Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Quantization of a Moduli Space of Parabolic Higgs Bundles

by

Indranil Biswas and Avijit Mukherjee

Preprint no.: 17 2003

QUANTIZATION OF A MODULI SPACE OF PARABOLIC HIGGS BUNDLES

INDRANIL BISWAS AND AVIJIT MUKHERJEE

Abstract. Let $\mathcal{M}^s_H$ be a moduli space of stable parabolic Higgs bundles of rank two over a Riemann surface X. It is a smooth variety over $\mathbb{C}$ equipped with a holomorphic symplectic form. Fix a projective structure $\mathcal{P}$ on X. Using $\mathcal{P}$, we construct a quantization of a certain Zariski open dense subset of the symplectic variety $\mathcal{M}^s_H$.

1. Introduction

Let X be a compact connected Riemann surface and $S \subset X$ a fixed finite subset. A parabolic vector bundle E_* of rank two over X consists of a rank two holomorphic vector bundle E, a line $F_s \subset E_s$ and $\lambda_s \in (0\,,1)$ for each $s \in S$. A Higgs field on E_* is a holomorphic section θ of $\mathrm{End}(E) \otimes K_X \otimes \mathcal{O}_X(S)$ with $\theta(s)$ nilpotent for the flag $0 \subset F_s \subset E_s$ for each $s \in S$. The pair $(E_*\,,\theta)$ is called a parabolic Higgs bundle. In [7] the notion of a Higgs bundle was introduced and stability of a Higgs bundle was defined.

Let $\mathcal{M}^s_H$ denote the moduli space of all stable parabolic Higgs bundles of rank two over X of fixed degree and parabolic weights. The moduli space $\mathcal{M}^s_H$ is a smooth irreducible quasiprojective complex variety. The variety $\mathcal{M}^s_H$ has a natural algebraic symplectic structure, which we will denote by Ω.

The symplectic form Ω defines a Poisson structure on $\mathcal{O}_{\mathcal{M}^s_H}$, the sheaf of complex valued algebraic functions on $\mathcal{M}^s_H$. A quantization of Ω is a one–parameter family of associative algebra structures on $\mathcal{O}_{\mathcal{M}^s_H}$ deforming the abelian algebra structure defined by pointwise multiplication, with the infinitesimal deformations of the pointwise multiplication structure being governed by the Poisson structure.

Any symplectic structure admits a quantization, but there is no uniqueness of quantization in the sense that the space of all quantizations of a symplectic structure is infinite dimensional. The main result here is to produce a canonical quantization of the symplectic form Ω on a Zariski open dense subset $\mathcal{U}$ of the moduli space $\mathcal{M}^s_H$ once a projective structure on X has been chosen (Theorem 3.2).

A projective structure on X is defined by giving a covering of X by holomorphic coordinate charts such that all the transition functions are Möbius transformations. The space of all projective structures on X is nonempty. The Zariski open subset $\mathcal{U} \subset \mathcal{M}^s_H$ over which the quantization of Ω is constructed does not depend on the choice of the projective structure needed in the construction.

2. Quantization, parabolic Higgs bundles, and projective structure

2.1. **Quantization.** Let M be a complex manifold. Its holomorphic tangent bundle will be denoted by TM. Let Θ be a holomorphic symplectic form on M and $\tau\,:\,T^*M\,\longrightarrow\,TM$ the holomorphic isomorphism defined by Θ. So $\tau^{-1}(v)(w)=\Theta(w,v)$, where $v,w\in T_xM$ and $x\in M$.

Let f and g be any two holomorphic functions defined on some open subset U on M. Sending the pair $(f\,,g)$ to the holomorphic function

$$\{f\,,g\}\,:=\,\Theta(\tau(df)\,,\tau(dg))$$

over U defines a *holomorphic Poisson structure* on the space of all locally defined holomorphic functions on M. Let $\mathcal{H}(M)$ denote the algebra of all locally defined holomorphic functions on M. Let $\mathcal{A}(M)\,:=\,\mathcal{H}(M)[[h]]$ be the space of all formal Taylor series

$$f\,:=\,\sum_{j=0}^{\infty}h^jf_j$$

where $f_j\in\mathcal{H}(M)$ and h is a formal parameter.

