# Max-Planck-Institut
## für Mathematik
## in den Naturwissenschaften
## Leipzig


**A Note on the Poincaré Inequality for Convex Domains**

by


*Mario Bebendorf*



Preprint no.: 24 2003

# A Note on the Poincaré Inequality for Convex Domains

Mario Bebendorf *

March 13, 2003

In this article a proof for the Poincaré inequality with explicit constant for convex domains is given. This proof is a modification of the original proof [5], which contains a mistake.

**AMS Subject Classification:** 26D10

## 1 Introduction

The classical proof for the Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq c_\Omega \|\nabla u\|_{L^2(\Omega)},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u \in H^1(\Omega)$ with vanishing mean value over $\Omega$, is based on the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ which is valid under quite general assumptions on $\Omega$ (cf. [6]). However, the constant $c_\Omega$ depends on the domain $\Omega$, and the proof based on compactness does not provide insight into this dependency.

For practical purposes it is important to know an explicit expression for this constant (see for example [2], [7]). Therefore, the special case of convex domains is interesting, since in [5] this constant is proved to be $d/\pi$, where $d$ is the diameter of $\Omega$. Though this proof is elegant, it contains a mistake. The same mistake can also be found in [1], in which the $L^1$ estimate is considered.

The goal of this article is to fix the error made in [5]. Luckily, the constant $d/\pi$ in the Poincaré inequality remains valid.

*Max Planck Institute for Mathematics in the Sciences, Inselstraße 22–26, 04103 Leipzig, Germany (bebendorf@mis.mpg.de)

## 2 The one-dimensional case

We first prove the Poincaré inequality for the one-dimensional case. In fact we will prove a generalization which the multidimensional case can be reduced to.

**Lemma 2.1** *Let $m \in \mathbb{N}$ and $\rho$ be a non-negative concave function on the interval $[0, L]$. Then for all $u \in H^1(0, L)$ satisfying*

$$\int_0^L \rho^m(x)u(x)\,\mathrm{d}x = 0 \tag{2.1}$$

*it holds that*

$$\int_0^L \rho^m(x)|u(x)|^2\,\mathrm{d}x \le \frac{L^2}{\pi^2}\int_0^L \rho^m(x)|u'(x)|^2\,\mathrm{d}x. \tag{2.2}$$

*Furthermore, the constant $L^2/\pi^2$ is optimal.*

*Proof.* (a) Let us first assume that $\rho$ is strictly positive and twice differentiable. Then each non-zero function $v$ minimizing the quotient

$$\frac{\int_0^L \rho^m(x)|u'(x)|^2\,\mathrm{d}x}{\int_0^L \rho^m(x)|u(x)|^2\,\mathrm{d}x} \tag{2.3}$$

and satisfying (2.1) must satisfy the Sturm-Liouville system (cf. [3])

$$[\rho^m v']' + \lambda\rho^m v = 0 \qquad \text{with } v'(0) = v'(L) = 0, \tag{2.4}$$

where $\lambda$ is the minimum of the quotient (2.3). After dividing (2.4) by $\rho^m$ and differentiating, we introduce the new variable $w = \rho^{m/2}v'$ and obtain

$$w'' + \frac{m}{2}\left[\frac{\rho''}{\rho} - \left(1 + \frac{m}{2}\right)\frac{(\rho')^2}{\rho^2}\right]w + \lambda w = 0 \qquad \text{with } w(0) = w(L) = 0.$$

Since $\rho$ is concave, $\rho'' \le 0$. Hence, $w'' + \lambda w \ge 0$ and integration by parts leads to

$$\lambda \ge \frac{\int_0^L |w'(x)|^2\,\mathrm{d}x}{\int_0^L |w(x)|^2\,\mathrm{d}x}.$$

The last quotient is bounded by the first eigenvalue of the vibrating string with fixed ends, which gives $\lambda \ge \pi^2/L^2$.

