out that these Banach manifolds may not be connected (but clearly are locally connected), this does not however affect our following discussion. Let $\mathcal{X}$ be a vector field of $B(M;X)$ along $B_{g,\mathcal{H}}(M;X)$. Corresponding to the horizontal distribution $\mathcal{H}$, one has an orthogonal projection to $\mathcal{H}$, still denoted by $\mathcal{H}$. Accordingly, one can also define the projection of $\mathcal{X}$, denoted by $\mathcal{H}\mathcal{X}$, which is a vector field of $B_{g,\mathcal{H}}(M;X)$ and the value of which at any point of $B_{g,\mathcal{H}}(M;X)$ is actually a horizontal vector field of X (again, if necessary, it can be considered as a section of a certain pull-back bundle).

In this note, we first give some examples of Carnot-Caratheodory spaces, in which we are really interested. These spaces are actually a class of (locally) complex homogeneous manifolds which fibre over the corresponding symmetric spaces of noncompact type and the fiberations are Riemannian submersion under the standard invariant metrics. We will show that this class of spaces satisfies the Conditions 1) and 2) above; on the other hand, such homogeneous spaces, as Riemannian manifolds, are complete and by the definition of Carnot-Caratheodory distance, the Riemannian distance is not greater than the Carnot-Caratheodory distance, so this class of homogeneous spaces, as Carnot-Caratheodory spaces, are also complete under the corresponding Carnot-Caratheodory distance. Thus we can apply the Banach spaces defined above to this class of homogeneous complex manifolds. We next consider the following heat flow from $M \times [0,\infty)$ into X

$$(*) \qquad \mathcal{H}\tau(u) - \frac{\partial u}{\partial t} = 0,$$

with the initial data $u(\cdot,0) = g(\cdot)$, here $\tau(u)$ is the stress-energy tensor of u with respect to the space variable, $g(\cdot)$ is a smooth horizontal map. We show that one can always deform horizontally any smooth horizontal map into a horizontal harmonic map. It is worth noting that the operator $\mathcal{H}\tau$, as applied to the Banach space $B(M;X)$, is not elliptic in general, but if applied to the Banach space $B_{g,\mathcal{H}}(M;X)$, it is indeed elliptic, i.e., the symbol of its linearization is an isomorphism from the horizontal tangent subbundle of X to itself, and hence one can apply the implicit function theorem to the Banach space $B_{g,\mathcal{H}}(M;X)$ to obtain the short-time existence of a (unique) solution of $(*)$ with the initial map g.

Acknowlegements. The second named author wants to thank Dr Guofang Wang for valuable discussion. The main part of this work was completed when he was visiting the Max-Planck-Institute for Mathematics in the Sciences. He would like also to thank the institute for its hospitality and good working conditions.

2 A class of Carnot-Caratheodory spaces

In this section, we will show some concrete examples for Carnot-Caratheodory spaces, which are actually the objects in which we are really interested. These examples are a class of (locally) complex homogeneous manifolds [3, 6]: Let G be a connected noncompact real semisimple Lie group satisfying that it has a compact Cartan subgroup; as a consequence, if K is a maximal compact subgroup of G, then G and K have the same rank; moreover G/K is not a Hermitian symmetric space. Denote such a Cartan subgroup by H, and choose a suitable subgroup Z of K containing H, which is actually the centralizer in G of a certain circle subgroup T of H. Taking the quotients G/Z and G/K, one has then that G/Z is a homogeneous complex manifold and G/K is a symmetric space of noncompact type; moreover G/Z is a fiberation over G/K with the fiber K/Z; under the standard invariant metrics [6], the fibration $\pi : G/K \to G/Z$ is a Riemannian submersion, and hence it has a horizontal distribution $\mathcal{H}$, which satisfies all the assumptions mentioned in the preceeding section, as shown in the following. Let Γ be a discrete subgroup of G. Because of the discreteness of Γ and the compactness of K, one can assume that $\Gamma \cap K = \emptyset$. Thus we have the Riemannian submersion $\Gamma \setminus G/Z \to \Gamma \setminus G/K$. Similarly, one has the horizontal distribution which is the discrete quotient of $\mathcal{H}$ and hence also satisfies the assumption in the preceeding section, denoted by $\mathcal{H}'$. In the remaining part of this section, we will show that the distribution $\mathcal{H}$, and hence $\mathcal{H}'$, does satisfy those assumptions. First, we check the assumption for sectional curvature in the horizontal direction $\mathcal{H}$; actually, one generally has the following

Proposition 1 *Let $\pi : X \to B$ be a Riemannian submersion. If B has non-positive sectional curvature, then X, in the horizontal direction $\mathcal{H}$, also has* *non-positive sectional curvature.*

Since G/K is a symmetric space of noncompact type, so it, and hence G/Z in the horizontal direction $\mathcal{H}$, has non-positive sectional curvature.

Proof of Proposition 1. The proof is a simple consequence of the O'Neill formulae: Denote the curvature tensors of X and B by R and R' respectively; then one of O'Neill's formulae says, for horizontal tangent vectors Y, Z, U, V of X,

$$< R(Y,Z)U,V >=< R'(Y,Z)U,V > -2 < A(Y,Z), A(U,V) >$$
$$+ < A(Z,U), A(Y,V) > - < A(Y,U), A(Z,V) > .$$

Here, $Y, \cdots$ are also regarded as tangent vectors of B; the definition of A refers to the proof of the Lemma 2 in the next section; the key point is that A is skew-symmetric with respect to horizontal vectors. So

$$< R(Y,Z)Y,Z >=< R'(Y,Z)Y,Z > -2 < A(Y,Z), A(Y,Z) > +$$
$$< A(Z,Y), A(Y,Z) >=< R'(Y,Z)Y,Z > -3 < A(Y,Z), A(Y,Z) > \leq 0.$$

We now turn to check the Chow condition. By Cartan's classification theorem for simple groups [4], the simple Lie groups satisfying the conditions stated in the beginning of this section are as follows :

$\mathrm{SO}(p,2q)\ q \geq 2$	$\mathbf{e}_{8(8)}$
$\mathrm{S}p(p,q)$	$\mathbf{e}_{8(-24)}$
$\mathbf{e}_{6(2)}$	$\mathbf{f}_{4(4)}$
$\mathbf{e}_{7(7)}$	$\mathbf{f}_{4(-20)}$
$\mathbf{e}_{7(-5)}$	$\mathbf{g}_{2(2)}$

