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Hierarchical Matrices based on a Weak Admissibility Criterion

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by

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Abstract

In preceding papers [8, 11, 12, 6], a class of matrices ($\mathcal{H}$-matrices) has been developed which are data-sparse and allow to approximate integral and more general nonlocal operators with almost linear complexity. In the present paper, a weaker admissibility condition is described which leads to a coarser partitioning of the hierarchical $\mathcal{H}$-matrix format. A coarser format yields smaller constants in the work and storage estimates and thus leads to a lower complexity of the $\mathcal{H}$-matrix arithmetic. On the other hand, it preserves the approximation power which is known in the case of the standard admissibility criterion. Furthermore, the new weak $\mathcal{H}$-matrix format allows to analyse the accuracy of the $\mathcal{H}$-matrix inversion and multiplication.

1 Introduction

The hierarchical matrix technique allows an efficient treatment of dense matrices arising, e.g., from boundary element methods (BEM). At the same time it provides matrix formats which enable the computation and storage of inverse FE stiffness matrices corresponding to elliptic problems. It is shown (e.g., in [8], [11], [12]) that the storage of $n \times n$ $\mathcal{H}$-matrices as well as the matrix-vector multiplication and matrix-matrix addition have a cost of order $\mathcal{O}\left(kn \log n\right)$, where the local rank k is the parameter determining the approximation error. Moreover, the matrix-matrix-multiplication and the inversion take $\mathcal{O}(k^2 n \log^2 n)$ operations.

The hierarchical matrices are represented by means of a certain block partitioning. Figure 2.1 shows a typical block structure. Each block is filled by a submatrix of a rank not exceeding k. Then, for the mentioned class of matrices, it can be shown that the exact dense matrix A and the hierarchical matrix $A_{\mathcal{H}}$ differ by $\|A - A_{\mathcal{H}}\| \leq \mathcal{O}(\eta^k)$ for a certain number $\eta < 1$. This exponential decrease allows to obtain an error ε by the choice $k = \mathcal{O}\left(\log(1/\varepsilon)\right).$

Although the bounds for computational cost are almost linear in the matrix size n, one has to take care of the constant suppressed by the notation $\mathcal{O}\left(\ldots\right).$ In fact, the matrix-*matrix* multiplication needs a much larger CPU time than the matrix-*vector* multiplication. The reason is not only the extra factor $k \log n$ but a larger constant associated with the matrix-matrix multiplication.

This gives rise to consider in more detail the involved constants. In fact the arising constants are identified and described in the papers [12, 6]. The more interesting question is whether there is a possibility to find modifications with smaller constants.

It is not surprising that the mentioned constants can be decreased when we use a simpler block partitioning. Figure 2.3 shows such a coarser partitioning, which is already discussed in [8] as first model problem. However, in [8] we rejected this simple partitioning since the proof of the approximation result $\|A - A_{\mathcal{H}}\| \leq \mathcal{O}(\eta^k)$ requires the finer partitioning shown in Figure 2.1.

Surprisingly, recent numerical tests have shown that nevertheless $\|A - A_{\mathcal{H}}\| \leq \mathcal{O}(\tilde{\eta}^k)$ with $\eta \leq \tilde{\eta} < 1$ seems to be valid also for the simpler partitioning from Figure 2.3 so that the same accuracy can be obtained with less storage and shorter CPU times for the matrix operations.

The partitionings shown in Figures 2.1 and 2.3 belong to one spatial dimension (or an integral operator defined over a curve). The present considerations are even more relevant when we turn to 2D or 3D problems. In these cases the involved constants are much larger than in 1D. Therefore a reduction of the constant is essential. A generalisation to higher space dimensions is the subject of a forthcoming paper [14].

1.1 Overview

In Section 2 we introduce the hierarchical matrices ($\mathcal{H}$-matrices). However, we simplify the explanations by concentrating on a critical model case described in §2.1. The definition of the set $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$ of $\mathcal{H}$-matrices in §2.2.4 makes use of a cluster tree $T(I)$ (see §2.2.1), a block cluster tree $T(I \times I)$ (see §2.2.2) and a partitioning $\mathcal{P}$ (see §2.2.3).

The standard admissibility condition (now called 'strong admissibility condition') used in the all previous papers and corresponding to Figure 2.1 is recalled in §2.3.

The so-called sparsity constant $C_{\mathrm{sp}}(\mathcal{P})$ is introduced in §2.4. We show that the storage estimate involves $C_{\mathrm{sp}}(\mathcal{P})$ (see §2.4.2). The dependence on the spatial dimension d is discussed in §2.4.3.

As a remedy for a possibly too large constant $C_{\mathrm{sp}}(\mathcal{P})$, we introduce a 'weak admissibility condition' in §2.5 corresponding to Figure 2.3.

The purpose of both admissibility conditions is to construct a partitioning $\mathcal{P}$, which guarantees an error bound for $\|A - A_k\|$ (A: exact dense matrix, $A_k \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$). Such an error estimate in discussed in §3.1 for the strong admissibility condition. In the case of the weak admissibility condition (§3.2), we first show that an $\mathcal{H}$-matrix with local rank increased from k to $3Lk$ ($L = \log_2 n$, see §3.2.1) can lead to the same error bound as in the case of strong admissibility. The number $3Lk$ can even be reduced to Lk, as shown in §3.2.2.

In Subsection 3.4 we collect approximation results and their theoretical estimations. Unfortunately, they do not prove our conjecture but lead to weaker bounds.

The behaviour of the error $\|A - A_k\|$ with respect to k can also be observed by numerical computations. This is done in Subsection 3.3. The best approximation of a BEM matrix (corresponding to a Nyström discretisation) is computed by means of singular value decomposition (SVD) in §3.3.1. The discussion of the results in §3.3.2 and §3.4 leads to Conjecture 3.2, which is the core of this paper. It claims that *even with the weak admissibility condition we get almost the same error behaviour as usually obtained by the strong one.* In §3.3.3 similar results are presented for a collocation discretisation instead of the Nyström method used before. In practice, the singular value decomposition is replaced by simpler methods which are explained in §3.3.4.

The observed properties inspire the definition of matrix families $\mathcal{M}_{k,\tau}$ and $\mathcal{M}_k(\varepsilon)$ in Section 4. The Conjecture 3.2 implies that BEM matrices corresponding to a one-dimensional manifold belong to $\mathcal{M}_k(\varepsilon)$ (defined in §4.4) with $k = \mathcal{O}(\log(1/\varepsilon))$. The interesting properties of matrices from $\mathcal{M}_k(\varepsilon)$ are that

i) they can be approximated by matrices from $\mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ corresponding to the weak admissibility (see Theorem 4.16) and

ii) products and inverse matrices remain in this class (exact statement in Theorem 4.17). This allows to state that meromorphic functions $f(\cdot)$ applied to a matrix A can be approximated in $\mathcal{M}_{\mathcal{H},\mathbf{k}'}(I \times I, \mathcal{P}_W)$ for certain local ranks.

A particular application of the last result is the inversion of $\mathcal{H}$-matrices. In Section 5 we describe the inversion algorithm (see §5.1) and require certain stability assumptions (see §5.2). Then we are able to show that $A_{\mathrm{BEM}} \in \mathcal{M}_{\mathbf{k}}(c\eta^k)$ has an inverse in the same class and that the inversion algorithm determines an approximation of the desired accuracy (Theorem 5.2).

Finally, we show in Section 6 a similar result for the product of hierarchical matrices (see Theorem 6.2).

2 $\mathcal{H}$-Matrices

In the Sections §§2-3 we consider the case of only one spatial dimension. Furthermore, we restrict the discussion to a model case. This simplifies the following definitions and helps to concentrate on the main topic.

2.1 Model Problem (BEM)

We consider an integral equation $\mathcal{A}u = f$ or $(\lambda I - \mathcal{A})u = f$ with the integral operator

$$(\mathcal{A}u)(x) := \int_0^1 s(x,y)u(y)dy \qquad \text{for } x \in [0,1], \tag{2.1}$$

where a typical kernel function may be

$$s(x, y) = \log|x - y| \qquad \text{for } x, y \in [0, 1].$$ (2.2)

A Galerkin discretisation leads to the fully populated matrix

$$A = (a_{ij})_{i,j \in I} \qquad \text{with } a_{ij} = \int_0^1 \int_0^1 s(x, y)\phi_i(x)\phi_j(y)dxdy,$$ (2.3)

where $\mathcal{B} = \{\phi_i : i \in I\}$ is the finite-element basis. We may consider two different examples of $\mathcal{B}$ based on an equidistant grid $x_\nu = \nu h$ ($\nu = 0, \ldots, N$) for the step size $h = 1/N$.

Standard examples are the *piecewise constant elements*

$$\phi_i(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in (x_i, x_{i+1}) \\ 0 & \text{otherwise} \end{array} \right\} \quad \text{for } i \in I := \{0, \ldots, N-1\},$$ (2.4)

as well as the *piecewise linear elements*

$$\phi_i(x) \text{ linear on each interval } (x_{\nu-1}, x_\nu) \text{ and } \phi_i(x_\nu) = \delta_{i\nu} \quad \text{for } i, \nu \in I := \{0, \ldots, N\}.$$ (2.5)

The index set I is different in both cases. Its cardinality is denoted by $n := \#I$ ($n = N$ for (2.4) and $n = N + 1$ for (2.5)). To simplify the discussion, we assume that n is a power of 2,

$$n = 2^L.$$ (2.6)

The support of the basis functions is denoted by

$$X(i) := \text{supp}(\phi_i) \qquad \text{for } i \in I.$$ (2.7)

In the case of (2.4) the supports are essentially disjoint (i.e., intersection of measure zero is neglected), whereas in the case of (2.5) $X(i) \cap X(i+1) = [x_i, x_{i+1}]$.

2.2   Definition of Hierarchical Matrices

In the considered 1D-case, the definitions are as simple as in the introductory paper [8]. To define the matrix set $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$ – hierarchical matrices of local rank k based on the partitioning $\mathcal{P}$ – we have to consider the cluster tree $T(I)$ (see §2.2.1), the block cluster tree $T(I \times I)$ (see §2.2.2), the admissibility condition and the corresponding partitioning P_2 (see §2.2.3). This defines the set $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$ introduced in §2.2.4. Since the admissibility condition is not yet fixed in §2.2.3, we introduce the traditional admissibility condition (standard admissibility or S-admissibility) in §2.3. The discussion in §2.4 will lead us to a weaker admissibility condition (weak admissibility or W-admissibility) which is introduced in §2.5.

2.2.1   Cluster Tree $T(I)$

The vertices of the cluster tree $T(I)$ are called 'clusters' and are subsets of the index set I. I is the root of the tree and sons $\sigma_1, \ldots, \sigma_s$ of a cluster τ are disjoint subsets of τ so that $\tau = \cup_{i=1}^s \sigma_i$. Furthermore, the leaves τ satisfy $\#\tau = 1$ (for the formal definition of the hierarchical cluster tree $T(I)$ we refer to [8]).

We give a concrete example for our model case from §2.1. The *level* introduced in the following example is the distance of the vertex from the root.

Example 2.1 *Let $I = \{0, ..., n-1\}$ and $n = 2^L$. The clusters of level L are the one-element subsets*

$$\tau_1^L = \{0\}, \ \tau_2^L = \{1\}, \ldots, \ \tau_n^L = \{n-1\}.$$

On level $L-1$, two subsets from level L are combined:

$$\tau_1^{L-1} = \{0, 1\}, \tau_2^{L-1} = \{2, 3\}, \ldots, \ \tau_{n/2}^{L-1} = \{n-2, n-1\}.$$

Similarly, we obtain 4-element subsets τ_i^{L-2} of level $L-2$, etc. Finally, at level 0, the whole index set $\tau_i^0 = I$ is the only cluster. This defines a binary tree $T(I)$ with the vertices ('clusters') $\{\tau_i^\ell : 0 \le \ell \le L, \ 1 \le i \le 2^\ell\}$, where

$$\tau_i^\ell = \left\{ (i-1)*2^{L-\ell}, (i-1)*2^{L-\ell} + 1, \ldots, i*2^{L-\ell} - 1 \right\}.$$

I is the root. The vertices at level L are leaves. The sons of τ_i^ℓ ($\ell < L$) are $\tau_{2i-1}^{\ell+1}$ and $\tau_{2i}^{\ell+1}$.

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For later use, the supports $X(i)$ from (2.7) are generalised to clusters by the definition

$$X(\tau) := \bigcup_{i \in \tau} X(i) \qquad \text{for } \tau \in T(I). \tag{2.8}$$

Obviously, we have

$$X(\tau_i^\ell) = \begin{cases} \left[(i-1) * 2^{L-\ell}h, i * 2^{L-\ell}h\right] \subset [0,1] & \text{for the case of (2.4),} \\[2mm] \left[\left((i-1) * 2^{L-\ell} - 1\right) h, \left(i * 2^{L-\ell} + 1\right) h\right] \cap [0,1] & \text{for (2.5).} \end{cases} \tag{2.9}$$

The upper index of τ_i^ℓ refers to the level number, which may be regarded as distance from the root in the tree. This gives rise to the notation

$$T^\ell(I) = \left\{\tau_i^\ell : 1 \le i \le 2^\ell\right\}. \tag{2.10}$$

2.2.2 Block Cluster Tree $T(I \times I)$

While the vector components are indexed by $i \in I$, the entries of a (square) matrix have indices from the index set $I \times I$. The block-cluster tree is nothing but the cluster tree for $I \times I$ instead of I. In a canonical way (cf. [11]), the block-cluster tree $T(I \times I)$ can be constructed from $T(I)$, where all vertices ('blocks') $b \in T(I \times I)$ are of the form $b = \tau \times \sigma$ with $\tau, \sigma \in T(I)$. The construction starts with stating that $I \times I$ is the root of $T(I \times I)$. Then the sons of $b = \tau \times \sigma \in T(I \times I)$ form the set of all blocks $b' := \tau' \times \sigma'$, where τ' (σ') are the sons of τ (σ) provided that these exist.

In the case of the tree $T(I)$ from Example 2.1, the block cluster tree $T(I \times I)$ is

$$T(I \times I) = \left\{\tau_i^\ell \times \tau_j^\ell : 0 \le \ell \le L, \ 1 \le i, j \le 2^\ell\right\}. \tag{2.11}$$

The leaves of the tree $T(I \times I)$ are the 1×1-blocks $\tau_i^L \times \tau_j^L = \{(i-1, j-1)\}$.

By definition, both clusters τ, σ in $b = \tau \times \sigma \in T(I \times I)$ must belong to the same level, so that $T(I \times I)$ decomposes into the sets $T^\ell(I \times I)$ corresponding to the respective level ℓ:

$$T^\ell(I \times I) = \left\{\tau_i^\ell \times \tau_j^\ell : 1 \le i, j \le 2^\ell\right\} \qquad (0 \le \ell \le L). \tag{2.12}$$

2.2.3 Partitioning $\mathcal{P}$

The blocks of $T(I \times I)$ are not disjoint (each index pair (ν, μ) belongs to $L+1$ different blocks of different size). A *partitioning* $\mathcal{P}$ of $I \times I$ is a *disjoint* decomposition of $I \times I$ into blocks b_ι, $\iota \in J$, such that $\bigcup_{\iota \in J} b_\iota = I \times I$. Moreover, we require $b_\iota \in T(I \times I)$ for all $\iota \in J$, i.e., $\mathcal{P} \subset T(I \times I)$.

The finest partitioning is $\mathcal{P} = \left\{\tau_i^L \times \tau_j^L : 1 \le i, j \le n\right\}$ which corresponds to the format of standard full matrices. The coarsest partitioning is $\mathcal{P} = \{I \times I\}$ consisting of only one block. Since later we shall fill the blocks by low-rank matrices, the size of the block must be controlled by a criterion called '*admissibility condition*'. The concrete description will be given in §2.3 and modified in §2.5. Here, we only need some Boolean function

$$Adm : T(I \times I) \to \{true, false\} \tag{2.13}$$

with the consistency requirement $Adm\,(b) \Rightarrow Adm\,(b')$ for all sons b' of $b \in T(I \times I)$ and the property $Adm\,(b) = true$ for all leaves $b \in T(I \times I)$.

