Max-Planck-Institut
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Nirmalendu Chaudhuri

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Nirmalendu Chaudhuri

Max-Planck Institute for Mathematics in the Sciences
Inselstrasse 22-26, 04103 Leipzig, Germany
chaudhur@mis.mpg.de

ABSTRACT: In this note we show that the best constant C in the improved Hardy-Sobolev inequality of Adimurthi, Chaudhuri and Ramaswamy [1] for $2 \le p < n$, is bounded by $\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \le C \le \frac{p-1}{2}\left(\frac{n-p}{p}\right)^{p-2}$.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n, n \ge 2$ with $0 \in \Omega$. Adimurthi, Chaudhuri and Ramaswamy in [1] have obtained the following improved Hardy-Sobolev inequality. Let $1 < p < n$ and let $R \ge e^{2/p} \sup_\Omega |x|$, then there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla u|^p dx \ge \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p}\, dx + C\int_\Omega \frac{|u|^p}{|x|^p}\left(\log\frac{R}{|x|}\right)^{-2} dx \tag{1.1}$$

holds for all $u \in W_0^{1,p}(\Omega)$. In his book on *Sobolev Spaces* [14] (see section 2.1.6) Maz'ja discovered that the classical multidimensional Hardy type inequalities with sharp constant can be improved by adding different additional positive integrals. However the above inequality have applications in proving existence, nonexistence and regularity of solutions for differential equations involving the potential $\frac{1}{|x|^p}$, see [1, 3, 10, 11, 12, 15]. Adimurthi and Esteban [2] extended the above inequality for $W^{1,p}$ functions and found interesting applications to Schrödinger operator. However, finding the *best constant* in the inequality (1.1) remains open. In this article we find an interesting bounds for the best constant $C(n, p, R, \Omega)$, defined in (1.4). In [Theorem 1.2, 1], it has been shown that for $0 < \mu < \left(\frac{n-p}{p}\right)^p$, the eigenvalue problem

$$\begin{cases} -\left(\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \dfrac{\mu}{|x|^p}|u|^{p-2}u\right) = \lambda \dfrac{|u|^{p-2}}{|x|^p\left(\log\frac{R}{|x|}\right)^2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

admits a positive weak solution $u \in W_0^{1,p}(\Omega)$ corresponding to the eigenvalue $\lambda = \lambda_\mu^1 > 0$. Moreover, $\lambda_\mu^1 \to C(n,p,R,\Omega)$, as $\mu \to \left(\frac{n-p}{p}\right)^p$. Thus the bounds on the best constant in the inequality (1.1) gives bounds on the limiting behaviour of the first eigenvalue for the eigenvalue problem (1.2). In [1], the following n-dimensional version of the Hardy-Sobolev inequality is also been established. For any bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with $0 \in \Omega$,

$$\int_\Omega |\nabla u|^n dx \geq \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u|^n}{|x|^n}\left(\log \frac{R}{|x|}\right)^{-n} dx \tag{1.3}$$

holds for every $u \in W_0^{1,n}(\Omega)$. Adimurthi and Sandeep [3] proved that the best constant is indeed $\left(\frac{n-1}{n}\right)^n$. For some interesting improvements of the classical Hardy-Sobolev inequality and their applications see [5, 6, 7, 8, 9, 13].

Before stating our theorem we define the *best constant* $C(n,p,R,\Omega)$ in the inequality (1.1) by

$$C(n,p,R,\Omega) := \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} Q_{\Omega,R}(u), \tag{1.4}$$

where ,

$$Q_{\Omega,R}(u) := \frac{\int_\Omega |\nabla u|^p \, dx - \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p}\, dx}{\int_\Omega \frac{|u(x)|^p}{|x|^p}\left(\log \frac{R}{|x|}\right)^{-2} dx}. \tag{1.5}$$

It is also known (see [1]) that the best constant in $C(n,p,R,\Omega)$ is not achieved. In this article we prove the following theorem.