(b) If $\rho$ is a non-negative concave function, we may represent it as the $L^\infty$-limit of strictly positive concave $C^2$-functions $\rho_k$, cf. [4]. From Part (a) one has

$$\int_0^L \rho_k^m(x)|\hat{u}(x)|^2\,\mathrm{d}x \le \frac{L^2}{\pi^2}\int_0^L \rho_k^m(x)|u'(x)|^2\,\mathrm{d}x,$$

where $\hat{u}(x) := u(x) - \overline{u}$ and

$$\overline{u} := \frac{\int_0^L \rho_k^m(x) u(x)\,\mathrm{d}x}{\int_0^L \rho_k^m(x)\,\mathrm{d}x}.$$

Hence,

$$\int_0^L \rho_k^m(x)|u(x)|^2\,\mathrm{d}x \le \frac{L^2}{\pi^2}\int_0^L \rho_k^m(x)|u'(x)|^2\,\mathrm{d}x + \overline{u}\int_0^L \rho_k^m(x)u(x)\,\mathrm{d}x.$$

In the limit $k \to \infty$ we obtain (2.2).

(c) To see that the constant $L^2/\pi^2$ is optimal, choose $\rho^m \equiv 1$, $L = 1$ and $u(x) = \cos(\pi x)$. Then $\int_0^1 \rho^m(x)u(x)\,\mathrm{d}x = 0$ and

$$\frac{\int_0^1 \rho^m(x)|u(x)|^2\,\mathrm{d}x}{\int_0^1 \rho^m(x)|u'(x)|^2\,\mathrm{d}x} = \frac{1}{\pi^2}\frac{\int_0^1 \cos^2(\pi x)\,\mathrm{d}x}{\int_0^1 \sin^2(\pi x)\,\mathrm{d}x} = \frac{1}{\pi^2}.$$

■

# 3 The $n$-dimensional case

In the rest of this article we will consider the case $n \ge 2$. By the following lemma we are able to reduce the $n$-dimensional problem to the one-dimensional case.

**Lemma 3.1** *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter $d$. Assume that $u \in L^1(\Omega)$ satisfies $\int_\Omega u(x)\,\mathrm{d}x = 0$. Then for any $\delta > 0$ there are disjoint convex domains $\Omega_i$, $i = 1, \dots, k$, such that*

$$\overline{\Omega} = \bigcup_{i=1}^k \overline{\Omega}_i, \qquad \int_{\Omega_i} u(x)\,\mathrm{d}x = 0, \quad i = 1, \dots, k,$$

*and for each $\Omega_i$ there is rectangular coordinate system such that*

$$\Omega_i \subset \{(x,y) \in \mathbb{R}^n : 0 \le x \le d \text{ and } |y_j| \le \delta,\ j = 1, \dots, n-1\}.$$

*Proof.* For each $\alpha \in [0, 2\pi]$ there is a unique hyperplane $H_\alpha \subset \mathbb{R}^n$ with normal $(0, \dots, 0, \cos(\alpha), \sin(\alpha))$ that divides $\Omega$ into two convex sets $\Omega'_\alpha$ and $\Omega''_\alpha$ of equal volume. Since $I(\alpha) = -I(\alpha + \pi)$, where $I(\alpha) = \int_{\Omega'_\alpha} u(x)\,\mathrm{d}x$, by continuity there is $\alpha_0$ such that $I(\alpha_0) = 0$. Applying this procedure recursively to each of the parts $\Omega'_{\alpha_0}$ and $\Omega''_{\alpha_0}$, we are able to subdivide $\Omega$ into convex sets $\Omega_i$ such that each of the sets is contained between two parallel hyperplanes with normal of the form $(0, \dots, 0, \cos(\beta), \sin(\beta))$ at distance at most $\delta$, and the average of $u$ vanishes on each of them.