A partitioning $\mathcal{P}$ is *admissible* if $Adm\,(b) = true$ for all $b \in \mathcal{P}$.

Since we are interested in a partitioning with as few blocks as possible, we define the *minimum admissible partitioning* as the admissible partitioning $\mathcal{P}_{\min}$ with smallest cardinality. It can be obtained by the following procedure: Start with $\mathcal{P} = \{I \times I\}$ and replace the blocks $b \in \mathcal{P}$ by their sons as long as $Adm\,(b) = false$ (cf. [11, Algorithm 3.8]).

2.2.4 Hierarchical Matrices $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$

For some partitioning $\mathcal{P}$ and a natural number k, we define the set $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}) \subset \mathbb{R}^{I \times I}$ of (real) hierarchical matrices by

$$\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}) := \{M \in \mathbb{R}^{I \times I} : \operatorname{rank}(M|_b) \leq k \text{ for all } b \in \mathcal{P}\}.$$

Here, $M|_b = (m_{ij})_{(i,j) \in b}$ denotes the matrix block of $M = (m_{ij})_{i,j \in I}$ corresponding to $b \in \mathcal{P}$. The matrices from $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$ are implemented by means of the list $\{M|_b : b \in \mathcal{P}\}$ of matrix blocks, where each[1] $M|_b$ ($b = \tau \times \sigma$ with $\tau, \sigma \in T(I)$) is represented by $\sum_{\nu=1}^k a_\nu b_\nu^\top$ with vectors $a_\nu \in \mathbb{R}^\tau, b_\nu \in \mathbb{R}^\sigma$. The number k is called the local rank.

We remark that besides the matrix-vector multiplication, also the matrix-matrix addition and multiplication as well as the inversion can be performed approximately (see [8]).

There are two conflicting issues to be satisfied. First, it must be possible to approximate the true full matrix A (e.g., from (2.3)) sufficiently well by some $A_\mathcal{H} \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$, i.e.,

$$\|A - A_\mathcal{H}\| \leq \varepsilon \tag{2.14}$$

must be reachable (at least for an appropriate k). Second, the related costs should be as small as possible. The most important costs are $\mathcal{N}_{st}$ (number of storage) and $\mathcal{N}_{MV}$ (number of arithmetic operations for the matrix-vector multiplication). For $\mathcal{P}$ resulting from the admissibility conditions discussed in §2.3 and §2.5, we obtain for both costs the respective order

$$\mathcal{N}_{st}, \mathcal{N}_{MV} = \mathcal{O}(kn \log n) \quad \text{and} \quad \mathcal{O}(kn \log n \log(\log n)). \tag{2.15}$$

2.3 Standard Admissibility

We recall that any cluster τ is associated with a set $X(\tau)$ defined in (2.8). This allows the notations

$$\operatorname{diam}(\tau) := \operatorname{diam} X(\tau), \qquad \operatorname{dist}(\sigma, \tau) := \operatorname{dist}(X(\sigma), X(\tau)). \tag{2.16}$$

The admissibility condition (2.13) already introduced in [8] reads as follows, where $\eta > 0$ is some parameter:

$$Adm_\eta(b) = true \quad \text{for } b = \tau \times \sigma \in T(I \times I) \quad :\Longleftrightarrow \tag{2.17}$$
$$(b \text{ is a leaf}) \quad \text{or} \quad \min\{\operatorname{diam}(\sigma), \operatorname{diam}(\tau)\} \leq 2\eta \operatorname{dist}(\sigma, \tau).$$

Sometimes the minimum in (2.17) is replaced by a maximum (i.e., $\max\{\operatorname{diam}(\sigma), \operatorname{diam}(\tau)\} \leq 2\eta \operatorname{dist}(\sigma, \tau)$). In general, this is a stronger condition, but for the present 1D-model case both conditions coincide.

A simple choice of $\eta > 0$ is $\eta = 1/2$ leading to $Adm_{1/2}$. In the 1D-model case (2.4), we have

$$\operatorname{diam}(\tau_i^\ell) = 2^{L-\ell}h, \qquad \operatorname{dist}(\tau_i^\ell, \tau_j^\ell) = \max\{0, |i-j|-1\} 2^{L-\ell}h. \tag{2.18}$$

Hence a block $b = \tau_i^\ell \times \tau_j^\ell$ is admissible if $\ell = L$ (i.e., b is a leaf) or if $|i-j| \geq 2$. The partitioning $\mathcal{P}$ generated by $Adm_{1/2}$ is shown in Figure 2.1.

2.4 Sparsity Constant

In the following we investigate the fine-structure of the cost functions $\mathcal{N}_{st}, \mathcal{N}_{MV}$. Instead of the asymptotic description (2.15) we want information about the constant $C_\mathcal{P}$ in

$$\max\{\mathcal{N}_{st}, \mathcal{N}_{MV}\} \leq C_\mathcal{P} * knL, \tag{2.19}$$

where $L = \log_2(n)$ is the dual logarithms and also the depth of the tree $T(I)$.

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[1] In practice, one uses a standard full-matrix representation for blocks of small enough size (see [6]).
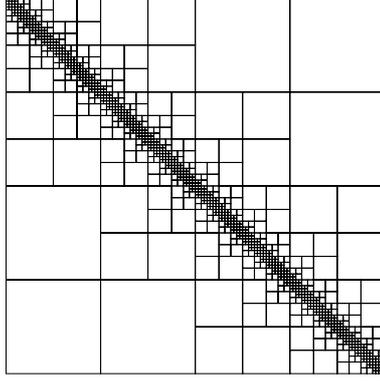
Figure 2.1: Partitioning by the standard admissibility condition $Adm_{1/2}$

2.4.1 Definition

The key quantity is the so-called *sparsity constant*

$$C_{\mathrm{sp}}(\mathcal{P}) := \max \left\{ \begin{array}{c} \max_{\tau \in T(I) \setminus T^L(I)} \#\{\sigma \in T(I) : \tau \times \sigma \in \mathcal{P}\}, \\ \max_{\sigma \in T(I) \setminus T^L(I)} \#\{\tau \in T(I) : \tau \times \sigma \in \mathcal{P}\} \end{array} \right\} \tag{2.20}$$

(see [6] or slightly different definitions of the same quantity in [12, 15]). The number $\#\{\sigma \in T(I) : \tau \times \sigma \in \mathcal{P}\}$ counts how often the cluster τ is used as *row* block, while $\#\{\tau \in T(I) : \tau \times \sigma \in \mathcal{P}\}$ counts how often σ is used as *column* block.

A look at Figure 2.1 shows that the number of σ with $\tau \times \sigma \in \mathcal{P}$ is 2, if we restrict to the upper triangular part; e.g., the two biggest blocks at the upper right corner share the same τ. In general (not for the biggest blocks), there are also up to 2 blocks in the same row belonging to the lower triangular part of the matrix. However, one observes that 2 blocks in the right part correspond to at most 1 block in the left part so that the sum is always ≤ 3. The resulting conjecture $C_{\mathrm{sp}}(\mathcal{P}) = 3$ is true as stated in

Remark 2.2 *Consider the 1D-model case (2.4). Then the partitioning $\mathcal{P}$ generated by $Adm_{1/2}$ has the sparsity constant $C_{\mathrm{sp}}(\mathcal{P}) = 3$.*
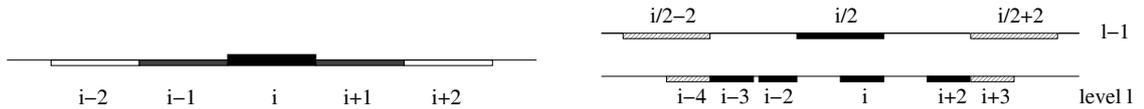

Figure 2.2: *Left:* The closest admissible neighbours are $i+2$ and $i-1$. *Right:* The neighbours $i+3$ and $i-4$ are contained in admissible blocks of level $\ell - 1$.

Proof. Take $\tau = \tau_i^\ell$ with $\ell > 0$. σ with $\tau \times \sigma \in \mathcal{P}$ must belong to the same level, i.e., $\sigma = \tau_j^\ell$. We check the various cases for j.

Case $j = i, i+1, i-1$: Since $\mathrm{dist}(X(\tau_i^\ell), X(\tau_j^\ell)) = 0$ (cf. (2.18)), $b = \tau \times \sigma$ is not admissible; hence, $\tau \times \sigma \notin \mathcal{P}$ (see left part of Figure 2.2).

Case $j > i+1$ or $j < i-1$: Since $\mathrm{dist}(\tau_i^\ell, \tau_j^\ell) \geq \mathrm{diam}(\tau_i^\ell)$, $b = \tau \times \sigma$ is admissible ($Adm_{1/2}(b) = true$). However, we have to check whether b is contained in a bigger admissible block of level $< \ell$. For this purpose, we assume first that i is even, i.e., $\tau' = \tau_{i/2}^{\ell-1}$ is the father of τ_i^ℓ and the interval $X(\tau_i^\ell)$ is the right half of $X(\tau_{i/2}^{\ell-1})$ (see right part of Figure 2.2). Let $\sigma' = \tau_{i/2+\delta}^{\ell-1}$ be the father of $\sigma = \tau_j^\ell$. $\tau' \times \sigma'$ is admissible if $|\delta| \geq 2$. One checks that this happens for $j \geq i+3$ and $j \leq i-4$, but not for $j = i-3, i-2, i+2$. Hence, we have identified the three clusters $\sigma = \tau_j^\ell$ of the set $\{\sigma \in T(I) : \tau \times \sigma \in \mathcal{P}\}$. If $j \notin [1, 2^\ell]$, there are even less than $C_{\mathrm{sp}}(\mathcal{P}) = 3$ elements.

For an odd index i, one finds that $\tau \times \sigma = \tau \times \tau_j^\ell \in \mathcal{P}$ for $j = i-2, i+2, i+3$, so that again $C_{\mathrm{sp}}(\mathcal{P}) = 3$. ∎

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Definition (2.20) excludes $\tau, \sigma \in T^L(I)$, since for the leaves the property $\tau \times \sigma \in \mathcal{P}$ does not require the inequality $\min\{\operatorname{diam}(\sigma), \operatorname{diam}(\tau)\} \le 2\eta \operatorname{dist}(\sigma, \tau)$. We compensate this by the extra constant

$$C_{\mathrm{sp}}^L (\mathcal{P}) := \max \left\{ \max_{\tau \in T^L(I)} \#\{\sigma \in T(I) : \tau \times \sigma \in \mathcal{P}\}, \max_{\sigma \in T^L(I)} \#\{\tau \in T(I) : \tau \times \sigma \in \mathcal{P}\} \right\}.$$

Remark 2.3 *Consider the 1D-model case (2.4). Then the partitioning $\mathcal{P}$ generated by $Adm_{1/2}$ has the L-level sparsity constant $C_{\mathrm{sp}}^L (\mathcal{P}) = 6$.*

Proof. In addition to the j's found in the proof of Remark 2.2, also $j = i, i+1, i-1$ lead to $\tau_i^\ell \times \tau_j^\ell \in \mathcal{P}$. ∎

2.4.2 Cost Estimates

The interesting fact is that the storage cost is directly connected with the sparsity constant (cf. [5]). The following lemma is not restricted to the 1D-case.

Lemma 2.4 *Let $n = \#I$. Then (a) and (b) hold.*
(a) The number of blocks is bounded by $n C_{\mathrm{sp}}^L (\mathcal{P}) + (n-1) C_{\mathrm{sp}} (\mathcal{P})$.
(b) The storage requirements $\mathcal{N}_{st}$ for an $\mathcal{H}$-matrix $M \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$ is bounded by

$$\mathcal{N}_{st} \le \left[2k(L-1) C_{\mathrm{sp}} (\mathcal{P}) + C_{\mathrm{sp}}^L (\mathcal{P}) \right] n.$$

Whenever $C_{\mathrm{sp}}^L (\mathcal{P}) \le 2k C_{\mathrm{sp}} (\mathcal{P})$, which happens at least if k is sufficiently large, the storage can be estimated by

$$\mathcal{N}_{st} \le 2k L C_{\mathrm{sp}} (\mathcal{P}) n. \tag{2.21}$$

Proof. a) The tree $T(I)$ has at most $2n - 1$ vertices, in particular, $\#T(I) \backslash T^L(I) \le n - 1$ and $\#T^L(I) = n$. Therefore,

$$\#\mathcal{P} = \sum_{\tau \times \sigma \in \mathcal{P}} 1 = \sum_{\tau \in T(I)} \#\{\sigma \in T(I) : \tau \times \sigma \in \mathcal{P}\}$$

$$\le \sum_{\tau \in T^L(I)} C_{\mathrm{sp}}^L (\mathcal{P}) + \sum_{\tau \in T(I) \backslash T^L(I)} C_{\mathrm{sp}} (\mathcal{P}) \quad \le \quad n C_{\mathrm{sp}}^L (\mathcal{P}) + (n-1) C_{\mathrm{sp}} (\mathcal{P}).$$

b) The storage needed for a block $b = \tau \times \sigma$ is $k(\#\tau + \#\sigma)$, since k vectors from $\mathbb{R}^\tau$ and k vectors from $\mathbb{R}^\sigma$ are to be stored. An exception holds for level L, where only one number is to be stored; hence, storage size $= 1$.

In the following, the symbol $\sum^*$ refers to the summation restricted to *level* $< L$. Furthermore, we exclude the level L, since $\mathcal{P}$ cannot contain an admissible block from level L. We have

$$\mathcal{N}_{st}^* = \sum_{\tau \times \sigma \in \mathcal{P}}^* k(\#\tau + \#\sigma) \le k \left(\sum_{\tau \times \sigma \in \mathcal{P}}^* \#\tau + \sum_{\tau \times \sigma \in \mathcal{P}}^* \#\sigma \right)$$

$$\le C_{\mathrm{sp}} (\mathcal{P}) k \left(\sum_{\tau \in T(I)}^* \#\tau + \sum_{\sigma \in T(I)}^* \#\sigma \right) \le 2 C_{\mathrm{sp}} (\mathcal{P}) k \sum_{\tau \in T(I)}^* \#\tau.$$

Since $\sum_{\tau \in T(I)}^* \#\tau = \sum_{\ell=1}^{L-1} \sum_{\tau \in T^\ell(I)} \#\tau$ and $\sum_{\tau \in T^\ell(I)} \#\tau = n$ for all ℓ, the result $\mathcal{N}_{st}^* \le 2 C_{\mathrm{sp}} (\mathcal{P}) k (L-1) n$ follows.

For level L, we obtain $\mathcal{N}_{st}^L = \sum_{\tau \times \sigma \in \mathcal{P}}^{level=L} 1 \le C_{\mathrm{sp}}^L (\mathcal{P}) n$. ∎

In the case of a dense matrix stored in the usual way, the matrix-vector multiplication requires one multiplication and one addition per matrix element, i.e., the cost for a matrix-vector multiplication is bounded by twice the storage cost. The same holds for hierarchical matrices:

Lemma 2.5 $\mathcal{N}_{st} \le \mathcal{N}_{MV} \le 2\mathcal{N}_{st}$, *where $\mathcal{N}_{MV}$ is the number of arithmetic operations for the multiplication of a matrix from $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$ by a vector.*

We mention the costs $\mathcal{N}_{MM}$ and $\mathcal{N}_{inv}$ for the matrix-matrix multiplication and the inversion only quite briefly. As can be seen from [6], another constant is involved which we are not going to explain. The dependence on C_{sp}, however, is given by

$$\mathcal{N}_{MM}, \mathcal{N}_{inv} = \mathcal{O} \left(C_{\mathrm{sp}}^2 k^2 n \log^2 n \right). \tag{2.22}$$

2.4.3 Dependence on the Spatial Dimension d

Instead of the 1D case (2.4), one can also consider model cases in 2D and 3D (then the simplest trees are no more binary, but quad-trees ($d = 2$) or octree ($d = 3$), respectively). Due to a result from [12] for tensor-grids, we have a bound of the sparsity constant by

$$C(\mathcal{P}, d, \eta) = (2^d - 1)(1 + \sqrt{d}/\eta)^d, \tag{2.23}$$

which yields $C(\mathcal{P}, 1, \eta) = 3$, $C(\mathcal{P}, 2, \eta) = 27$ and $C(\mathcal{P}, 3, \eta) = 189$ for $\eta = \frac{\sqrt{d}}{2}$.