The above list is called *groups of Hodge type but not of Hermitian type* in Simpson's paper[10]. In order to show the Chow condition, we can actually turn the problem into a Lie-theoretic problem. To this end, we first need to give the relation between the Lie bracket of left invariant vector fields and the Lie bracket of the Lie algebras in question when considering left invariant vector fields as elements of the Lie algebra. We use the notations of [7]. Denote the Lie algebra of G and Z by $\mathfrak{g}$ and $\mathfrak{z}$ respectively, then it is easy to see that we have a direct sum decomposition of vector spaces

$$\mathfrak{g} = \mathfrak{z} + \mathfrak{m}$$

with $[\mathfrak{z}, \mathfrak{m}] \subset \mathfrak{m}$. Here $\mathfrak{m}$ can be identified with the tangent space of G/Z at the origin or the set of all G-invariant vector fields on G/Z. Theorem 2.10 of

[7] tells us that there exists a unique torsion-free G-invariant affine connection ∇ with

$$\nabla_Y Z = \frac{1}{2}[Y, Z]_{\mathfrak{m}}, \text{ for } Y, Z \in \mathfrak{m},$$

here by Y, Z on the left-hand side we mean vector fields on G/Z while Y, Z on the right-hand side mean elements in $\mathfrak{g}$; $[Y, Z]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $[Y, Z]$. Thus, one has

$$[Y, Z] = [Y, Z]_{\mathfrak{m}},$$

here by the left-hand side we mean the Lie bracket of vector fields; afterwards we will not point out this since it should be clear from the context. As before, one has a Cartan subgroup H contained in Z, the Lie algebra of which is a maximal abelian subalgebra, denoted by $\mathfrak{h}$. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, here $\mathfrak{k}$ is the Lie algebra of K. We then have the following relations $\mathfrak{h} \subset \mathfrak{z} \subset \mathfrak{k} \subset \mathfrak{g}$ and $\mathfrak{p} \subset \mathfrak{m}$. Again, $\mathfrak{p}$, as a vector subspace of $\mathfrak{m}$, can be indentified with the horizontal tangent subspace at the origin with respect to the Riemannian submersion $G/Z \to G/K$ and its left translation forms the horizontal distribution $\mathcal{H}$ of the Riemannian submersion; furthermore, its elements can be identified with G-invariant horizontal vector fields of G/K. By the previous relation of two Lie brackets, in order to show that the horizontal distribution $\mathcal{H}$ satisfies the Chow condition, it is sufficient to show that $\mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}]$ span $\mathfrak{m}$. To this end, we use the root system of the complexification $\mathfrak{g}^{\mathbf{C}}$ of $\mathfrak{g}$ correspoding to the Cartan subalgebra $\mathfrak{h}$. Let Δ be the root system of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{h}$, $\mathfrak{g}^{\alpha}$ the root space corresponding to $\alpha \in \Delta$, $\mathfrak{g}^{\mathbf{C}} = \mathfrak{k}^{\mathbf{C}} + \mathfrak{p}^{\mathbf{C}}$ the Cartan decomposition, θ the Cartan involution, σ the conjugation of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{g}$. Since $\mathfrak{h}$ lies in $\mathfrak{k}$ while $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, so the root space $\mathfrak{g}^{\alpha}$ lies in either $\mathfrak{k}^{\mathbf{C}}$ or $\mathfrak{p}^{\mathbf{C}}$. In the first case, we call α a *compact root*; denote the set of all compact roots by $\Delta(\mathfrak{k})$; in the last case, a *noncompact root*; denote the set of noncompact roots by $\Delta(\mathfrak{p})$. On the other hand, we also have the direct sum decomposition for vector spaces $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}'$, obviously $\mathfrak{m} \subset \mathfrak{m}'$; furthermore one has the direct sum $\mathfrak{m}' = \mathfrak{k}' + \mathfrak{p}$ with $\mathfrak{h} + \mathfrak{k}' = \mathfrak{k}$. So if we can show that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}'$, equivalently $[\mathfrak{p}^{\mathbf{C}}, \mathfrak{p}^{\mathbf{C}}] = \mathfrak{k}'^{\mathbf{C}}$, then the Chow condition is obtained. ¿From the root theory, we has

$$\mathfrak{k}'^{\mathbf{C}} = \sum_{\alpha \in \Delta(\mathfrak{k})} \mathfrak{g}^{\alpha} \text{ and } \mathfrak{p}^{\mathbf{C}} = \sum_{\alpha \in \Delta(\mathfrak{p})} \mathfrak{g}^{\alpha}.$$

Note that $\sigma(\mathfrak{g}^{\alpha}) = \mathfrak{g}^{-\alpha}$ while $\sigma(\mathfrak{k}'^{\mathbf{C}}) = \mathfrak{k}'^{\mathbf{C}}$ and $\sigma(\mathfrak{p}^{\mathbf{C}}) = \mathfrak{p}^{\mathbf{C}}$, so if $\alpha \in \Delta(\mathfrak{k})$ (resp. $\Delta(\mathfrak{p})$), then so is $-\alpha$. We now state the following

Proposition 2 *For any root $\alpha \in \Delta(\mathfrak{k})$, there exist two noncompact roots β and γ with $\beta + \gamma = \alpha$.*

Clearly if the proposition is true, then the Chow codition is obtained. In the following, we will case by case write down compact roots and noncompact roots of $\mathfrak{g}^{\mathbf{C}}$ for the above simple groups list and then easily check that the above assertion is true.

$SO(p, 2q), q \geq 2$: we have two cases to consider. $SO(2p, 2q), p, q \geq 2$: It is the noncompact real form of $SO(2(p+q), \mathbf{C})$ with the maximal compact subgroup $K = SO(2p) \times SO(2q)$. The root system of $\mathfrak{so}(2(p+q), \mathbf{C})$ is $D_{p+q} = \{\pm e_i \pm e_j, 1 \leq i < j \leq p+q\}$, here $\{e_i\}$ is the standard basis of $\mathbf{R}^{p+q}$, while the root systems of $\mathfrak{so}(2p, \mathbf{C})$ and $\mathfrak{so}(2q, \mathbf{C})$, embedded in D_{p+q}, are

$$D_p = \{\pm e_i \pm e_j, 1 \leq i < j \leq p\}$$

and

$$D_q = \{\pm e_i \pm e_j, p+1 \leq i < j \leq p+q\}$$

respectively. Therefore, corresponding to the noncompact real form $SO(2p, 2q)$ and its compact Cartan subalgebra, $\mathfrak{so}(2(p+q), \mathbf{C})$ has noncompact roots