Theorem 1.1 *The constant $C(n,p,R,\Omega)$ defined by (1.4) is independent of the domain Ω and the choice of R. For $2 \leq p < n$*

$$\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \leq C(n,p) \leq \frac{p-1}{2}\left(\frac{n-p}{p}\right)^{p-2}. \tag{1.6}$$

It appears to me that for the case $2 \leq p < n$, the constant $C(n,p)$ is indeed $\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2}$.

2 Proof of Theorem 1.1

PROOF : We prove the independence and the bounds for the best constant through the following steps.

Step 1. We first prove that if B_1 and B_2 are concentric balls centered at origin of radii T_1 and T_2 respectively then $C(n,p,R_1,B_1) = C(n,p,R_2,B_2)$, where $R_i = \alpha T_i$ with $\alpha \geq e^{2/p}$,

$i = 1, 2$. Take $u \in W_0^{1,p}(B_2)$ and define, $v(x) = u\left(\frac{T_2}{T_1}x\right)$ for $|x| < T_1$. Then

$$\begin{aligned} Q_{B_1,R_1}(v) &= \frac{\int_{B_1} |\nabla v|^p dx - \left(\frac{n-p}{p}\right)^p \int_{B_1} \frac{|v|^p}{|x|^p}\, dx}{\int_{B_1} \frac{|v|^p}{|x|^p} \left(\log \frac{\alpha T_1}{|x|}\right)^{-2} dx} \\ &= \frac{\int_{B_2} |\nabla u|^p dx - \left(\frac{n-p}{p}\right)^p \int_{B_2} \frac{|u|^p}{|x|^p}\, dx}{\int_{B_2} \frac{|u|^p}{|x|^p} \left(\log \frac{\alpha T_2}{|x|}\right)^{-2} dx} \\ &= Q_{B_2,R_2}(u), \end{aligned} \tag{2.7}$$

and hence $C(n, p, R_1, B_1) = C(n, p, R_2, B_2)$.

Step 2. Now we prove that $C(n, p, R, \Omega) = C(n, p, R, \Omega^*)$, where $\Omega^* = B(0, T)$ is the ball of radius $T = \left(\frac{|\Omega|}{|B(0,1)|}\right)^{1/n}$, $|.|_n$ denotes the n-dimensional Lebesgue measure. Take Ω^* as above, then for any $u \in W_0^{1,p}(\Omega)$, $|u|^* \in W_0^{1,p}(\Omega^*)$, where $|u|^*$ be the symmetric decreasing rearrangement of the function $|u|$. By the standard symmetrization arguments, see [4] we conclude that for any $u \in W_0^{1,p}(\Omega)$, $Q_{\Omega,R}(u) \geq Q_{\Omega^*,R}(u^*)$ and hence $C(n, p, R, \Omega) \geq C(n, p, R, \Omega^*)$. To prove the other inequality, take $s > 0$ such that the ball $B_s = B(0, s) \subseteq \Omega$. Then clearly, $C(n, p, R, \Omega) \leq C(n, p, R, B_s)$ and hence by step 1, $C(n, p, R, \Omega) = C(n, p, R, \Omega^*)$.

Now if Ω_1 and Ω_2 are two bounded domains with $R_i \geq e^{2/p} \sup_{\Omega_i} |x|$, by step 1 and step 2, $C(n, p, R_1, \Omega_1) = C(n, p, R_2, \Omega_2)$ and hence the constant is independent of the domain and the choice of R. We shall denote this constant simply by $C(n, p)$.

Step 3. Lower Bound: The lower bound for the best constant $C(n, p)$ essentially follows from the proof of Theorem 1.1 in [1]. But for the sake of completeness we include a proof. Since $C(n, p)$ is independent of the domain, without loss of generality we assume Ω to be the unit ball $B := B(0, 1)$. Let $R \geq e^{2/p}$. For $u \in C_0^2(B)$, $u > 0$, radially nonincreasing, we define

$$v(r) := u(r)\, r^{(n-p)/p}\ , \quad r = |x|. \tag{2.8}$$

Here without loss of generality we as well assume $u'(r) < 0$, (replacing u by $u + \epsilon(1 - r)$ for $\epsilon > 0$, sufficiently small). Now we observe that

