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Parabolic systems with nowhere smooth solutions

by

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Abstract

We construct smooth 2×2 parabolic systems with smooth initial data and C^α right hand side which admit solutions that are nowhere C^1. The elliptic part is in variational form and the corresponding energy ϕ is strongly quasiconvex and in particular satisfies a uniform Legendre-Hadamard (or strong ellipticity) condition.

1 Main results

In this paper we construct smooth 2×2 parabolic systems with smooth initial data and C^α right hand side which admit solutions which are nowhere

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C^1. The elliptic part is in variational form and the corresponding energy ϕ is strongly quasiconvex and in particular satisfies a uniform Legendre-Hadamard (or strong ellipticity) condition.

Theorem 1 *Let Ω be the unit ball in $\mathbb{R}^2$. Let $\eta > 0, T > 0, \alpha \in (0,1)$. Then there exists a function $\phi : \mathbb{R}^{2\times 2} \to \mathbb{R}$ such that ϕ is strongly quasiconvex, smooth and $|D^2\phi| \le C$, a function $f \in C^\alpha(\Omega \times [0,T]; \mathbb{R}^2)$ with $||f||_{C^\alpha} < \eta$ and a Lipschitz solution $w : \Omega \times [0,T] \to \mathbb{R}$ of the parabolic system*

$$\partial_t w - \operatorname{div} D\phi(\nabla w) = f \quad \text{in } \Omega \times (0,T) \tag{1}$$

and

$$w(\cdot, 0) \equiv 0, \quad w(t,x) = 0 \ \text{for}\, x \in \partial\Omega \tag{2}$$

such that w is nowhere C^1 in $\Omega \times (0,T)$.

Other unusual features such as non-uniqueness and failure of the energy inequality are discussed in Corollaries 2 and 3 below. We first briefly review the rôle of the assumptions on ϕ and the connections with variational problems.

For $f = 0$ equation (1) is formally the L^2 gradient flow of the functional

$$I(w) = \int_\Omega \phi(\nabla w)\, dx.$$

In the study of minimizers of I, (strong) quasiconvexity plays a crucial rôle. A function $\phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called *strongly quasiconvex* if there exists $C > 0$ such that

$$\int_\Omega \phi(F + \nabla\eta) - \phi(F)\, dx \ge C \int_\Omega |\nabla\eta|^2\, dx \tag{3}$$

for all $\eta \in C_0^\infty(\Omega, \mathbb{R}^m)$, all matrices $F \in \mathbb{R}^{m\times n}$ and all domains Ω with $|\partial\Omega| = 0$ (by a covering argument it suffices to consider a fixed domain Ω, e.g. a ball or a cube). If the inequality holds with $C = 0$ we say that ϕ is quasiconvex.

The importance of quasiconvexity was first realized by Morrey [Mo 52] who showed that (under suitable growth conditions) quasiconvexity is a necessary and sufficient condition for weak lower semicontinuity of the functional I in the Sobolev space $W^{1,p}$ (see also [AF 84, Ma 85]). Thus quasiconvexity of I is closely related to the existence of minimizers of I (see e.g. [Da 89, Mu 99]). Quasiconvexity is also closely related to (partial) regularity. Evans [Ev 86] showed that minimizers of I are smooth outside a closed null set if ϕ satisfies the assumptions of Theorem 1, i.e. if ϕ is strictly quasiconvex, smooth and $|D^2\phi| \le C$. For a recent extension to local minimizers see [KT 01]. More general stationary points of I, however, can be nonsmooth everywhere (see [MS 99] and Section 2 below) and this is a crucial ingredient in the proof of Theorem 1. We finally remark that for C^2-functions strong quasiconvexity implies the Legendre-Hadamard condition (also known as strong ellipticity or uniform rank-1 convexity)

$$D^2\phi(F)(a \otimes b, a \otimes b) \ge C|a|^2|b|^2, \quad C > 0. \tag{4}$$

To see this it suffices to take

$$\eta(x) = \eta_0(x)\frac{\varepsilon a}{k}\sin kbx,$$

where $\eta_0 \in C_0^\infty(\Omega)$ and to study the limit $k \to \infty$, $\varepsilon \to 0$.

Since the subtraction of an affine function from ϕ does not affect its properties we may assume that $\phi(0) = 0$, $D\phi(0) = 0$. This immediately yields a nonuniqueness result:

Corollary 2 *Let ϕ, η, T, α, Ω be as in Theorem 1, with η sufficiently small. Then the initial-boundary value problem* (1), (2), *i.e.*

$$\begin{aligned} \partial_t w - \operatorname{div} D\phi(\nabla w) &= f \quad \text{in } \Omega \times (0, T), \\ w(x,0) = 0, \quad w(t,x) &= 0 \quad \text{for } x \in \partial\Omega \end{aligned}$$

has at least two solutions.

Indeed the Implicit Function Theorem, the C^α theory for linear parabolic systems and the strong ellipticity condition (4) imply that there exists a (smooth) solution w of the initial boundary value problem as long as f is sufficiently small in the C^α-norm.

Our example also shows that Lipschitz solutions of (1), (2) need not satisfy the energy identity

$$\int_\Omega \phi(\nabla w(t,x))\, dx|_{t=0}^T = -\int_0^T \int_\Omega |\partial_t w|^2 \, dx\, dt + \int_0^T \int_\Omega f \partial_t w \, dx\, dt$$

or the energy inequality

$$\int_\Omega \phi(\nabla w(t,x))\, dx|_{t=0}^T \le \frac{1}{4}\int_0^T \int_\Omega |f|^2 \, dx\, dt.$$

Corollary 3 *Under the assumptions of Theorem 1 we can achieve in addition that w satisfies*

$$\int_\Omega \phi(\nabla w(T,x)) - \phi(0)\, dx \geq C \ > \ 0, \quad \text{for all } T > 0.$$

Indeed we will construct a w such that ∇w is uniformly away from 0 and then the assertion follows from strong quasiconvexity.

An interesting open question is whether one can construct (for a general quasiconvex ϕ) solutions of (1), (2) which do satisfy an energy inequality (e.g. by discretization in time and minimization at each time step) and whether such solutions have better regularity properties.

2 Review of the elliptic counterexample

In this section we briefly review the construction of a similar counterexample in the elliptic context. This will allow us to introduce the key ideas and the necessary notation in a simpler setting.

2.1 Reduction to first order

Let Ω be the unit ball in $\mathbb{R}^2$. We seek Lipschitz, nowhere C^1 solutions $w : \Omega \to \mathbb{R}^2$ of the 2×2 system

$$-\operatorname{div} D\phi(\nabla w) = 0 \quad \text{in } \Omega \tag{5}$$

where

$$\phi \text{ strongly quasiconvex, smooth, } |D^2\phi| \leq C. \tag{6}$$

In particular ϕ is strongly elliptic, i.e.

$$D^2\phi(F)(a \otimes b, a \otimes b) \geq c|a|^2|b|^2, \quad c > 0. \tag{7}$$

We first reduce the problem to a first order system. Equation (5) is equivalent to the existence of a potential W such that $D\phi(\nabla w)J = \nabla W$, where J is the 90° rotation. If we introduce

$$u = \begin{pmatrix} w \\ W \end{pmatrix}, \quad u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^4 \tag{8}$$

then (5) is equivalent to

$$\nabla u \in K \subset \mathbb{R}^{4\times 2}, \tag{9}$$

where

$$K = \left\{ \left(\begin{array}{c} X \\ Y \end{array} \right) : Y = D\phi(X)J, X \in \mathbb{R}^{2\times 2} \right\}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \qquad (10)$$

2.2 Rank-1 connections and T_4 configurations

Our goal is to construct highly oscillatory solutions of the partial differential relation $\nabla u \in K$, where K is a given set in $\mathbb{R}^{m\times n}$. This is easy if K contains a rank-1 connection, i.e. if there are matrices $A, B \in K$ with $\mathrm{rk}(B-A) = 1$. In this case $B - A = a \otimes n$ and we can take $u = Ax + ah(x \cdot n)$, where h is a Lipschitz function with $h' \in \{0, 1\}$ a.e.

