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Rigidity Estimate for Two Incompatible Wells

by

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1 Introduction

Recently, Friesecke, James and Müller [8, 9] obtained the following interesting rigidity estimate in connection to their study in nonlinear plate theory.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \geq 2$. There exists a constant $C(\Omega)$ with the property that for each $u \in W^{1,2}(\Omega, \mathbb{R}^n)$, there exists an associated rotation $R \in SO(n)$, such that*

$$\|\nabla u - R\|_{L^2(\Omega)} \leq C(\Omega) \|\mathrm{dist}(\nabla u, SO(n))\|_{L^2(\Omega)} . \qquad (1)$$

This generalizes a classical result of F. John [11] who derived an estimate of $\|\nabla u - R\|_{L^2}$ in terms of $\|\mathrm{dist}(\nabla u, SO(n)\|_{L^\infty}$ for locally Bilipschitz maps u. In connection with mathematical models for materials undergoing solid-solid phase transformations [1, 2, 4, 7, 17], one is interested in deformations u whose gradient is close to a set $K := \cup_{i=1}^m SO(n)U_i$, which consists of several copies of $SO(n)$ (so-called energy wells). Here we consider the two-well problem for two *strongly incompatible* wells. For further information on the two-well problem see [6, 15, 22]. Rigidity for a linearized version of the two-well problem is discussed in [5, 12]. We prove an estimate of the type (1) for two strongly incompatible wells.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \geq 2$ and $K := SO(n) \cup SO(n)\,H$, where $H = \mathrm{diag}\,(\lambda_1, \cdots \lambda_n)$, $\lambda_i > 0$ such that $\sum_{i=1}^n (1 - \lambda_i)\,(1 - \det H/\lambda_i) > 0$. There exists a positive constant $C(\Omega, H)$ with the following property. For each $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ there is an associated $R := R(u, \Omega) \in K$ such that*

$$\|\nabla u - R\|_{L^2(\Omega)} \leq C(\Omega, H)\, \|\mathrm{dist}(\nabla u\,, K\,)\|_{L^2(\Omega)} . \qquad (2)$$

Theorem 2 has interesting consequences for the scaling of the energy in thin martensitic films [3, 20] which will be discussed in a forthcoming paper.

2 Preliminary Results

To prove Theorem 2, we need some preliminary lemmas. The first lemma is due to J. P. Matos [15] and concerns construction of smooth uniformly convex function, which have quadratic growth and whose gradient is the cofactor on the set $K := SO(n) \cup SO(n)\, H$.

Lemma 1 (Matos [15]). *Let* $K := SO(n) \cup SO(n)\, H,\ H = \mathrm{diag}\,(\lambda_1, \cdots \lambda_n)$, $\lambda_i > 0$. *Then there exits a smooth function* $W : \mathbb{R}^{n\times n} \to \mathbb{R}$, *which is uniformly convex and has quadratic growth and satisfies* $\nabla W = \nabla \det = \mathrm{cof}$ *in* K, *if and only if* $\sum_{i=1}^{n} (1 - \lambda_i)\,(1 - \det H/\lambda_i) > 0$.

The following lemma is a version of the generalized Poincaré inequality, see Theorem 3.6.5 in [16].

Lemma 2. $\Omega \subset \mathbb{R}^n$, $n \geq 2$ *be a bounded Lipschitz domain and* $0 < \delta \leq 1$. *Suppose that* $u \in W^{1,1}(\Omega)$ *and* $\mathcal{L}^n\left(\{x \in \Omega : u(x) = 0\}\right) \geq \delta \mathcal{L}^n(\Omega)$. *Then there exists* $C(n, \delta, \Omega) > 0$ *such that*

$$\|u\|_{L^{n/(n-1)}(\Omega)} \leq C(n, \delta, \Omega)\, \|\nabla u\|_{L^1(\Omega)}\,.$$

Next we state a variant of a lemma by Luckhaus [14] for bounded domains. This lemma is an important ingredient in the proof of our main Theorem.