Obviously, we recover $C_{\mathrm{sp}}(\mathcal{P}) = 3$ from Remark 2.2. But the annoying fact is that the sparsity constant increases significantly with the dimension d.

We remark that a similar dimensional dependence holds for another sparsity constant: When we deal with usual sparse matrices, we may define $C_{\mathrm{sp}}(A)$ by the maximum or average number of non-zero entries per row. A FE method in a tensor grid leads to $C_{\mathrm{sp}} = 3$ for $d = 1$, $C_{\mathrm{sp}} = 9$ for $d = 2$, and $C_{\mathrm{sp}} = 27$ for $d = 3$. Again the increase is exponential like 3^d, but not as dramatic as for $C(\mathcal{P}, d, \eta)$ from above.

2.5 Weak Admissibility

In order to get a simpler partitioning $\mathcal{P}$, we have to weaken the admissibility condition Adm_η. Instead of (2.17), we define

$$Adm_W(b) = true \quad \text{for } b = \tau \times \sigma \in T(I \times I) \quad :\Longleftrightarrow \quad ((b \text{ is a leaf}) \quad \text{or} \quad \sigma \neq \tau), \tag{2.24}$$

where τ, σ are assumed to belong to the same level. Note that the definition of Adm_W does not depend on kind of basis functions ((2.4) or (2.5)).

We call (2.24) the 'weak admissibility' and say that a block is 'W-admissible'.
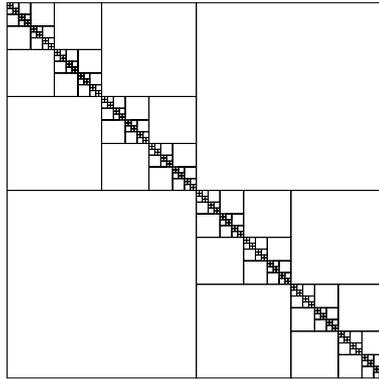

Figure 2.3: Partitioning $\mathcal{P}_W$ corresponding to the weak admissibililty.

Remark 2.6 *a) Let Adm_η and Adm_W be as in (2.17) and (2.24). Then for all $\tau \times \sigma \in T(I \times I)$ it holds that $Adm_\eta(b) \Longrightarrow Adm_W(b)$ (independently of $\eta > 0$).*

b) Let $\mathcal{P}_\eta$ and $\mathcal{P}_W$ be the respective partitionings generated by Adm_η and Adm_W. Then $\mathcal{P}_W$ has fewer and bigger blocks in the sense that if $b \in \mathcal{P}_\eta$ then there is a block $b' \in \mathcal{P}_W$ with $b \subset b'$.

c) In the case of Example 2.1, $b = \tau_i^\ell \times \tau_j^\ell$ is W-admissible if $i \neq j$ or $\ell = L$. In the case of piecewise constant basis functions (see (2.4)), a block $b = \tau \times \sigma \in T(I \times I)$ is W-admissible if the intervals $X(\tau)$ and $X(\sigma)$ intersect at most in a point.

Remark 2.7 *Note that in the case of piecewise linear basis functions (see (2.5)), $X(\tau)$ and $X(\sigma)$ may overlap by more than a point: $X(\tau_i^\ell) \cap X(\tau_{i+1}^\ell) = \left[\left(i * 2^{L-\ell} - 1\right) h, \left(i * 2^{L-\ell} + 1\right) h\right]$. The clusters $\tau_i^\ell, \tau_{i+2}^\ell$ overlap in the point $\left(i * 2^{L-\ell} + 1\right) h$, while for $|i - j| > 2$ the clusters τ_i^ℓ, τ_j^ℓ are disjoint.*

The coarse partitioning observed in Remark 2.6 can be expressed by means of the sparsity constant.

Lemma 2.8 *The partitioning $\mathcal{P}_W$ has the minimum sparsity constant $C_{st}(\mathcal{P}_W) = 1$. Further, $C_{st}^L(\mathcal{P}_W) = 2$. This implies $\mathcal{N}_{st} \leq [2k(L-1)+2]\,n \leq 2kLn$ for the storage size.*

The generalisation to the 2D and 3D case will be considered in a forthcoming paper [14].

3   Accuracy

The purpose of the hierarchical matrix format is the approximation of a dense matrix A as in (2.3) by a matrix $A_k \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P})$. The typical dependence of $\|A - A_k\|$ on the local rank k is discussed below.

3.1   Standard Case

For the integral operator (2.3) with the kernel (2.2), it is shown in [8] that

$$\|A - A_k\| = \mathcal{O}\left(\eta^k\right) \tag{3.1}$$

with $\eta = 1/2$ in the case of the partitioning of Figure 2.1. In [11] and [12] it is proved that (3.1) holds for asymptotically smooth kernels. A similar estimate is true for Green's functions corresponding to elliptic differential operators (cf. [3]). The proof is based on the fact that the kernel function $s(x, y)$ can be approximated by a degenerate kernel

$$s_{k,b}(x, y) = \sum_{i=1}^{k} \Phi_{i,b}(x)\Psi_{i,b}(y), \tag{3.2}$$

in the sense that $|s(x, y) - s_{k,b}(x, y)| = \mathcal{O}\left(\eta^k\right)$ for $x \in X(\tau)$ and $y \in X(\sigma)$, where $b = \tau \times \sigma$ is an Adm_η-admissible cluster (cf. (2.17)).

Estimate (3.1) ensures exponential convergence for increasing k. In particular,

$$\|A - A_k\| \leq \varepsilon$$

can be obtained by the choice $k = \mathcal{O}(|\log \varepsilon|)$. Usually, we want to have ε smaller than the discretisation error, which is expected to be of the size $\mathcal{O}(n^{-\kappa})$ with $\kappa > 0$ depending on the consistency order of the discretisation. Therefore, the practical choice of k is

$$k = \mathcal{O}(\log n). \tag{3.3}$$

Estimate (3.1) has a theoretical and a practical aspect. Both are discussed below.

Theoretical aspect: If we use the Frobenius norm, there is a projection from the set of (dense) matrices onto hierarchical ones,

$$\pi_k : \mathbb{R}^{I \times I} \to \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_\eta). \tag{3.4}$$

For any $A|_b$, $b \in \mathcal{P}$, the singular-value decomposition (SVD) defines[2] the *optimal* truncation to a block matrix $(\pi_k A)|_b$ of rank $\leq k$ (see [8]). Hence, the estimate (3.1) can be understood in the sense that $\|A - \pi_k A\| = \mathcal{O}(\eta^k)$ holds for *best approximation* $A_k := \pi_k A$ in $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_\eta)$.

Practical aspect: In particular for BEM applications, one must avoid the set-up of the matrix A, since this costs at least $O(n^2)$ operations. Therefore the projection π_k cannot be applied. Instead one needs efficient methods to determine $A_k \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_\eta)$ directly. There are at least two different approaches.

The *first approach* starts from the kernel approximation (3.2), which can, e.g., be obtained either by Taylor expansion or interpolation (e.g., see [13, 10]). Integration of $s_{k,b}(x, y)$ multiplied by the basis functions yields the desired candidate $A_k \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_\eta)$. An exception is a leaf b (1×1 block), in which case the entries a_{ij} of A_k are the original entries of A (in the 1D model case this happens only for $|i - j| \leq 1$).

A *second approach* is described in [2], [4]. It uses the information of few entries a_{ij} of A to construct $A_k \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_\eta)$.

---

[2]Let $A|_b = U\Sigma V^\top$ be the SVD, i.e., U, V unitary, $\Sigma = \text{diag}\{\sigma_1, \ldots, \sigma_n\}$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$. Set $\Sigma_k := \text{diag}\{\sigma_1, \ldots, \sigma_k, 0, \ldots, 0\}$. Then $(\pi_k A)|_b$ is defined by $U\Sigma_k V^\top$.

3.2 Case of Weak Admissibility

Now we consider the partitioning $\mathcal{P}_W$ due to the weak admissibility condition of §2.5. The obvious advantage of $\mathcal{P}_W$ is the fact that the partitioning is simpler (fewer and bigger blocks). However, this makes sense only if we can obtain the same accuracy $\|A - A_k\| \le \varepsilon$ by a similar local rank k as before. Here, a problem arises which may be formulated as follows. Let $\pi_k : \mathbb{R}^{I \times I} \to \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$ be the projection defined via SVD (3.4). The critical question is

$$\text{How does } \varepsilon_k := \|A - \pi_k A\| \text{ behave for increasing local rank } k? \tag{3.5}$$

An equivalent question is:

$$\text{What is the minimum } k = k(\varepsilon) \text{ to ensure } \|A - \pi_k A\| \le \varepsilon? \tag{3.6}$$

A second question is how to determine A_k practically.

In the case of Adm_η, one could use Taylor's expansion of $s(x,y) = \log|x - y|$ for $x \in X(\tau)$ and $y \in X(\sigma)$. The distance between $X(\tau)$ and $X(\sigma)$ enables an estimate of the Taylor remainder. Now, in the case of Adm_W, the intervals $X(\tau)$ and $X(\sigma)$ may touch as, e.g., $X(\tau_1^1) = [0, 1/2]$ and $X(\tau_2^1) = [1/2, 1]$ for the case (2.4). But then the Taylor remainder becomes unbounded at $x = y = 1/2$ and the standard proof does not work.

3.2.1 Blockwise Agglomeration

Due to Remark 2.6b, any block $b \in \mathcal{P}_W$ is a (disjoint) union of blocks $b_\iota \in \mathcal{P}_{1/2}$. Figure 3.1 shows on the right side the block $b = \tau_1^1 \times \tau_2^1 \in \mathcal{P}_W$ (upper right quarter in Figure 2.3) and on the left the blocks $b_\iota \subset b$ from $\mathcal{P}_{1/2}$. One sees that the blocks b_ι consists of three blocks of the levels $2, 3, \ldots, L-1$ and four blocks of level L. Hence, the number of blocks is $3(L-2) + 4 < 3L$.
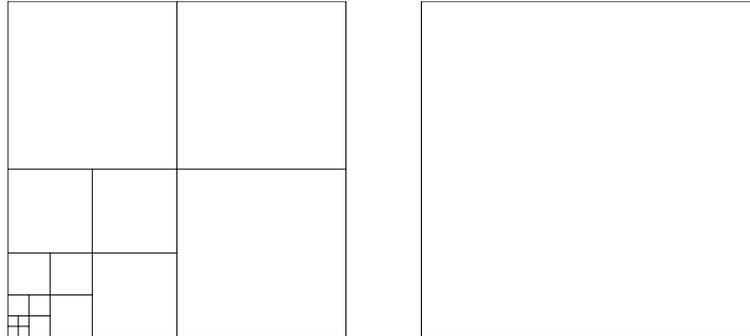

Figure 3.1: *Left:* the blocks from $\mathcal{P}_{1/2}$ contained in $b = \tau_1^1 \times \tau_2^1 \in \mathcal{P}_W$. *Right:* block $b \in \mathcal{P}_W$

Next, we consider question (3.6). Let π_k^s be the projection (3.4) associated with $\mathcal{P}_{1/2}$, while π_k^W refers to the partitioning $\mathcal{P}_W$. For a given ε, we choose k such that $\|A - \pi_k^s A\| \le \varepsilon$ for $\pi_k^s A \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_{1/2})$. From (3.1) we know that $k = k_s(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$. Now consider the block $b = \tau_1^1 \times \tau_2^1 \in \mathcal{P}_W$ discussed above. The block matrix $(\pi_k^s A)|_b$ consists of less than $3L$ block matrices $(\pi_k^s A)|_{b_\iota}$ $(b_\iota \in \mathcal{P}_{1/2})$ which have a rank $\le k$ due to the definition of $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_{1/2})$. Hence, $(\pi_k^s A)|_b$ is a sum of not more than $3L$ matrices of rank $\le k$ and therefore has a rank bounded by $3Lk$. Since this bound holds also for the other blocks of $\mathcal{P}_W$, this proves

$$\pi_k^s A \in \mathcal{M}_{\mathcal{H},3Lk}(I \times I, \mathcal{P}_W). \tag{3.7}$$

Because of $\|A - \pi_{3Lk}^W A\| \le \|A - \pi_k^s A\|$, (3.6) holds for the rank

$$k(\varepsilon) := 3L k_s(\varepsilon), \tag{3.8}$$

where $k_s(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$ refers to the (strong) admissibility $Adm_{1/2}$. However, the question concerning a *minimum* $k(\varepsilon)$ in (3.6) is not yet answered.

The result (3.8) is not sufficient to encourage the use of the weak admissibility, since we pay with a logarithmic factor $L = \log_2 n$ for an improvement of the constant C_{sp}. But Section 3.3 will show that the bound $3Lk_s(\varepsilon)$ is too pessimistic.

The following Section 3.2.2 shows that the bound $3Lk_s(\varepsilon)$ can be replaced by $Lk_s(\varepsilon)$. It will be explained why the gain by the factor 3 is remarkable (cf. Remark 3.1 below).

3.2.2 Improved Agglomeration

The standard admissibility condition $Adm_{1/2}$ in (2.17) involves the *minimum* of the diameters. This allows to define a completely anisotropic block subpartitioning[3] for the block $b = \tau_1^1 \times \tau_2^1 \in \mathcal{P}_W$ discussed in Figure 3.1. We let $\sigma = \tau_2^1$ unchanged but partition τ_1^1 into $\tau_1^2, \tau_1^3, \ldots, \tau_1^L, \tau_2^L$. Note that the union of these clusters yield τ_1^1 and that all blocks $\tau_1^\ell \times \tau_2^1$ ($\ell = 2, \ldots, L$) and $\tau_2^L \times \tau_2^1$ are (strongly) admissible in the sense of $Adm_{1/2}$. The left side of Figure 3.2 shows this partitioning. Similar constructions can be defined for the other parts of
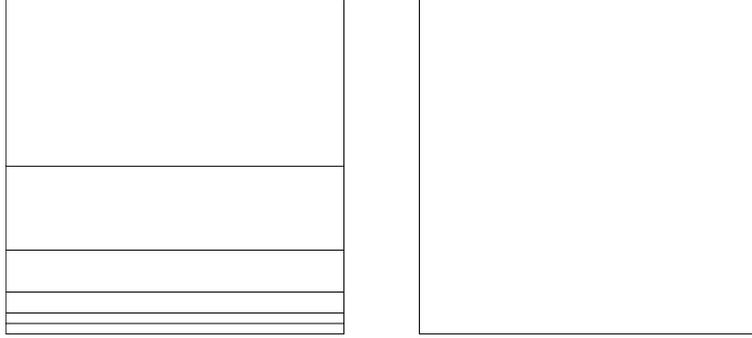

Figure 3.2: *Left:* anisotropic subpartitioning of $b = \tau_1^1 \times \tau_2^1 \in \mathcal{P}_W$. *Right:* block $b \in \mathcal{P}_W$

the matrix. Denote the resulting subpartitioning by $\mathcal{P}_{\mathrm{anisotropic}}$. Since $Adm_{1/2} = true$ for the new blocks, we get the same error estimate of $\|A - A_k\|$ for $A_k = \pi_k^{\mathrm{anisotropic}} A \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_{\mathrm{anisotropic}})$ as for the previous partitioning $\mathcal{P}_{1/2}$. The number of the blocks $\{\tau_1^2 \times \tau_2^1, \ldots, \tau_1^L \times \tau_2^1, \tau_2^L \times \tau_2^1\}$ (see Figure 3.2) is L. Hence, $\mathrm{rank}\, A_k|_b \leq Lk$ holds and proves that the best approximation $\pi_k^{\mathrm{anisotropic}} A \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_{\mathrm{anisotropic}})$ with local rank k can be regarded as hierarchical matrix with local rank Lk with respect to the weak admissibility condition:

$$\pi_k^{\mathrm{anisotropic}} A \in \mathcal{M}_{\mathcal{H},Lk}(I \times I, \mathcal{P}_W).$$

Remark 3.1 *(a) Although the gain by a factor 3 compared with (3.7) seems to be quite modest, the result becomes more significant when we look for generalisations to more than one spatial dimension. The factor 3 in $3Lk$ from the 1D case becomes a 'constant' $C_{\mathrm{aggl}}(d)$ which increases with the dimension d. On the other hand, the factor 1 in $Lk = 1Lk$ remains unchanged when d increases.*

(b) The derivation from above shows that the rank of $(\pi_k^{\mathrm{anisotropic}} A)|_b$ for $b = \tau_i^\ell \times \tau_{i+1}^\ell \in \mathcal{P}_W$ is even bounded by $(L - \ell + 1)\,k$.