$$\{\pm e_i \pm e_j, 1 \leq i \leq p, p+1 \leq j \leq p+q\};$$

the second case is $SO(2p+1, 2q), p, q \geq 2$: it is the noncompact real form of $SO(2(p+q)+1, \mathbf{C})$ with the maximal compact subgroup $K = SO(2p+1) \times SO(2q)$. The root system of $\mathfrak{so}(2(p+q)+1, \mathbf{C})$ is $B_{p+q} = \{\pm e_i, \pm e_i \pm e_j, 1 \leq i, j \leq p+q, i \neq j\}$ while the root systems of $\mathfrak{so}(2p+1, \mathbf{C})$ and $\mathfrak{so}(2q, \mathbf{C})$, embedded in B_{p+q}, are

$$B_p = \{\pm e_i, \pm e_i \pm e_j, 1 \leq i, j \leq p, i \neq j\}$$

and

$$D_q = \{\pm e_i \pm e_j, p+1 \leq i < j \leq p+q\}$$

respectively. Therefore, corresponding to the noncompact real form $SO(2p+1, 2q)$ and its compact Cartan subalgebra, $\mathfrak{so}(2(p+q)+1, \mathbf{C})$ has noncompact roots

$$\{\pm e_i \pm e_j, \pm e_j, 1 \leq i \leq p, p+1 \leq j \leq p+q\}.$$

$Sp(p, q)$: It is the noncompact real form of $Sp(p+q, \mathbf{C})$ with the maximal compact subgroup $K = Sp(p) \times Sp(q)$. The root system of $\mathfrak{sp}(p+q, \mathbf{C})$ is $C_{p+q} = \{\pm 2e_i, \pm e_i \pm e_j, 1 \leq i, j \leq p+q, i \neq j\}$, while the root systems of $\mathfrak{sp}(p, \mathbf{C})$ and $\mathfrak{sp}(q, \mathbf{C})$, embedded in C_{p+q}, are

$$C_p = \{\pm 2e_i, \pm e_i \pm e_j, 1 \leq i, j \leq p, i \neq j\}$$

and

$$C_q = \{\pm 2e_i, \pm e_i \pm e_j, p+1 \leq i, j \leq p+q, i \neq j\}$$

respectively; therefore, corresponding to the noncompact real form $Sp(p,q)$ and its compact Cartan subalgebra, $\mathfrak{sp}(p+q,\mathbf{C})$ has noncompact roots

$$\{\pm e_i \pm e_j, 1 \le i \le p, p+1 \le j \le p+q\}.$$

$\mathbf{e}_{6(2)}$: It is the noncompact real form of $\mathbf{e}_6$ with the maximal compact subgroup $K = SU(6)\times SU(2)$. The root system of $\mathbf{e}_6$ is

$$\begin{aligned} E_6 &= \{e_i - e_j, i \ne j, 1 \le i,j \le 6\} \cup \{\pm(e_7 - e_8)\} \cup \\ &\quad \{\frac{1}{2}(e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)} - e_{\sigma(4)} - e_{\sigma(5)} - e_{\sigma(6)} \pm (e_7 - e_8)), \sigma \in P(6)\}, \end{aligned}$$

where $P(6)$ is the permutation group of $\{1,2,3,4,5,6\}$. The root system of $\mathfrak{sl}(6,\mathbf{C}) + \mathfrak{sl}(2,\mathbf{C})$, embedded in E_6, is

$$A_5 + A_1 = \{e_i - e_j, i \ne j, 1 \le i,j \le 6\} \bigcup \{\pm(e_7 - e_8)\}.$$

Thus, corresponding to the noncompact real form $\mathbf{e}_{6(2)}$ and its compact Cartan subalgebra, $\mathbf{e}_6$ has noncompact roots

$$\{\frac{1}{2}(e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)} - e_{\sigma(4)} - e_{\sigma(5)} - e_{\sigma(6)} \pm (e_7 - e_8)), \sigma \in P(6)\}.$$

$\mathbf{e}_{7(7)}$: It is the noncompact real form of $\mathbf{e}_7$ with the maximal compact subgroup $K = SU(8)$. The root system of $\mathbf{e}_7$ is

$$\begin{aligned} E_7 &= \{e_i - e_j, 1 \le i,j \le 8, i \ne j\} \cup \\ &\quad \{\frac{1}{2}(e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)} + e_{\sigma(4)} - e_{\sigma(5)} - e_{\sigma(6)} - e_{\sigma(7)} - e_{\sigma(8)}), \sigma \in P(8)\}, \end{aligned}$$

here $P(8)$ is the permutation group of $\{1,2,3,4,5,6,7,8\}$. The root system of $\mathfrak{sl}(8,\mathbf{C})$, embedded in E_7, is

$$A_7 = \{e_i - e_j, i \ne j, 1 \le i,j \le 8\}.$$

Thus, corresponding to the noncompact real form $\mathbf{e}_{7(7)}$ and its compact Cartan subalgebra, $\mathbf{e}_7$ has noncompact roots

$$\{\frac{1}{2}(e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)} + e_{\sigma(4)} - e_{\sigma(5)} - e_{\sigma(6)} - e_{\sigma(7)} - e_{\sigma(8)}), \sigma \in P(8)\}.$$

$\mathbf{e}_{7(-5)}$: It is the noncompact real form of $\mathbf{e}_7$ with the maximal compact subgroup $K = SO(12)\times SU(2)$. The root system of $\mathbf{e}_7$ is

$$\begin{aligned} E_7 &= \{e_i - e_j, 1 \le i,j \le 8, i \ne j\} \cup \\ &\quad \{\frac{1}{2}(e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)} + e_{\sigma(4)} - e_{\sigma(5)} - e_{\sigma(6)} - e_{\sigma(7)} - e_{\sigma(8)}), \sigma \in P(8)\}, \end{aligned}$$

here $P(8)$ is the permutation group of $\{1,2,3,4,5,6,7,8\}$. The root system of $\mathfrak{so}(12,\mathbf{C})+\mathfrak{sl}(2,\mathbf{C})$, embedded in E_7, is

$$\begin{aligned} D_6+A_1 &= \{e_i-e_j, 1\le i,j\le 6, i\neq j\}\cup \\ &\{\pm\frac{1}{2}(e_{\sigma(1)}+e_{\sigma(2)}+e_{\sigma(3)}+e_{\sigma(4)}-e_{\sigma(5)}-e_{\sigma(6)}-e_7-e_8)\}\bigcup \\ &\{\pm(e_7-e_8)\}. \end{aligned}$$