A *quantization* of the Poisson structure is an associative algebra operation on $\mathcal{A}(M)$, which is denoted by $\star$, satisfying the following conditions; see [9], [1], [4], [5] for the details. For any element $g\,:=\,\sum_{j=0}^{\infty}h^jg_j\,\in\,\mathcal{A}(M)$ the product

$$f\star g\,=\,\sum_{j=0}^{\infty}h^j\phi_j$$

satisfies the following conditions

(1) each $\phi_i\,\in\,\mathcal{H}(M)$ is some polynomial (independent of f and g) in derivatives (of arbitrary order) of $\{f_j\}_{j\geq 0}$ and $\{g_j\}_{j\geq 0}$;
(2) $\phi_0\,=\,f_0g_0$;
(3) $1\star f\,=\,f\star 1\,=\,f$ for every $f\,\in\,\mathcal{H}(M)$;
(4) $f\star g\,-\,g\star f\,=\,\sqrt{-1}h\{f_0\,,g_0\}\,+\,h^2\beta$, where $\beta\in\mathcal{A}(M)$ depends on f,g.

We will now describe the Moyal–Weyl quantization which will be used here.

Let V be a complex vector space of dimension $2n$. Let Θ be a constant symplectic form on V. In other word, $\Theta\,\in\,\bigwedge^2V^*$ defining a nondegenerate skew-symmetric bilinear form on V. As before, $\mathcal{H}(V)$ is the space of all locally defined holomorphic functions on V equipped with the Poisson structure.

Let

$$\Delta\,:\,V\,\longrightarrow\,V\times V$$

denote the diagonal homomorphism defined by $v\,\longmapsto\,(v\,,v)$. There exists a unique differential operator

$$D\,:\,\mathcal{H}(V\times V)\,\longrightarrow\,\mathcal{H}(V\times V) \tag{2.1}$$

with constant coefficients such that for any pair $f, g \in \mathcal{H}(V)$,

$$\{f\,, g\} \,=\, \Delta^* D(f \otimes g)$$

where $f \otimes g$ is the function on $V \times V$ defined by $(u\,, v) \longmapsto f(u)g(v)$ [9], [5].

The *Moyal–Weyl algebra* is defined by

$$f \star g \,=\, \Delta^* \exp(\sqrt{-1}hD/2)(f \otimes g) \,\in\, \mathcal{A}(V) \tag{2.2}$$

for $f, g \in \mathcal{H}(V)$, and it is extended to a multiplication operation on $\mathcal{A}(V)$ using the bilinearity condition with respect to h. In other words, if $f := \sum_{j=0}^{\infty} h^j f_j$ and $g := \sum_{j=0}^{\infty} h^j g_j$ are two elements of $\mathcal{A}(V)$, then

$$f \star g \,=\, \sum_{i,j=0}^{\infty} h^{i+j}(f_i \star g_j) \,\in\, \mathcal{A}(V)\,.$$

It is known that this $\star$ operation makes $\mathcal{A}(V)$ into an associative algebra that quantizes the symplectic structure Θ. See [1], [9] for the details.

Let $\mathrm{Sp}(V)$ denote the group of all linear automorphism of V preserving the symplectic form Θ. The group $\mathrm{Sp}(V)$ acts on $\mathcal{A}(V)$ in an obvious way namely, $(\sum_{j=0}^{\infty} h^j f_j) \circ G = \sum_{j=0}^{\infty} h^j (f_j \circ G)$, where $G \in \mathrm{Sp}(V)$. The differential operator D in (2.1) evidently commutes with the diagonal action of $\mathrm{Sp}(V)$ on $V \times V$. This immediately implies that

$$(f \circ G) \star (g \circ G) \,=\, (f \star g) \circ G\,. \tag{2.3}$$

for any $G \in \mathrm{Sp}(V)$ and $f, g \in \mathcal{A}(V)$.

2.2. **Projective structure.** Let $\mathbb{P}(V)$ denote the projective line consisting of all one–dimensional subspaces of a complex vector space V of dimension two. The group of all automorphisms of $\mathbb{P}(V)$ coincides with $\mathrm{PSL}(V) := \mathrm{SL}(V)/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ is the center of $\mathrm{SL}(V)$ consisting of $\pm\mathrm{Id}_V$. Note that choosing a basis of V, the Möbius group (the group of all fractional linear transformations of $\widehat{\mathbb{C}} = \mathbb{CP}^1$) gets identified with $\mathrm{PSL}(V)$.

By a *holomorphic coordinate function* on X we will mean a pair of the form $(U\,, \phi)$, where $U \subset X$ is some open subset and

$$\phi \,:\, U \longrightarrow \mathbb{P}(V) \tag{2.4}$$

a holomorphic embedding. By a *holomorphic atlas* on X we will mean a collection of holomorphic coordinate functions $\{(U_i\,, \phi_i)\}_{i\in I}$ such that

$$\bigcup_{i\in I} U_i \,=\, X\,.$$

Let $\{(U_i\,, \phi_i)\}_{i\in I}$ be a holomorphic atlas satisfying the condition that for each pair $(i\,, j) \in I \times I$ with $U_i \cap U_j \,\neq\, \emptyset$ there is an element $T_{i,j} \,\in\, \mathrm{Aut}(\mathbb{P}(V))$ such that the transition function $\phi_i \circ \phi_j^{-1}$ coincides with the restriction of $T_{i,j}$ to $\phi_j(U_i \cap U_j)$.