Lemma 3. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain,* $n \geq 2$ *and let* $\chi : \Omega \to \{0, 1\}$ *be a characteristic function. Then there exists a constant* $C(\Omega) > 0$, *such that for any* $u \in W^{1,2}(\Omega)$

$$\min\left(\int_\Omega \chi\,, \int_\Omega 1 - \chi\right) \leq 16 \int_\Omega |u - \chi|^2 + C(\Omega) \left(\int_\Omega |u - \chi|^2 \int_\Omega |\nabla u|^2\right)^{n/2(n-1)}.$$

Proof. Let $u \in W^{1,2}(\Omega)$ and let $A := \{x \in \Omega : u(x) \leq 1/2\}$. Suppose first that $\mathcal{L}^n(A) \geq 1/2\, \mathcal{L}^n(\Omega)$. Define, $E := \{x \in \Omega : \chi = 1\}$ and $E_u := \{x \in E : u \geq 3/4\}$. On $E \setminus E_u$ the inequality $u < 3/4$ implies $4\,|u - \chi| \geq \chi$ and hence

$$\int_\Omega \chi = \int_{E_u} \chi + \int_{E\setminus E_u} \chi \leq \int_{E_u} \chi + 16 \int_\Omega |u - \chi|^2\,. \tag{3}$$

To estimate the integral $\int_{E_u} \chi$, we define the function $\psi : \Omega \to \mathbb{R}$ by

$$\psi(x) := \left(u(x) - \frac{1}{2}\right)_+ \wedge \frac{1}{4},$$

where $a \wedge b := \min(a,b)$ and $a_+ := \max(a,0)$. Observe that $\nabla\psi \equiv 0$ on $\{x \in \Omega : u(x) \geq 3/4\} \cup A$ and $\psi = 0$ on A. Hence by Lemma 2, we have

$$\begin{aligned}
\int_{E_u} \chi &= \mathcal{L}^n(E_u) \\
&= 4^{n/(n-1)} \int_{E_u} |\psi|^{n/(n-1)}\, dx \\
&\leq 4^{n/(n-1)} \int_{\Omega} |\psi|^{n/(n-1)}\, dx \\
&\leq C \left(\int_\Omega |\nabla\psi|\right)^{n/(n-1)} dx \\
&= C \left(\int_{\{1/2 \leq u \leq 3/4\}} |\nabla u|\, dx\right)^{n/(n-1)} \\
&\leq C \left(\mathcal{L}^n(\{1/2 \leq u \leq 3/4\}) \int_\Omega |\nabla u|^2\right)^{n/2(n-1)} \\
&\leq 4^{n/(n-1)} C \left(\int_\Omega |u-\chi|^2 \int_\Omega |\nabla u|^2\right)^{n/2(n-1)}. \qquad (4)
\end{aligned}$$

Hence for the case $\mathcal{L}^n(A) \geq 1/2\, \mathcal{L}^n(\Omega)$ we obtain from (3) and (4)

$$\int_\Omega \chi \leq 16 \int_\Omega |u-\chi|^2 + C \left(\int_\Omega |u-\chi|^2 \int_\Omega |\nabla u|^2\right)^{n/2(n-1)}. \qquad (5)$$

If $\mathcal{L}^n(A) < 1/2\, \mathcal{L}^n(\Omega)$, it suffices to replace u by $1-u$ and χ by $1-\chi$. ☐

Lemma 4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded Lipschitz domain and let K_1, K_2 be compact disjoint subsets of $\mathbb{R}^{n\times n}$. Define, $\mathrm{d}_P(\,\cdot\,) := \mathrm{dist}(\,\cdot, P)$ and $K := K_1 \cup K_2$. Then there exists a constant $C := C(K_1, K_2, \Omega) > 0$, such that for any $w \in W^{2,2}(\Omega, \mathbb{R}^n)$*

$$\begin{aligned}
\min\left(\int_\Omega \mathrm{d}^2_{K_1}(\nabla w), \int_\Omega \mathrm{d}^2_{K_2}(\nabla w)\right) &\leq C \left(\int_\Omega \mathrm{d}^2_K(\nabla w) \int_\Omega |\nabla^2 w|^2\right)^{n/2(n-1)} \\
&\quad + C \int_\Omega \mathrm{d}^2_K(\nabla w). \qquad (6)
\end{aligned}$$

Proof. Let $f : \mathbb{R}^{n\times n} \to [0,1]$ be the Lipschitz function defined by

$$f(F) := \frac{\mathrm{dist}(F, K_1)}{\mathrm{dist}(F, K_1) + \mathrm{dist}(F, K_2)} \,.$$