(Note that if letting $\mathbf{R}^n$ have the standard basis $\{f_1,\cdots f_n\}$, $D_n=\{\pm f_i\pm f_j, i\neq j\}$; so we need to construct an isomorphism between D_6 and $\{e_i-e_j, 1\le i,j\le 6, i\neq j\}\cup\{\pm\frac{1}{2}(e_{\sigma(1)}+e_{\sigma(2)}+e_{\sigma(3)}+e_{\sigma(4)}-e_{\sigma(5)}-e_{\sigma(6)}-e_7-e_8)\}$. This is done by the uniqueness: $\{e_i-e_j, 1\le i,j\le 6, i\neq j\}\cup\{\pm\frac{1}{2}(e_{\sigma(1)}+e_{\sigma(2)}+e_{\sigma(3)}+e_{\sigma(4)}-e_{\sigma(5)}-e_{\sigma(6)}-e_7-e_8)\}$ indeed is a root system of cardinality 60; on the other hand, the root system of cardinality 60 is only D_n by the Cartan classification theorem.)
Therefore, corresponding to the noncompact real form $\mathbf{e}_{7(-5)}$ and its compact Cartan subalgebra, $\mathbf{e}_7$ has noncompact roots

$$\begin{aligned} &\{\pm(e_i-e_7),\pm(e_i-e_8), 1\le i\le 6\}\cup \\ &\{\pm\frac{1}{2}(e_{\sigma(1)}+e_{\sigma(2)}+e_{\sigma(3)}-e_{\sigma(4)}-e_{\sigma(5)}-e_{\sigma(6)}+e_7-e_8)\}. \end{aligned}$$

$\mathbf{e}_{8(8)}$: It is the noncompact real form of $\mathbf{e}_8$ with the maximal compact subgroup $K=SO(16)$. The root system of $\mathbf{e}_8$ is

$$E_8=\{\pm e_i\pm e_j, \frac{1}{2}\sum_{i=1}^{8}(-1)^{m(i)}e_i \text{ with } \sum m(i) \text{ being even}, 1\le i,j\le 8\},$$

where $m(i)$ is 0 or 1. The root system of $\mathfrak{so}(16,\mathbf{C})$, embedded in E_8, is $D_8=\{\pm e_i\pm e_j, 1\le i,j\le 8\}$. Therefore, corresponding to the noncompact real form $\mathbf{e}_{8(8)}$ and its compact Cartan subalgebra, $\mathbf{e}_8$ has noncompact roots

$$\{\frac{1}{2}\sum_{i=1}^{8}(-1)^{m(i)}e_i \text{ with } \sum m(i) \text{ being even}, 1\le i,j\le 8\}.$$

$\mathbf{e}_{8(-24)}$: It is the noncompact real form of $\mathbf{e}_8$ with the maximal compact subgroup $K=\mathbf{e}_{7(-133)}\times SU(2)$. The root system of $\mathbf{e}_8$ is

$$E_8=\{\pm e_i\pm e_j, \frac{1}{2}\sum_{i=1}^{8}(-1)^{m(i)}e_i \text{ with } \sum m(i) \text{ being even}, 1\le i,j\le 8\}.$$

The root system of $\mathbf{e}_7 + sl(2, \mathbf{C})$, embedded in E_8, is

$$E_7 + A_1 = \{e_i - e_j, 1 \leq i, j \leq 8, i \neq j\} \cup$$
$$\{\frac{1}{2}(e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)} + e_{\sigma(4)} - e_{\sigma(5)} - e_{\sigma(6)} - e_{\sigma(7)} - e_{\sigma(8)}), \sigma \in P(8)\} \bigcup$$
$$\{\pm\frac{1}{2}(e_1 + e_2 + \cdots + e_8)\}.$$

Thus, corresponding to the noncompact real form $\mathbf{e}_{8(-24)}$ and its compact Cartan subalgebra, $\mathbf{e}_8$ has noncompact roots

$$\{\pm(e_i + e_j), 1 \leq i < j \leq 8\} \cup$$
$$\{\frac{1}{2}(e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)} + e_{\sigma(4)} + e_{\sigma(5)} + e_{\sigma(6)} - e_{\sigma(7)} - e_{\sigma(8)}), \sigma \in P(8)\}.$$

$\mathbf{f}_{4(4)}$: It is the noncompact real form of $\mathbf{f}_4$ with the maximal compact subgroup $Sp(3) \times SU(2)$. The root system of $\mathbf{f}_4$ is

$$F_4 = \{\pm e_i, \pm e_i \pm e_j \ (1 \leq i, j \leq 4, i \neq j), \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\};$$

while the root system of $\mathfrak{sp}(3, \mathbf{C}) + \mathfrak{sl}(2, \mathbf{C})$, embedded in F_4, is

$$C_3 + A_1 = \{\pm 2f_i, \pm f_i \pm f_j, 1 \leq i, j \leq 3, i \neq j\} \bigcup \{\pm(e_3 + e_4)\}$$

where $f_1 = \frac{1}{2}(e_1 - e_2), f_2 = \frac{1}{2}(e_1 + e_2), f_3 = \frac{1}{2}(e_3 - e_4)$. Thus, corresponding to the noncompact real form $\mathbf{f}_{4(4)}$ and its compact Cartan subalgebra, $\mathbf{f}_4$ has noncompact roots

$$\{\pm e_3, \pm e_4, \pm e_i \pm e_j, i = 1, 2, j = 3, 4\} \cup$$
$$\{\frac{1}{2}(\pm(e_1 - e_2) \pm (e_3 + e_4)), \frac{1}{2}(\pm(e_1 + e_2) \pm (e_3 + e_4))\}.$$

$\mathbf{f}_{4(-20)}$: It is the noncompact real form of $\mathbf{f}_4$ with the maximal compact subgroup $SO(9)$. The root system of $\mathbf{f}_4$ is

$$F_4 = \{\pm e_i, \pm e_i \pm e_j \ (1 \leq i, j \leq 4, i \neq j), \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\};$$

while the root system of $\mathfrak{so}(9, \mathbf{C})$, embedded in F_4, is $B_4 = \{\pm e_i, \pm e_i \pm e_j, 1 \leq i, j \leq 4, i \neq j\}$. Therefore, corresponding to the noncompact real form $\mathbf{f}_{4(-20)}$ and its compact Cartan subalgebra, $\mathbf{f}_4$ has noncompact roots

$$\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

$\mathbf{g}_{2(2)}$: It is the noncompact real form of $\mathbf{g}_2$ with the maximal compact subgroup $SU(2) \times SU(2)$. The root system of $\mathbf{g}_2$ is

$$G_2 = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\},$$

where $\alpha = e_1, \beta = -\frac{3}{2}e_1 + \frac{\sqrt{3}}{2}e_2$; while the root system of $\mathfrak{sl}(2,\mathbf{C}) + \mathfrak{sl}(2,\mathbf{C})$, embedded in G_2, is $A_1 + A_1 = \{\pm\beta\} \bigcup \{\pm(2\alpha+\beta)\}$. Therefore the noncompact root system is

$$\{\pm\alpha, \pm(\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}.$$

Summing the above all up, it is easy to check that the noncompact roots can generate the compact roots, i.e. for any compact root α there exist two noncompact roots β and γ satisfying $\alpha = \beta + \gamma$.