$$\begin{aligned} \int_B |\nabla u|^p dx - \left(\frac{n-p}{p}\right)^p \int_B \frac{|u(x)|^p}{|x|^p} dx &= \omega_n \int_0^1 \left| \frac{n-p}{p} r^{-n/p} v(r) - r^{1-n/p} v'(r) \right|^p r^{n-1} dr \\ &\quad - \left(\frac{n-p}{p}\right)^p \omega_n \int_0^1 \frac{v^p(r)}{r}\, dr \\ &= \omega_n \left(\frac{n-p}{p}\right)^p \int_0^1 v^p(r) \left\{ \left| 1 - \frac{p v'(r)\, r}{(n-p) v(r)} \right|^p - 1 \right\} \frac{dr}{r}, \end{aligned}$$

where ω_n be volume of the (n-1)-dimensional sphere. Since u is a decreasing function, we have from (2.8), $v'(r) - \dfrac{(n-p)v(r)}{p\,r} < 0$ and we call $x(r) := -\dfrac{pv'(r)\,r}{(n-p)v(r)}$ so that, $x(r) > -1$. By using $(1+x)^p \geq 1 + px + (p-1)x^2$, for all $x \geq -1$ and for all $p \geq 2$, we obtain.

$$\begin{aligned}
\int_B |\nabla u|^p - \left(\frac{n-p}{p}\right)^p \int_B \frac{|u(x)|^p}{|x|^p} &\geq \omega_n (p-1)\left(\frac{n-p}{p}\right)^{p-2} \int_0^1 v^{p-2}(r)|v'(r)|^2 r\,dr \\
&\quad - \omega_n p \left(\frac{n-p}{p}\right)^{p-1} \int_0^1 v^{p-1}(r) v'(r) dr \\
&= \frac{4\omega_n (p-1)}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \int_0^1 \left|\left(v^{p/2}(r)\right)'\right|^2 r\,dr\,, \quad (2.9)
\end{aligned}$$

since $v \in C_0^1(0,T)$. By applying the n-dimensional Hardy inequality (1.3) with $n = 2$ for the function $v^{p/2}$, we obtain

$$\begin{aligned}
\int_0^1 \left|\left(v^{p/2}(r)\right)'\right|^2 r\,dr &\geq \frac{1}{4}\int_0^1 \left(\frac{v^{p/2}(r)}{r\log R/r}\right)^2 r\,dr \\
&= \frac{1}{4}\int_0^1 \frac{u^p(r)}{r^p} \left(\log R/r\right)^{-2} r^{n-1} dr \\
&= \frac{1}{4\omega_n}\int_B \frac{|u(x)|^p}{|x|^p}\left(\log R/|x|\right)^{-2} dx\,. \quad (2.10)
\end{aligned}$$

Hence for all $u \in C_0^2(B)$, $u > 0$, radially nonincreasing functions, we have

$$\int_B |\nabla u|^p - \left(\frac{n-p}{p}\right)^p \int_B \frac{|u(x)|^p}{|x|^p} \geq \frac{(p-1)}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \int_B \frac{|u(x)|^p}{|x|^p}\left(\log R/|x|\right)^{-2} dx\,. \quad (2.11)$$

Now by standard approximation and symmetrization the inequality (2.11) holds for all $u \in W_0^{1,p}(B)$ and hence $C(n,p) \geq \dfrac{(p-1)}{p^2}\left(\dfrac{n-p}{p}\right)^{p-2}$.

Step 3. Upper Bound: Here our idea is to construct a family of functions in $W_0^{1,p}(B)$, where $B := B(0,1)$ is the unit ball and then estimate $Q_{B,R}$ for such a family. We take the following family of functions similar to the one found in [1], in the following manner. For any $0 < \epsilon < 1$ and for $k \geq 2$, an integer we define

$$u_{\epsilon,k}(r) := \begin{cases} 0, & \text{for } r \leq \epsilon^k\,, \\ \dfrac{\log r/\epsilon^k}{(k-1) r^{(n-p)/p}\log 1/\epsilon}, & \text{for } \epsilon^k \leq r \leq \epsilon\,, \\ \dfrac{\log 1/r}{r^{(n-p)/p}\log 1/\epsilon}, & \text{for } \epsilon \leq r \leq 1\,. \end{cases} \quad (2.12)$$