Then $f = 0$ in K_1 and $f = 1$ in K_2. Let $u \in W^{1,2}(\Omega)$ and let χ be a characteristic function on Ω. Then by Lemma 3, we have

$$\begin{aligned}
\int_\Omega \mathrm{d}^2(u, \{0\}) \bigwedge \int_\Omega \mathrm{d}^2(u, \{1\}) &= \int_\Omega |u|^2 \bigwedge \int_\Omega |u-1|^2 \\
&= \int_\Omega |u-\chi+\chi|^2 \bigwedge \int_\Omega |u-\chi+\chi-1|^2 \\
&\le 2\int_\Omega \big(|u-\chi|^2 + |\chi|\big) \bigwedge \int_\Omega \big(|u-\chi|^2 + |\chi - 1|\big) \\
&= 2\left[\int_\Omega |u-\chi|^2 + \min\left(\int_\Omega \chi\,, \int_\Omega 1-\chi\right)\right] \\
&\le 2\int_\Omega |u-\chi|^2 + 16\int_\Omega |u-\chi|^2 \\
&\quad + C(\Omega)\left(\int_\Omega |u-\chi|^2 \int_\Omega |\nabla u|^2\right)^{n/2(n-1)} \qquad (7)
\end{aligned}$$

Let $w \in W^{2,2}(\Omega, \mathbb{R}^n)$, define $u : \Omega \to \mathbb{R}$ by $u(x) := f(\nabla w(x))$. Since f is Lipschitz, $u \in W^{1,2}(\Omega)$. Define,

$$\chi(x) := \begin{cases} 0, & \text{if } \ u(x) \le 1/2 \\ 1, & \text{if } \ u(x) > 1/2\,. \end{cases}$$

Hence $\mathrm{dist}(u(x), \{0,1\}) = |u(x) - \chi(x)|$. Now observe that for any $F \in \mathbb{R}^{n\times n}$, $\mathrm{dist}(f(F), \{0,1\}) = \mathrm{dist}(f(F), f(K)) \le \mathrm{Lip}(f)\,\mathrm{dist}(F,K)$. Let $M := \max(\mathrm{diam}(K), |K|_\infty)$, $|K|_\infty := \max_K |F|$, $B(0,M) := \{F \in \mathbb{R}^{n\times n} : |F| \le M\}$ and $C = C(K_1, K_2) := \sup_{B(0,2M)} [\mathrm{dist}(F, K_1) + \mathrm{dist}(F, K_2)]$. Then on $B(0, 2M)$, $\mathrm{dist}(\,\cdot\,, K_1) \le C\,f$ and $\mathrm{dist}(\,\cdot\,, K_2) \le C\,(1-f)$. Note that for $|F| \ge 2\,M$, $\mathrm{dist}(F,K) \ge M$ and hence $\mathrm{dist}(F, K_i) \le 2\,\mathrm{dist}(F,K)$ $\ i = 1, 2$. Therefore by taking $u = f(\nabla w)$, $w \in W^{2,2}(\Omega, \mathbb{R}^n)$, we obtain

$$\begin{aligned}
\int_\Omega \mathrm{d}^2_{K_1}(\nabla w) &= \int_{\{x\in\Omega:\,|\nabla w(x)|\le 2M\}} \mathrm{d}^2_{K_1}(\nabla w) + \int_{\{x\in\Omega:\,|\nabla w(x)|> 2M\}} \mathrm{d}^2_{K_1}(\nabla w) \\
&\le C\int_{\{x\in\Omega:\,|\nabla w(x)|\le 2M\}} |f(\nabla w)|^2 + 4\int_{\{x\in\Omega:\,|\nabla w(x)|> 2M\}} \mathrm{d}^2_{K}(\nabla w) \\
&\le C\int_\Omega |u|^2 + 4\int_\Omega \mathrm{d}^2_K(\nabla w)\,. \qquad (8)
\end{aligned}$$

Similarly, we obtain

$$\int_\Omega \mathrm{d}^2_{K_2}(\nabla w) \le C \int_\Omega |1-u|^2 + 4\int_\Omega \mathrm{d}^2_K(\nabla w)\,. \tag{9}$$

Hence the lemma follows from (7)–(9). □

Remark 5. *One easily sees that the best constant C in Lemma 4 is invariant under uniform scaling and translation of the domain.*

3 The Rigidity Theorem

We begin with an interior estimate.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \ge 2$, and $U \subset\subset \Omega$. Let $K := SO(n) \cup SO(n)\,H$,where $H = \operatorname{diag}(\lambda_1, \cdots \lambda_n)$, $\lambda_i > 0$ is such that $\sum_{i=1}^n (1-\lambda_i)\,(1-\det H/\lambda_i) > 0$. Then there exists a positive constant $C(U, \Omega, H)$ with the following property. For each $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ there is an associated $R \in K$ such that*

$$\|\nabla u - R\|_{L^2(U)} \le C(U, \Omega, H)\,\|\mathrm{dist}(\nabla u\,, K\,)\|_{L^2(\Omega)}\ . \tag{10}$$