We recall that the storage cost given in (2.21) is $\mathcal{N}_{st} \leq 2kLC_{\mathrm{sp}}(\mathcal{P})\,n$. In the case of $\mathcal{P} = \mathcal{P}_{1/2}$ we get $\mathcal{N}_{st} \leq 2kLC_{\mathrm{sp}}(\mathcal{P}_{1/2})n$ with $L = \log_2 n$, while $\mathcal{P}_W$ yields a comparable accuracy with $k' = Lk$ so that the storage cost is $\mathcal{N}_{st} \leq 2kL^2 C_{\mathrm{sp}}(\mathcal{P}_W)n$ (or even $\mathcal{N}_{st} \leq L\,(L+1)\,C_{\mathrm{sp}}(\mathcal{P}_W)n$ due to Remark 3.1b). Hence, $C_{\mathrm{sp}}(\mathcal{P}_{1/2})$ has to be compared with $\frac{1}{2}C_{\mathrm{sp}}(\mathcal{P}_W)\log_2 n$. Since $2C_{\mathrm{sp}}(\mathcal{P}_{1/2})/C_{\mathrm{sp}}(\mathcal{P}_W)$ is rather large for $d \geq 2$, the inequality $C_{\mathrm{sp}}(\mathcal{P}_W)\log_2 n \leq 2C_{\mathrm{sp}}(\mathcal{P}_{1/2})$ can be expected to be valid for realistic sizes of n.

Nevertheless, the foregoing bounds are too pessimistic as shown next by means of numerical examples.

3.3 Numerical Results

The previous constructions can only yield upper bounds for the true rank $k(\varepsilon)$, which can be computed exactly by means of SVD.

---

[3]This partitioning is not used in practice, since this format is less efficient for matrix operations.

3.3.1 A Numerical Study by SVD

We consider the kernel $\log|x-y| = \log(|x|+|y|)$ for $x \in I_1 := [-1,0]$ and $y \in I_2 := [\delta, 1+\delta]$. If $\delta > 0$, the intervals I_1, I_2 are (strongly) admissible in the sense of $Adm_{1/(2\delta)}$; in particular, $Adm_{1/2} = true$ holds for $\delta \geq 1$. On the other hand, for $\delta = 0$ the intervals I_1 and I_2 are only weakly admissible.

The Nyström discretisation (cf. [1] for more details) leads to the submatrix

$$A = (\log|\xi_i - \eta_j|)_{i,j=1}^n \quad \text{with } \xi_i = -\frac{i-1/2}{n}, \ \eta_j = \delta + \frac{j-1/2}{n}$$

corresponding to I_1, I_2. The argument closest to the singularity is $|\xi_1 - \eta_1| = \delta + \frac{1}{n}$.

The following results corresponding to $n = 128$ show the singular values σ_k of A for various values of δ. The relation to the question (3.5) is given by the following result. Let $A = U\Sigma V^\top$ be the SVD of A and define $A_k := U\Sigma_k V^\top$ with $\Sigma_k := \text{diag}\{\sigma_1, \ldots, \sigma_k, 0, \ldots, 0\}$. Then $\text{rank}(A_k) = k$ and $\|A - A_k\| = \sigma_{k+1}$ hold with respect to the spectral norm. Hence, $\varepsilon_k = \sigma_{k+1}$ are the corresponding errors and can be seen from Table 3.1.

k	1	2	3	4	5	10	15	20	25	30
$n=128, \delta=0$	10.1	4.69	6.2-01	9.7-02	1.8-02	2.2-06	1.2-10	2.7-15	1.3-16	1.1-16
$n=128, \delta=0.1$	7.4	5.42	1.9-01	1.1-02	8.6-04	4.4-09	2.6-14	1.4-16	1.2-16	1.1-16
$n=128, \delta=1$	14.3	1.10	5.8-03	6.4-05	8.3-07	2.3-15	2.9-16	1.9-16	1.7-16	1.5-16
$n=256, \delta=0$	20.3	9.38	1.26	0.21	4.5-02	1.7-05	3.2-09	3.7-13	2.8-16	2.3-16

Table 3.1: Singular values σ_k of the matrix A for different values of δ

We see that in all cases the approximant A_k converges to A exponentially with respect to k, but the coefficients in the exponents depend on the distance parameter δ (see also Figure 3.3).

The next result corresponds to a BEM application in 2D, where the integration is performed over a curve $\Gamma \subset \mathbb{R}^2$. We assume that Γ contains two straight lines joining at $0 \in \mathbb{R}^2$ by a $90°$ angle:

$$I_1 := \{(x,y) \in \mathbb{R}^2 : y = 0, x \in [0,1]\}, \qquad I_2 := \{(x,y) \in \mathbb{R}^2 : x = 0, y \in [\delta, 1+\delta]\}.$$

Again, $A = (\log\|\xi_i - \eta_j\|)_{i,j=1}^n$ with $\xi_i = ((i-\frac{1}{2})h, 0) \in I_1$ and $\eta_j = (0, \delta + (j-\frac{1}{2})h) \in I_2$ is the Nyström approximation with step size $h = 2/n$, where $n = 128$.
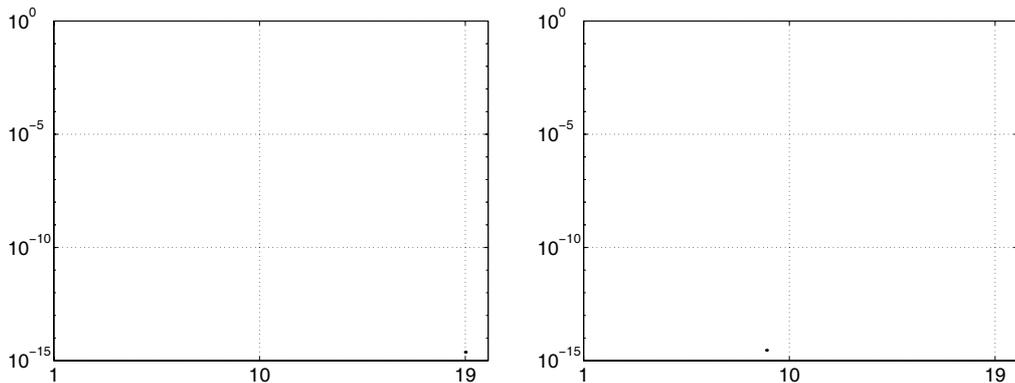

Figure 3.3: Singular values of a Nyström matrix A (left: $\delta = 0$, right: $\delta = 1$)

The comparison of the numbers for $\delta = 0$ (corresponding to Adm_W) and $\delta = 1$ (corresponding to $Adm_{1/2}$) seems to show that a certain accuracy ε is obtained for the choice $k_\delta(\varepsilon)$, where the k_δ for $\delta = 0, 1$ are proportional (see Figure 3.3):

$$k_0(\varepsilon) \sim c_p k_1(\varepsilon) \text{ with } c_p \approx 2. \tag{3.9}$$

3.3.2 Decay of Singular Values

A constant of size $c_p \in [2,3]$ would be much more desirable than the logarithmic factor from (3.8). In particular, a constant factor would imply that the singular values decay with the same speed as in the strongly admissible case, namely $\sigma_k \lesssim \exp(-ck)$ (cf. (3.1), where $-c = \log(\eta)$).

The relation (3.9) is almost true, as the following proposition is stating.

Conjecture 3.2 *a) For discretisations of the 1D single layer potential operator, the partitioning $\mathcal{P}_W$ is appropriate in the following sense: All submatrices $A|_b$, $b \in \mathcal{P}_W$, have singular values decaying like $\sigma_k \lesssim C_1 \exp(-ck)$ independent of the matrix size. As a consequence, the answer to question (3.6) would be $k(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$.*

Proposition 3.3 *b) The local ranks $k_W(\varepsilon), k_s(\varepsilon)$ for the blocks of the respective partitioning $\mathcal{P}_W, \mathcal{P}_{1/2}$ are related by $k_W(\varepsilon) \leq c' k_s(\varepsilon)$ with a moderate constant c' independent of n and ε (the experiments indicate a value of about $c' \in [2, 3.5]$).*

In §3.4 we list several approaches which, unfortunately, do not prove our conjecture. Instead they yield either $\sigma_k \lesssim \exp(-c\sqrt{k})$ or $\sigma_k \lesssim \exp(-ck/\log(1/h))$

3.3.3 Example of a Collocation Method

To observe the behaviour of σ_k more carefully, we consider a simple model problem. Let A be the matrix arising from a collocation BEM in $\Omega = [0,1]$ with the kernel function $s(x,y) := \log|x - y|$. We use the piecewise constant elements from (2.4). The entries of A are

$$a_{ij} = \int_{(j-1)h}^{jh} \log\left|\left(i - \tfrac{1}{2}\right)h - y\right| dy \qquad \text{for } i, j = 1, \ldots, n,$$

where $h = 1/n$. The matrix $A_{\mathcal{H}}$ is computed using the best approximation of the matrix blocks of A by SVD. The integrals needed for a_{ij} are evaluated exactly.

The hierarchical matrix approximating A is denoted by $A_{\mathcal{H}}$. The upper part of Table 3.2 compares $A_{\mathcal{H}}$ obtained by $Adm_{1/2}$ and $A_{\mathcal{H}}$ obtained by Adm_W. The storage size of $A_{\mathcal{H}}$ is given in Megabyte.

		$Adm_{1/2}$			Adm_W	
n	k	$\frac{\|A - A_{\mathcal{H}}\|_F}{\|A\|_F}$	storage	k	$\frac{\|A - A_{\mathcal{H}}\|_F}{\|A\|_F}$	storage
256	2	2.0_{10}-5	0.1	5	9.1_{10}-6	0.1
512	2	1.5_{10}-5	0.3	5	1.1_{10}-5	0.3
1024	2	1.0_{10}-5	0.7	5	1.1_{10}-5	0.7
2048	2	7.4_{10}-6	1.7	5	8.8_{10}-6	1.5
4096	2	5.3_{10}-6	3.8	5	6.7_{10}-6	3.3
8192	2	3.7_{10}-6	8.3	5	5.0_{10}-6	7.4
8192	2	3.8_{10}-6	8.3	5	5.0_{10}-6	7.4
16384	2	2.8_{10}-6	18.2	5	3.7_{10}-6	16.1
32768	2	2.0_{10}-6	39.5	5	2.7_{10}-6	34.8

Table 3.2: Accuracy and storage size in the strongly and weakly admissible case

We see that the storage size is less in the case of weak admissibility. This is due to the fact that $C_{sp}(\mathcal{P}_W) = 1 = C_{sp}(\mathcal{P}_{1/2})/3$ holds for the sparsity constants, while the ratio $5/2$ of the involved local ranks is ≤ 3.

We consider the blocks depicted in Figure 3.4, which are the biggest strongly or respectively weakly admissible blocks. The respective dimension is chosen such that the blocks b from Figure 3.4 have identical sizes $n_b \times n_b$. In Figure 3.5, we present the first singular values σ_k of these weakly and strongly admissible blocks for increasing size n_b.
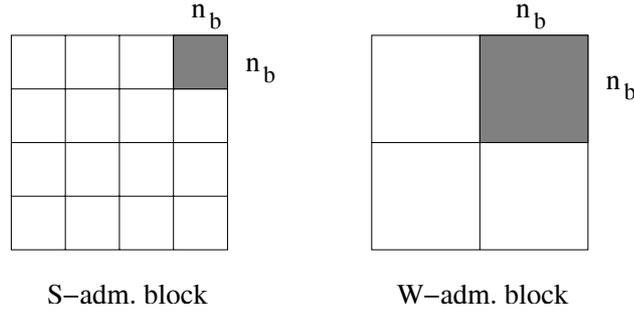
S−adm. block   W−adm. block

Figure 3.4: Admissible blocks used in Figure 3.5

In the strongly admissible case, the σ_k values are near independent of n_b (see left part of Figure 3.5). The exponential decay $\sigma_k \sim \exp(-ck)$ corresponds to the linear decrease shown in Figure 3.6 (label "standard") because of the logarithmic scaling.

In the weakly admissible case, for fixed k, the σ_k values increase as $n_b \to \infty$ (see right part of Figure 3.5). Obviously, for smaller n_b the singular values decay faster as $k \to \infty$ than for larger n_b. If we fix an error bound ε, $k_W(\varepsilon)$ increases as $n_b \to \infty$: whenever the graph of some σ_k crosses the horizontal line $\sigma = \varepsilon$, $k_W(\varepsilon)$ increases by one. However, $k_W(\varepsilon)$ must remain bounded, since the provable inequality $\sigma_k \lesssim \exp(-ck/\log(k))$ implies $k_W(\varepsilon) \lesssim \log|\log\varepsilon| \cdot |\log\varepsilon|$ independent of n_b. But even for the size $n_b = 4096$, the σ_k's show the behaviour $\sigma_k \sim \exp(-ck)$ due to the linear graph in Figure 3.6 (with label "weak"). On the other hand,
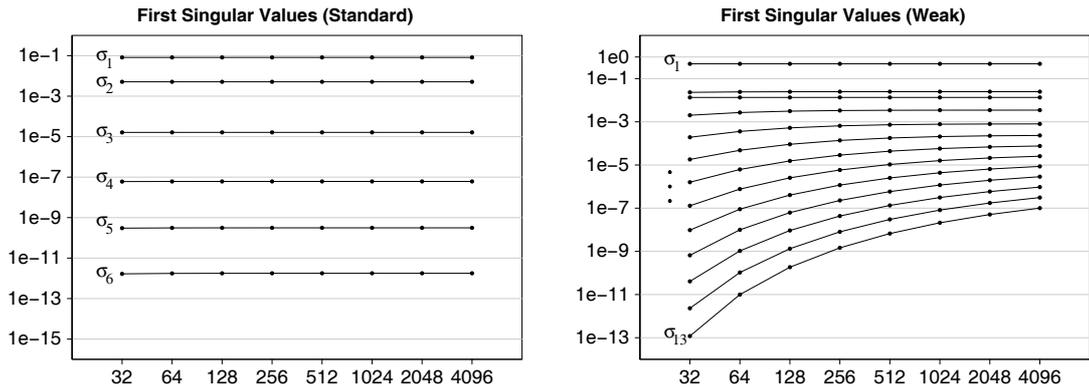

Figure 3.5: Singular values of S-admissible (left) and W-admissible (right) blocks for different blocksizes.

the constant c in $\sigma_k \sim \exp(-ck)$ seem to increase with $n_b \to \infty$. For a closer analysis, we make the ansatz $\sigma_k = C_1 \exp(-ck^\alpha)$ with constants C_1, c, α and consider the ratios $\sigma_1/\sigma_k = \exp(c(k^\alpha - 1))$, i.e., $\log(\sigma_1/\sigma_k) = c(k^\alpha - 1)$. The determination of the remaining constants c, α is done by minimising

$$\sum_{k=1}^{J} \left(\log\frac{\sigma_1}{\sigma_k} - c(k^\alpha - 1) \right)^2.$$

In the weakly admissible case with $n = 256$, the least squares minimisation leads to $c = 1.56$, $\alpha = 1.02$, while the strongly admissible case yields $c = 4.3$, $\alpha = 1.05$. The corresponding results for $n = 4096$ are $c = 1.23$, $\alpha = 1.05$ (W-admissibility) and $c = 4.2$, $\alpha = 1.05$ (S-admissibility). The ratio c' mentioned in Conjecture 3.2b corresponds to the ratios of the c-values: $4.3/1.56 = 2.74$ and $4.2/1.23 = 3.4$ confirm $c' \in [2, 3.5]$.