Clearly, $u_{\epsilon,k} \in W_0^{1,p}(B)$ is continuous and differentiable a.e. and its derivative is given by

$$u'_{\epsilon,k}(r) = \begin{cases} 0, & \text{for } 0 \le r \le \epsilon^k\,, \\ \dfrac{1}{(k-1)r^{n/p}\log 1/\epsilon}\left[1 - \dfrac{n-p}{p}\log r/\epsilon^k\right], & \text{for } \epsilon^k \le r \le \epsilon\,, \\ -\dfrac{1}{r^{n/p}\log 1/\epsilon}\left[1 + \dfrac{n-p}{p}\log 1/r\right], & \text{for } \epsilon \le r \le 1\,. \end{cases}$$

Since $\epsilon > 0$ is sufficiently small, we have the following estimates, after a change of variables and the use of Neumann series:

$$\begin{aligned}
\int_B |\nabla u_{\epsilon,k}|^p dx &= \frac{\omega_n}{(\log 1/\epsilon)^p}\left[\frac{1}{(k-1)^p}\int_{\epsilon^k}^{\epsilon}\left|\frac{n-p}{p}\log\frac{r}{\epsilon^k} - 1\right|^p \frac{dr}{r} + \int_\epsilon^1 \left|1 + \frac{n-p}{p}\log\frac{1}{r}\right|^p \frac{dr}{r}\right] \\
&= \frac{\lambda_{n,p}\omega_n}{(p+1)}(\log 1/\epsilon)\left[(k-1)\left(1 - \frac{p}{(k-1)(n-p)\log 1/\epsilon}\right)^{p+1}\right. \\
&\quad \left. + \left(1 + \frac{p}{(n-p)\log 1/\epsilon}\right)^{p+1}\right] \\
&= \frac{\lambda_{n,p}\omega_n}{(p+1)}(\log 1/\epsilon)\left[(k-1) - \frac{p(p+1)}{(n-p)\log 1/\epsilon}\right. \\
&\quad + \frac{p(p+1)}{2(k-1)}\left(\frac{p}{(n-p)\log 1/\epsilon}\right)^2 + O\left(\frac{1}{(k-1)^2(\log 1/\epsilon)^3}\right) \\
&\quad \left. + 1 + \frac{p(p+1)}{(n-p)\log 1/\epsilon} + \frac{p(p+1)}{2}\left(\frac{p}{(n-p)\log 1/\epsilon}\right)^2 + O\left(\frac{1}{\log 1/\epsilon}\right)^3\right] \\
&= \frac{k\,\lambda_{n,p}\omega_n}{(p+1)}\log 1/\epsilon + \frac{k\,p\,\omega_n}{2(k-1)}\left(\frac{n-p}{p}\right)^{p-2}(\log 1/\epsilon)^{-1} \\
&\quad + O\left(\frac{1}{(k-1)\log 1/\epsilon}\right)^2 + O\left(\frac{1}{\log 1/\epsilon}\right)^2. \qquad (2.13)
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_B \frac{|u_{\epsilon,k}|^p}{|x|^p}dx &= \frac{\omega_n}{(\log 1/\epsilon)^p}\left[\frac{1}{(k-1)^p}\int_{\epsilon^k}^{\epsilon}\left(\log r/\epsilon^k\right)^p\frac{dr}{r} + \int_\epsilon^1 (\log 1/r)^p\frac{dr}{r}\right] \\
&= \frac{\omega_n}{(p+1)(\log 1/\epsilon)^p}\left[\frac{1}{(k-1)^p}\int_{\epsilon^k}^{\epsilon}\frac{d}{dr}\left(\log r/\epsilon^k\right)^{p+1}dr - \int_\epsilon^1 \frac{d}{dr}(\log 1/r)^{p+1}dr\right] \\
&= \frac{k\,\omega_n}{(p+1)}(\log 1/\epsilon)\,. \qquad (2.14)
\end{aligned}$$