Proof. First we note that, $|K|_\infty := \max_{F\in K}|F| = \max\left(\sqrt{n}\,, (\sum_{i=1}^n \lambda_i^2)^{1/2}\right)$. Throughout this proof C is a generic absolute constant depending only on n, the λ_i, Ω and U. Its value can vary from line to line, but each line is valid with C being a pure positive number. By a truncation argument, see Proposition A.1 in [9] it is enough to prove the inequality (10) for maps with $\|\nabla u\|_{L^\infty(\Omega)} \le M$, for some constant M depending only on Ω and the set K. To see this, first observe that $|F| \le 2\,\mathrm{dist}(F, K)$ if $|F| \ge 2\,|K|_\infty$. Hence by Proposition A.1 in [9] applied with $\lambda = 4\,|K|_\infty$, for each $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ there exists a map $v \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ satisfying

$$\|\nabla v\|_{L^\infty(\Omega)} \le 4C|K|_\infty := M\,,$$

$$\begin{aligned}\|\nabla v - \nabla u\|^2_{L^2(\Omega)} &\le C\int_{\{x\in\Omega\,:\,|\nabla u(x)|>2|K|_\infty\}} |\nabla u|^2 dx \\ &\le 4\,C\int_\Omega \mathrm{dist}^2(\nabla u,\,K)\,dx\,.\end{aligned}$$

This in particular implies that $\|\mathrm{dist}(\nabla v, K)\|_{L^2(\Omega)} \le (2\sqrt{C}+1)\|\mathrm{dist}(\nabla u, K)\|_{L^2(\Omega)}$. Hence, if we prove the inequality (10) for v the assertion for u follows by the triangle inequality.

Step 1. Elliptic estimate:

Let

$$\epsilon := \|\text{dist}(\nabla u,\, K)\|_{L^2(\Omega)}\,. \tag{11}$$

Without loss of generality we may assume $\epsilon \leq 1$. By Lemma 1, there exists a smooth function $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ such that W is uniformly convex and satisfies $|\nabla W(F)| \leq C(1+|F|)$, $|\nabla^2 W(F)| \leq C$ for all $F \in \mathbb{R}^{n\times n}$ and $\nabla W = \text{cof}$ on $K = SO(n) \cup SO(n)\,H$. Define $A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $A := \nabla W$. Then A is a uniformly monotone vector field, i.e. $A(F) - A(G) : F - G \geq C\,|F-G|^2$, where $A : B := \text{tr}(A^t\,B)$. Now define $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by

$$f(F) := \text{cof}(F) - A(F)\,.$$

Since $f = 0$ on K and $\text{div}\,\text{cof}\nabla u = 0$ (where div is taken by rows) we obtain

$$-\,\text{div}\,A(\nabla u) = \text{div}\,f(\nabla u) \tag{12}$$

and

$$|f(F)|^2 \leq C\,\text{dist}^2(F,K) \quad \text{whenever} \quad |F| \leq M\,. \tag{13}$$

Let $w \in W^{1,2}(\Omega,\mathbb{R}^n)$ be a solution to,

$$\begin{cases} \text{div}\,A(\nabla w) &= 0 \quad \text{in } \Omega\,, \\ w &= u, \quad \text{on } \partial\Omega\,. \end{cases} \tag{14}$$

To see that (14) has a solution it suffices to minimize $v \mapsto \int_\Omega W(\nabla v)$ subject to $v = u$ on $\partial\Omega$. By the standard elliptic regularity (see e.g. Theorem 1.1, Chapter II in [10]), $w \in W^{2,2}_{\text{loc}}(\Omega,\mathbb{R}^n)$ and for each $x \in \Omega$, $0 < r < \frac{1}{2}\,\text{dist}(x,\partial\Omega)$, we have

$$\int_{B(x,r)} |\nabla^2 w|^2\,dx \leq \frac{C}{r^2}\int_{B(x,2r)} |\nabla w|^2\,dx\,. \tag{15}$$

Let $z := u - w$, then $z = 0$ on $\partial\Omega$. Since

$$-\,[\text{div}\,A(\nabla u) - A(\nabla w)] = \text{div}\,f(\nabla u) \quad \text{in} \quad \Omega\,,$$

we obtain, by testing with $z = u - w$

$$\begin{aligned}\int_\Omega A(\nabla u) - A(\nabla w) : \nabla u - \nabla w\,dx &= \int_\Omega f(\nabla u) : \nabla w - \nabla u\,dx \\ &\leq \left(\int_\Omega |f(\nabla u)|^2\,dx \int_\Omega |\nabla u - \nabla w|^2\,dx\right)^{1/2}.\end{aligned}$$