The fit of the points $(k, \log\frac{\sigma_1}{\sigma_k})$ by $\varphi_{c,\alpha}(k) = c(k^\alpha - 1)$ with the values c, α from above is presented in Figure 3.7. The perfect match is a clear indication that the Conjecture 3.2 might be true with small enough constant c in the exponent.
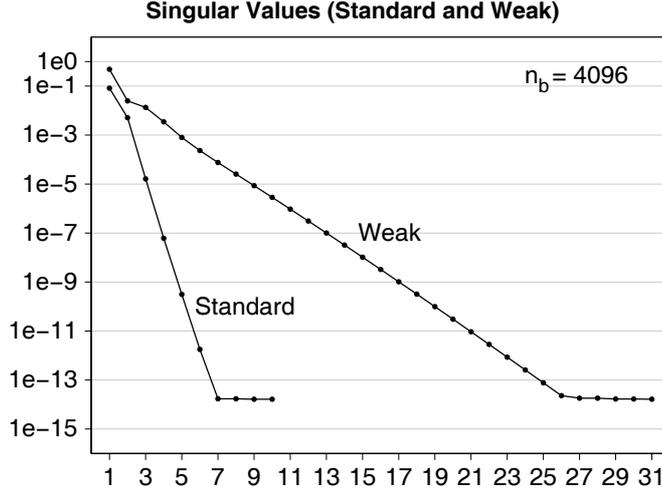
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Singular Values (Standard and Weak)

Figure 3.6: Comparison of the singular values of S- and W-admissible blocks (right) from Figure 3.5

		$Adm_{1/2}$			Adm_W	
n	k	$\left\|I - AA_{\mathcal{H}}^{-1}\right\|_F$	CPU	k	$\left\|I - AA_{\mathcal{H}}^{-1}\right\|_F$	CPU
256	2	$8.0_{10}\text{-}5$	0.2	5	$1.8_{10}\text{-}5$	0.04
512	2	$8.1_{10}\text{-}5$	0.4	5	$3.4_{10}\text{-}5$	0.1
1024	2	$8.1_{10}\text{-}5$	1.1	5	$4.6_{10}\text{-}5$	0.3
2048	2	$8.1_{10}\text{-}5$	2.8	5	$1.4_{10}\text{-}4$	0.7
4096	2	$8.1_{10}\text{-}5$	6.7	5	$1.5_{10}\text{-}4$	1.8
8192	2	$8.0_{10}\text{-}5$	15.9	5	$1.5_{10}\text{-}4$	4.4
16384	2	$8.0_{10}\text{-}5$	37.3	5	$1.5_{10}\text{-}4$	10.5
32768	2	$8.1_{10}\text{-}5$	86.0	5	$1.5_{10}\text{-}4$	25.2

	$Adm_{1/2}$		Adm_W		$\left\|A_{\mathcal{H},s} - A_{\mathcal{H},w}\right\|_2 / \left\|A_{\mathcal{H},s}\right\|_2$
n	k	CPU	k	CPU	
131072	4	1031	12	475	$1_{10} - 6$

Table 3.3: Error of the inverse and CPU time needed

The hierarchical matrix technique enables an approximate inversion (see [8]). We perform this inversion with respect to the different formats. Table 3.3 shows the relative error $\left\|I - AA_{\mathcal{H}}^{-1}\right\|$ and the computing time in seconds.

Table 3.3 indicates two remarkable facts. First it shows that comparable starting errors $\left\|A - A_{\mathcal{H}}\right\|$ lead to comparable errors $\left\|I - AA_{\mathcal{H}}^{-1}\right\|$ (this norm bounds the relative error $\|A^{-1} - A_{\mathcal{H}}^{-1}\|/\|A^{-1}\|$) although the computation is performed in different formats. Second, the CPU time is much better in the case of the weak admissibility. The reason is that the CPU time has a bound where C_{sp} appears quadratically (see (2.22)). This explains a gain of more than a factor $3 = C_{\text{sp}}(\mathcal{P}_{1/2})/C_{\text{sp}}(\mathcal{P}_W)$.

The upper part of Table 3.3 can be completed by results for the higher dimension $n = 2^{17}$. Since the computation of the SVD is too expensive, we give only the relative error $\left\|A_{\mathcal{H},s} - A_{\mathcal{H},w}\right\|_2 / \left\|A_{\mathcal{H},s}\right\|_2$.

3.3.4 Practical Implementation

For BEM matrices, it remains to explain how to compute the blocks $A|_b$, $b \in \mathcal{P}_W$.

First, we consider $Adm_{1/2}$: Instead of using SVD to fill the rank-k-matrices, we use an algorithm described in [2], [4], which approximates the respective low-rank blocks by evaluating few block entries. In order to increase the accuracy, the procedure is used to produce a local rank $k' > k$ (standard choice in later examples
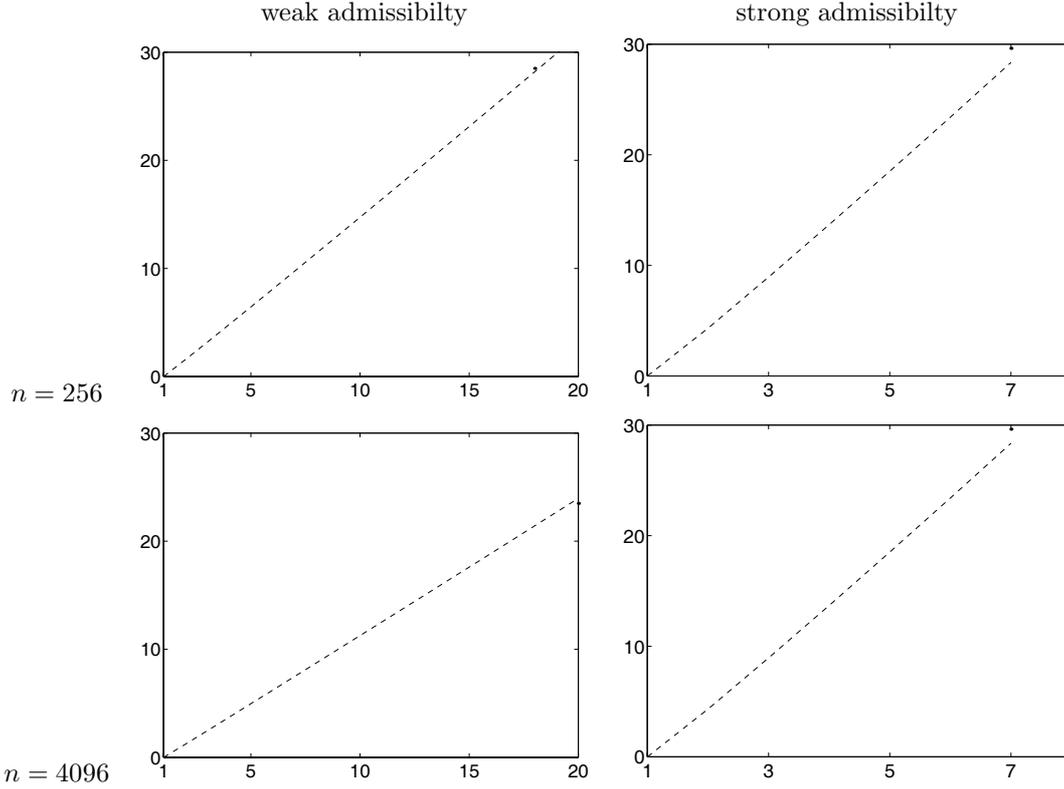
Figure 3.7: Least square fitting of singular values for W- and S-admissible blocks ($n = 256, 4096$)

is $k' = 3k$) followed by an optimal truncation to rank k.

This procedure (as well as the proof of its accuracy) applies only to the S-admissible case.

In the case of the W-admissibility, consider a block $b \in \mathcal{P}_W$. It is the union $b = \dot{\cup}_i b_i$ of S-admissible blocks $b_i \in \mathcal{P}_{1/2}$. While the agglomeration in §3.2.1 maps rank-k-matrices $A|_{b_i}$ into $A|_b$ with rank $\leq 3Lk$ without loss of information, the following computation of $A|_b$ uses a recursive truncation, where k is a given positive input number. The following algorithm actually generalises the so-called ACA-approximation (cf. [2, 4] and [18]) to the case of W-admissibility. It will be called the *weak adaptive cross approximation (WACA)* and it can be implemented as follows.

1. Compute $A|_{b_i}$ with rank $\leq k$ for all $b_i \in \mathcal{P}_{1/2}$, $b_i \subset b \in \mathcal{P}_W$, as described above.

Let $\ell := level(b)$. Then the blocks b_i belong to the levels $\ell + 1, \ldots, L$; more precisely, there are four b_i's of level L, and three b_i's for the levels $\ell + 1, \ldots, L - 1$.

2. For $m = L, L - 1, \ldots, \ell + 1$ do: Agglomerate the four blocks of level m into one block of rank $4k$ (belonging to level $m + 1$) and truncate to rank k.

Note that Step 2 produces a new block of level $m + 1$ so that in the next step of the loop, 4 blocks of level $m + 1$ are available. When the loop finishes, $A|_b$ with rank $\leq k$ is computed.

The accuracy of the $\mathcal{H}$-matrix obtained by this procedure is almost identical to the accuracy of the exact singular value decomposition. The second part of the tables 3.2 and 3.3 shows the results of the approximate algorithm for larger problem sizes. Although the accuracy is the same, it comes with a much smaller cost, because the complexity of the present method is only $\mathcal{O}(k'n \log n)$ compared to the complexity $\mathcal{O}(n^3)$ of the exact SVD.

3.4 Approximation Results

In this section, we consider the case of the kernel $s(x, y)$ for $x \in I_1$, $y \in I_2$ in two intervals with one common end point, say $I_1 = [-1, 0]$ and $I_2 = [0, 1]$. Since only a constant number of basis functions contains the common end point 0 in their support, we are allowed to reduce the intervals to $I_1 = [-1, -h]$ and $I_2 = [h, 1]$, where h is the local step size (here we assume that $\mathcal{O}(\log 1/h) = \mathcal{O}(\log n)$). The omitted degrees of freedom can increase the rank of the corresponding block only by a constant.

Problem 3.4 *The resulting problem is the separable approximation (3.2) for $\log (x + y)$ in $x, y \in [h, 1]$.*

Remark 3.5 *Instead we can also look for a separable approximation of $1/(x + y)$ for $x, y \in [h, 1]$. A separable approximation of the latter kernel yields a separable approximation of $\log (x + y)$ after integration with respect of one of the unknowns x or y.*

Let $\varepsilon > 0$ be a given accuracy. We are looking for a separable approximation (3.2) with $k = k(\varepsilon)$ such that the error is below ε. There are various approaches via interpolation, which however have not yet led to the desired accuracy, i.e., the asymptotic of $k(\varepsilon)$ is clearly worse than $\log(1/\varepsilon)$, which is the Conjecture 3.2. Here we list several approaches together with their results.

(1) The *hp-adaptive piecewise polynomial interpolation* leads to $k(\varepsilon) = \mathcal{O}(\log^2(1/\varepsilon))$ terms. This estimate is independent of h.

(2) The optimal *rational approximation* of $1/(x + y)$ (see Remark 3.5) is (implicitly) used for the optimal ADI parameters (see [7, Section 7.5.3]) and leads to $k(\varepsilon) = \mathcal{O}(\log(1/h) \cdot \log(1/\varepsilon))$ terms.

(3) The *sinc interpolation* needs a mapping ϕ from $\mathbb{R}$ onto $[0, 1]$, e.g., $\phi(\zeta) := 1/\cosh(\zeta)$. Then $\phi^\alpha(\zeta) \log (\phi(\zeta) + y)$ with $\alpha > 0$ can be interpolated by sinc functions w.r.t. ζ. Due to the fast decay of ϕ as $|\zeta| \to \infty$, the infinite sum can be replaced by a finite one with $k(\varepsilon) = \mathcal{O}(\log^2(1/\varepsilon))$ term (see [17]).

(4) The quadratic behaviour in the previous approach is due to the fact that ϕ decays "only" exponentially. The function $\phi(\zeta) := 1/\cosh(\sinh(\zeta))$ used in [16] decays *twice* exponentially so that a function like $\phi^\alpha(\zeta) \log (\phi(\zeta))$ can be interpolated by $k(\varepsilon) = \mathcal{O}(\log(1/\varepsilon) \cdot \log\log(1/\varepsilon))$ sinc terms. Unfortunately, $\log (\phi(\zeta) + y)$ with small but positive y leads to singularities quite close to the real axis. An involved analysis then leads to $k(\varepsilon) = \mathcal{O}(\log(1/\varepsilon) \cdot \min\{\log(1/\varepsilon), \log(1/h)\})$ (cf. [9]).[4]

(5) The *agglomeration technique* from §§3.2.1-3.2.2 yields the rank $k(\varepsilon) = \mathcal{O}(\log(1/\varepsilon) \cdot \log(n))$.

Although the results of these five approaches look different, they all coincide with $k(\varepsilon) = \mathcal{O}(\log^2 n)$ for the usual choice $\log(1/\varepsilon) = \mathcal{O}(\log n)$ because of $\log(1/h) = \mathcal{O}(\log n)$. Interestingly, the approaches (1) and (3) work for $h = 0$, in particular, the separable approximation can be used for the continuous problem.

Finally, we discuss a further approach by *sinc quadrature*:

(6) The integrand in $\frac{1}{r} = \int_{-\infty}^{\infty} e^{\sinh(x) - re^{\sinh(x)}} \cosh(x) dx$ is of the form $e^{rF(t)} G(t) dt$. The sinc quadrature (with step size $\mathfrak{h}$) yields the sum $\sum_{\nu \in \mathbb{Z}} e^{rF(\nu\mathfrak{h})} G(\nu\mathfrak{h})$ (cf. [17]). Since $e^{rF(t)} G(t)$ is decaying very fast for $t \to \pm\infty$, one gets a finite sum $1/r \approx \sum_\nu e^{rF(\nu\mathfrak{h})} G(\nu\mathfrak{h})$. Setting $r = x + y$, we obtain the separable approximation $\frac{1}{x+y} \approx \sum_\nu G(\nu\mathfrak{h}) e^{xF(\nu\mathfrak{h})} e^{yF(\nu\mathfrak{h})}$. Although the numerical results prove to be rather accurate, the theoretical estimate does not imply the desired estimate.

4 The Matrix Families $\mathcal{M}_{k,\tau}$ and $\mathcal{M}_{\mathbf{k}}(\varepsilon)$

The previous results give rise to the following definitions of matrix sets $\mathcal{M}_{k,\tau}$ and $\mathcal{M}_k(\varepsilon)$. The relation to the hierarchical matrix sets $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$ and $\mathcal{M}_{\mathcal{H}^2,k}(I \times I, \mathcal{P}_W)$ will be mentioned in §§4.2-4.3. The interesting properties can be used to describe approximations to meromorphic functions of A.

4.1 The Set $\mathcal{M}_{k,\tau}$

Definition 4.1 *Let $\tau \in T(I)$ be a cluster and $k \in \mathbb{N}$. Set $\tau' := I \backslash \tau$. A matrix A belongs to $\mathcal{M}_{k,\tau}(I)$ if* $\operatorname{rank}(A|_{\tau \times \tau'}) \leq k$ *and* $\operatorname{rank}(A|_{\tau' \times \tau}) \leq k$. *If I is fixed, we write $\mathcal{M}_{k,\tau}$ instead of $\mathcal{M}_{k,\tau}(I)$.*

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[4]Numerical tests show a much better behaviour than indicated by the theoretical error bounds.

For an illustration, we may numerate the indices of τ first. Then we get the partitioning

$$A = \begin{array}{c} \\ \\ A = \end{array} \begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \begin{array}{c} \tau \\ \\ \tau' \end{array} \tag{4.1}$$

and Definition 4.1 states $\operatorname{rank} A_{12} \le k$ and $\operatorname{rank} A_{21} \le k$.

The following simple results are of importance. Note that the operations $*, ^{-1}, +$ are exact without any truncation.