Thus (2.13) and (2.14) yields

$$\int_B |\nabla u_{\epsilon,k}|^p - \left(\frac{n-p}{p}\right)^p\int_B \frac{|u_{\epsilon,k}|^p}{|x|^p} = \frac{kp\omega_n}{2(k-1)}\left(\frac{n-p}{p}\right)^{p-2}(\log 1/\epsilon)^{-1} + O\left(\frac{1}{\log 1/\epsilon}\right)^2 \qquad (2.15)$$

Finally, let us try find a"good" estimate of the following integral

$$\begin{aligned} I_p : &= \int_B \frac{|u_{\epsilon,k}|^p}{|x|^p} \left(\log R/|x|\right)^{-2} dx \\ &= \frac{\omega_n}{(\log 1/\epsilon)^p} \Big[\frac{1}{(k-1)^p} \int_{\epsilon^k}^{\epsilon} \frac{\left(\log r/\epsilon^k\right)^p}{r\,\left(\log R/r\right)^2} dr + \int_\epsilon^1 \frac{(\log 1/r)^p}{r\,\left(\log R/r\right)^2} dr \Big]. \end{aligned}$$

Now by change of variable, $r \mapsto \log R/r$ and denoting, $a_\epsilon := \log R/\epsilon$, $b_\epsilon := \log R/\epsilon^k$ and $c := \log R$, we have

$$I_p = \frac{\omega_n}{((k-1)\log 1/\epsilon)^p} \int_{a_\epsilon}^{b_\epsilon} \frac{\left(\log\, R\, e^{-r}/\epsilon^k\right)^p}{r^2} dr \;+\; \frac{\omega_n}{(\log 1/\epsilon)^p} \int_c^{a_\epsilon} \frac{(\log\, e^r/R)^p}{r^2} dr\,. \tag{2.16}$$

We now call the first and second integral of (2.16) by I_p^1 and I_p^2 respectively and we do the following estimations:

$$\begin{aligned} I_p^1 &= \int_{a_\epsilon}^{b_\epsilon} \left(\log\, R/\epsilon^k - r\right)^p \frac{dr}{r^2} \\ &= b_\epsilon^p \int_{a_\epsilon}^{b_\epsilon} \left(1 - \frac{r}{b_\epsilon}\right)^p \frac{dr}{r^2} \\ &\geq b_\epsilon^p \int_{a_\epsilon}^{b_\epsilon} \left(1 - \frac{pr}{b_\epsilon} + \frac{(p-1)r^2}{b_\epsilon^2}\right) \frac{dr}{r^2} \\ &= \frac{b_\epsilon^p}{a_\epsilon} \left[\left(1 - \frac{a_\epsilon}{b_\epsilon}\right) \left(1 + (p-1)\frac{a_\epsilon}{b_\epsilon}\right) - \frac{p\, a_\epsilon}{b_\epsilon} \log \frac{b_\epsilon}{a_\epsilon} \right], \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} I_p^2 &= \int_c^{a_\epsilon} (r - \log R)^p \frac{dr}{r^2} \\ &= \int_c^{a_\epsilon} r^{p-2} \left(1 - \frac{c}{r}\right)^p dr \\ &\geq \int_c^{a_\epsilon} r^{p-2} \left(1 - \frac{pc}{r} + \frac{(p-1)c^2}{r^2}\right) dr \\ &= \begin{cases} a_\epsilon \Big[\left(1 - \dfrac{c}{a_\epsilon}\right) - 2\dfrac{c}{a_\epsilon} \log \dfrac{a_\epsilon}{c} + o(1) \Big], & \text{for } p = 2\,, \\ a_\epsilon^2 \left[\dfrac{1}{2} \left(1 - (\dfrac{c}{a_\epsilon})^2\right) + 2 \left(\dfrac{c}{a_\epsilon}\right)^2 \log \dfrac{a_\epsilon}{c} + o(1) \right], & \text{for } p = 3\,, \\ a_\epsilon^{p-1} \left[\dfrac{1}{p-1} \left(1 - (\dfrac{c}{a_\epsilon})^{p-1}\right) + o(1) \right] & \text{for } p \neq 2,\ p \neq 3 \end{cases} \\ &= a_\epsilon^{p-1} \left[\frac{1}{p-1} + o(1) \right], \end{aligned} \tag{2.18}$$