By a result of Ball [Ba 80], however, sets of the form (9) cannot contain a rank-1 connection if ϕ is uniformly rank-1 convex, i.e. if (4) holds. A crucial observation is that there are simple sets K which have a nontrivial rank-1 convex hull (defined through separation by rank-1 convex functions) even if they contain no rank-1 connections. In the present context of elliptic counterexamples it was first noted by Scheffer in his (unpublished) thesis [Sch 74]. He used this fact to construct highly oscillatory solutions in the Sobolev space $W^{1,1}$ (or $W^{1,2}$ for nonvariational examples). In other contexts such sets where obtained independently by a number of authors [AH 86, CT 93, NM 91, Ta 93].

Definition 4 *Consider quadruples* $\mathbf{M} = (M_1, M_2, M_3, M_4)$ *of matrices* $M_i \in \mathbb{R}^{m\times n}$. *We say that* $\mathbf{M}$ *is a* T_4 *configuration if there exist rank-1 matrices* C_1, C_2, C_3, C_4 *with* $\sum_{j=1}^{4} C_j = 0$, *scalars* $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ *with* $\kappa_i > 0$ *and matrices* $P_j \in \mathbb{R}^{m\times n}$ *such that the relations*

$$\begin{aligned} P_{j+1} - P_j &= C_j \\ M_j - P_{j+1} &= \kappa_j C_j \end{aligned}$$

hold (see Fig. 1).

The simplest example arises already in diagonal 2×2 matrices. One may take

$$M_1 = -M_3 = \left(\begin{array}{cc} 3 & 0 \\ 0 & -1 \end{array} \right), M_2 = -M_4 = \left(\begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right). \qquad (11)$$

We emphasize that in general a T_4 configuration need not lie in a plane.

Extending Gromov's technique of convex integration [Gr 86] one can show that there exist nontrivial maps whose gradients lie arbitrarily close

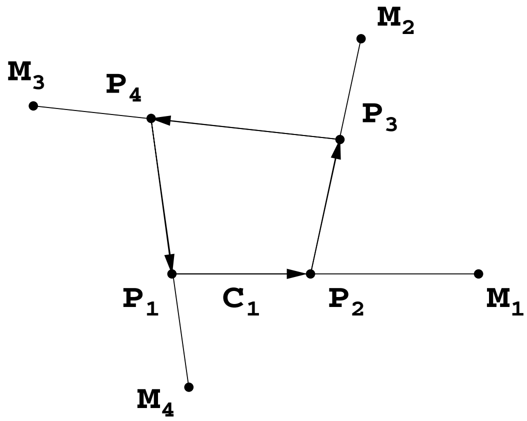

Figure 1: T_4 configuration with $P_1 = P$, $P_2 = P + C_1$, $P_3 = P + C_1 + C_2$, $P_4 = P + C_1 + C_2 + C_3$. The lines indicate rank-1 connections. Note that the figure need not be planar.

to a T_4 configuration (see [MS 99], Theorem 3.1; for other approaches see [DM 99, Ki 01, MSy 01]). We do not repeat the proof here since we use this result only for motivation and will give a self-contained proof of Theorem 1 below.

Proposition 5 *Let $\varepsilon > 0$ and let (M_1, M_2, M_3, M_4) be a T_4 configuration. Let P_1 be as in the definition of a T_4 configuration and let u_0 be the affine map*

$$u_0(x) = P_1 x.$$

Let $B_\varepsilon(M_i)$ denote the ball of radius ε around M_i. Then, for any open bounded set $\Omega \subset \mathbb{R}^n$, there exists a Lipschitz map $u : \Omega \to \mathbb{R}^m$ such that

$$\nabla u \in \cup_{i=1}^4 B_\varepsilon(M_i) \quad a.e.\ in\ \Omega, \tag{12}$$

$$\nabla u = u_0 \quad on\ \partial\Omega, \tag{13}$$

$$\sup_\Omega |u - u_0| < \varepsilon. \tag{14}$$

Remark. Note that for sufficiently small ε there are no rank-1 connections between the four components
$B_\varepsilon(M_i)$.

Our strategy will now be to identify a strongly quasiconvex ϕ such that the set K given by (10) contains many T_4 configurations.

2.3 Embedding many T_4 configurations in K

The first crucial observation is that there exists a strongly quasiconvex and smooth function $\phi : \mathbb{R}^{2\times 2} \to \mathbb{R}$ (with $|D^2\phi| \le C$) such that the set

$$K = \left\{ \begin{pmatrix} X \\ D\phi(X)J \end{pmatrix} : X \in \mathbb{R}^{2\times 2} \right\} \subset \mathbb{R}^{4\times 2}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

admits a T_4 configuration $\mathbf{M}^0$ with $M_i^0 \in K$, see [MS 99], Lemma 4.3. One may take

$$M_1^0 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \\ 0 & -1 \\ 3 & 0 \end{pmatrix}, \quad M_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}, \quad M_3^0 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \\ 0 & 1 \\ -3 & 0 \end{pmatrix}, \quad M_4^0 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \\ 0 & -3 \\ -1 & 0 \end{pmatrix}.$$

This result per se is not enough to conclude the existence of interesting solutions of $\nabla u \in K$ since Proposition 5 guarantees only that we can find solutions whose gradient stays close to the M_i^0, but K is not open, in fact it has codimension 4. To overcome this difficulty we show that $K \times K \times K \times K$ contains not only the special T_4 configuration $\mathbf{M}^0$ but an eight-dimensional family of T_4 configurations and that the corresponding corner points P_i cover an open set in the eight dimensional space $\mathbb{R}^{4\times 2}$. This will give us enough flexibility to carry out the construction both in the elliptic and the parabolic case.