Another holomorphic atlas $\{(U_j\,,\phi_j)\}_{j\in J}$ satisfying this condition on transition functions is called *equivalent* to $\{(U_i\,,\phi_i)\}_{i\in I}$ if the above condition on transition functions holds also for the union $\{(U_k\,,\phi_k)\}_{k\in I\cup J}$. A *projective structure* on X is an equivalence class of holomorphic atlases satisfying the above condition on transition functions [6].

Any Riemann surface admits a projective structure. The uniformization theorem says that the universal cover of X is biholomorphic to either $\mathbb{C}$ or $\mathbb{CP}^1$ or the upper half plane $\mathbb{H}$. Since the group of all automorphisms of each of these three Riemann surfaces is contained in the Möbius group, X has a natural projective structure. The space of all projective structures on X is an affine space for $H^0(X,\, K_X^{\otimes 2})$, the space of quadratic differentials.

Let

$$\mathcal{Z} \,:=\, K_X \setminus 0_X \tag{2.5}$$

be the complement of the zero section of the total space of the holomorphic cotangent bundle. The total space of K_X has a natural algebraic symplectic structure. (We recall that the total space of K_X has a tautological one–form on it; its exterior derivative defines the symplectic structure.) Let

$$\theta_0 \,\in\, H^0(\mathcal{Z},\, \Omega^2_{\mathcal{Z}}) \tag{2.6}$$

be the algebraic symplectic form on $\mathcal{Z}$ obtained by restricting the natural symplectic form on the total space of K_X.

Given a projective structure on X, there is a natural quantization of the symplectic form θ_0 on $\mathcal{Z}$ [2]. We will briefly recall the construction of the quantization.

First consider the special case $X \,=\, \mathbb{P}(V)$. The symplectic surface, defined in (2.5), corresponding to $\mathbb{P}(V)$ will be denoted by $\mathcal{Z}_0$. After fixing a nonzero element in the line $\Theta \,\in\, \bigwedge^2 V^*$, we have an isomorphism $\mathcal{Z}_0 \,=\, (V\setminus\{0\})/\sigma$, where σ is the involution that sends any $v\in V$ to $-v$. Since the symplectic structure on V defined Θ is preserved by the involution σ, it descends to a symplectic structure on the quotient $(V\setminus\{0\})/\sigma$. The above identification of $(V\setminus\{0\})/\sigma$ with $\mathcal{Z}_0$ takes this descended symplectic structure to θ_0 defined in (2.6). Consider the Moyal–Weyl quantization of the symplectic form Θ on V (see (2.2)). Using the identity (2.3) for $\sigma \,\in\, \mathrm{Sp}(V)$ the Moyal–Weyl quantization descends to a quantization of the symplectic structure θ_0 on $\mathcal{Z}_0$.

Let X be a Riemann surface equipped with a projective structure $\mathcal{P}$. Let ϕ be an embedding as in (2.4) for $\mathcal{P}$. Let p_0 (respectively, p) be the projection of $\mathcal{Z}_0$ (respectively, $\mathcal{Z}$) to $\mathbb{P}(V)$ (respectively, X). The differential $d\phi$ identifies $p_0^{-1}(\phi(U))$ with $p^{-1}(U)$, and the map clearly preserves the symplectic forms. Therefore, the quantization over $p_0^{-1}(\phi(U))$, obtained by restricting the above quantization of $\mathcal{Z}_0$, gives a quantization of θ_0 over $p^{-1}(U)$.

For another map $\phi' : U' \longrightarrow \mathbb{P}(V)$ as in (2.4) for $\mathcal{P}$, we have $\phi' \circ \phi^{-1} =: G \in \mathrm{PSL}(V)$. Now the identity (2.3) ensures that the two quantizations on $p^{-1}(U)$ and $p^{-1}(U')$ coincide over $p^{-1}(U \cap U')$. Therefore, we have constructed a quantization of the symplectic manifold $\mathcal{Z}$. See [2] for more details.

2.3. **Parabolic bundles.** Let X be a compact connected Riemann surface of genus g_X. Fix a finite subset

$$S := \{s_1, s_2, \cdots, s_n\} \subset X .$$

Assume that $n = \#S \geq 4$ if $g_X = 0$, and if $g_X = 1$ then $n \geq 1$. If $g \geq 2$, then n is allowed to take any value in $\mathbb{N}$.