where $o(1) \to 0$ as $\epsilon \to 0$. Thus from (2.17) and (2.18) we obtain

$$\begin{aligned} I_p \geq J_{k,\epsilon} : \;&=\; \frac{\omega_n}{((k-1)\log 1/\epsilon)^p} \frac{b_\epsilon^p}{a_\epsilon} \left[\left(1 - \frac{a_\epsilon}{b_\epsilon}\right) \left(1 + (p-1)\frac{a_\epsilon}{b_\epsilon}\right) - \frac{p\, a_\epsilon}{b_\epsilon} \log \frac{b_\epsilon}{a_\epsilon} \right] \\ &\quad + \frac{\omega_n}{(\log 1/\epsilon)^p} a_\epsilon^{p-1} \left[\frac{1}{p-1} + o(1) \right], \end{aligned}$$

and hence from (2.15), we obtain

$$\begin{aligned} Q_{B,R}(u_{\epsilon,k}) \;&\leq\; \frac{p\,k}{2(k-1)} \left(\frac{n-p}{p}\right)^{p-2} (\log 1/\epsilon)^{p-1} \\ &\quad \times \left[\frac{b_\epsilon^p}{(k-1)^p a_\epsilon} \left\{ \left(1 - \frac{a_\epsilon}{b_\epsilon}\right) \left(1 + (p-1)\frac{a_\epsilon}{b_\epsilon}\right) \right\} + a_\epsilon^{p-1} \left\{ \frac{1}{p-1} + o(1) \right\} \right]^{-1} \\ &\quad + J_{k,\epsilon}^{-1} \left[O\left(\frac{1}{\log 1/\epsilon}\right)^2 \right] \\ &= \frac{p\,k}{2(k-1)} \left(\frac{n-p}{p}\right)^{p-2} \\ &\quad \times \left[\frac{(k-1)^{-p} b_\epsilon^p}{a_\epsilon \left(\log \frac{1}{\epsilon}\right)^{p-1}} \left\{ \left(1 - \frac{a_\epsilon}{b_\epsilon}\right) \left(1 + (1-p)\frac{a_\epsilon}{b_\epsilon}\right) \right\} + \left(\frac{a_\epsilon}{\log \frac{1}{\epsilon}}\right)^{p-1} \left\{ \frac{1}{p-1} + o(1) \right\} \right]^{-1} \\ &\quad + J_{k,\epsilon}^{-1} \left[O\left(\frac{1}{\log 1/\epsilon}\right)^2 \right]. \end{aligned} \tag{2.19}$$

Here we note that $b_\epsilon^p / a_\epsilon (\log 1/\epsilon)^{p-1} \to k^p$ as $\epsilon \to 0$ and hence $J_{k,\epsilon}^{-1} \left[O\left(\frac{1}{\log 1/\epsilon}\right)^2 \right] \to 0$ as either $\epsilon \to 0$ or $k \to \infty$. Thus we have

$$\begin{aligned} Q_{B,R}(u_{\epsilon,k}) \;&\to\; \frac{p\,k}{2(k-1)} \left(\frac{n-p}{p}\right)^{p-2} \\ &\quad \times \left[\left(\frac{k}{k-1}\right)^p \left\{ \left(1 - \frac{1}{k}\right) \left(1 + \frac{p-1}{k}\right) + \frac{p}{k} \log \frac{1}{k} \right\} + \frac{1}{p-1} \right]^{-1}, \quad \text{as } \epsilon \to 0 \\ &\to \frac{p}{2} \left(\frac{n-p}{p}\right)^{p-2} \left[1 + \frac{1}{p-1} \right]^{-1}, \quad \text{as } k \to \infty \\ &= \frac{p-1}{2} \left(\frac{n-p}{p}\right)^{p-2}. \end{aligned}$$

Since $C(n,p) \leq Q_{B,R}(u_{\epsilon,k})$, for all $k \geq 2$ and for any sufficiently small $\epsilon > 0$, we have by passing through the limits as $\epsilon \to 0$ and $k \to \infty$,

$$C(n,p) \leq \frac{p-1}{2} \left(\frac{n-p}{p}\right)^{p-2}$$

and hence the theorem. ∎

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