We first recall that the set

$$\mathcal{M} = \{\mathbf{M} \in (\mathbb{R}^{4\times 2})^4 : \mathbf{M} \text{ is a } T_4 \text{ configuration}\} \subset \mathbb{R}^{32}.$$

is a 24-dimensional manifold near $\mathbf{M}^{\mathbf{0}}$(see [MS 99]). Moreover for $\mathbf{M} \in \mathcal{M}$ near $\mathbf{M}^0$ the points P_j and hence C_j and κ_i (introduced in Definition 4) are uniquely determined by $\mathbf{M}$. We also introduce the set

$$\mathcal{K} = \{\mathbf{M} \in (\mathbb{R}^{4\times 2})^4 : M_i \in K\} = K \times K \times K \times K$$

which forms a 16-dimensional manifold near $\mathbf{M}^0$. We denote by π_j and μ_j the maps

$$\begin{aligned} \pi_j : \mathcal{M} \cap \mathcal{K} &\longrightarrow \mathbb{R}^{4\times 2} \\ (M_1, M_2, M_3, M_4) &\longmapsto P_j \\ \mu_j : \mathcal{M} \cap \mathcal{K} &\longrightarrow \mathbb{R}^{4\times 2} \\ (M_1, M_2, M_3, M_4) &\longmapsto M_j \end{aligned}$$

Let $T_{M_j^0}K$ be the tangent space of K at M_j^0, let $Q_j^\perp$ denote the projection onto its orthogonal complement and define the map

$$\begin{aligned} \psi_j : \mathcal{M} \cap K &\longrightarrow \mathbb{R}^{4\times 2} \\ (M_1, M_2, M_3, M_4) &\longmapsto (M_j, Q_j^\perp(P_j - P_j^0)) \end{aligned}$$

Proposition 6 *([MS 99]) There exists a choice of ϕ such that $\mathbf{M}^0 \in K$ (where K and $\mathbf{M}^0$ are given above) and*

(i) *in a neighbourhood of $\mathbf{M^0}$ the manifolds $\mathcal{M}$ and K intersect transversely in an eight dimensional manifold,*

(ii) *π_j and ψ_j are local diffeomorphisms from a neighbourhood of $\mathbf{M}^0$ in $\mathcal{M} \cap K$ to open sets in $\mathbb{R}^{4\times 2}$.*

We now choose an increasing sequence of (small) neighbourhoods $\mathbf{M}^0 \in \mathcal{O}_1 \subset \mathcal{O}_2 \subset ...$ in $\mathcal{M} \cap K$ (which are diffeomorphic to an eight dimensional ball). For $\frac{1}{2} < \lambda_1 < \lambda_2 < ... < 1$ we consider the maps $\lambda_i\mu_j + (1 - \lambda_j)\pi_j$ and we define the sets (see Fig. 2)

$$\mathcal{U}_i^j = (\lambda_i\mu_j + (1 - \lambda_i)\pi_j)(\mathcal{O}_i) \subset \mathbb{R}^{4\times 2}, \quad \mathcal{U}_i = \bigcup_{j=1}^{4} \mathcal{U}_i^j. \tag{15}$$

Using the nondegeneracy of ψ_j one can show that $\mathcal{U}_i^j$ is open (if λ_1 is chosen sufficiently close to 1), see [MS 99]. We also define

$$\mathcal{V}_i^j = \pi_j(\mathcal{O}_i) \subset \mathbb{R}^{4\times 2}, \quad \mathcal{V}_i = \bigcup_{j=1}^{4} \mathcal{V}_i^j \tag{16}$$

and these sets are open by Proposition 6. The i-th order lamination

convex hull $E^{lc,i}$ of a set E is defined by inductively adding rank-1 segments. More precisely we set

$$\begin{aligned} E^{lc,0} &= E \\ E^{lc,i+1} &= E^{lc,i} \cup \{[A,B] : A, B \in E^{lc,i}, \mathrm{rk}(B - A) = 1\}. \end{aligned}$$

Proposition 7 *The sets $\mathcal{U}_i^j$ and $\mathcal{V}_i^j$ have the following properties (see Figs. 2 and 3).*

(i) *If $F \in \mathcal{U}_i^j$ then there exist $A \in \mathcal{U}_{i+1}^j, B \in \mathcal{V}_i^j$, such that* $\mathrm{rk}(B - A) = 1$,

$$F = \frac{\lambda_i}{\lambda_{i+1}} A + (1 - \frac{\lambda_i}{\lambda_{i+1}})B.$$

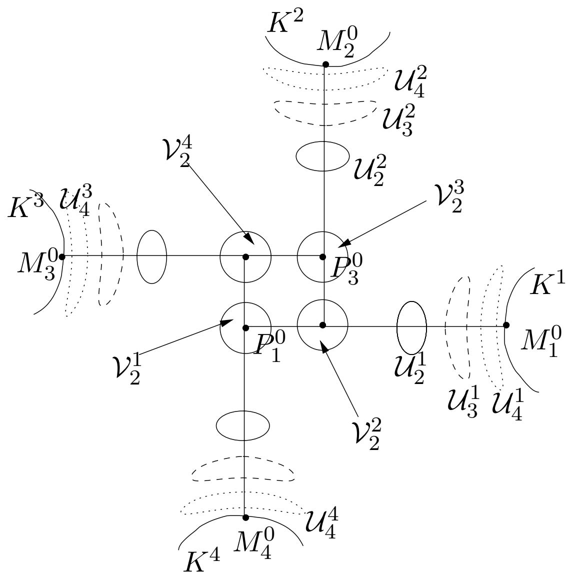

Figure 2: Schematic illustration of the sets $\mathcal{U}_i^j$, $\mathcal{V}_2^j$ and K^j. All the open sets $\mathcal{V}_i = \cup_j \mathcal{V}_i^j$ contain the planar rank-one square with corner points $P_1^0, \ldots P_4^0$. Hence the second order lamination convex hull $\mathcal{V}_i^{lc,2}$ contains an open neighbourhood of this square and in particular an open neighbourhood of 0.

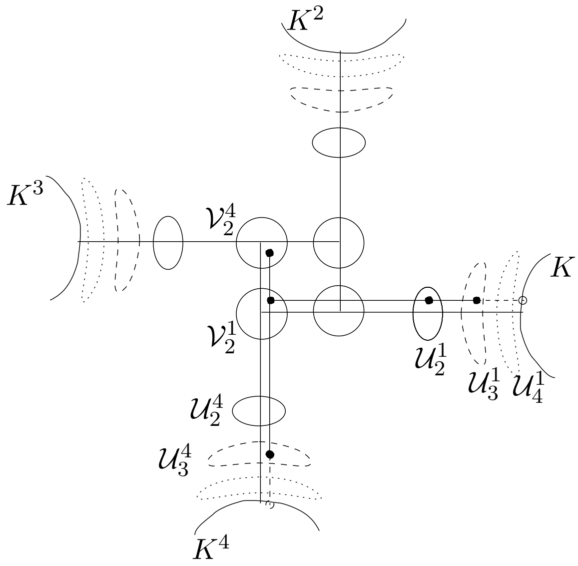

Figure 3: Each point in $\mathcal{U}_i^1$ is a rank-one convex combination of a point in $\mathcal{U}_{i+1}^1$ and a point in $\mathcal{V}_i^1$; in turn each point in $\mathcal{V}_i^1$ is a rank-one convex combination of points in $\mathcal{V}_i^4$ and $\mathcal{U}_{i+1}^4$.

(ii) *If* $F \in \mathcal{V}_i^{j+1}$ *then there exist* $A \in \mathcal{U}_{i+1}^j$, $B \in \mathcal{V}_i^j$, *such that* $\mathrm{rk}(B-A) = 1$,

$$F = \mu A + (1-\mu)B, \quad \frac{1}{4} < \mu < \frac{3}{4} \quad (\textit{if } \lambda_1 > \frac{3}{4}).$$

(iii) $\mathcal{U}_i^j \to K^j$ *as* $i \to \infty$, *where* $K^j \subset K$ *is contained in a small neighbourhood of* M_j^0.