By monotonicity we have

$$\int_\Omega |\nabla u - \nabla w|^2\,dx \;\le\; C\int_\Omega |f(\nabla u)|^2\,dx$$
$$\le\; C\int_\Omega \mathrm{dist}^2(\nabla u,\,K)\,dx$$
$$=\; C\,\epsilon^2\,. \tag{16}$$

Therefore it is enough to prove that there exists $R\in\, K$, such that

$$\int_\Omega |\nabla w - R|^2\,dx \;\le\; C\,\epsilon^2\,. \tag{17}$$

Step 2. Estimates in measure:

Let us define $E := \{x\in\Omega : \mathrm{dist}(\nabla w(x), SO(n)H) \le \rho\}$, where $2\rho := \mathrm{dist}(SO(n),\, SO(n)H\,)$. Therefore $\mathrm{dist}(\nabla w(x), SO(n)) \ge \rho$ on the set E and $\mathrm{dist}(\nabla w(x), SO(n)) \le C\,\mathrm{dist}(\nabla w(x),\,K)$ in $\Omega\setminus E$. If $\mathcal{L}^n(E) = 0$, then by Theorem 1 (Theorem 3.1 in [9]), there exists $R\in SO(n)$ satisfying (17) and hence we are done in this case. Let U be a relatively compact subset of Ω. If $\mathcal{L}^n(E\cap U) = 0$, trivially we obtain (10) and hence we assume $\mathcal{L}^n(E\cap U) > 0$. Choose $0 < s_0 < 1/2$, let α_n be the volume of the unit ball in $\mathbb{R}^n$ and let $\delta = \delta(U) := 1/3\,\mathrm{dist}(U\,,\partial\Omega)$. From (15) and (16) we obtain

$$\int_U |\nabla^2 w|^2 \le C(\delta,\Omega)\int_\Omega |\nabla w|^2 \le C(\delta,\Omega)\int_\Omega \big(|K|_\infty^2 + \mathrm{dist}^2(\nabla w, K)\big) \le C(\delta,\Omega,K)\,.$$

Let $K_1 := SO(n)$, $K_2 := SO(n)\,H$ and $\mathrm{d}_P(\,\cdot\,) := \mathrm{dist}(\,\cdot\,P)$. Therefore by Lemma 4, we have

$$\begin{aligned}
\mathcal{L}^n(E\cap U)\bigwedge\mathcal{L}^n(U\setminus E) &\le \frac{1}{\rho^2}\left(\int_U \mathrm{d}^2_{K_1}(\nabla w)\bigwedge\int_U \mathrm{d}^2_{K_2}(\nabla w)\right)\\
&\le C(n,U,K)\left[\epsilon^2 + \left(\epsilon^2\int_U |\nabla^2 w|^2\right)^{n/2(n-1)}\right]\\
&\le C(n,\delta,U,\Omega,K)\,\epsilon^{n/(n-1)}\\
&\le \begin{cases} \alpha_n s_0\,\delta^n, & \text{if } \;\epsilon\le\epsilon_0\\[2ex] (\alpha_n s_0/C)^{-(n-2)/n}\,\delta^{2-n}\,\epsilon^2, & \text{if } \;\epsilon\ge\epsilon_0\,,\end{cases}
\end{aligned} \tag{18}$$

where $\epsilon_0 := (\alpha_n s_0/C)^{(n-1)/n}\,\delta^{n-1}$. If $\epsilon\ge\epsilon_0$, then we have a bound for $\int_U \mathrm{d}^2_{K_1}(\nabla w)$ or $\int_U \mathrm{d}^2_{K_2}(\nabla w)$ with the optimal scaling ϵ^2 and hence the assertion follows from Theorem 1. Therefore, suppose $\epsilon\le\epsilon_0$ and hence either $\mathcal{L}^n(E\cap U)$ or $\mathcal{L}^n(U\setminus E)$ is small.