3 Heat flow for horizontal harmonic maps

Let $\pi : X \to B$ be a Riemannian submersion, $\mathcal{H}$ the corresponding horizontal distribution, and M a compact Riemannian manifold. Assume that $\pi : X \to B$ satisfies the conditions stated in the Introduction, i.e. the Chow condition and B having non-positive sectional curvature and that X is complete under both the Carnot-Caratheodory distance and the Riemannian metric. Consider the following heat equation on M

$$(*) \qquad \mathcal{H}\tau(u) - \frac{\partial u}{\partial t} = 0,$$

where $\mathcal{H}$ represents the projection to $\mathcal{H}$, and τ is the tension field (nonlinear Laplacian) of u. Assume that u has initial data $u(\cdot, 0) = g(\cdot)$. We always assume that g is a smooth horizontal map from M to X. We wish to obtain some horizontal harmonic map from M into X by solving the above heat equation for the initial data g, when X is considered as a Riemannian manifold.

Lemma 1 *There exists a positive number T, such that the equation $(*)$ with the initial data g has a smooth solution $u(x,t)$ for $t \in [0,T)$ satisfying $u(\cdot,t) \in B_{g,\mathcal{H}}(M;X)$. Furthermore, if $u(x,t)$ is a solution of $(*)$ with $u(\cdot,0) = g(\cdot)$ for $t \in [0,T'), T' > 0$, then $u(\cdot,t) \in B_{g,\mathcal{H}}(M;X)$ and hence $\frac{u(\cdot,t)}{\partial t}$ is a horizontal tangent vector field of X for any $t \in [0,T')$.*

Proof. The first part of the lemma is essentially a standard result if one restricts the problem to the space $B_{g,\mathcal{H}}(M;X)$: The symbol of the linearization of the operator $\mathcal{H}\tau$ is just an isomorphism from the horizontal tangent subbundle of X to itself, so $\mathcal{H}\tau$ is elliptic. Thus one can still apply the implicit function theorem to the present case, as one applies the implicit function theorem to the usual harmonic map heat flow, to obtain the short-time existence. As for the second part, it is also easy to see from the following discussion. Since $\mathcal{H}\tau(u)$ is a horizontal vector on $B(M;X)$, i.e. a horizontal vector field on X, so $\frac{\partial u}{\partial t}$ is also horizontal. Fix a point $x \in M$ and take arbitrarily a curve $\gamma(s)$ starting from x for $s \in [0, s_0]$ and a vertical tangent vector V at $g(x)$, translate parallelly V along the t-curve $u(x,t)$ and then the s-curves $u(\gamma(s),t)$, still denoted by V. Note that V is not necessarily parallel, even not continuous, along the t-curves $u(\gamma(s),t)$ for $s \neq 0$. Compute $\frac{\partial}{\partial t} < \frac{\partial}{\partial s} u(\gamma(0),t), V >$

$$\begin{aligned} &\frac{\partial}{\partial t} < \frac{\partial}{\partial s} u(\gamma(0),t), V > = < \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} u(\gamma(0),t), V > \\ = \ & < \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} u(\gamma(0),t), V > = \frac{\partial}{\partial s} < \frac{\partial}{\partial t} u(\gamma(0),t), V > = 0. \end{aligned}$$

Since $< \frac{\partial}{\partial s} u(\gamma(0),t), V > |_{t=0} = < \frac{\partial}{\partial s} g(\gamma(0)), V > = 0$, so $< \frac{\partial}{\partial s} u(\gamma(0),t), V > = 0$. Thus $u(\cdot,t)$ is horizontal. Then, the horizontality of $\frac{u(\cdot,t)}{\partial t}$ implies $u(\cdot,t) \in B_{g,\mathcal{H}}(M;X)$. The lemma is obtained.

Let $e(u)(x,t) = \frac{1}{2}|\nabla u|^2(x,t)$ be the energy density of $u(\cdot,t)$ for $t \in [0,T)$. Denote the Laplace operator of M by Δ and take $\{e_i\}$ as a normal frame of M; denote the Ricci tensor of M by Ric^M and the curvature tensor of X by R^X. By $\mathcal{V}$ we mean to take the vertical component of vectors. Then compute $(\Delta - \frac{\partial}{\partial t})e(u)$:

$$\begin{aligned} (\Delta - \frac{\partial}{\partial t})e(u) &= < \nabla_{e_i}\nabla_{e_i}\mathrm{d}u, \mathrm{d}u > + |\nabla \mathrm{d}u|^2 - < \nabla \frac{\partial u}{\partial t}, \mathrm{d}u > \\ &= < \nabla(\mathcal{V}\tau(u)), \mathrm{d}u > + |\nabla \mathrm{d}u|^2 + < \mathrm{Ric}^M(\mathrm{d}u(e_i), \mathrm{d}u(e_i)) > \\ &\quad - < R^X(\mathrm{d}u(e_i), \mathrm{d}u(e_j))\mathrm{d}u(e_i), \mathrm{d}u(e_j) > \\ &= -|\mathcal{V}\tau(u)|^2 + |\nabla \mathrm{d}u|^2 + < \mathrm{Ric}^M(\mathrm{d}u(e_i), \mathrm{d}u(e_i)) > \\ &\quad - < R^X(\mathrm{d}u(e_i), \mathrm{d}u(e_j))\mathrm{d}u(e_i), \mathrm{d}u(e_j) > . \end{aligned}$$

In the second equality above we used the Weitzenböck formula and the equation $(*)$; in the last equality we used the horizontality of u. The following observation is important for the present study.

Lemma 2 *Let $\pi : X \to B$ be a Riemannian submersion. Then, for any horizontal map u from a Riemannian manifold M into X, the vertical part $\mathcal{V}\tau(u)$ of its stress-energy tensor $\tau(u)$ vanishes.*

Remark. Since the horizontal distribution $\mathcal{H}$ is generally not integrable, so the vertical part of the Hessian of a horizontal map u does not necessarily vanish.