A *parabolic vector bundle* E_* *of rank two* over X with parabolic structure over S consists of the following [8]:

(1) a holomorphic vector bundle E of rank two over X;
(2) for each point $s \in S$, a line $F_s \subset E_s$ (called the *quasiparabolic flag*);
(3) real numbers $\lambda_s \in (0\,,1)$, $s \in S$ (called *parabolic weights*).

A Higgs structure on the parabolic vector bundle E_* is a holomorphic section

$$\theta \in H^0(X, \mathrm{End}(E) \otimes K_X \otimes \mathcal{O}_X(S)) \tag{2.7}$$

with the property that for each $s \in S$, the image of the homomorphism

$$\theta(s) : E_s \longrightarrow (E \otimes K_X \otimes \mathcal{O}_X(S))_s$$

is contained in the subspace $F_s \otimes (K_X \otimes \mathcal{O}_X(S))_s$ and $\theta(s)(F_s) = 0$ [7], [3].

For a general rank two parabolic bundle E_* as in [8], at each point $s \in S$ there are two parabolic weights. However given a general set of weights, say $0 \leq \alpha_s < \beta_s < 1$, if we replace it by the single nonzero weight $\beta_s - \alpha_s$, then the parabolic (semi)stability condition remains unchanged. So the moduli space of parabolic Higgs bundles remains unchanged. Hence there is no loss of generality due to our assumption that at each $s \in S$ there is exactly one nonzero parabolic weight.

We fix real numbers $\{\lambda_s\}_{s \in S}$ and an integer d. Unless specified otherwise, henceforth by a parabolic vector bundle we will always mean a parabolic vector bundle of rank two and degree d and parabolic weights λ_s, $s \in S$.

The moduli space of stable parabolic Higgs bundles is an irreducible smooth quasiprojective variety of dimension $8g_X - 6 + 2n$. This moduli space will be denoted by $\mathcal{M}_H^s$. There is a canonically defined algebraic symplectic form on the moduli space $\mathcal{M}_H^s$, which we will denote by Ω. The construction of this symplectic form Ω can be found in [7], [3].

3. Quantization of Ω

Fix one of the points, say $s_1 \in S$, over which the parabolic structure is defined. We start with the following lemma:

Lemma 3.1. *There is a nonempty Zariski open dense subset $U_1 \subset \mathcal{M}_H^s$ such that for all parabolic Higgs bundle (E_*, θ) in U_1*

(1) $\dim H^0(X,\, E \otimes \mathcal{O}_X((g_X - 1/2 - d/2)s_1)) = 1$ *if the degree d of E is odd;*
(2) *if $d = 2d'$, $d' \in \mathbb{Z}$, then*

$$\dim\{\beta \in H^0(X,\, E \otimes \mathcal{O}_X((g_X - d')s_1)) \,|\, \beta(s_1) \subset F_{s_1}\} = 1$$

where $F_{s_1} \subset E_{s_1}$ is the line defining the quasiparabolic structure over the fixed point s_1.

Proof. First consider the case where the degree d is odd. Then

$$\mathrm{degree}(E \otimes \mathcal{O}_X((g_X - 1/2 - d/2)s_1)) = 2g_X - 1\,.$$

Hence $\dim H^0(X, (E\otimes\mathcal{O}_X((g_X-1/2-d/2)s_1) - \dim H^1(X, (E\otimes\mathcal{O}_X((g_X-1/2-d/2)s_1) = 2g_X - 1 - 2g_X + 2 = 1$. So

$$\dim H^0(X, (E \otimes \mathcal{O}_X((g_X - 1/2 - d/2)s_1) \geq 1\,. \tag{3.1}$$

But for the general stable vector bundle V of rank two and degree $2g_X - 1$ we have $H^1(X,\, V) = 0$. (The general stable bundle W of degree d_0 with $d_0 \geq \mathrm{rank}(W)(g_X - 1)$ has $H^1(X,\, W) = 0$; all W with $H^1(X,\, W) \neq 0$ form the generalized theta divisor on the moduli space when $d_0 = \mathrm{rank}(W)(g_X - 1)$.) Therefore, there are stable parabolic Higgs bundles (E_*, θ) with $\dim H^0(X,\, E) = 1$. Now the inequality (3.1) combined with the semicontinuity of $\dim H^0$ shows that there is a nonempty Zariski open subset $U_1 \subset \mathcal{M}_H^s$ with $\dim H^0(X,\, E\otimes\mathcal{O}_X((g_X - 1/2 - d/2)s_1)) = 1$ for all (E_*, θ) in U_1. (The semicontinuity of $\dim H^0$ says that in a family of vector bundles where $\dim H^0$ attains minimum is Zariski open.) Since $\mathcal{M}_H^s$ is irreducible, U_1 is Zariski dense as well.