Consider one of these sets. By rotating the coordinate system we can assume that the normal of the enclosing hyperplanes is $(0, \ldots, 0, 1)$. In these coordinates we apply the above arguments using hyperplanes with normals of the form $(0, \ldots, 0, \cos(\alpha), \sin(\alpha), 0)$. Continuing this procedure we end up with the desired decomposition of $\Omega$. ■

**Theorem 3.2** *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter $d$. Then*

$$\|u\|_{L^2(\Omega)} \leq \frac{d}{\pi} \|\nabla u\|_{L^2(\Omega)}$$

*for all $u \in H^1(\Omega)$ satisfying $\int_\Omega u(x)\,\mathrm{d}x = 0$.*

*Proof.* Let us first assume that $u$ is twice continuously differentiable. According to the previous Lemma 3.1 we are able to decompose $\Omega$ into convex subsets $\Omega_i$ such that for each $\Omega_i$ there is a rectangular coordinate system in which $\Omega_i$ is contained in

$$\{(x, y) \in \mathbb{R}^n : 0 \leq x \leq d_i,\ |y_j| \leq \delta \text{ for } j = 1, \ldots, n-1\}.$$

We may assume that the interval $[0, d_i]$ on the $x$-axis is contained in $\Omega_i$. Let $R(t)$ be the $(n-1)$-volume of the intersection of $\Omega_i$ with the hyperplane $x = t$. In polar coordinates $R(t)$ can be written in the form

$$R(t) = \int_{\mathbb{S}^{n-2}} \int_0^{\rho(t,\omega)} r^{n-2}\,\mathrm{d}r\,\mathrm{d}\omega = \frac{1}{n-1} \int_{\mathbb{S}^{n-2}} \rho^{n-1}(t, \omega)\,\mathrm{d}\omega,$$

where $\rho(t, \omega)$ is the distance of the boundary point of $\Omega_i$ at $(t, \omega)$ to the $x$-axis. Since $\Omega_i$ is convex, $\rho$ is a concave function with respect to $t$.[1]

From the smoothness of $u$ it can be seen that there are constants $c_1$, $c_2$ and $c_3$ such that

$$\left| \int_{\Omega_i} u(x, y)\,\mathrm{d}x\,\mathrm{d}y - \int_0^{d_i} u(x, 0) R(x)\,\mathrm{d}x \right| \leq c_1 |\Omega_i| \delta \tag{3.1}$$

$$\left| \int_{\Omega_i} |\frac{\partial u}{\partial x}(x, y)|^2\,\mathrm{d}x\,\mathrm{d}y - \int_0^{d_i} |\frac{\partial u}{\partial x}(x, 0)|^2 R(x)\,\mathrm{d}x \right| \leq c_2 |\Omega_i| \delta \tag{3.2}$$

$$\left| \int_{\Omega_i} |u(x, y)|^2\,\mathrm{d}x\,\mathrm{d}y - \int_0^{d_i} |u(x, 0)|^2 R(x)\,\mathrm{d}x \right| \leq c_3 |\Omega_i| \delta \tag{3.3}$$

Let $\omega \in \mathbb{S}^{n-2}$. Since $u(\cdot, 0) \in H^1(0, d_i)$, we can apply Lemma 2.1 to $\hat{u}_\omega(x) := u(x, 0) - \overline{u}_\omega$, where

$$\overline{u}_\omega := \frac{\int_0^{d_i} u(x, 0) \rho^{n-1}(x, \omega)\,\mathrm{d}x}{\int_0^{d_i} \rho^{n-1}(x, \omega)\,\mathrm{d}x}.$$

[1]In [5] it is claimed that $R(t)$ is a concave function, which is not true for $n \geq 3$.