Lemma 4.2 *(a) Let $A \in \mathcal{M}_{k_A, \tau}$ and $B \in \mathcal{M}_{k_B, \tau}$. Then $A * B \in \mathcal{M}_{k, \tau}$ for $k = k_A + k_B$.*
(b) Let $A \in \mathcal{M}_{k, \tau}$ be invertible. Then $A^{-1} \in \mathcal{M}_{k, \tau}$ holds with the same k.
(c) Let $A \in \mathcal{M}_{k, \tau}$. Then $A + aI \in \mathcal{M}_{k, \tau}$ for any $a \in \mathbb{C}$.
(d) Let $A \in \mathcal{M}_{k, \tau}(I)$ and $\tau \subset I' \in T(I)$, where $I' \subset I$ is a proper subset. Then the principal submatrix $A|_{I' \times I'}$ belongs to $\mathcal{M}_{k, \tau}(I')$. The same holds for the Schur complement $S_{I'} = A|_{I' \times I'} - A|_{I' \times I''} * (A|_{I'' \times I''})^{-1} * A|_{I'' \times I'}$ $(I'' := I \backslash I')$, provided that $(A|_{I'' \times I''})^{-1}$ exists.

Proof. a) Using the notation (4.1) for A, B and $C := AB$ correspondingly, we have $C_{12} = A_{11}B_{12} + A_{12}B_{22}$. From $\operatorname{rank}(A_{11}B_{12}) \le \operatorname{rank}(B_{12}) \le k_B$ and $\operatorname{rank}(A_{12}B_{22}) \le \operatorname{rank}(A_{12}) \le k_A$ we conclude that $\operatorname{rank}(C_{12}) \le k_A + k_B$. Similarly for $\operatorname{rank}(C_{21})$.

b1) Assume that the block matrix A_{11} is invertible, too. Then the Schur complement $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is also invertible and the inverse of A from (4.1) is given by

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{bmatrix}. \tag{4.2}$$

Obviously, $\operatorname{rank}(A^{-1}|_{\tau \times \tau'}) = \operatorname{rank}(-A_{11}^{-1}A_{12}S^{-1}) \le \operatorname{rank}(A_{12}) \le k$ as well as $\operatorname{rank}(A|_{\tau' \times \tau}) \le \operatorname{rank}(A_{12}) \le k$ proving $A^{-1} \in \mathcal{M}_{k, \tau}$.

b2) If A_{11} is singular, the matrix $A_\varepsilon := A + \varepsilon I$ is regular for $\varepsilon \ne 0$ sufficiently small. Since $\operatorname{rank}(A_\varepsilon^{-1}|_{\tau \times \tau'}) \le \operatorname{rank} A_{12}$ independently of ε, it follows that $A_\varepsilon^{-1} \in \mathcal{M}_{k, \tau}$. The limit $\lim_{\varepsilon \to 0} A_\varepsilon^{-1}$ is A^{-1}, since by assumption A is regular. The rank satisfies $\operatorname{rank}(A^{-1}|_{\tau \times \tau'}) = \operatorname{rank}(\lim_{\varepsilon \to 0} A_\varepsilon^{-1}|_{\tau \times \tau'}) \le \lim_{\varepsilon \to 0} \operatorname{rank}(A_\varepsilon^{-1}|_{\tau \times \tau'}) \le \operatorname{rank}(A_{12}) \le k$. Together with the similar estimate $\operatorname{rank}(A^{-1}|_{\tau' \times \tau}) \le k$, the statement $A^{-1} \in \mathcal{M}_{k, \tau}$ follows.

c) The operation $A \mapsto A + aI$ does not change the parts $*|_{\tau \times \tau'}$ and $*|_{\tau' \times \tau}$.

d) Restricting a matrix to $I' \times I' \subset I \times I$, one can only reduce the rank. Due to Part b), $A^{-1} \in \mathcal{M}_{k, \tau}(I)$. Note that $(S_{I'})^{-1}$ is the principal $I' \times I'$-submatrix of A^{-1} implying $(S_{I'})^{-1} \in \mathcal{M}_{k, \tau}(I')$. Applying again Part b) yields $S_{I'} \in \mathcal{M}_{k, \tau}(I')$. ∎

A consequence is the following

Theorem 4.3 *Let $R(x)$ be the rational function $R(x) = P^I(x)/P^{II}(x)$, where P^I, P^{II} are polynomials of the respective degrees $d_I, d_{II} \in \mathbb{N}_0$. Let $A \in \mathcal{M}_{k, \tau}$ be a matrix with eigenvalues distinct from the poles of R. Then $R(A)$ belongs to $\mathcal{M}_{k_R, \tau}$ with $k_R = k * d_R$, where $d_R := \max(d_I, d_{II})$ is the degree of R.*

Proof. For theoretical reasons we may factorise P^I, P^{II} into $P^I(x) = a_I \prod_{i=1}^{d_I} (x - x_i^I)$ and $P^{II}(x) = a_{II} \prod_{i=1}^{d_{II}} (x - x_i^{II})$ with possibly complex x_i^I, x_i^{II}. For $i \le \min(d_I, d_{II})$, the rational factors $\frac{x - x_i^I}{x - x_i^{II}}$ equal $1 + (x_i^{II} - x_i^I)/(x - x_i^{II})$. Replacing x by $A \in \mathcal{M}_{k, \tau}$, we get $I + (x_i^{II} - x_i^I)(A - x_i^{II}I)^{-1} \in \mathcal{M}_{k, \tau}$ by Lemma 4.2c. Hence $R(A)$ is a product of $\min(d_I, d_{II})$ rational factors and $\max(d_I, d_{II}) - \min(d_I, d_{II})$ linear factors which all belong to $\mathcal{M}_{k, \tau}$. Due to Lemma 4.2a, the product is in $\mathcal{M}_{k_R, \tau}$ with $k_R = kd_R$. ∎

4.2 Relation to the $\mathcal{H}$-Matrix Format with Weak Admissibility

Remark 4.4 *Let $A \in \mathcal{M}_{k, \tau}$ for all clusters $\tau \in T(I)$. Then $A \in \mathcal{M}_{\mathcal{H}, k}(I \times I, \mathcal{P}_W)$.*

Proof. Let τ', τ'' be the sons of I. Then $A|_{\tau' \times \tau''}$ coincides with the submatrix A_{12} of (4.1) proving $\text{rank}(A|_{\tau' \times \tau''}) \le k$. To check the further hierarchical structure, we consider the structure of the principal submatrix $A|_{\tau' \times \tau'}$. Let $\dot{\tau}$ and $\ddot{\tau}$ be the sons of τ'. We have to prove that $\text{rank}(A|_{\dot{\tau} \times \ddot{\tau}}) \le k$. This, however, is a simple consequence of $A \in \mathcal{M}_{k,\dot{\tau}}$ and $\ddot{\tau} \subset I \backslash \dot{\tau}$ implying $\text{rank}(A|_{\dot{\tau} \times \ddot{\tau}}) \le \text{rank}(A|_{\dot{\tau} \times (I \backslash \dot{\tau})}) \le k$. ∎

Note that, in general, the reverse statement of Remark 4.4 is not valid. Instead, we have

Lemma 4.5 *Let* $A \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$. *Then* $A \in \mathcal{M}_{\ell \cdot k, \tau}$ *for all clusters* $\tau \in T^\ell(I)$ *of level* ℓ *(cf. (2.10))*.

Proof. Let $\tau \in T^\ell(I)$ and $\tau' = I \backslash \tau$. Then $\tau \times \tau'$ has an intersection with ℓ blocks $b \in \mathcal{P}_W$, for which by definition $\text{rank} A|_b \le k$. Thus, the rank of $A|_{\tau \times \tau'}$ is bounded by $\ell \cdot k$. Similar for $A|_{\tau' \times \tau}$. ∎

A stronger result holds for the more special class of $\mathcal{H}^2$-matrices discussed in §4.3.

A trivial case, where $A \in \mathcal{M}_{k,\tau}$ holds for all $\tau \in T(I)$, is stated below.

Remark 4.6 *Any band matrix belongs to* $\mathcal{M}_{k,\tau}$ *for all* $\tau \in T(I)$, *where* k *is the band width (i.e.,* $a_{ij} = 0$ *for* $|i - j| > k$).

In the following, we consider a generalisation of the format $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$. Replace the constant k by a function $k : \mathcal{P}_W \to \mathbb{N}_0$, i.e., the condition $\text{rank} A|_b \le k$ is replaced by $\text{rank} A|_b \le k(b)$ for all $b \in \mathcal{P}_W$. This defines the set $\mathcal{M}_{\mathcal{H},k(\cdot)}(I \times I, \mathcal{P}_W)$.

A particular choice of $k(\cdot)$ is

$$k(b) := k_\ell \qquad \text{for all } b \in T^\ell(I \times I) \tag{4.3}$$

(cf. (2.12)), i.e., the value $k(b)$ depends only on the level number $\ell = level(b)$. Introducing the vector $\mathbf{k} = (k_\ell)_{\ell=1}^L$, we can define

$$\mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W) = \mathcal{M}_{\mathcal{H},k(\cdot)}(I \times I, \mathcal{P}_W) \quad \text{for } k(\cdot) \text{ from (4.3)}.$$

Note that $\mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W) = \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$ for the (constant) choice $k_\ell = k$.

A conclusion from Lemmata 4.2b and 4.5 is

Corollary 4.7 *Set* $\mathbf{k} = (k_\ell)_{\ell=1}^L$ *with* $k_\ell := \ell k$. *Let* $A \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$. *Then* $A^{-1} \in \mathcal{M}_{\ell \cdot k, \tau}$ *for all clusters* $\tau \in T^\ell(I)$. *In particular,* $A^{-1} \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ *holds*.

4.3 Relation to the $\mathcal{H}^2$-Matrix Format

In [15] (see also [10]) the $\mathcal{H}^2$-matrices are introduced. Here, we need families $(V_\tau)_{\tau \in T(I)}$, $(W_\tau)_{\tau \in T(I)}$ of subspaces $V_\tau, W_\tau \subset \mathbb{R}^\tau$ with the consistency condition $V_\tau|_{\tau'} \subset V_{\tau'}$ and $W_\tau|_{\tau'} \subset W_{\tau'}$ for any son τ' of τ. Then, we say that $A \in \mathcal{M}_{\mathcal{H}^2,k}(I \times I, \mathcal{P}_W)$ if

$$\dim V_\tau \le k, \ \dim W_\sigma \le k \qquad \text{for all } \tau, \sigma \in T(I),$$
$$A|_b \in V_\tau \otimes W_\sigma \qquad \text{for all blocks } b = \tau \times \sigma \in \mathcal{P}_W$$

(the tensor space $V_\tau \otimes W_\sigma$ is the span of all matrices $vw^\top$ with $v \in V_\tau$ and $w \in W_\sigma$). Note that this implies $A \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$; hence, $\mathcal{M}_{\mathcal{H}^2,k}(I \times I, \mathcal{P}_W) \subset \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$.

Remark 4.8 *Let* $A \in \mathcal{M}_{\mathcal{H}^2,k}(I \times I, \mathcal{P}_W)$. *Then* $A \in \mathcal{M}_{k,\tau}$ *holds for all* $\tau \in T(I)$.

Proof. First, one finds that for all $\tau \in T(I)$, the block $\tau \times (I \backslash \tau)$ is the (disjoint) union of blocks $b_i = \tau \times \sigma_i \subset \tau_i \times \sigma_i \in \mathcal{P}_W$. By consistency, the restriction of the tensor space $V_{\tau_i} \otimes V_{\sigma_i}$ to $\tau \subset \tau_i$ is contained in $V_\tau \otimes V_{\sigma_i}$. Hence, all blocks $A|_{b_i}$ have an image contained in V_τ so that this holds also for the sum $A|_{\tau \times (I \backslash \tau)}$ implying $\text{rank} A|_{\tau \times (I \backslash \tau)} \le \dim V_\tau \le k$. ∎

The reverse statement is also true.

Remark 4.9 *Let* $A \in \mathcal{M}_{k,\tau}$ *for all* $\tau \in T(I)$. *Then* $A \in \mathcal{M}_{\mathcal{H}^2,k}(I \times I, \mathcal{P}_W)$ *holds with subspaces* V_τ *as constructed in the proof*.

Proof. a) We remark that $B \in V_\tau \otimes W_\sigma$ is equivalent to $image(B) \subset V_\tau$ and $image(B^\top) \subset W_\sigma$, where $image(\cdot)$ is the image space, i.e., the span of the columns of the matrix.

b) For $\tau \in T(I)$ set $V_\tau := image(A|_{\tau \times \tau'})$ and $W_\tau := image((A|_{\tau' \times \tau})^\top)$, where $\tau' := I \backslash \tau$. We have $\dim V_\tau = \operatorname{rank} A|_{\tau \times \tau'} \leq k$ and $\dim W_\tau \leq k$. Let $\dot\tau$ be a son of τ with the complement $\dot\tau' = I \backslash \dot\tau$. Then $V_\tau|_{\dot\tau} = image(A|_{\tau \times \tau'})|_{\dot\tau} = image(A|_{\dot\tau \times \tau'})$. Since $\dot\tau \subset \tau$ implies $\tau' \subset \dot\tau'$, we have $V_\tau|_{\dot\tau} \subset image(A|_{\dot\tau \times \dot\tau'}) = V_{\dot\tau}$. Similarly, $W_\tau|_{\dot\tau} \subset W_{\dot\tau}$ holds, proving the consistency conditions.

c) Let $b = \tau \times \sigma \in \mathcal{P}_W$ and set $\tau' := I \backslash \tau$, $\sigma' := I \backslash \sigma$. Since $\sigma \subset \tau'$, $image(A|_b) \subset image(A|_{\tau \times \tau'}) = V_\tau$. Analogously, $image((A|_b)^\top) \subset W_\sigma$ holds, proving $A|_b \in V_\tau \otimes W_\sigma$ and thus $A \in \mathcal{M}_{\mathcal{H}^2, k}(I \times I, \mathcal{P}_W)$. ∎

Combining the Remarks 4.8 and 4.9, we obtain

Lemma 4.10 *Let $A \in \mathcal{M}_{\mathcal{H}^2, k}(I \times I, \mathcal{P}_W)$ with respect to the subspace families $\{V_\tau, W_\tau : \tau \in T(I)\}$. Assume that all principal submatrices $A|_{\tau \times \tau}$ are invertible. Then $A^{-1} \in \mathcal{M}_{\mathcal{H}^2, k}(I \times I, \mathcal{P}_W)$ holds with respect to the subspace families $\{\hat{V}_\tau := (A|_{\tau \times \tau})^{-1} V_\tau, \hat{W}_\tau := (A|_{\tau \times \tau})^{-\top} W_\tau : \tau \in T(I)\}$.*

Proof. Let $\tau \in T(I)$ and represent $A^{-1}|_{\tau \times \tau'}$ as in (4.2) by $-A_{11}^{-1} A_{12} S^{-1}$ (note that $A_{11} = A|_{\tau \times \tau}$ and $A_{12} = A|_{\tau \times \tau'}$). Hence, $\hat{V}_\tau = image(A^{-1}|_{\tau \times \tau'}) \subset image(A_{11}^{-1} A_{12}) = A_{11}^{-1} image(A_{12}) = (A|_{\tau \times \tau})^{-1} image(A|_{\tau \times \tau'}) = (A|_{\tau \times \tau})^{-1} V_\tau$. ∎

Corollary 4.11 *Remarks 4.8, 4.9 and Lemma 4.10 are easily generalisable to $\mathcal{M}_{\mathcal{H}^2, \mathbf{k}}(I \times I, \mathcal{P}_W)$ with any vector $\mathbf{k} = (k_\ell)_{\ell=1}^L$.*

4.4 The Set $\mathcal{M}_{\mathbf{k}}(\varepsilon)$

The previous results use the property that A satisfies $A \in \mathcal{M}_{k, \tau}$ simultaneously for all $\tau \in T(I)$. For other considerations it suffices to find for all $\tau \in T(I)$ different $A_\tau \in \mathcal{M}_{k, \tau}$ which are close to a matrix $A \in \mathbb{C}^{I \times I}$.