Now consider the case $d = 2d'$, with $d' \in \mathbb{Z}$. For a parabolic Higgs bundle (E_*, θ), with E as the underlying vector bundle, let V be the vector bundle over X that fits in the following exact sequence of sheaves

$$0 \longrightarrow V \longrightarrow E \otimes \mathcal{O}_X((g_X - d')s_1) \longrightarrow (E_{s_1}/F_{s_1}) \otimes \mathcal{O}_X((g_X - d')s_1)_{s_1} \longrightarrow 0\,,$$

where $F_{s_1} \subset E_{s_1}$ is the flag for the parabolic structure and $\mathcal{O}_X((g_X - d')s_1)_{s_1}$ is the fiber of the line bundle $\mathcal{O}_X((g_X - d')s_1)$ over s_1; the (right–hand side) surjective map in the above exact sequence is simply the one dimensional quotient of the restriction of the vector bundle $E \otimes \mathcal{O}_X((g_X - d')s_1)$ to the point $s_1 \in X$.

So $H^0(X,\, V) \subset H^0(X,\, E \otimes \mathcal{O}_X((g_X - d')s_1))$, and from the above exact sequence it is immediate that

$$H^0(X,\, V) \cong \{\beta \in H^0(X,\, E \otimes \mathcal{O}_X((g_X - d')s_1)) \,|\, \beta(s_1) \subset F_{s_1}\}\,.$$

On the other hand, degree$(V) = 2g_X - 1$, and hence $\dim H^0(X, V) - \dim H^1(X, V) = 1$. Since for the general V with degree$(V) = 2g_X - 1$ we have $H^1(X, V) = 0$, it follows immediately using the semicontinuity of $\dim H^0$ that on a Zariski open dense subset U_1 of $\mathcal{M}^s_H$ the condition $\dim H^0(X, V) = 1$ holds. This completes the proof of the lemma. $\square$

Fix a positive integer δ such that $2(\delta + g_X) - 1 > \max\{0\,, 6(g_X - 1) + \#S\}$. Set

$$\delta_0 := 2(\delta + g_X) - 1$$

if d is odd, and if d is even then set

$$\delta_0 := 2(\delta + g_X)\,.$$

Let $\mathcal{M}^{\delta_0}_H$ denote the moduli space of all rank two stable parabolic Higgs bundles $(E_*\,, \theta)$, where E_* is a parabolic vector bundle of degree δ_0 with parabolic weights λ_s at $s \in S$ (same as the ones fixed in Section 2.3) and θ a Higgs structure on E_*. Let $\mathcal{M}^{\delta_0}_T$ denote the moduli space of triples of the form $(E_*\,, \theta\,, s)$, where $(E_*\,, \theta) \in \mathcal{M}^{\delta_0}_H$ and $s \in H^0(X, E) \setminus \{0\}$ a nonzero section (see [3]). The projection

$$p : \mathcal{M}^{\delta_0}_T \longrightarrow \mathcal{M}^{\delta_0}_H \tag{3.2}$$

that sends any $(E_*\,, \theta\,, s)$ to $(E_*\,, \theta)$ is a smooth projective bundle with the fiber over the point in $\mathcal{M}^{\delta_0}_H$ corresponding to $(E_*\,, \theta)$ being $\mathbb{P}H^0(X, E)$, the projective space of lines in $H^0(X, E)$; the numerical condition on δ_0 ensures that $\dim H^0(X, E)$ is independent of E (see [3]).

Given a parabolic vector bundle E_* and any line bundle L, the tensor product $E \otimes L$, where E is the underlying vector bundle for E_*, has a natural parabolic structure induced by the parabolic structure of E_*; this parabolic vector bundle will be denoted by $E_* \otimes L$. The parabolic weights remain unchanged and $F_s \otimes L_s \subset E_s \otimes L_s$ is the quasiparabolic flag for $E_* \otimes L$ at the parabolic point s, where $F_s \subset E_s$ is the flag for E_*. If θ is a Higgs structure on E_*, then θ defines a Higgs structure on $E_* \otimes L$ in an obvious way. The parabolic Higgs bundle $(E_* \otimes L\,, \theta)$ is stable if and only if $(E_*\,, \theta)$ is so.