Proof. We first review an idea of B. O'Neill [2, 8]. According to B. O'Neill, one can define a type $(2,1)$-tensor field on X, denoted by A, as follows: for any two vector fields Y, Z on X,

$$A(Y,Z) = \mathcal{H}\nabla_{\mathcal{H}Y}\mathcal{V}Z + \mathcal{V}\nabla_{\mathcal{H}Y}\mathcal{H}Z,$$

here $\mathcal{H}$ and $\mathcal{V}$ mean taking the horizontal part and the vertical part respectively, as mentioned before. An easy calculation shows that A indeed is a tensor field on X, namely, the value of $A(Y,Z)$ at any fixed point x depends only on the values of Y and Z at x, although its definition does depend on the value of Y and Z on a small neighborhood of x; moreover, it has the following key property (here we state slightly more than we actually need):

$$A(Y,Z) = -A(Z,Y) = \frac{1}{2}\mathcal{V}[Y,Z]$$

for any two horizontal vectors Y and Z. The proof of this property is simple: It is sufficient to show $A(Y,Y) = 0$. Namely if this is the case, $A(Y+Z, Y+Z) = A(Y,Z) + A(Z,Y) = 0$; and, by the definition of A,

$$\mathcal{V}[Y,Z] = \mathcal{V}\nabla_Y Z - \mathcal{V}\nabla_Z Y = A(Y,Z) - A(Z,Y).$$

Since A is a tensor, one can take the horizontal vector field Y being the unique lift of a vector field Y' on B, i.e. $\pi_*(Y) = Y'$. Let U be any vertical vector field on X. Then we have $\pi_*[Y,U] = [\pi_* Y, \pi_* U] = 0$, namely $[Y,U]$ is a vertical vector field on X. Thus one has, by the torsion-freeness of the connection ∇,

$$\begin{aligned} & < A(Y,Y), U > = < \nabla_Y Y, U > = - < Y, \nabla_Y U > \\ = & - < Y, [Y,U] + \nabla_U Y > = - < Y, \nabla_U Y > = -\frac{1}{2} U|Y|^2. \end{aligned}$$

Since Y is the lift of a vector field Y of B, so $|Y|^2$ is constant on any fiber of $\pi : X \to B$, and hence $< A(Y,Y), U > = 0$ for any vertical vector U. On the other hand, by the definition, $A(Y,Y)$ is a vertical vector, so $A(Y,Y) = 0$.

We now turn to the proof of the lemma. Take a normal frame $\{e_i\}$ of M and a orthogonal frame of X as follows: $\{e_\alpha, e_\beta, e_\gamma, \cdots, e_\mu, e_\nu, \cdots\}$ with the properties $\{e_\alpha, \cdots\}$ being horizontal and $\{e_\mu, \cdots\}$ vertical (note that, under such a restriction, one cannot get a normal frame in general). Then, under these frames, the stress-energy tensor of the horizontal map u can be written as

$$\tau(u) = \sum_i \nabla \mathrm{d}u(e_i, e_i) = \sum_{i,\alpha} u^\alpha_{ii} e_\alpha + \sum_{i,\alpha,\beta} u^\alpha_i u^\beta_i \nabla_{e_\beta} e_\alpha,$$

and hence its vertical part is $\sum_{i,\alpha,\beta} u^\alpha_i u^\beta_i \mathcal{V}\nabla_{e_\beta} e_\alpha$, which, by the previous discussion, is just

$$\mathcal{V}\tau(u) = \sum_{i,\alpha,\beta} u^\alpha_i u^\beta_i A(e_\alpha, e_\beta) = \sum_i A(\mathrm{d}u(e_i), \mathrm{d}u(e_i)) = 0.$$

This completes the proof of the lemma.

The lemma 1 tells us that the solution $u(\cdot,t)$ to $(*)$ is horizontal for any $t\in[0,T)$, so $\mathcal{V}\tau(u(\cdot,t))=0$ for $t\in[0,T)$. Thus, by the previous computation, we actually obtain

$$\begin{aligned}(\Delta-\frac{\partial}{\partial t})e(u) &= |\nabla \mathrm{d}u|^2+<\mathrm{Ric}^M(\mathrm{d}u(e_i),\mathrm{d}u(e_i))> \\ &\quad -<R^X(\mathrm{d}u(e_i),\mathrm{d}u(e_j))\mathrm{d}u(e_i),\mathrm{d}u(e_j)>.\end{aligned}$$

By the assumption on $\pi:X\to B$, X has non-positive sectional curvature in the horizontal direction, so we have

$$(\Delta-\frac{\partial}{\partial t})e(u)\geq ce(u),$$

for some constant c, which only depends on M. Denote the total energy of $u(\cdot,t)$ by $E(u(\cdot,t))$ for $t\in[0,T)$, i.e. $E(u(\cdot,t))=\int_M e(u(\cdot,t))\mathrm{d}x$. Then, one has

$$\begin{aligned}&\frac{d}{dt}E(u(\cdot,t))=\frac{d}{dt}\int_M <du,du>dx=\int_M <\nabla_{\frac{\partial}{\partial t}}du,du>dx\\ =&\int_M <\nabla\frac{\partial u}{\partial t},du>dx=-\int_M <\frac{\partial}{\partial t}u,\tau(u)>dx=-\int_M |\mathcal{H}\tau(u)|^2dx\leq 0.\end{aligned}$$

Summing all the above up, we have

Lemma 3 *Suppose $u(x,t)$ is a solution of $(*)$. Then for some constant c,*

$$(\Delta-\frac{\partial}{\partial t})e(u)\geq ce(u);$$

furthermore, the total energy $E(u(\cdot,t))$ is a decreasing function of t.

Combining the above lemma with Lemma 2.3.1 in [5], one has

Lemma 4 *Let $t>0$, $0<R<\min(i(M),\frac{\pi}{2\Lambda})$, where $i(M)$ is the injective radius of M, and Λ^2 is an upper bound for the sectional curvature of M. Then, for all $x\in M$,*

$$e(u)(x,t)\leq c(tR^{-m-2}+t^{-\frac{m}{2}})\int_M e(g)(y)dy,$$

where $m=dimM$ and c is some constant depending only on the geometry of M; and for any $t_0<t$, in particular $t_0=0$,

$$e(u)(x,t)\leq cR^{-2}\sup_{x\in M}e(u)(y,t_0).$$

In the following, we want to derive a stability lemma. Let $g(x,s)$ be a smooth horizontal family of smooth horizontal maps from M to X with parameter $s \in [0,s_0]$, i.e. both $g(\cdot,s)$ for any $s \in [0,s_0]$ and $\frac{\partial g}{\partial s}$ being horizontal. Suppose that $u(x,t,s)$ is a family of solutions of $(*)$ with initial data $g(x,s)$ for $0 \le s \le s_0$. As pointed out before, $\frac{\partial u}{\partial t}$ is horizontal; using the same discussion as in Lemma 1, we now show that $\frac{\partial u}{\partial s}$ is also horizontal: Fixing $x \in M$ and $s_1 \in [0,s_0]$, one can then consider $u(x,t,s)$ as a variation of the curve $u(x,t,s_1)$. Take arbitrarily a vertical tangent vector V at $u(x,0,s_1)$ and translate parallelly V along the t-curve $u(x,t,s_1)$ and then the s-curves $u(x,t,s)$, still denoted by V. (Note that V is not necessarily parallel along other t-curves $u(x,t,s)$ for $s \neq s_1$.) Compute $\frac{\partial}{\partial t} < \frac{\partial u}{\partial s}, V >$:

$$\frac{\partial}{\partial t} < \frac{\partial u}{\partial s}, V > = < \nabla_{\frac{\partial}{\partial t}} \frac{\partial u}{\partial s}, V > = < \nabla_{\frac{\partial}{\partial s}} \frac{\partial u}{\partial t}, V > = \frac{\partial}{\partial s} < \frac{\partial u}{\partial t}, V > = 0;$$

on the other hand, $< \frac{\partial u}{\partial s}, V > |_{t=0} = 0$, therefore $< \frac{\partial u}{\partial s}, V > = 0$. Thus, for any fixed $t \in [0,T)$ and $s \in [0,s_0]$, the derivative of u with respect to s, $\frac{\partial u}{\partial s}(\cdot,t,s)$, can be considered as a horizontal vector field of X (if necessary, it can be considered as some section of a certain pull-back bundle). Using the horizontality of $\frac{\partial u}{\partial s}(\cdot,t,s)$, we then have

Lemma 5 *For every $s \in [0,s_0]$, the quantity*

$$\sup_{x\in M} |\frac{\partial u}{\partial s}|^2(x,t,s)$$

is decreasing in t. Hence also the quantity

$$\sup_{x\in M, s\in[0,s_0]} |\frac{\partial u}{\partial s}|^2(x,t,s)$$

is a decreasing function in t.

Proof. As before, one can compute under a normal frame $\{e_i\}$

$$\begin{aligned} &(\Delta - \frac{\partial}{\partial t})|\frac{\partial u}{\partial s}|^2 \\ = \;& 2|\nabla \frac{\partial u}{\partial s}|^2 - 2\sum_i < R(\frac{\partial u}{\partial s}, \mathrm{d}u(e_i))\frac{\partial u}{\partial s}, \mathrm{d}u(e_i) > . \end{aligned}$$

Here we use the heat eqaution and $\frac{\partial u}{\partial s}$'s horizontality. Thus, by the assumption on the sectional curvature in the horizontal direction, we have

$$(\Delta - \frac{\partial}{\partial t})|\frac{\partial u}{\partial s}|^2 \ge 0.$$

The lemma then follows from the maximum principle for parabolic equations.

In order to apply the regularity theorems for elliptic equations, we have to make sure that the solution of $(*)$ with the given initial data g lies in a suitable coordinate chart of X when the domain considered is small enough and the time interval enough short. We have obtained a point-wise upper bound for the derivatives of u with respect to the space variables, so we still have to derive a bound for the time derivative of the solution. This can be done by applying the above lemma.

Lemma 6 *Suppose that $u(x,t)$ is a solution of $(*)$ with the initial data g for $t \in [0,T)$. Then for all $t \in [0,T)$ and $x \in M$*

$$|\frac{\partial u(x,t)}{\partial t}| \leq \sup_{y \in M} |\frac{\partial u(y,0)}{\partial t}|.$$

Proof. Setting $u(x,t,s) = u(x,t+s)$, then $u(x,t,s)$ can be considered a family of solutions to $(*)$ with a family of initial data $u(x,s)$. Applying the preceeding lemma to $u(x,t,s)$, we then get the present lemma.

Fix $x \in M$ and $t \in [0,T)$. As before, we take a normal frame $\{e_i\}$ at x, and a orthogonal frame $\{e_\alpha, e_\beta, e_\gamma, \cdots, e_\mu, e_\nu, \cdots\}$ at $u(x,t)$ with the property that $\{e_\alpha, \cdots\}$ are horizontal and $\{e_\mu, \cdots\}$ are vertical; as pointed out before, under such a restriction, one cannot get a normal frame at $u(x,t)$ in general. Then, the heat equation $(*)$ can be rewritten under such frames at (x,t) as

$$(*') \qquad \sum_i u^\alpha_{ii} + \sum_{i,\beta,\gamma} \Gamma^\alpha_{\beta\gamma} u^\beta_i u^\gamma_i = \frac{\partial u^\alpha}{\partial t}.$$

Remark. Note that the solution with the initial data g is horizontal for both the space variable and the time variable, as seen in Lemma 1. So by Lemma 2, $\mathcal{V}\tau(u) = 0$, i.e. $\mathcal{H}\tau(u) = \tau(u)$. Thus we can actually omit $\mathcal{H}$ in the equation $(*)$ and think that u just satisfies the usual heat equation for harmonic maps, $\tau(u) - \frac{\partial u}{\partial t} = 0$. In the following estimate, we will actually adopt this point of view although it will not be pointed out explicitly.

Lemma 7 *Suppose that $u(x,t)$ is a solution of $(*)$ (or $(*')$) with the initial data g for $t \in [0,T)$. Then for every $\alpha \in (0,1)$*

$$\|u(\cdot,t)\|_{C^{2+\alpha}(M;X)} + \|\frac{\partial u}{\partial t}(\cdot,t)\|_{C^\alpha(M;X)} \leq c,$$

where c depends on α, the initial data $g(x)$, and the geometry of M and X, but not on t.

Proof. Rewrite $(*')$ as

$$\sum_i u^\alpha_{ii} = -\sum_{i,\beta,\gamma} \Gamma^\alpha_{\beta\gamma} u^\beta_i u^\gamma_i + \frac{\partial u^\alpha}{\partial t}.$$

If we restrict the solution u to a suitable small coordinate chart at the point $x_0 \in M$, say $B(x_0, \rho)$ with ρ enough small, and a suitable small time interval $[t_0, t_1]$, $u(x,t)$ will stay in a certain coordinate chart of X by the lemma 4 and the lemma 6; moreover, those two lemmata also imply that the right-hand side of the above equation is bounded (note that the bound does not depend on t), this, by the elliptic regularity theory, then implies a bound (again not depend on t) for $\|u(\cdot,t)\|_{C^{1+\alpha}(M;X)}$ on a smaller coordinate chart, say $B(x_0, \frac{\rho}{2})$ (see [5], Theorem 2.2.1). Thus, the right-hand side of the following parabolic equation

$$\frac{\partial u^\alpha}{\partial t} - \sum_i u^\alpha_{ii} = \sum_{i,\beta,\gamma} \Gamma^\alpha_{\beta\gamma} u^\beta_i u^\gamma_i$$

is bounded (the bound being independent of t) in $C^\alpha(M;X)$, and hence the Schauder estimate for parabolic equations then implies the estimate in the lemma, at least in the above small coordinate chart; but M is compact, so the estimate is valid on M.