Hence,

$$\int_0^{d_i} |\hat{u}_\omega(x)|^2 \rho^{n-1}(x,\omega)\,\mathrm{d}x \le \frac{d_i^2}{\pi^2}\int_0^{d_i} |\frac{\partial u}{\partial x}(x,0)|^2 \rho^{n-1}(x,\omega)\,\mathrm{d}x.$$

Applying Fubini's theorem we obtain

$$\begin{aligned}
\frac{d_i^2}{\pi^2}\int_0^{d_i} |\frac{\partial u}{\partial x}(x,0)|^2 R(x)\,\mathrm{d}x &= \frac{1}{n-1}\int_{\mathbb{S}^{n-2}} \frac{d_i^2}{\pi^2}\int_0^{d_i} |\frac{\partial u}{\partial x}(x,0)|^2 \rho^{n-1}(x,\omega)\,\mathrm{d}x\,\mathrm{d}\omega \\
&\ge \frac{1}{n-1}\int_{\mathbb{S}^{n-2}} \int_0^{d_i} |\hat{u}_\omega(x)|^2 \rho^{n-1}(x,\omega)\,\mathrm{d}x\,\mathrm{d}\omega \\
&= \frac{1}{n-1}\int_{\mathbb{S}^{n-2}} \int_0^{d_i} \hat{u}_\omega(x) u(x,0) \rho^{n-1}(x,\omega)\,\mathrm{d}x\,\mathrm{d}\omega \\
&\ge \int_0^{d_i} |u(x,0)|^2 R(x)\,\mathrm{d}x - M\left|\int_0^{d_i} u(x,0)R(x)\,\mathrm{d}x\right|,
\end{aligned}$$

where $M := \max_{\omega\in\mathbb{S}^{n-2}} |\overline{u}_\omega|$. By $\int_{\Omega_i} u(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$, (3.1) and (3.2) we are lead to

$$\begin{aligned}
\int_0^{d_i} |u(x,0)|^2 R(x)\,\mathrm{d}x &\le \frac{d_i^2}{\pi^2}\int_0^{d_i} |\frac{\partial u}{\partial x}(x,0)|^2 R(x)\,\mathrm{d}x + c_1|\Omega_i| M\delta \\
&\le \frac{d_i^2}{\pi^2}\int_{\Omega_i} |\nabla u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y + \left(c_1 M + c_2\frac{d_i^2}{\pi^2}\right)|\Omega_i|\delta.
\end{aligned}$$

From (3.3) and the summation over $i$ we obtain

$$\int_\Omega |u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \le \frac{d^2}{\pi^2}\int_\Omega |\nabla u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y + (c_1 M + c_2\frac{d^2}{\pi^2} + c_3)|\Omega|\delta$$

and, since $\delta > 0$ is arbitrary, the desired estimate is proven. The assertion follows from the density of $C^\infty(\overline{\Omega})$ in $H^1(\Omega)$. ■

# Acknowledgement

Thanks go to C. Carstensen for pointing out the mistake in [5].

# References

[1] G. Acosta, and R. G. Durán: *An optimal Poincaré inequality in $L^1$ for convex domains.* Preprint, University of Buenos Aires, 2003.

[2] M. Bebendorf, and W. Hackbusch: *Existence of $\mathcal{H}$-Matrix Approximants to the Inverse FE-Matrix of Elliptic Operators with $L^\infty$-Coefficients.* to appear in Numer. Math., 2003.

[3] R. Courant, and D. Hilbert: *Methoden der mathematischen Physik. I*, Dritte Auflage, Springer, Berlin, 1968.

[4] H. G. Eggleston: *Convexity.* Cambridge Univ. Press, New York, 1958.

[5] L. E. Payne, and H. F. Weinberger: *An optimal Poincaré inequality for convex domains.* Arch. Rational Mech. Anal. **5**, 286–292, 1960.

[6] J. Wloka: *Partial differential equations*, Translated from the German by C. B. Thomas and M. J. Thomas, Cambridge Univ. Press, Cambridge, 1987.

[7] R. Verfürth: *A note on polynomial approximation in Sobolev spaces.* M2AN Math. Model. Numer. Anal. **33**, no. 4, 715–719, 1999.