For any $(E_*\,, \theta) \in \mathcal{M}^s_H$ define

$$E'_* := E_* \otimes \mathcal{O}_X((g_X - 1/2 - d/2 + \delta)s_1) \tag{3.3}$$

if d is odd, and define

$$E'_* := E_* \otimes \mathcal{O}_X((g_X - d' + \delta)s_1) \tag{3.4}$$

if $d = 2d'$. So $(E'_*\,, \theta) \in \mathcal{M}^{\delta_0}_H$. The resulting isomorphism

$$\psi_0 : \mathcal{M}^s_H \longrightarrow \mathcal{M}^{\delta_0}_H \tag{3.5}$$

that sends any $(E_*\,, \theta)$ to $(E'_*\,, \theta)$ preserves the symplectic structures of the two moduli spaces of stable parabolic Higgs bundles, that is,

$$\psi_0^*\Omega' = \Omega$$

where Ω (respectively, Ω') is the algebraic symplectic form on $\mathcal{M}^s_H$ (respectively, $\mathcal{M}^{\delta_0}_H$) (see Section 2.3).

Take any $(E_*, \theta) \in U_1$, where U_1 is the open subset in Lemma 3.1. Let β be a nonzero section of $H^0(X,\, E\otimes\mathcal{O}_X((g_X-1/2-d/2)s_1))$ (respectively, a nonzero section of $H^0(X,\, E\otimes\mathcal{O}_X((g_X-d/2)s_1))$ with $\beta(s_1)\subset F_{s_1}$) if d is odd (respectively, even). Lemma 3.1 ensures that there is such a section and any two choices of β differ by the multiplication with an element in $\mathbb{C}^*$.

Let

$$\psi : U_1 \longrightarrow \mathcal{M}^{\delta_0}_T \tag{3.6}$$

be the map that sends any (E_*, θ) to the triple $(E'_*, \theta, \beta\otimes s_\delta)$, where s_δ is the section of the line bundle $\mathcal{O}_X(\delta s_1)$ defined by the constant function 1 (recall that $\delta > 0$); the parabolic vector bundle E'_* is defined in (3.3) and def.-E-prime-2. Since any two choices of β differ by a nonzero scalar multiplication, the above map ψ is well defined. Note that $p\circ\psi = \psi_0$ over U_1, where p and ψ_0 are defined in (3.2) and (3.5) respectively.

Let Z denote the total space of the line bundle $K_X\otimes\mathcal{O}_X(S)$. In [3] we considered a map from $\mathcal{M}^{\delta_0}_T$ to the Hilbert scheme $\mathrm{Hilb}^l(Z)$, where l is $4g_X+2\delta+n-3$ (respectively, $4g_X+2\delta+n-2$) if d is odd (respectively, even) with $n = \#S$ (see Section 3.1 of [3]). Let

$$\psi_1 : U_1 \longrightarrow \mathrm{Hilb}^l(Z) \tag{3.7}$$

be the composition of this map with ψ defined in (3.6). Using this map ψ_1 we will construct a map from a nonempty Zariski open subset of U_1 to $\mathrm{Hilb}^{4g_X+n-3}(Z)$.

For this first note that the above section s_δ of the line bundle $\mathcal{O}_X(\delta s_1)$ vanishes at the point s_1 of order δ. Let $0_{s_1}\in(K_X\otimes\mathcal{O}_X(S))_{s_1}\subset Z$ be the zero vector in the fiber over s_1 of the line bundle $K_X\otimes\mathcal{O}_X(S)$. If d is odd, then for any $y\in U_1$ the zero dimensional subscheme

$$\psi_1(y) \in \mathrm{Hilb}^l(Z)$$

has the property that 0_{s_1} occurs in $\psi_1(y)$ of multiplicity at least 2δ (as s_δ vanishes at s_1 of order δ); this means that the image of $\psi_1(y)$ by the forgetful map from $\mathrm{Hilb}^l(Z)$ to the symmetric product $\mathrm{Sym}^l(Z)$ has the property that 0_{s_1} occurs with multiplicity at least 2δ. If $\psi_1(y) = (E'_*, \theta, \beta\otimes s_\delta)$, then the spectral curve corresponding to the parabolic Higgs bundle (E'_*, θ) is (totally) ramified over s_1 and it passes through $0_{s_1}\in(K_X\otimes\mathcal{O}_X(S))_{s_1}$ (see [3]). Therefore, as $\beta\otimes s_\delta$ vanishes at the point s_1 of order δ, it follows immediately that 0_{s_1} occurs in $\psi_1(y)$ of multiplicity at least 2δ.

If d is even, then 0_{s_1} occurs in $\psi_1(y)$ with multiplicity at least $2\delta+1$. The extra multiplicity is due to the condition that if $\psi_1(y) = (E'_*, \theta, \beta\otimes s_\delta)$, then the evaluation $\beta\otimes s_\delta(s_1)$ lies in the line in the fiber E'_{s_1} defining the quasiparabolic structure over s_1.