(iv) $0 \in \mathcal{V}_1^{lc,2}$ *and* $\mathcal{V}_1^{lc,2}$ *is open and hence contains an open neighbourhood of the two-dimensional square given by the convex hull of* $P_1^0, P_2^0, P_3^0, P_4^0$.

Using these properties one can construct an elliptic counterexample satisfying (1) and (2) by starting with a map u^0 with $\nabla u^0 \equiv 0$ and iteratively splitting the gradient along rank-1 segments (using the 'elliptic' counterpart of Lemma 8 below). Since the construction in the parabolic case is very similar (with an additional complication arising from the need to control the time derivative) we do not give the details here and refer the reader to [MS 99].

Our construction will be carried out in an open set $\mathcal{U}$ which involves $\mathcal{U}_i, \mathcal{V}_i$ and suitable rank-1 segments between these sets:

$$\begin{aligned} \mathcal{U} \;\; = \;\; & \cup\, \mathcal{U}_i \quad \cup\, \mathcal{V}_i \quad \cup\, \mathcal{V}_1^{lc,2} \\ & \cup \{[A,B] : \mathrm{rk}(B-A) = 1, A \in \mathcal{U}_{i+1}^j, B \in \mathcal{V}_i^j\}. \end{aligned} \tag{17}$$

The reason for this choice of $\mathcal{U}$ will become clear in the proof of Lemma 9. In the following we will always assume that the sets $\mathcal{O}_i$ have been chosen sufficiently small. In particular $\mathcal{U}$ is contained in a very small neighbourhood of the segment $[P_{i+1}^0, M_i^0]$ and the square with corners P_i^0 (see also Figures 2 and 3).

3 The parabolic counterexample

Solutions to the parabolic system (1)–(2) are constructed using an iterative splitting along rank-1 segments and Proposition 7. The main ingredient is Lemma 12 which shows how to make a small perturbation of an affine function such that the result satisfies $\nabla u \in K$ in a set of fixed volume fraction and such that u is piecewise affine on the complement. This is done in two steps. First we achieve $\nabla u \in K$ modulo a C^∞ remainder (see Lemma 9). Then we approximate C^∞ maps by piecewise affine C^1 maps. Both Lemma 9 and Lemma 11 are obtained by iteration of very simple modifications, which are stated in Lemma 8 and Lemma 10, respectively.

Lemma 8 *(smooth splitting along a rank-1 segment). Given*

$$\begin{aligned} G \subset \Omega \times (0,T) \;\; &\textit{open}, \\ A, B \in \mathbb{R}^{m\times n}, \;\; &\mathrm{rk}(A-B) = 1, \\ F = \lambda A + (1-\lambda)B, \;\; &\lambda \in (0,1), \\ \alpha \in (0,1), \;\; \varrho > 0, \;\; R > 0, \;\; &i \in \mathbb{N}, \end{aligned}$$

and u Lipschitz in G with

$$\nabla u = F \;\; \textit{in} \;\; G$$

there exists $v \in C^\infty(G;\mathbb{R}^m)$ and open sets $G_A \subset G$, $G_B \subset G$ with $|\partial G_A| =$

$|\partial G_B| = 0$ *such that*

$$v - u \in C_0^\infty(G;\mathbb{R}^m), \tag{18}$$

$$||(v-u)_t||_{C^\alpha} < 2^{-i}, \tag{19}$$

$$\mathrm{dist}(\nabla v, [A,B]) < \varrho, \tag{20}$$

$$\nabla v = A \ \ on\, G_A, \ \nabla v = B \ \ on\, G_B, \tag{21}$$

$$|G_A| \geq (1-2^{-i})\lambda|G|, \ |G_B| \geq (1-2^{-i})(1-\lambda)|G|, \tag{22}$$

$$|G_B \cap B(x,R)| > 0, \quad \forall x \in \bar{G}_A. \tag{23}$$

Remark. The assumption that u be Lipschitz is not empty since G is in general not connected.

Proof. We may assume without loss of generality $F = 0$. In a suitable orthonormal coordinate system we thus have $A = (1-\lambda)a \otimes e_n, B = -\lambda\, a \otimes e_n$. It suffices to consider the special case $m = 1$ and $a = 1$. Indeed if for this case we have constructed a function $\widetilde{v} \in C^\infty(G)$ which satisfies conditions (18) - (22) with ϱ replaced by $\widetilde{\varrho} = \varrho/|a|$ and i replaced by $\widetilde{i}$ such that $2^{-\widetilde{i}} \leq 2^{-i}\min(1, 1/|a|)$ then $v = a\widetilde{v}$ satisfies (18) - (22).

We thus assume from now on $m = 1,\ F = 0,\ A = (1-\lambda)e_n,\ ,\ B = -\lambda e_n$. To construct v we exhaust most of G by rectangles, define v as a C_0^∞ function on these rectangles and extend v by zero outside.

Let $\eta = \frac{1}{K}$, where K is an integer. Then there exist finitely many disjoint cubes $Q_k = a_k + (0,l_k)^{n+1} \subset G$ whose total measure is larger than $(1-\eta)|G|$. Each cube can be subdivided into disjoint rectangles $R_{k,j} = a_{k,j} + (0,w_k)^n \times (0,l_k)$ where l_k/w_k is an integer. On each such rectangle we define v by

$$v(a_{k,j} + x) = \frac{w_k}{MK}\,\Theta\left(\frac{x}{w_k}\right)\ \Psi\left(\frac{t}{l_k}\right)\ h\left(MK\frac{x_n}{w_k}\right),$$

where M is an integer, $\phi \in C_0^\infty((0,1)^n)$, $\Psi \in C_0^\infty(0,1)$ and $h : \mathbb{R} \to \mathbb{R}$ is smooth and periodic with period one. We now specify the choice of Θ, Ψ, h, M and w_k. We require that

$$0 \leq \Theta \leq 1, \quad \Theta_{|(\eta,1-\eta)^n} \equiv 1, \quad |\nabla\Theta| \leq \frac{C}{\eta}, \tag{24}$$

$$0 \leq \Psi \leq 1, \quad \Psi_{|(\eta,1-\eta)} \equiv 1, \quad |\Psi'| \leq \frac{C}{\eta}, \quad |\Psi''| \leq \frac{C}{\eta^2}$$
$$h' \in [-\lambda, 1-\lambda], \quad 0 \leq h \leq 1$$

and that there exist intervals $I^A \subset (0,1)$ and $I^B \subset (0,1)$ such that

$$h'_{|I^A} = (1-\lambda), \quad h'_{|I^B} = -\lambda, \quad |I^A| = (1-\eta)\lambda, \quad |I^B| = (1-\eta)(1-\lambda).$$

Let $R^A = [(\eta, 1-\eta)^n \times (\eta, 1-\eta)] \cap \{\xi : MK\xi_n - [MK\xi_n] \in I^A\}$ where $[s]$ denotes the integer part of a real number s. Since $\eta = \frac{1}{K}$ we obtain

$$|R^A| = (1-2\eta)^{n+1}|I^A| = (1-2\eta)^{n+1}(1-\eta)\lambda$$

Similarly one defines R^B and obtains

$$|R^B| = (1-2\eta)^{n+1}|I^B| = (1-2\eta)^{n+1}(1-\eta)(1-\lambda).$$

If we define scaled sets $R^A_{k,j}$ by stretching by w_k in x-direction, by l_k in t-direction and translating by $a_{k,j}$ we find

$$\nabla v = A \quad \text{in } R^A_{k,j}, \quad \nabla v = B \quad \text{in } R^B_{k,j}.$$

Taking $G_A = \underset{k,j}{\cup} R^A_{k,j}$, $G_B = \underset{k,j}{\cup} R^B_{k,j}$ we obtain $|\partial G_A| = |\partial G_B| = 0$ as well as (21) and (22) as long as η is chosen such that $(1-2\eta)^{n+1}(1-\eta)^2 > 1 - 2^{-i}$.