Based on the local existence for solutions and the above Schauder estimate, one has the following global existence theorem for $(*)$ with the initial data g.

Theorem 1 *The solution $u(x,t)$ of the heat equation $(*)$ with the horizontal initial data g exists for all $t \in [0,\infty)$, if the Riemannian submersion $\pi : X \to B$ satisfies the Chow condition and B has non-positive sectional curvature.*

In the following, we will show that the global solution $u(\cdot,t)$ in the theorem above converges to a horizontal harmonic map as t goes to infinity. As seen before, we have shown the energy decay formula, namely

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(\cdot,t)) = -\int_M |\frac{\partial u(x,t)}{\partial t}|^2 \mathrm{d}x = -\int_M |\mathcal{H}\tau(u)|^2 \mathrm{d}x;$$

observe also that the energy function $E(u(\cdot,t))$ in t is nonnegative for $t \in [0,\infty)$, so there exists a sequence $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ as $n \to \infty$ satisfying $\frac{\mathrm{d}}{\mathrm{d}t} E(u(\cdot,t))|_{t_n} \to 0$ as $n \to \infty$, this is just equivalent to $\int_M |\frac{\partial u}{\partial t}(x,t_n)|^2 \mathrm{d}x \to 0$ as $n \to \infty$. On the other hand, as seen in Lemma 7, $\frac{\partial u}{\partial t}(\cdot,t)$ has a C^α-bound independent of the time t, so we obtain

Lemma 8 *There exists a sequence $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ as $n \to \infty$, for which $\frac{\partial u}{\partial t}(x,t_n)$ converges to zero uniformly in $x \in M$ as $n \to \infty$.*

Lemma 7 also tells us that $u(\cdot, t)$ has a time-independent $C^{2+\alpha}$-bound, so one obtains, by possibly passing to a subsequence of $\{t_n\}$, that $u(\cdot, t_n)$ converges at least C^2-uniformly to a map $u : M \to X$, which then is also horizontal; furthermore, since $\{u(\cdot, t_n)\}$ is at least C^2-uniformly convergent to u and both $u(\cdot, t_n)$ and u are horizontal, so by the Hopf-Rinow theorem, as mentioned in the Introduction, some $u(\cdot, t_n)$, and hence $g(\cdot)$, is homotopic to $u(\cdot)$ by some horizontal homotopy $h(\cdot, s)$ for $s \in [0, 1]$ with $h(\cdot, 0) = u(\cdot, t_n)$ and $h(\cdot, 1) = u(\cdot)$. Here by the homotopy $h(\cdot, s)$ being horizontal we mean that $h(\cdot, s)$ for each $s \in [0, 1]$ is a horizontal map and the s-curves are also horizontal. Again since $\{u(\cdot, t_n)\}$ uniformly converges to u, w.l.o.g., we can assume that the lengths of the s-curves $h(x, s)$ have a sufficiently small upper bound $\epsilon > 0$ independent of $x \in M$. Now, consider the family of the solutions $u'(x, t, s)$ ($s \in [0, 1]$) to $(*)$ with $h(x, s)$ as the family of initial maps. It is clear that $u'(x, t, 1) = u(x)$ since $\mathcal{H}\tau(u) = 0$ and $h(x, 1) = u(x)$; while $u'(x, t, 0) = u(x, t + t_n)$. By the Lemma 5, the supermum with respect to x of the length of s-curves $u'(\cdot, t, s)$ is a decreasing function in t and hence less than ϵ. Since ϵ is arbitrary, we have that $u(x, t)$ converges uniformly to $u(x)$ in t in the sense of C^0, not only for a subsequence $\{t_n\}$. Applying this to the heat equation $(*)$, one obtains

$$\mathcal{H}\tau(u) = 0.$$

Finally, the horizontality of $u(x)$ and the Lemma 2 tell us that $\mathcal{V}\tau(u) = 0$, and hence

$$\tau(u) = 0,$$

i.e. the limit u is a horizontal harmonic map. Thus we have

Theorem 2 *Suppose that $\pi : X \to B$ is a Riemannian submersion satisfying the Chow conditions and that B has non-positive sectional curvature. Let M be a compact Riemannian manifold and $g : M \to X$ a horizontal smooth map from M to X. Then there exists a horizontal harmonic map $u : M \to X$ from M into X that is homotopic to g by a horizontal homotopy.*

Remark. The theorem above is actually valid in a more general setting, namely the equivariant one: Let $\phi : \pi_1(M) \to \pi_1(X)$ be a homomorphism and g a ϕ-equivariant map from M into X, then one can solve the corresponding heat equation $(*)$ and obtain similar results, e.g. the existence for ϕ-equivariant horizontal harmonic maps. We omit this, but point out that in applications we shall just use that setting. We will come back to this in [6].

References

[1] A. Bellaïche, The tangent space in sub-Riemannian geometry, in: *Sub-Riemannian Geometry*, Progress in Mathematics, Birkhäuser, 1996, 1-78.

[2] A. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge Band 10, Springer-Verlag, 1987.

[3] P. Griffiths and W. Schmid, Locally homogeneous complex manifolds, Acta Mathematica, 123, 1969, 253-302.

[4] S. Helgason, *Differential Geometry and Symmetric Spaces*, New York, Academic Press, 1962.

[5] J. Jost, *Nonlinear Methods in Riemannian and Kählerian Geometry*, DMV Seminar, Band 10, Birkhäuser, 1988.

[6] J. Jost and Y.-H. Yang, Horizontal harmonic maps into locally homogeneous complex manifolds, in preparation.

[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol.II, Interscience Publishers, 1969.

[8] B. O'Neill, The fundamental equations of a submersion, Michigan Mathematical Journal, **13**, 1966, 459-469.

[9] J.-P. Serre, *Complex Semisimple Lie Algebras*, Springer-Verlag, New York, 1987.

[10] C. Simpson, Higgs bundles and Local systems, Publ. I. H. E. S. 75, 1992, 5-95.

Jürgen Jost:
Max-Planck-Institute for Mathematics in the Sciences, Leipzig, Germany
and
Yi-Hu Yang:
Department of Applied Mathematics, Tongji University, Shanghai, China
e-mail: yhyang@mail.tongji.edu.cn