Step 3. Covering argument and the final estimate:

Let us first assume that $\mathcal{L}^n(E\cap U) \le \alpha_n s_0\, \delta^n$. In this case we will prove that there exists a constant C, depending only on n, Ω and K, such that

$$\mathcal{L}^n(E\cap U) \,\le\, C\int_\Omega \operatorname{dist}^2(\nabla w(x),\, K)\, dx\,. \tag{19}$$

By $\fint_E f\,dx$ we denote the mean value $(\mathcal{L}^n(E))^{-1}\int_E f\,dx$. Let M be the *Hardy maximal operator* defined by

$$Mf(x) \,:=\, \sup_{0<r<\infty} \fint_{B(x,r)} |f|\,dx\,.$$

Let $x\in\Omega$ and $0<r<\frac{1}{2}\operatorname{dist}(x,\partial\Omega)$ and $B(x,r)\subset\Omega$, be the ball of radius r, centered at x. Then by Remark 5 there exists $C:=C(n,K)>0$, such that

$$\begin{aligned} C\int_{B(x,r)} \mathrm{d}^2_{K_1}(\nabla w) \bigwedge \int_{B(x,r)} \mathrm{d}^2_{K_2}(\nabla w) \le &\left(\int_{B(x,r)} \mathrm{d}^2_K(\nabla w) \int_{B(x,r)} |\nabla^2 w|^2\right)^{n/2(n-1)} \\ &+ \int_{B(x,r)} \mathrm{d}^2_K(\nabla w)\,. \end{aligned}\tag{20}$$

Substituting (15) in (20) and dividing both sides by $\mathcal{L}^n(B(x,r))$, we obtain

$$\begin{aligned} C\fint_{B(x,r)} \mathrm{d}^2_{K_1}(\nabla w) \bigwedge \fint_{B(x,r)} \mathrm{d}^2_{K_2}(\nabla w) \le &\left(\fint_{B(x,r)} \mathrm{d}^2_K(\nabla w) \fint_{B(x,2r)} |\nabla w|^2\right)^{n/2(n-1)} \\ &+ \fint_{B(x,r)} \mathrm{d}^2_K(\nabla w) \\ \le &\left(M(|\nabla w|^2)(x) \fint_{B(x,r)} \mathrm{d}^2_K(\nabla w)\right)^{n/2(n-1)} \\ &+ \fint_{B(x,r)} \mathrm{d}^2_K(\nabla w) \end{aligned}\tag{21}$$

Here and in the following we extend $|\nabla w|^2$ by zero outside Ω. Define the set $A_\infty := \{x\in\Omega : M(|\nabla w|^2(x)) \ge R^2\}$, where $R := 2\sqrt{2}|K|_\infty$. We claim $A_\infty \subset \{x\in\Omega : M(\operatorname{dist}^2(\nabla w(x),\,K)) \ge R^2/10\}$. Indeed observe that for each $x\in\Omega$, $|\nabla w(x)|^2 \le \left(|\nabla w(x)|^2 - \frac{R^2}{2}\right)_+ + \frac{R^2}{2}$ and hence $M(|\nabla w(x)|^2) \le M\left(|\nabla w(x)|^2 - \frac{R^2}{2}\right)_+ + \frac{R^2}{2}$. Therefore, for each $x\in A_\infty$,

$M\left(|\nabla w(x)|^2 - \frac{R^2}{2}\right)_+ \geq \frac{R^2}{2}$. By the definition of R, it is easy to verify that $\left(|\nabla w(x)|^2 - \frac{R^2}{2}\right)_+ \leq 4\,\mathrm{dist}^2(\nabla w(x),\, K)$. This yields the claim. Therefore by the weak L^1 estimate for the maximal function (e.g. see Theorem 7.4 in [19])

$$\mathcal{L}^n(A_\infty) \leq \mathcal{L}^n\left(\{M(\mathrm{dist}^2(\nabla w(x), K)) \geq R^2/10\}\right) \leq C \int_\Omega \mathrm{dist}^2(\nabla w, K)\,. \tag{22}$$

If $\mathcal{L}^n(E \cap U \setminus A_\infty) = 0$, then $\mathcal{L}^n(E \cap U) = \mathcal{L}^n(A_\infty)$ and hence (19) follows from (22). Suppose $\mathcal{L}^n(E \cap U \setminus A_\infty) > 0$. By the Lebesgue point Theorem, there exists a set N of measure zero, such that for each $x \in (E \cap U \setminus A_\infty) \setminus N$ there exists $r_x > 0$ satisfying

$$\frac{\mathcal{L}^n(E \cap U \cap B(x, r_x))}{\mathcal{L}^n(B(x, r_x))} = s_0\,. \tag{23}$$