Let $\|\cdot\|$ be a suitable matrix norm. Assume that a cluster tree $T(I)$ is given. Due to the result of Corollary 4.7, we are interested in a level-dependent rank $\mathbf{k} = (k_\ell)_{\ell=1}^L$.

Definition 4.12 *Let $\mathbf{k} = (k_\ell)_{\ell=1}^L$. A matrix $A \in \mathbb{C}^{I \times I}$ belongs to $\mathcal{M}_{\mathbf{k}}(\varepsilon)$ with $\varepsilon \geq 0$, if for all clusters $\tau \in T(I)$ there exists a matrix $A_\tau \in \mathcal{M}_{k_\ell, \tau}$ ($\ell = level(\tau)$) with $\|A - A_\tau\| \leq \varepsilon$.*

The following statements are trivial:

Remark 4.13 *(a) If $A \in \mathcal{M}_{\mathbf{k}}(\varepsilon_1)$ and $\|A - A'\| \leq \varepsilon_2$, then $A' \in \mathcal{M}_{\mathbf{k}}(\varepsilon_1 + \varepsilon_2)$.*
(b) If $A \in \mathcal{M}_{\mathbf{k}}(0)$, then $A \in \mathcal{M}_{k_\ell, \tau}$ for all $\tau \in T^\ell(I)$.

Applications to BEM matrices are mentioned in the parts a,b of

Remark 4.14 *(a) Assume that Conjecture 3.2 applies to the kernel of an integral operator on a curve (i.e., on a one-dimensional manifold). Then the BEM matrices belong to $\mathcal{M}_{\mathbf{k}}(\varepsilon)$ with constant $k_\ell = k = \mathcal{O}(|\log \varepsilon|)$.*
(b) Consider BEM matrices as in (a). Without use of Conjecture 3.2, the agglomeration technique used in Remark 3.1b yields an approximation in $\mathcal{M}_{\mathbf{k}}(\varepsilon)$, where

$$\mathbf{k} = (k_\ell)_{\ell=1}^L, \ k_\ell = (L - \ell + 1) k \tag{4.4}$$

and $k = \mathcal{O}(|\log \varepsilon|)$.
(c) Another example is the Fredholm integral equation $(\lambda I + K) u = f$ with a kernel $\kappa(x, y)$ of K being sufficiently smooth in the two triangular parts $0 \leq x < y \leq 1$ and $0 \leq y < x \leq 1$. Due to the assumed smoothness, the discretised problem leads to a matrix in the set $\mathcal{M}_{\mathbf{k}}(\varepsilon)$ with constant $k_\ell = k$.

We notice that also a Volterra integral equation with a smooth kernel satisfies the assumptions of Remark 4.14c. Moreover, Green's kernels also possess a similar property.

Due to (3.1), the stiffness matrix A_{BEM} of a boundary element problem in $\mathbb{R}^2$ (i.e., the boundary is a curve) can be approximated by $A_k \in \mathcal{M}_{\mathcal{H}, k}(I \times I, \mathcal{P}_\eta)$ such that $\|A - A_k\| \leq c' \eta^k$ for all $0 < \eta \leq \eta_0$ with some constant c'. We can use the construction of the agglomeration technique from Remark 3.1b. Then Remark 4.14b with $\varepsilon = c\eta^k$ yields $A_{\text{BEM}} \in \mathcal{M}_{\mathbf{k}}(c\eta^k)$ with $k_\ell = (L - \ell + 1) k$ and proves part (a) of

Corollary 4.15 *(a) Set* $\mathbf{k} = (k_\ell)_{\ell=1}^L$ *with* $k_\ell := (L - \ell + 1)\}k$. *Let* A_{BEM} *be the stiffness matrix of a boundary element problem in* $\mathbb{R}^2$. *Then for all* $0 < \eta \leq \eta_0$ *there is a constant* c *such that* $A_{\mathrm{BEM}} \in \mathcal{M}_{\mathbf{k}}(c\eta^k)$.
(b) If Conjecture 3.2 applies, $A_{\mathrm{BEM}} \in \mathcal{M}_{\mathbf{k}}(c\eta^k)$ *holds even for the constant choice* $k_\ell = k$.

Proof. For part (b) use Remark 4.14a. ∎

An important property of $A \in \mathcal{M}_{\mathbf{k}}(\varepsilon)$ is the approximability by a $B \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$.

Theorem 4.16 *Let* $\|\cdot\|$ *be the Frobenius norm and let* $A \in \mathcal{M}_{\mathbf{k}}(\delta)$ *with* $\mathbf{k} := (k_\ell)_{\ell=1}^L$. *Then there is* $B \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ *with*

$$\|A - B\| \leq \delta\sqrt{n_{\mathcal{P}}/2}, \quad \text{where } n_{\mathcal{P}} := \# \left\{ b \in \mathcal{P}_W : b = \tau \times \sigma \text{ with } \min(\#\tau, \#\sigma) > k_{level(b)} \right\}. \tag{4.5}$$

Proof. For blocks $b = \tau \times \sigma \in T^\ell(I \times I) \cap \mathcal{P}_W$ of size $\min\{\#\tau, \#\sigma\} \leq k_\ell$, we may set $B|_b := A|_b$ causing no approximation error. Consider a block $b = \tau \times \sigma$ of a size exceeding k_ℓ. Due to $A \in \mathcal{M}_{\mathbf{k}}(\delta)$, there is some $A_\tau \in \mathcal{M}_{k_\ell, \tau}$ with $\|A - A_\tau\| \leq \delta$. We set $B|_b := A_\tau|_b$. This shows $B \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ and $\|A|_b - B|_b\|^2 + \|A|_{b'} - B|_{b'}\|^2 \leq \|A - A_\tau\|^2 \leq \delta^2$, where $b' := \sigma \times \tau$ is the transposed block. Summation over all blocks yields $\|A - B\|^2 \leq \delta^2 n_{\mathcal{P}}/2$, from which the statement follows. ∎

We have used the Frobenius norm, since then blockwise estimates turn easily into a global estimate. Here, improvements of the estimation may be possible. If we are interested in the exponentially asymptotic behaviour $\delta = C \exp(-\beta k)$, a factor $\sqrt{n_{\mathcal{P}}/2} = \mathcal{O}(\#I^{1/2})$ with polynomial dependence on the dimension $\#I$ is irrelevant.

4.5 Approximation of Meromorphic Functions of A

Next we make use of the exact representation of rational functions in Theorem 4.3.

Theorem 4.17 *Let* $\mathbf{k} := (k_\ell)_{\ell=1}^L$. *Let* $\|\cdot\|$ *be the Frobenius norm and fix* $\varepsilon \geq 0$. *Further, let* R *be a rational function of degree* d_R *which has no poles on the spectrum of the matrix* A. *Furthermore, assume* $A \in \mathcal{M}_{\mathbf{k}}(\delta)$ *with* δ *such that* $\|R(A) - R(A_\delta)\| \leq \varepsilon$ *for all* A_δ *with* $\|A - A_\delta\| \leq \delta$. *Set* $\mathbf{k}_R = \mathbf{k} \cdot d_R := (k_\ell d_R)_{\ell=1}^L$. *Then (a)* $R(A) \in \mathcal{M}_{\mathbf{k}_R}(\varepsilon)$ *and (b) there is a matrix* B *with*

$$B \in \mathcal{M}_{\mathcal{H},\mathbf{k}_R}(I \times I, \mathcal{P}_W) \quad \text{such that } \|R(A) - B\| \leq \varepsilon\sqrt{n_{\mathcal{P}}/2} \quad (n_{\mathcal{P}} \text{ from } (4.5))$$

Proof. Due to $A \in \mathcal{M}_{\mathbf{k}}(\delta)$, for any $\tau \in T(I)$ there is $A_\tau \in \mathcal{M}_{k_\ell, \tau}$ with $\|A - A_\tau\| \leq \delta$. Hence $\|R(A) - R(A_\tau)\| \leq \varepsilon$ and $R(A_\tau) \in \mathcal{M}_{k_\ell d_R, \tau}$ due to Theorem 4.3. This proves part (a): $R(A) \in \mathcal{M}_{\mathbf{k}_R}(\varepsilon)$. Finally, Theorem 4.16 implies part (b). ∎

The approximation of a meromorphic function by a rational one is considered in

Remark 4.18 *Let* A *be a matrix with spectrum* $\sigma(A)$ *and* f *a meromorphic function defined on* $\Omega \supset \sigma(A)$. *Assume that there is a rational function* R *of degree* d_R *such that* $\|f(A) - R(A)\| \leq \varepsilon_R$. *Then, under the assumptions of Theorem 4.17, there is a matrix* $B \in \mathcal{M}_{\mathcal{H},\mathbf{k}_R}(I \times I, \mathcal{P}_W)$ *with* $\|R(A) - B\| \leq \varepsilon_B$. *Together, we obtain* $\|f(A) - B\| \leq \varepsilon$ *with* $\varepsilon = \varepsilon_R + \varepsilon_B$.

5 Accuracy of the $\mathcal{H}$-Inversion in $\mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$-Format

A simple example of a rational function is $f(x) = 1/x$, so that Theorem 4.17 can be applied to the inverse A^{-1}. In [8], the inverse is computed by means of the block-Gauss elimination in the case of a general partition $\mathcal{P}$ generated by a binary tree $T(I)$. Here, we analyse the inversion algorithm in the case of the partitioning $\mathcal{P}_W$. Due to the special format $\mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$, we modify the algorithm from [8]. We start from the representation of the inverse of $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ by (4.2). In [8], we recursively computed approximations $X_{11} \approx A_{11}^{-1}$ and $T \approx S^{-1}$ and evaluated (4.2) with these replacements using the formatted addition and multiplication. Now we treat the inversion of the Schur complement S differently.

5.1 Inversion Algorithm

The approximate inversion procedure Inv is defined recursively. On level $\ell = L$ the inversion is done exactly (since $\#\tau_i^L = 1$, level L corresponds to 1×1 matrices!).

Assume that Inv is defined for all levels $> \ell$ and let $A \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(\tau \times \tau, \mathcal{P}_W^\tau)$ with $level(\tau) = \ell$. Here, we use the notation $\mathcal{P}_W^\tau = \{b \in \mathcal{P}_W : b \subset \tau \times \tau\}$ for the partitioning restricted to $\tau \times \tau$.

The submatrices A_{11}, A_{22} (corresponding to the two sons τ_1, τ_2 of τ) belong to level $\ell + 1$. Therefore, $X_{11} := Inv(A_{11})$ and $Inv(A_{22})$ are defined, provided that the inverses exist. The Schur complement $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is approximated by $\tilde{S} = A_{22} - A_{21}X_{11}A_{12}$ and is of the form $A_{22} + UV^\top$, e.g., with $U := -B_{21}$ and $V^\top := X_{11}B_{12}$. Due to the assumption $A \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(\tau \times \tau, \mathcal{P}_W^\tau)$, the off-diagonal blocks B_{21} and B_{12} have rank $\leq k_{\ell+1}$ implying that also U, V have rank $\leq k_{\ell+1}$. The Sherman-Morrison-Woodbury formula yields $\tilde{S}^{-1} = A_{22}^{-1} - A_{22}^{-1}U(I + V^\top A_{22}^{-1}U)^{-1}V^\top A_{22}^{-1}$. Replacing A_{22}^{-1} by $X_{22} := Inv(A_{22})$, we obtain

$$Y_{22} := X_{22} - X_{22}U(I + V^\top X_{22}U)^{-1}V^\top X_{22}. \tag{5.1}$$

The inversion $(I + V^\top X_{22}U)^{-1}$ is done explicitly, since it is a matrix of the small size $k_{\ell+1} \times k_{\ell+1}$. Finally, we obtain

$$X = \begin{bmatrix} X_{11} + X_{11}A_{12}Y_{22}A_{21}X_{11} & -X_{11}A_{12}Y_{22} \\ -Y_{22}A_{21}X_{11} & Y_{22} \end{bmatrix} \tag{5.2}$$

as approximation of A^{-1}. $Inv^*(A) := X$ is the intermediate result, where all multiplications and additions are performed exactly. Next we have to project the diagonal blocks of X to the desired format $\mathcal{M}_{\mathcal{H},\mathbf{k}}(\tau_i \times \tau_i, \mathcal{P}_W^{\tau_i})$, $i = 1, 2$, resulting in

$$Inv(A) := truncation \circ Inv^*(A) \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(\tau \times \tau, \mathcal{P}_W^\tau). \tag{5.3}$$

Note that the off-diagonal blocks are already of the required rank. This completes the recursive definition of Inv on level ℓ.

Next we study the error $Inv(A) - A^{-1}$. Note that a possible error is only caused by the truncation. Obviously, the error would be large if the inverse cannot be well represented by some $Z \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(\tau \times \tau, \mathcal{P}_W^\tau)$. However, the previous results guarantee the existence of some $Z \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(\tau \times \tau, \mathcal{P}_W^\tau)$ approximating A^{-1}. This property enables the main Theorem 5.2 below that is based on quite natural stability and approximation assumptions explained next.

5.2 Stability Assumptions

Let the matrix A be the argument of $Inv(A)$. The later assumptions must be valid not only for A but also for its principal submatrices. Accordingly, we define the set $\mathcal{A} = \{A|_{\tau \times \tau} : \tau \in T(I)\}$.

The stability assumptions (5.4a,b) are restricted to errors $\|E\|, \varepsilon$ small enough:
a) if $B \in \mathcal{A}$, then for any perturbation E :

$$\|(B + E)^{-1} - B^{-1}\| \leq C_S \|E\|. \tag{5.4a}$$

b) Let $B \in \mathcal{A}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{R}^{\tau \times \tau}$ (block partitioning with respect to $S(\tau) = \{\tau_1, \tau_2\}$). Then for any $X_{11} \in \mathbb{R}^{\tau_1 \times \tau_1}$ and $X_{22} \in \mathbb{R}^{\tau_2 \times \tau_2}$ with

$$\|X_{11} - B_{11}^{-1}\| \leq \varepsilon, \quad \|X_{22} - B_{22}^{-1}\| \leq \varepsilon,$$

the approximation of B^{-1} by $X = Inv^*(B)$ (X from (5.2)) satisfies

$$\|X - B^{-1}\| \leq C_{\text{inv}}\varepsilon. \tag{5.4b}$$

5.3 Approximability in $\mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$

Let A_{BEM} be the stiffness matrix of a boundary element problem in $\mathbb{R}^2$. We claim that

$$A_{\text{BEM}} \in \mathcal{M}_{\mathbf{k}}(c\eta^k) \text{ and } A_{\text{BEM}}^{-1} \in \mathcal{M}_{\mathbf{k}}(C_S c\eta^k) \quad \text{for } \mathbf{k} \text{ from } (4.4) \tag{5.5}$$

for some constants c, C_S and all $0 < \eta \le \eta_0$. Indeed, $A_{\mathrm{BEM}} \in \mathcal{M}_{\mathbf{k}}(c\eta^k)$ is stated in Corollary 4.15a. Applying Theorem 4.17 with $\delta = c\eta^k$ and $R(A) = A^{-1}$, we conclude from (5.4a) that $A_{\mathrm{BEM}}^{-1} \in \mathcal{M}_{\mathbf{k}}(C_S c\eta^k)$.

In addition to $A_{\mathrm{BEM}} \in \mathcal{M}_{\mathbf{k}}(c\eta^k)$ from (5.5), let A be an approximation satisfying $\|A - A_{\mathrm{BEM}}\| \le \mathcal{O}(\eta^k)$. Then Remark 4.13a states that $A \in \mathcal{M}_{\mathbf{k}}(c\eta^k)$ for another constant c. Applying Theorem 4.17 with $\delta = c\eta^k$ and $R(A) = A^{-1}$, we conclude from (5.4a) that $A^{-1} \in \mathcal{M}_{\mathbf{k}}(C_S c\eta^k)$. We summarise:

Lemma 5.1 *Let (5.5) hold and choose any $A \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ with $\|A - A_{\mathrm{BEM}}\| \le c\eta^k$. Assume (5.4a) for $B := A$. Then there is a matrix $Z \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ with*

$$\left\| A^{-1} - Z \right\| \le C(I)\, c\eta^k \qquad \text{for some } Z \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W) \tag{5.6}$$

with $C(I) := C_S \left(1 + \sqrt{n_\mathcal{P}/2} \right)$ with $n_\mathcal{P}$ from (4.5).