Let $Z' := Z \setminus \{0_{s_1}\} \subset Z$ be the complement of the point. There is a Zariski open subset $U_2 \subset U_1$ and a (unique) map

$$\psi_2 : U_2 \longrightarrow \mathrm{Hilb}^{4g_X+n-3}(Z') \tag{3.8}$$

satisfying the condition that $\psi_1(y) \cap Z' = \psi_2(y)$ for all $y \in U_2$, where ψ_1 is defined in (3.7). That U_2 is a Zariski open subset follows immediately from the fact that Z' is Zariski open dense in Z. To prove that U_2 is nonempty first note that if U_2 is an empty set then the above remarks on the multiplicity of $\psi_1(y)$ at 0_{s_1} imply that

$$\dim \mathrm{image}(\psi_1) < 2l - 4\delta - 1 - (-1)^d = 8g_X + 2n - 6\,.$$

On the other hand, $\dim U_1 = 8g_X + 2n - 6$ and $\psi_1^*\Theta = \Omega$, where Θ is the meromorphic symplectic form on $\mathrm{Hilb}^l(Z)$ (see [3, Theorem 3.2]). Consequently, $\dim \mathrm{image}(\psi_1) = \dim U_1$.

So we have a map ψ_2 as in (3.8) where U_2 is a nonempty Zariski open dense subset of U_1. Note that $\dim U_2 = \dim \mathrm{Hilb}^{4g_X+n-3}(Z')$. Also, the map ψ_2 is dominant. Indeed, this is an immediate consequence of the main result of [3] (see [3, Theorem 3.2]) that the meromorphic symplectic form on $\mathrm{Hilb}^{4g_X+n-3}(Z')$ pulls back to the symplectic form on U_2.

Let $U'' \subset \mathrm{Hilb}^{4g_X+n-3}(Z')$ be the Zariski open dense subset corresponding to *distinct* $4g_X + n - 3$ points of Z'. Let $p_X : Z' \longrightarrow X$ be the natural projection. Define

$$U := \{y \in U'' \mid y \cap (p_X^{-1}(S) \cup 0_X) = \emptyset\}$$

where 0_X is the image of the zero section of the line bundle $K_X \otimes \mathcal{O}_X(S)$. So U is a Zariski open dense subset of $\mathrm{Hilb}^{4g_X+n-3}(Z')$. It is easy to see that the complement of U is a divisor of $\mathrm{Hilb}^{4g_X+n-3}(Z')$. Set

$$\mathcal{U} := \psi_2^{-1}(U)$$

which is a Zariski open dense subset of U_2, where ψ_2 is defined in (3.8). Let

$$\widehat{\psi} : \mathcal{U} \longrightarrow U \tag{3.9}$$

be the restriction to U of the map ψ_2.

Given a projective structure on X, we will show that the above defined Zariski open dense subset $\mathcal{U} \subset \mathcal{M}_H^s$ equipped with the symplectic form Ω (see Section 2.3) has a natural quantization.

The meromorphic symplectic form Θ on $\mathrm{Hilb}^{4g_X+n-3}(Z)$ defines a symplectic form on the open subset U, as the pole of Θ is supported on the divisor of $\mathrm{Hilb}^{4g_X+n-3}(Z)$ consisting of all zero–dimensional subschemes with support intersecting $p_X^{-1}(S)$. We have $\widehat{\psi}^*\Theta = \Omega$ [3, Theorem 3.2]. Therefore, to quantize Ω over $\mathcal{U}$ it suffices to quantize Θ over U, as using $\widehat{\psi}$ a quantization of Θ over U gives a quantization of Ω over $\mathcal{U}$. We will construct a quantization of Θ over U.

Recall the variety $\mathcal{Z}$ defined in (2.5). Set

$$\mathcal{Z}^0 \, := \, \mathcal{Z} \setminus p^{-1}(S) \subset \mathcal{Z}$$

where p, as in Section 2.2, is the projection of $\mathcal{Z}$ to X. Let

$$\widehat{\mathcal{Z}} \, \subset \, (\mathcal{Z}^0)^{4g_X+n-3}$$

be the Zariski open dense subset of the Cartesian product parametrizing all distinct $4g_X + n - 3$ points of $\mathcal{Z}^0$. Let Σ be the permutation group of $\{1, 2, \cdots, 4g_X + n - 3\}$. So Σ acts freely on $\widehat{\mathcal{Z}}$ and the quotient

$$\widehat{\mathcal{Z}}/\Sigma \, = \, U\, . \tag{3.10}$$

The identification of U with $\widehat{\mathcal{Z}}/\Sigma$ follows immediately from the definitions of U and $\widehat{\mathcal{Z}}$.