By smallness of measure of $E \cap U$, it follows that $B(x, 2r_x) \subset \Omega$. By Besicovitch covering Theorem there exists countable number of disjoint balls $B(x_i, r_i)$ satisfying (23) such that

$$\mathcal{L}^n(E \cap U \setminus A_\infty) \leq C \sum_{i \geq 1} \mathcal{L}^n(B(x_i, r_i))\,. \tag{24}$$

Since for each $i \geq 1$ we have $\mathrm{dist}(\nabla w(x), SO(n)H) \geq \rho$ on $B(x_i, r_i) \setminus E$ we deduce from (21) that for each $x \in (E \cap U) \setminus A_\infty$

$$\begin{aligned} s_0\rho^2 &\leq \min(s_0\rho^2\,,\, (1 - s_0)\rho^2) \\ &\leq \min\left(⨍_{B(x_i,r_i)} \mathrm{d}^2_{K_1}(\nabla w)\,, ⨍_{B(x_i,r_i)} \mathrm{d}^2_{K_2}(\nabla w)\right) \\ &\leq C\left(⨍_{B(x_i,r_i)} \mathrm{d}^2_K(\nabla w)\right)^{n/2(n-1)} + ⨍_{B(x_i,r_i)} \mathrm{d}^2_K(\nabla w) \end{aligned} \tag{25}$$

Since r_i can be chosen smaller than 1 and $\int_{B(x_i,r_i)} \mathrm{d}^2_K(\nabla w) \leq C\epsilon^2$, from the above inequality we obtain

$$\int_{B(x_i,r_i)} \mathrm{d}^2_K(\nabla w) \geq C\left(s_0\rho^2\right)^{2(n-1)/n} \mathcal{L}^n(B(x_i, r_i)\,. \tag{26}$$

Hence by summing over all i and by (24), we obtain

$$\mathcal{L}^n(E \cap U \setminus A_\infty) \leq C \int_\Omega \mathrm{d}^2_K(\nabla w)\,. \tag{27}$$

Therefore the inequality (19) follows from (22) and (27). Now from (19), we obtain

$$\begin{aligned}\int_U \operatorname{dist}^2(\nabla w, SO(n)) &= \int_{U\setminus E} \operatorname{dist}^2(\nabla w, SO(n)) + \int_{U\cap E} \operatorname{dist}^2(\nabla w, SO(n)) \\ &\leq C\int_{U\setminus E} \operatorname{dist}^2(\nabla w, K) + C\left[\mathcal{L}^n(U\cap E) + \int_{U\cap E} \operatorname{dist}^2(\nabla w, K)\right] \\ &\leq C\int_\Omega \operatorname{dist}^2(\nabla w, K)\,. \qquad (28)\end{aligned}$$

Now the desired estimate follows from Theorem 1. If $\mathcal{L}^n(U\setminus E) \leq \alpha_n s_0\,\delta^n$, we obtain the inequality (28) with $SO(n)H$ instead of $SO(n)$. This finishes the proof of the Theorem 3.1. □

Proof of Theorem 2.

To establish the estimate up to the boundary we proceed as in [9] and make use of the following cube decomposition of Ω (see Theorem 1 and Proposition 3, Chapter VI in [21]).

Proposition 3.2. *There exists a constant N, which depends only on the dimension n and a collection $\mathcal{F} = \{Q_1, Q_2 \cdots\}$ of closed cubes, whose sides are parallel to the axes and having disjoint interiors so that*

(i) $\Omega = \cup_k Q_k$

(ii) $\operatorname{diam} Q_k \leq \operatorname{dist}(Q_k, \partial\Omega) \leq 4\operatorname{diam} Q_k$

(iii) *each point in Ω is contained in at most N of the enlarged concentric cubes Q_k^*, where $Q_k^* := x_k + \frac{9}{8}(Q_k - x_k)$ and where x_k is the center of Q_k.*

As in the proof of Theorem 3.1 we may assume $\|\nabla u\|_{L^\infty(\Omega)} \leq M$, M being a constant depending on the domain Ω and the λ_i. We again use the decomposition $w = u - z$ as in the proof of Theorem 3.1. We now establish a weighted estimate for $\nabla^2 w$ and then conclude by a weighted Poincaré inequality. Fix one of the cubes $Q := \operatorname{int} Q_k = \bar{x} + \left(-\frac{r}{2}, \frac{r}{2}\right)^n$ of the above family $\mathcal{F}$ and denote $Q^\mu := \bar{x} + \mu(Q - \bar{x})$ the concentric cube enlarged by a factor $\mu > 1$. From the assertion (ii) of Proposition 3.2 it follows that the enlarged cube Q^μ is contained in Ω for every $1 < \mu < 2$. We choose $\mu > 1$

such that $\mu^2 < 2$. Now apply the local estimate of Theorem 3.1 to $\Omega = Q^{\mu^2}$ and $U = Q^\mu$. Since the estimate (10) is invariant under dilations we get

$$\int_{Q^\mu} |\nabla u - R_Q|^2\, dx \;\le\; C(H,\mu) \int_{Q^{\mu^2}} \operatorname{dist}^2(\nabla u, K)\, dx\,. \tag{29}$$