Since (18) is obvious it only remains to verify (19), (20) and (23). Regarding (20) we note that

$$\nabla v = \Theta(\frac{x}{w_k})\Psi(\frac{t}{l_k})h'(MK\frac{x_n}{w_k})e_n + \frac{1}{MK}(\nabla\Theta)(\frac{x}{l_k})\Psi(\frac{t}{l_k})h(Mk\frac{x_n}{w_k})$$

Since the first term on the right hand side belongs to $[A, B]$ it suffices to assure that

$$\frac{1}{MK}|\nabla\Theta| < \varrho.$$

In view of (24) and since $\eta = \frac{1}{K}$ this can always be achieved by choosing $M \geq M_0(\varrho)$.

In view of the triangle inequality it suffices to verify (19) within a single rectangle (with the sharper bound $2^{-(i+1)}$). We have

$$v_t = \frac{w_k}{MKl_k} \ \Theta(\frac{x}{w_k}) \ \Psi'(\frac{t}{l_k}) \ h(MK\frac{x_n}{w_k})$$

Hence

$$|v_t| \leq C \ \frac{w_k}{MKl_k} \ \frac{1}{\eta} \leq C \ \frac{w_k}{Ml_k}.$$

To estimate the Hölder seminorm of u_t note that

$$[\phi]_\alpha \leq (\mathrm{Lip}\phi)^\alpha (\mathrm{osc}\phi)^{1-\alpha} \leq \frac{\mathrm{C}}{\eta^\alpha}, \quad [\Psi']_\alpha \leq \frac{\mathrm{C}}{\eta^{1+\alpha}}.$$

Taking into account that $\eta = \frac{1}{K}$ we find

$$\begin{aligned} [u_t]_\alpha &\leq C \ \frac{w_k}{MKl_k} \ \left[\frac{1}{\eta^\alpha}\ \frac{1}{w_k^\alpha}\ \frac{1}{\eta} + \frac{1}{\eta^{1+\alpha}}\ \frac{1}{l_k^\alpha} + \frac{1}{\eta}\left(\frac{MK}{w_k}\right)^\alpha\right] \\ &\leq C \ \frac{w_k}{MKl_k} \ [2K^{1+\alpha}w_k^{-\alpha} + K^{1+\alpha}M^\alpha w_k^{-\alpha}] \\ &\leq C \ \frac{w_k^{1-\alpha}\ K^\alpha}{M^{1-\alpha}\ l_k} \end{aligned}$$

Choosing

$$w_k \leq {l_k}^{\frac{1}{1-\alpha}} \quad \text{and} \ \ M \geq M_1(K,i) = \left(\frac{CK^\alpha}{2^{-(i+1)}}\right)^{\frac{1}{1-\alpha}}$$

we obtain (19). Finally to achieve (23) it suffices to choose $\max l_k \leq c_n R$. Then each ball $B(x,R)$ with $x \in \bar{G}_A$ contains a full cube Q_k and hence intersects G_B in a set of positive measure. □

Lemma 9 *(pushing ∇u to K by iterated splitting). Let $\mathcal{U}, \mathcal{U}_i^j, \mathcal{V}_i^j, K, K^j$ be given by (15), (16), (17), (10) and Proposition 7 (iii). Let $G \subset \Omega \times (0,T)$ be open, let u be Lipschitz in G with*

$$\nabla u = F \quad \text{in } G, \quad F \in \mathcal{U}. \tag{25}$$

Then there exists a decomposition

$$G = \widetilde{G} \cup H, \ \ \widetilde{G} \text{ open}, \ |H| \geq \frac{1}{32}|G|, \tag{26}$$

and a Lipschitz map v such that

(i)

$$v - u \in W_0^{1,\infty}(G;\mathbb{R}^m) \tag{27}$$

$$(v-u)_t \in C_0^\alpha(G;\mathbb{R}^m); \ \ ||(v-u)_t||_{C^\alpha} < 2^{-i}. \tag{28}$$

$$v_{|\widetilde{G}} \in C^\infty(\widetilde{G};\mathbb{R}^m), \quad \nabla u \in \mathcal{U} \ \text{ in } \tilde{G}, \tag{29}$$

$$\nabla v \in K^j \quad \text{a.e. in } H, \quad \text{for some } j \in \{1,2,3,4\}. \tag{30}$$

(ii) *If* $F \in \mathcal{U}_i^k$ *or* $\operatorname{dist}(F, \mathcal{V}^{k+1}) < \eta_0$, *where* $\mathcal{V}^{k+1} = \cup_i \mathcal{V}_i^{k+1}$ *and* $\eta_0 > 0$ *is a universal sufficiently small constant, then* $j = k$ *in (30).*

(iii) *For all* $x \in H$ *and all* $\delta > 0$

$$|\{y \in B(x, \delta) \setminus H : \nabla u(y) \in \mathcal{V}^j\}| > 0.$$

Remark. The combination of (ii) and (iii) will allow us to show that the solution we will construct has large oscillations of the gradient in every open subset.

Proof. The result is essentially a direct consequence of Proposition 7 and an iterative application of Lemma 8. In order to motivate the choice of the set $\mathcal{U}$ and to emphasize the role of assertion (iii) of the lemma we give a detailed proof.

Case 1: $F \in \mathcal{U}_i^j$. By Proposition 7(i) there exist $A_1 \in \mathcal{U}_{i+1}^i$ and $B_1 \in \mathcal{V}_i^j$ such that $\operatorname{rk}(B - A)$ and

$$F = \frac{\lambda_i}{\lambda_{i+1}} A_1 + (1 - \frac{\lambda_1}{\lambda_{i+1}}) B_1.$$

The set $\mathcal{U}$ is chosen such that it contains the segment $[A_1, B_1]$. Since $\mathcal{U}$ is open it also contains a ϱ_1-neighbourhood of that segment, for sufficiently small $\varrho_1 > 0$. Hence an application Lemma 8 (with $\varrho = \varrho_1$ and $R = R_1$) yields an open set $G_1 \subset G$ and a map u_1 with $u_1 - u_0 \in C_0^\infty(G; \mathbb{R}^m)$, $||(u_1 - u_0)_t||_{C^\alpha} < 2^{-(i+2)}$, $\nabla u_1 \in \mathcal{U}$ and

$$\nabla u_1 = A_1 \in \mathcal{U}_{i+1}^j \text{ in } G_1, \quad |G_1| > \left(\frac{\lambda_i}{\lambda_{i+1}} - 2^{-k_0-1}\right)|G|.$$