Proof. $\|A - A_{\mathrm{BEM}}\| \le c\eta^k$ and (5.4a) imply $\|A^{-1} - A_{\mathrm{BEM}}^{-1}\| \le C_S c\eta^k$. Furthermore, by Theorem 4.16 there is some $Z \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ with $\|Z - A_{\mathrm{BEM}}^{-1}\| \le C_S c\eta^k \sqrt{n_\mathcal{P}/2}$. Together, we get (5.6). ∎

5.4 Approximation to A^{-1}

In this section, we study the error of the approximate matrix operation $Inv(A)$ for $A \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$.

Theorem 5.2 *For a given $A \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ with $\mathbf{k}$ from (4.4) define the family $\mathcal{A} = \{A|_{\tau \times \tau} : \tau \in T(I)\}$. Assume the stability property (5.4b) and the approximation property (5.6) for all $B \in \mathcal{A}$. Then, with η small enough, the approximate $\mathcal{H}$-matrix inverse $Inv(A) \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ obtained by the algorithm from §5.1 satisfies*

$$\left\| Inv(A) - A^{-1} \right\| \le 2C(I)\, c \left(1 + C_{\mathrm{inv}} + C_{\mathrm{inv}}^2 + ... + C_{\mathrm{inv}}^{L-\ell-1} \right) \eta^k, \tag{5.7}$$

with $C(I)$ from (5.6), C_{inv} from (5.4b) and ℓ such that $A \in \mathbb{R}^{\tau \times \tau}$ with $\tau \in T^\ell(I)$. The computational cost of $Inv(A)$ has the standard bound

$$\mathcal{N}_{\mathrm{Inv}} = \mathcal{O}(k_1^2 L^2 n).$$

Proof. The matrices of $\mathcal{A}$ belong to $\mathbb{R}^{\tau \times \tau}$ for some $\tau \in T^\ell(I)$, $\ell \in \{0, \dots, L\}$. We prove the assertion by induction over the level number $\ell = L$ down to 0. The induction hypothesis reads

$$\left\| Inv(B) - B^{-1} \right\| \le \zeta_\ell \qquad \text{for all } B \in \mathcal{A} \text{ associated with } \tau \in T^\ell(I), \text{ where}$$
$$\zeta_\ell = 2C(I)\, c\,\eta^k \left(1 + C_{\mathrm{inv}} + C_{\mathrm{inv}}^2 + ... + C_{\mathrm{inv}}^{L-\ell-1} \right). \tag{5.8}$$

Since for $\ell = L$ the inversion is performed exactly, the induction hypothesis is trivially valid: $\zeta_L = 0$.

Let the hypothesis be valid for $\ell + 1$ and consider $B \in \mathcal{A}$, $B \in \mathbb{R}^{\tau \times \tau}$, $\tau \in T^\ell(I)$. $S(\tau) = \{\tau_1, \tau_2\}$ induces the block structure $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. By induction

$$\left\| X_{ii} - B_{ii}^{-1} \right\| \le \zeta_{\ell+1} \quad \text{for } X_{ii} := Inv(B_{ii}),\ i = 1, 2,$$

holds. The approximation of the Schur complement $B_{22} - B_{21} B_{11}^{-1} B_{12}$ by Y_{22} from (5.1) produces the intermediate approximation $X = Inv^*(B)$ defined in (5.2). Now, condition (5.4b) can be applied with $\varepsilon = \zeta_{\ell+1}$ and yields

$$\left\| Inv^*(B) - B^{-1} \right\| \le C_{\mathrm{inv}} \zeta_{\ell+1}.$$

By the approximability assumption (5.6), there is $Z \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ with $\|Z - B^{-1}\| \le C(I)\, c\eta^k$. Let Π be the truncation from (5.3) onto $\mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$. Then $Inv(B) = \Pi Inv^*(B)$ and

$$\begin{aligned}
\left\| Inv(B) - B^{-1} \right\| &\le \|\Pi Inv^*(B) - Z\| + \|Z - B^{-1}\| = \|\Pi\left(Inv^*(B) - Z\right)\| + \|Z - B^{-1}\| \\
&\le \|Inv^*(B) - Z\| + \|Z - B^{-1}\| \le \|Inv^*(B) - B^{-1}\| + 2\|Z - B^{-1}\| \\
&\le 2C(I)\, c\eta^k + C_{\mathrm{inv}} \zeta_{\ell+1} = \zeta_\ell.
\end{aligned}$$

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This completes the induction step.

The complexity estimate can be derived similarly to [8]. ∎

The condition "η small enough" from Theorem 5.2 has three purposes. First, it implies that $\eta \leq \eta_0$ (see Corollary 4.15). Second, the perturbation sizes $\|E\|$ and ε in (5.4a,b) must be small enough. Third, it is needed to interpret (5.7) in a positive sense. Note that C_{inv} is not necessarily bounded by 1. Assuming $C_{\text{inv}} > 1$, the right-hand side in (5.7) behaves like $\mathcal{O}(C_{\text{inv}}^{L-\ell-1} \eta^k) \leq \mathcal{O}(C_{\text{inv}}^L \eta^k)$. Since $L = \log_2 n$ and $k = c' \log_2 n$ (see (3.3)), $\mathcal{O}(C_{\text{inv}}^L \eta^k) = \mathcal{O}((C_{\text{inv}} \eta^{c'})^L) = \mathcal{O}(n^{\log_2(C_{\text{inv}} \eta^{c'})})$ describes the behaviour as $n \to \infty$. Choosing η small enough, $C_{\text{inv}} \eta^{c'} < 1$ ensures convergence.

So far, we have not made use of our Conjecture 3.2 which now allows stronger implications.

Corollary 5.3 *Assume that Conjecture 3.2 applies. Then Theorem 5.2 holds when $\mathbf{k}$ from (4.4) is replaced by the constant choice $k_\ell = k$. In particular, $Inv(A) \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$ has the standard format.*

Proof. Now we can apply Remark 4.14a. ∎

6 Accuracy of the $\mathcal{H}$-Matrix Product

In this section we study the error of an approximate $\mathcal{H}$-matrix product within the $\mathcal{P}_W$-format. First we give results about the exact product, which are easy to derive by induction from

$$AB = \left[\begin{array}{cc} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array} \right], \tag{6.1}$$

where $A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right], B = \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right] \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(\tau \times \tau, \mathcal{P}_W^\tau)$ according to the block structure corresponding to the sons $S(\tau) = \{\tau_1, \tau_2\}$.

Remark 6.1 *(a) Let $A, B \in \mathcal{M}_{\mathcal{H},k}(I \times I, \mathcal{P}_W)$ (constant k). Then $AB \in \mathcal{M}_{\mathcal{H},\mathbf{k}_{AB}}(I \times I, \mathcal{P}_W)$ with $\mathbf{k}_{AB} = (k_\ell)_{\ell=1}^L$, $k_\ell = (\ell + 1) k$.*
(b) Let $A, B \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(I \times I, \mathcal{P}_W)$ with $\mathbf{k}$ from (4.4). Then $AB \in \mathcal{M}_{\mathcal{H},\mathbf{k}_{AB}}(I \times I, \mathcal{P}_W)$ with $\mathbf{k}_{AB} = (k_\ell)_{\ell=1}^L$, $k_\ell = \left\{ 1 + (\ell + 1)(L - \frac{1}{2}\ell) \right\} k$.

Proof. In the case of (b), set $k_\ell^L := \left\{ 1 + (\ell + 1)(L - \frac{1}{2}\ell) \right\} k$ $(1 \leq \ell \leq L)$ for matrices of size $2^L \times 2^L$. Assume by induction that $A_{11}B_{11} \in \mathcal{M}_{\mathcal{H},\mathbf{k}_{AB}^{L-1}}(I \times I, \mathcal{P}_W)$ with $\mathbf{k}_{AB}^{L-1} = (k_\ell^{L-1})_{\ell=1}^{L-1}$. Since $A_{12}B_{21}$ has rank $\leq L$, we get the conditions $k_\ell^L \leq k_{\ell-1}^{L-1} + L$ for $\ell \geq 2$. The off-diagonal blocks in (6.1) belong to level $\ell = 1$ and give the condition $k_1^L \leq 2L$. Since the bound k_ℓ^L defined above satisfies these inequalities, the assertion is proved. ∎

The maximal values of k_ℓ in Remark 6.1 are of size $\mathcal{O}(Lk)$ (part a) or $\mathcal{O}(L^2 k)$ (part b), i.e., the bounds for the local ranks increase by a factor $\mathcal{O}(L)$, whereas a factor 2 would be more natural. Below we show that an approximation of AB in a format with double rank is possible.

6.1 Multiplication Algorithm

Again, the approximate matrix product $A \odot B$ is defined recursively. On level $\ell = L$ the product is done exactly (since $\#\tau_i^L = 1$, i.e., level L corresponds to 1×1 matrices). Assume that the product is defined for all levels $> \ell$ and let $A, B \in \mathcal{M}_{\mathcal{H},\mathbf{k}}(\tau \times \tau, \mathcal{P}_W^\tau)$ with $\tau \in T^\ell(I)$ and $\mathbf{k}$ from (4.4). $S(\tau) = \{\tau_1, \tau_2\}$ leads to the block structure $A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]$ and $B = \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right]$. The submatrices A_{ii}, B_{ii} belong to level $\ell + 1$. Therefore, $X_{ii} := A_{ii} \odot B_{ii}$ $(i = 1, 2)$ are already defined. We define the intermediate result C by

$$C := \left[\begin{array}{cc} X_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + X_{22} \end{array} \right]. \tag{6.2}$$

Next we have to project the diagonal blocks of $A \odot B$ to the desired format $\mathcal{M}_{\mathcal{H},2\mathbf{k}}(\tau_i \times \tau_i, \mathcal{P}_W)$, $i = 1, 2$, resulting in

$$A \odot B := truncation \circ C \in \mathcal{M}_{\mathcal{H},2\mathbf{k}}(\tau \times \tau, \mathcal{P}_W). \tag{6.3}$$

Note that the off-diagonal blocks are already of the required rank. This completes the recursive definition of $A \odot B$ on level ℓ.

6.2 Stability and Approximability for the Matrix Product

The following stability assumptions must be valid not only for the original matrix A but also for its principal submatrices. Accordingly, we define the set $\mathcal{A} = \{A|_{\tau \times \tau}, B|_{\tau \times \tau} : \tau \in T(I)\}$.

Let

$$A, B \in \mathcal{A}, \|A' - A\|, \|B' - B\| \leq \varepsilon \quad \Rightarrow \quad \|A'B' - AB\| \leq C_0 \varepsilon \tag{6.4}$$

hold for $\varepsilon > 0$ small enough. The second assumption reads as follows. If $A, B \in \mathcal{A}$, and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{R}^{\tau \times \tau}$. Then for any $X_{11} \in \mathbb{R}^{\tau_1 \times \tau_1}$ and $X_{22} \in \mathbb{R}^{\tau_2 \times \tau_2}$ ($S(\tau) = \{\tau_1, \tau_2\}$) with

$$\|X_{11} - A_{11}B_{11}\| \leq \varepsilon, \quad \|X_{22} - A_{22}B_{22}\| \leq \varepsilon,$$

the approximation of AB by $X := A \odot B$ (cf. (6.2), (6.3)) satisfies

$$\|X - AB\| \leq C_M \varepsilon. \tag{6.5}$$

Next, we need an approximability result of AB in $\mathcal{M}_{\mathcal{H}, 2\mathbf{k}}(I \times I, \mathcal{P}_W)$. By Remark 4.14b 2D-BEM matrices satisfy $A_{\text{BEM}}, B_{\text{BEM}} \in \mathcal{M}_{\mathbf{k}}(c_{\text{BEM}} \eta^k)$ with $\mathbf{k}$ from (4.4) for some $0 < \eta \leq \eta_0$. Let A, B be approximations of order $\mathcal{O}(\eta^k)$. Then by Remark 4.13a, $A, B \in \mathcal{M}_{\mathbf{k}}(c \eta^k)$ holds for another constant c.

Although Theorem 4.17 covers only the product AA (for the function $R(x) = x^2$), an analogous result holds for AB. Therefore, $AB \in \mathcal{M}_{\mathbf{k}}(C_0 c \eta^k)$ with C_0 from (6.4). As in Lemma 5.1, we get the following approximability result with a similar quantity $C(I) = \mathcal{O}(\sqrt{\#I})$:

$$A, B \in \mathcal{M}_{\mathbf{k}}(c \eta^k) \quad \Rightarrow \quad \exists X \in \mathcal{M}_{\mathcal{H}, 2\mathbf{k}}(I \times I, \mathcal{P}_W) : \|AB - X\| \leq C(I) c \eta^k. \tag{6.6}$$

6.3 Approximation to AB

Theorem 6.2 *Assume that $A, B \in \mathcal{M}_{\mathbf{k}}(c \eta^k)$ with $k_\ell = (L - \ell + 1) k$ for some $\eta < 1$ (see Remark 4.14b). Suppose that the stability and approximability assumptions (6.5) and (6.6) are valid. Then the approximate $\mathcal{H}$-matrix product $A \odot B \in \mathcal{M}_{\mathcal{H}, 2\mathbf{k}}(I \times I, \mathcal{P}_W)$ defined in §6.1) satisfies*

$$\|AB - A \odot B\| \leq 2C(I) \left(1 + C_M + C_M^2 + ... + C_M^{L - \ell - 1}\right) c \eta^k,$$

with $C(I)$ defined via (5.6) and with C_M from (6.5).

The complexity of multiplication is bounded by $\mathcal{N}_{A \odot B} = \mathcal{O}(k_1^2 L^2 n) = \mathcal{O}(k^2 L^4 n)$.

Proof. Again, we prove the result by induction over the level number $\ell = L$ down to 0. The induction hypothesis is $\|A \odot B - AB\| \leq \zeta_\ell$ for all $A, B \in \mathcal{A}$ associated with $\tau \in T^\ell(I)$, where

$$\zeta_\ell = 2C(I) c \eta^k \left(1 + C_M + C_M^2 + ... + C_M^{L - \ell - 1}\right). \tag{6.7}$$

Since for $\ell = L$ the product $\odot$ is exact, the induction hypothesis is valid: $\zeta_L = 0$. Let the hypothesis be valid for $\ell + 1$ and consider $A, B \in \mathcal{A}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{R}^{\tau \times \tau}$, $\tau \in T^\ell(I)$. By induction hypothesis

$$\|A_{11} \odot B_{11} - A_{11}B_{11}\| \leq \zeta_{\ell+1} \quad \text{and} \quad \|A_{22} \odot B_{22} - A_{22}B_{22}\| \leq \zeta_{\ell+1}$$

hold, which produce the intermediate approximation C from (6.2). Now, condition (6.5) can be applied with $\varepsilon = \zeta_{\ell+1}$ and yields

$$\|C - AB\| \leq C_M \zeta_{\ell+1}.$$

Furthermore, due to the approximability assumption, there is $X \in \mathcal{M}_{\mathcal{H}, 2\mathbf{k}}(I \times I, \mathcal{P}_W)$ with $\|X - AB\| \leq C(I) c \eta^k$. Let Π be the truncation in (6.3). Then $A \odot B = \Pi C \in \mathcal{M}_{\mathcal{H}, 2\mathbf{k}}(I \times I, \mathcal{P}_W)$ and

$$\|A \odot B - AB\| \leq \|\Pi(C - X)\| + \|X - AB\| \leq \|C - X\| + \|X - AB\|$$

$$\leq \|C - AB\| + 2\|X - AB\| \leq 2C(I) c \eta^k + C_M \zeta_{\ell+1} = \zeta_\ell,$$

completing the induction step. Again, the complexity bound can be derived similarly to [8]. ■

Again we improve the result by using Conjecture 3.2.

Corollary 6.3 *Assume that Conjecture 3.2 applies. Then Theorem 6.2 holds when* **k** *from (4.4) is replaced by the constant choice* $k_\ell = k$.

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