The symplectic structure θ_0 on $\mathcal{Z}$ (see (2.6)) defines a symplectic structure θ_m on the Cartesian product $\mathcal{Z}^m$ for any $m \geq 1$. For any point $\underline{z} \, := \, (z_1, \cdots, z_m) \, \in \, \mathcal{Z}^m$ and $v_i, w_i \, \in \, T_{z_i}$, $i \, \in \, [1\, , m]$, we have

$$\theta_m(\underline{v}\, , \underline{w}) \, := \, \sum_{i=1}^{m} \theta_0(z_i)(v_i\, , w_i)$$

with $\underline{v} \, := \, (v_1, \cdots, v_m)$ and $\underline{w} \, := \, (w_1, \cdots, w_m)$ in $T_{\underline{z}}\mathcal{Z}^m$.

Since the action of Σ preserves the symplectic form on $\mathcal{Z}^{4g_X+n-3}$, and Σ acts freely on $\widehat{\mathcal{Z}}$, the quotient $\widehat{\mathcal{Z}}/\Sigma$ gets a symplectic structure. In other words, the symplectic form on $\widehat{\mathcal{Z}}$ descends to $\widehat{\mathcal{Z}}/\Sigma$ The identification in (3.10) takes this symplectic form on $\widehat{\mathcal{Z}}/\Sigma$ to the symplectic form Θ on U. Indeed, this follows immediately from the definition of Θ.

Fix a projective structure on $\mathcal{P}$ on X. Using $\mathcal{P}$ we have a quantization of the symplectic structure θ_0 on $\mathcal{Z}$ (see Section 2.2). This quantization gives a quantization of the symplectic structure θ_m on $\mathcal{Z}^m$ for all $m \geq 1$. The action of the permutation group Σ on $(\mathcal{Z}^0)^{4g_X+n-3}$ preserves the quantization. Indeed, this is an immediate consequence of the identity in (2.3). (Note that for the direct sum of copies of a symplectic vector space, the induced symplectic form on the direct sum is preserved by the action of the permutation group that permutes the factors of the direct sum.) Consequently, the quantization of the symplectic form θ_{4g_X+n-3} on $\widehat{\mathcal{Z}}$ descends to a quantization of the symplectic variety $\widehat{\mathcal{Z}}/\Sigma$ (as the quantization is invariant under the action of Σ).

Thus we have constructed a quantization of the symplectic structure Θ over U. Therefore, we have proved the following theorem:

Theorem 3.2. *Let X be a compact Riemann surface equipped with a projective structure $\mathcal{P}$. The projective structure $\mathcal{P}$ gives a quantization of the Zariski open dense subset $\mathcal{U} \subset \mathcal{M}_H^s$ equipped with the symplectic form Ω.*

The map from the space of all projective structures on X to the space of all quantizations of $(\mathcal{Z}\, , \theta_0)$ that sends a projective structure to the quantization constructed in Section 2.2 is injective. From this it follows immediately that the map constructed in Theorem 3.2

from the space of all projective structures on X to the space of all quantizations of $(\mathcal{U}, \Omega)$ is injective.

References

[1] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz and D. Sternheimer, Deformation theory and quantizations I, II, *Ann. of Phys.* **111** (1978), 61–151.

[2] D. Ben-Zvi and I. Biswas, A quantization on Riemann surfaces with projective structure, *Lett. Math. Phys.* **54** (2000), 73–82.

[3] I. Biswas and A. Mukherjee, Symplectic structures on moduli spaces of parabolic Higgs bundles and Hilbert scheme, *Commun. Math. Phys.* (to appear) (http://arXiv.org/abs/math/0302207).

[4] M. De Wilde and P. B. A. Lecomte, Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifold, *Lett. Math. Phys.* **7** (1983), 487–496.

[5] B. V. Fedosov, A simple geometrical construction of deformation quantization, *Jour. Diff. Geom.* **40** (1994), 213–238.

[6] R. C. Gunning, *Lectures on Riemann surfaces*, Mathematical Notes 2, Princeton University Press, Princeton, New Jersey, 1966.

[7] N. J. Hitchin, The self–duality equations on a Riemann surface, *Proc. Lond. Math. Soc.* **55** (1987), 59–126.

[8] V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structure, *Math. Ann.* **248** (1980), 205–239.

[9] A. Weinstein, Deformation quantization, Séminaire Bourbaki, no. 789, 46ème année, (1994), 213–238.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany

E-mail address: avijit@mis.mpg.de