Since $B_1 \in \mathcal{V}_i^j$ assertion (23) yields

$$|\{y \in B(x, R_1) \setminus \bar{G}_1 : \nabla u_1(y) \in \mathcal{V}^j\}| > 0 \quad \forall x \in \bar{G}_1.$$

Now we can apply the same reasoning to u_1, A_1 and G_1. We thus obtain a map u_2 (originally defined on G_1) and an open set $G_2 \subset G_1$ with $u_2 - u_1 \in C_0^\infty(G_1; \mathbb{R}^m)$ and

$$\nabla u_2 = A_2 \in \mathcal{U}_{i+2}^j \text{ in } G_2, \quad |G_2| > \left(\frac{\lambda_{i+1}}{\lambda_{i+2}} - 2^{-k_0-2}\right)|G_1|.$$

It will be convenient to extend $u_2 - u_1$ by zero to G. Proceeding inductively we find maps u_k, open sets $G_k \subset G_{k-1}$, matrices A_k and numbers $R_k \to 0$ such that

$$\nabla u_k = A_k \in \mathcal{U}_{i+k}^j \text{ in } G_k, \quad |G_k| > \left(\frac{\lambda_{i+k-1}}{\lambda_{i+k}} - 2^{-k_0-k}\right)|G_{k-1}|,$$

$$|\{y \in B(x,R_k) \setminus \bar{G}_k : \nabla u_k(y) \in \mathcal{V}^j\}| > 0 \quad \forall x \in \bar{G}_k. \tag{31}$$

Moreover $u_k - u_{k-1} \in C_0^\infty(G;\mathbb{R}^m)$, $u_k = u_{k-1}$ on G_{k-1} and $||(u_k - u_{k-1})_t||_{C^\alpha} < 2^{-(i+k+1)}$. By Proposition 7(iii) we have $A_k \to A_\infty \in K^j$. Let $H = \bigcap_k \bar{G}_k, \tilde{G} = G \setminus H$. It is easy to show that $u_k \to v$ in $W^{1,1}(G;\mathbb{R}^m)$, that (27) and (28) hold and

$$\nabla v = A \in K^j \text{ a.e. in } H, \quad v = u_k \text{ in } G \setminus \bar{G}_k. \tag{32}$$

Hence (29) and (30) hold, too, and assertion (ii) of Lemma 9 is obvious. To prove assertion (iii) fix $x \in H$. Then there exists an l such that $x \in G_k$ for all $k \geqslant l$. Fix $k > l$ such that $R_k < \delta$. Then (31) and (32) imply assertion (iii). Finally the estimates on $|G_k|$ yield

$$|H| \geqslant (\lambda_1 - 2^{-k_0+2})|G| \geqslant \frac{1}{2}|G|,$$

for a suitable choice of k_0.

Case 2: $F \in \mathcal{V}_i^{j+1}$. By Proposition 7 (ii) there exist $A \in \mathcal{U}_{i+1}^j, B \in \mathcal{V}_i^j$ such that $\mathrm{rk}(B - A) = 1$ and

$$F = \mu A + (1-\mu)B, \quad \frac{1}{4} < \mu < \frac{3}{4}.$$

By definition $\mathcal{U}$ contains the segment $[A, B]$ and hence a ϱ_0-neighbourhood of it. Hence we can apply Lemma 8 (with $\varrho = \varrho_0$) and obtain an open set $G_0 \subset G$ and a map u_0 with

$$\nabla u_0 = A \in \mathcal{U}_{i+1}^j \text{ on } G_0, \quad |G| > \frac{1}{4}|G|.$$

Moreover $\nabla u_0 \in \mathcal{U}$ and the usual estimates hold for $u - u_0$. Now we can apply Case 1 to u_0 and G_0. This yields a map v which satisfies assertions (i)–(iii) of the lemma with $|H| \geqslant |G_0|/2 \geqslant |G|/8$.

Case 3: $F \in [A, B]$, where $A \in \mathcal{U}_{i+1}^j, B \in \mathcal{V}_i^j, \mathrm{rk}(A - B) = 1$. Then

$$F = \mu A + (1-\mu)B, \quad 0 < \mu < 1.$$

If $\mu > \frac{1}{4}$ we proceed as in Case 2 and we obtain a limit map with $\nabla v = A_\infty \in K^j$ in H, where $|H| \geqslant |G|/8$. If $\mu < \frac{1}{4}$ we first use Lemma 8 to obtain a map $\hat{u}$ and an open set $\hat{G} \subset G$ with

$$\nabla \hat{u} = B = \mathcal{V}_i^j \text{ on } \hat{G}, \quad |\hat{G}| > \frac{3}{4}|G|.$$

Moreover $\nabla \hat{u} \in \mathcal{U}$ and the usual estimates hold. Now we can apply Case 2 to $\hat{u}, \hat{G}$ and B.

It remains to verify that these constructions satisfy assertion (ii). Since the $\mathcal{V}^j$ is very close to the point P_j^0 and the sets $\mathcal{U}_i^j$ lie in a small neighbourhood of the segment $[P_{j+1}^0, M_j^0]$ the matrix F can only be close to $\mathcal{V}^j$ or to $\mathcal{V}^{j+1}$ (see Figs. 2 and 3). If F is close to $\mathcal{V}^{j+1}$ and hence to P_{j+1}^0 then μ must be close to or bigger than $\frac{1}{2}$. Hence assertion (ii) holds. If F is close to $\mathcal{V}^j$ then μ is close to zero. Application of Case 2 to $\hat{u}, \hat{G}$ and $\hat{B}$ yields a limit map v with $\nabla v = A_\infty \in K^{j-1}$ in H. Hence (ii) holds again.

Case 4: $F \in \mathcal{V}_1^{lc,1}$. Then there exist $A, B \in \mathcal{V}_1$ with $\mathrm{rk}(A-B) = 1$ and $F = \mu A + (1-\mu)B, \frac{1}{2} < \mu < 1$. If F is close to $\mathcal{V}^j$ (i.e. close to P_j^0) then so is A. Applying Lemma 8 we can easily reduce the situation to Case 2 and we obtain (i) to (iii) with $|H| \geqslant |G|/16$.

Case 5: $F \in \mathcal{V}_1^{lc,2}$. Then F is a rank-one convex combination $F = \mu A + (1-\mu)B$ of $A, B \in \mathcal{V}_1^{lc,1}$ with $\mu \geqslant \frac{1}{2}$. Again if F is close to $\mathcal{V}^j$ so is A. Hence Lemma 8 allows us to reduce the situation to Case 4 and we obtain (i)–(iii) with $|H| \geqslant |G|/32$. □

Lemma 10 *(partial approximation of C^∞ functions by piecewise affine functions). Let $\theta > 0$ be sufficiently small, let $G \subset \Omega \times (0,T)$ be open, let $\mathcal{U} \subset \mathbb{R}^{m\times n}$ be open and bounded, let $i \in \mathbb{N}$ and let $\eta \in C^0(G)$ with $\eta > 0$ in G. Denote $D := (\nabla, \partial_t)$. Suppose that*

$$u \in C^\infty(G;\mathbb{R}^m), \quad \nabla u \in \mathcal{U}.$$

Then there exist

$$\begin{aligned} \widetilde{G} \subset G \quad \textit{open}, \ |\widetilde{G}| \geq \theta |G|, \\ v \in C^\infty(G, \mathbb{R}^m), \ \nabla v \in \mathcal{U} \end{aligned} \tag{33}$$

such that

$$v - u \in C_0^\infty(G;\mathbb{R}^m), \ |D(v-u)|(x) < \eta(x) 2^{-i}, \tag{34}$$

$$||(v-u)_t||_{C^\alpha} < 2^{-i}, \tag{35}$$

$$\textit{and } v \textit{ is affine on each component of } \widetilde{G}. \tag{36}$$