By elliptic regularity we have

$$r^2 \int_Q |\nabla^2 w|^2\, dx \;\le\; \frac{C}{(\mu-1)^2} \min_{F\in\mathbb{R}^{n\times n}} \int_{Q_k} |\nabla w - F|^2\, dx\,. \tag{30}$$

Hence by using (29) and the decomposition $w = u - z$ we get

$$\int_Q r^2 |\nabla^2 w|^2\, dx \;\le\; C(\mu, H) \int_{Q^{\mu^2}} \left(\operatorname{dist}^2(\nabla u, K) \;+\; |\nabla z|^2\right)\, dx\,. \tag{31}$$

Now let $\mu = \sqrt{\dfrac{9}{8}}$. Then assertion (ii) of Proposition 3.2 implies that

$$\begin{aligned}\int_{Q_k} |\nabla^2 w|^2 \operatorname{dist}^2(x,\partial\Omega)\, dx &\le C(n,H) \int_{Q_k^*} \left(\operatorname{dist}^2(\nabla u, K) + |\nabla z|^2\right)\, dx \\ &= C(n,H) \int_\Omega \left(\operatorname{dist}^2(\nabla u, K) + |\nabla z|^2\right) \chi_{Q_k^*}(x)\, dx\,.\end{aligned} \tag{32}$$

Summation over k and the assertion (iii) of Proposition 3.2 and (16) yield

$$\begin{aligned}\int_\Omega |\nabla^2 w|^2 \operatorname{dist}^2(x,\partial\Omega)\, dx \;&\le\; C(n,H)\, N \int_\Omega \left(\operatorname{dist}^2(\nabla u, K) + |\nabla z|^2\right)\, dx \\ &\le C(n,\Omega,H) \int_\Omega \operatorname{dist}^2(\nabla u, K)\, dx\,.\end{aligned} \tag{33}$$

To conclude the proof we write $f = \nabla w$ and use a weighted Poincaré inequality of the form

$$\min_{F\in\mathbb{R}^{n\times n}} \int |f(x) - F|^2\, dx \;\le\; C(\Omega) \int_\Omega |\nabla f|^2 \operatorname{dist}^2(x,\partial\Omega)\, dx\,, \tag{34}$$

which is valid for $f \in W^{1,2}(\Omega, \mathbb{R}^{n\times n})$. This inequality is derived in [9] as an immediate consequence of the following estimate (see Theorem 1.5 of [18] or Theorem 8.8 of [13]):

$$\int_U |g|^2\, dx \;\le\; C(U) \int_U \left(|g|^2 + |\nabla g|^2\right) \operatorname{dist}^2(x,\partial\Omega)\, dx$$

for $g \in W^{1,2}_{\rm loc}(U) \cap L^2(U)$. Apply the inequality (34) to (33) to obtain $F \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} \int_\Omega |\nabla u - F|^2\,dx \;&\le\; 2\left(\int_\Omega |\nabla w - F|^2\,dx + \int_\Omega |\nabla z|^2\,dx\right) \\ &\le C\int_\Omega \mathrm{dist}^2(\nabla u, K)\,dx\,. \end{aligned} \tag{35}$$

If $F \in K$ we are done. Suppose $0 < \delta := \mathrm{dist}(F, K) = |F - R|$, $R \in K$. From (35) it easily follows that

$$\mathcal{L}^n(\Omega)\,\delta^2 \;\le\; C\int_\Omega \mathrm{dist}^2(\nabla u, K)\,dx\,,$$

and hence

$$\begin{aligned} \int_\Omega |\nabla u - R|^2\,dx \;&\le\; 2\int_\Omega |\nabla u - F|^2\,dx + 2\mathcal{L}^n(\Omega)\,\delta^2 \\ &\le C\int_\Omega \mathrm{dist}^2(\nabla u, K)\,dx\,. \end{aligned} \tag{36}$$

This finishes the proof of Theorem 2. □

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