Proof. We may assume that G is a small ball and $\mathrm{dist}(\nabla u, \partial \mathcal{U}) \geq c > 0$. Indeed in the general case we can always exhaust (a fixed compact subset of) G by balls (with maximal radius $R > 0$) whose total measure exceeds a fixed fraction of $|G|$. Without loss of generality we consider the ball $B_r \subset G$

with center 0 and radius $r \le R$ and assume $u(0)=0$. With the help of a cut off function $\psi \in C_0^\infty(B_1,[0,1])$ with $\psi|_{B_{1/2}} \equiv 1$ we define for $z \in B$:

$$v(z) := \psi\left(\frac{z}{r}\right) Du(0)z + \left(1-\psi\left(\frac{z}{r}\right)\right)u(z).$$

The function v is affine on the set where $\psi = 1$, i.e. on the ball $B_{r/2}$ and so we can set $\theta := |B_{r/2}|/|B_r|$, independent of r. Moreover using that $u \in C^\infty$ we can prove that $|\nabla u - \nabla v|$ is arbitrarily small if R was chosen sufficiently small. Hence (33) and (36) are satisfied and (34) will follow from (35).

To prove (35) we first calculate v_t as

$$\begin{aligned} v_t(z) &= \psi_t\left(\frac{z}{r}\right)\frac{1}{r}(Du(0)z-u(z)) \\ &\quad +\psi\left(\frac{z}{r}\right)(u_t(0)-u_t(z)) \\ &\quad +u_t(z). \end{aligned} \tag{37}$$

With $L := \mathrm{Lip}\,(Du) < \infty$ we can estimate the derivative of the first term in (37):

$$\begin{aligned} &\left|D\left(\psi_t\left(\frac{z}{r}\right)\frac{1}{r}(Du(0)z-u(z))\right)\right| \\ \le\ & \left|\psi_t\left(\frac{z}{r}\right)\frac{1}{r}(Du(0)-Du(z))\right| \\ &+|Du(0)-u(z)|\frac{1}{r^2}\left|D\psi_t\left(\frac{z}{r}\right)\right| \\ \le\ & C_1\left(\mathrm{Lip}(Du)\frac{|z|}{r}+\mathrm{Lip}(Du)\frac{|z|^2}{r^2}\right) \\ \le\ & 2C_1L. \end{aligned}$$

With the help of this inequality and a Taylor expansion we can estimate for $z_1 \neq z_2$ in the ball B:

$$\begin{aligned} &\left|\psi_t\left(\frac{z_1}{r}\right)\frac{1}{r}(Du(0)z_1-u(z_1))-\psi_t\left(\frac{z_2}{r}\right)\frac{1}{r}(Du(0)z_2-u(z_2))\right| \\ \le\ & 2C_1L|z_1-z_2| \ = \ 2C_1L\underbrace{|z_1-z_2|^{1-\alpha}}_{\le r^{1-\alpha}\le R^{1-\alpha}}|z_1-z_2|^\alpha. \end{aligned}$$

Choosing $R \le \left(2^{i+2}C_1L\right)^{-\frac{1}{1-\alpha}}$ and taking everything together we get

$$\begin{aligned} &\left|\psi_t\left(\frac{z_1}{r}\right)\frac{1}{r}(Du(0)z_1-u(z_1))-\psi_t\left(\frac{z_2}{r}\right)\frac{1}{r}(Du(0)z_2-u(z_2))\right| \\ \le\ & 2^{-i-1}|z_1-z_2|^\alpha. \end{aligned} \tag{38}$$

We can estimate the second term in (37) in a similar way:

$$\begin{aligned}
& \left|\psi\left(\frac{z_1}{r}\right)(u_t(0)-u_t(z_1))-\psi\left(\frac{z_2}{r}\right)(u_t(0)-u_t(z_2))\right| \\
\leq & \ \Big|\left(\psi\left(\frac{z_1}{r}\right)-\psi\left(\frac{z_2}{r}\right)\right)(u_t(0)-u_t(z_1)) \\
& -\psi\left(\frac{z_2}{r}\right)\Big((u_t(0)-u_t(z_2))-(u_t(0)-u_t(z_1))\Big)\Big| \\
\leq & \ \left(\frac{1}{r}\mathrm{Lip}\,(\psi)L|z_1|+L\right)|z_1-z_2| \\
\leq & \ \left(\mathrm{Lip}\,(\psi)LR^{1-\alpha}+LR^{1-\alpha}\right)|z_1-z_2|^{\alpha}.
\end{aligned}$$

Choosing $R \leq (2^{i+2}\mathrm{Lip}\,(\psi)L+2^{i+2}L)^{-\frac{1}{1-\alpha}}$ the last expression can be estimated from above by $2^{-i-1}|z_1-z_2|^{\alpha}$. From this and (38) we get

$$|(v_t-u_t)(z_1)-(v_t-u_t)(z_2)| < 2^{-i}|z_1-z_2|^{\alpha}.$$

Thus we have proved (35).

A straightforward iteration of Lemma 10 gives

Lemma 11 *(approximation of C^∞ functions by piecewise affine functions). Let $G \subset \Omega\times(0,T)$ be open, $\mathcal{U}\subset\mathbb{R}^{m\times n}$ be open, $i\in\mathbb{N}, \eta_0>0$. Suppose that*

$$u\in C^\infty(G;\mathbb{R}^m),\ \ \nabla u\in\mathcal{U}.$$

Then there exist disjoint open sets G_α, a Lebesgue null set N and a map v such that

$$\begin{aligned}
& G=\bigcup_\alpha G_\alpha \ \ \bigcup N, \\
& v\in C^1(G;\mathbb{R}^m),\ \ \nabla v\in\mathcal{U}, \\
& v-u\in W_0^{1,\infty}(G;\mathbb{R}^m),\ \ ||D(v-u)||_{L^\infty}<\eta_0, \\
& (v-u)_t\in C_0^\alpha(G;\mathbb{R}^m), ||(v-u)_t||_{C^\alpha}<2^{-i}, \\
& v \ \textit{affine on}\ G_\alpha.
\end{aligned}$$

Remark. To see that indeed $\nabla v\in\mathcal{U}$ and not only $\nabla v\in\bar{\mathcal{U}}$ it suffices to take $\eta(x)=\frac{1}{2}\operatorname{dist}(\nabla u(x),\partial\mathcal{U})>0$ in the iterative application of Lemma 10.

As an immediate consequence of Lemma 9 and Lemma 11 we obtain

Lemma 12 *(main iteration lemma). Let* $\mathcal{U}, \mathcal{U}_i^j, \mathcal{V}_i^j, K, K^j$ *be given by* (15), (16), (17), (10) *and Proposition 7 (iii). Let* $G \subset \Omega \times (0,T)$ *be open, let* $k \in \mathbb{N}$*, let* u *be Lipschitz in* G *with*

$$\nabla u = F \ \ in\ G, \quad F \in \mathcal{U}.$$

There exist a Lebesgue null set N *and a decomposition*

$$G = \bigcup G_\alpha \bigcup H \bigcup N, \ G_\alpha \ open, \ |H| \geq \frac{1}{32}|G|,$$

and a Lipschitz map v *such that*

(i)

$$\begin{aligned}
&v - u \in W_0^{1,\infty}(G;\mathbb{R}^m),\\
&(v-u)_t \in C_0^\alpha(G;\mathbb{R}^m), \ \ ||(v-u)_t||_{C^\alpha} < 2^{-i}.\\
&v \ affine\ on\, G_\alpha, \ \ \nabla v \in \mathcal{U}\\
&\nabla v \in K^j \ a.e.\ in\, H, \ \ for\ some\, j \in \{1,2,3,4\}.
\end{aligned}$$

In addition we may assume $\operatorname{diam} G_\alpha < 2^{-i}$*.*

(ii) If $F \in \mathcal{U}_i^k$ *or* $\operatorname{dist}(F, \mathcal{V}^{k+1}) < \eta_0$ *then* $j = k$ *in* (30).

(iii) For all $x \in H$ *and all* $\delta > 0$

$$|\{y \in B(x,\delta) \setminus H : \operatorname{dist}(\nabla v(y), \mathcal{V}^j) < \eta_0\}| > 0.$$

Proof of Theorem 1. Step 1: Construction of the solution.

Let $u_0 \equiv 0$. Let u_1 be the map v obtained from u_0 by application of Lemma 12 with $i = i_0$ sufficiently large (in particular $2^{-i_0} < 1/8$). Thus we obtain a decomposition $G = \cup G_\alpha \cup H_1 \cup N$ with $\nabla u_1 \in K$ in H_1. Now apply Lemma 12 to each of the subsets G_α, with $i = i_0 + 1$. This yields new maps u_2 on G_α and subsets $H_{2,\alpha}$ on which ∇u_2 belongs to K (a.e.). Since $u_2 - u_1 \in W_0^{1,\infty}(G_\alpha;\mathbb{R}^m)$ these maps are the restriction of a Lipschitz map u_2 defined on G and we set $H_2 = \cup H_{2,\alpha}$. Proceeding by induction (with

$i = i_0 + k$) we find Lipschitz maps u_k and disjoint sets H_k such that

$$\begin{aligned}
&u_k - u_0 \in W_0^{1,\infty}(G;\mathbb{R}^m) \;\text{ and} \\
&||(u_{k+1} - u_k)_t||_{C^\alpha} < 2^{-i_0+k}, \\
&u_{k+1} = u_k \quad \text{on } \; H_l, \text{ for all } l \le k, \qquad (39) \\
&\nabla u_{k+1} \in K \quad \text{on } \; H_{k+1}, \\
&\nabla u_{k+1} \in \mathcal{U} \quad \text{on } \; G \setminus \cup_{l=1}^{k+1} H_l, \\
&|G \setminus \cup_{l=1}^{k} H_l| \le \left(\frac{31}{32}\right)^k |G|. \qquad (40)
\end{aligned}$$

In particular we deduce that u_k is bounded in $W_0^{1,\infty}$ and Du_k and u_k converge a.e. Thus

$$u_k \longrightarrow u_\infty \quad \text{in } \; W^{1,1}(G;\mathbb{R}^m)$$

and by (39), (40)

$$\nabla u_\infty \in K \quad \text{a.e.}$$

Moreover

$$(u_k)_t \longrightarrow (u_\infty)_t \quad \text{in } \; C^\alpha, \;\; ||(u_\infty)_t||_{C^\alpha} \le 2^{-i_0}.$$

Recalling that

$$u_k = \begin{pmatrix} w_k \\ W_k \end{pmatrix}, \;\; \text{where } w_k \text{ and } W_k \in W^{1,\infty}(G;\mathbb{R}^2)$$

and taking into account the definition of K we see that w_∞ solves the parabolic system (1)–(2) with $f = (w_\infty)_t \in C^\alpha$.

Step 2: Lack of regularity. To prove that ∇w_∞ is nowhere continuous we will show that there exists a constant $c > 0$ such that whenever $B(x_0, 2\varrho) \subset \Omega \times (0,T)$ then

$$\operatorname*{ess\ osc}_{B(x_0,2\varrho)} \nabla w_\infty \ge c. \qquad (41)$$

First note that $|B(x_0,\varrho) \cap H_k| > 0$ for infinitely many k. Indeed if l was the largest value of k for which the estimate holds then $\nabla u_l \in K$ a.e. in $B(x_0,\varrho)$. Taking $x \in B(x_0,\varrho) \cap H_l$ and $B(x,\delta) \subset B(x_0,\varrho)$ we obtain a contradiction with Lemma 12 (iii) for $u = u_l$. Hence we may choose k such that $2^{-(k-1)} < \varrho$, $|B(x_0,\varrho) \cap H_k| > 0$.

By construction of u_k there exist open sets $G_{k-1,\alpha}$ such that u_{k-1} is affine on $G_{k-1,\alpha}$. Moreover $H_k = \cup H_{k,\alpha}$ where $H_{k,\alpha}$ arises from an application of Lemma 12 to $u_{k-1|G_{k-1,\alpha}}$. In particular $H_{k,\alpha} \subset G_{k-1,\alpha}$ and diam $G_{k-1,\alpha} < 2^{-(k-1)} < \varrho$. Choose α such that $|H_{k,\alpha} \cap B(x_0, \varrho)| > 0$. Then $G_{k-1,\alpha} \subset B(x_0, 2\varrho)$. Moreover by Lemma 12 (i)

$$\nabla u_\infty = \nabla u_k \in K^j \quad \text{in } H_{k,\alpha} \quad \text{for some } j \in \{1,2,3,4\} \tag{42}$$

and by Lemma 12 (iii)

$$|\{y \in G_{k-1,\alpha} \setminus H_{k,\alpha} : \operatorname{dist}(\nabla u_k(y), \mathcal{V}^j) < \eta_0\}| > 0.$$

Hence there exists an open set $G_{k,\beta} \subset G_{k-1,\alpha} \subset B(x_0, 2\varrho)$ such that

$$u_k \text{ affine on } G_{k,\beta}, \quad \nabla u_k = F \in \mathcal{U},\ \operatorname{dist}(F, \mathcal{V}^j) < \eta_0.$$

Thus Lemma 12 (ii) (applied to u_k and $G_{k,\beta}$) implies that there exists $H_{k+1,\beta} \subset G_{k,\beta}$ with

$$\nabla u_\infty = \nabla u_{k+1} \in K^{j-1} \text{ in } H_{k+1,\beta} \text{ and } |H_{k+1,\beta}| > 0. \tag{43}$$

Since the projection $\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto X$ from $\mathbb{R}^{4\times2}$ to $\mathbb{R}^{2\times2}$ maps K^j and K^{j-1} into two well separated sets in $\mathbb{R}^{2\times2}$ (in fact small neighbourhoods of two of the four points in (11)) the assertion (41) follows from (42) and (43). □

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