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The min–max construction of minimal surfaces

by

Tobias H. Colding and Camillo De Lellis

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THE MIN–MAX CONSTRUCTION OF MINIMAL SURFACES

TOBIAS H. COLDING AND CAMILLO DE LELLIS

Dedicated to Eugenio Calabi on occasion of his eightieth birthday

0. Introduction

In this paper we survey with complete proofs some well–known, but hard to find, results about constructing closed embedded minimal surfaces in a closed 3-dimensional manifold via min–max arguments. This includes results of J. Pitts, F. Smith, and L. Simon and F. Smith.

The basic idea of constructing minimal surfaces via min–max arguments and sweep-outs goes back to Birkhoff, who used such a method to find simple closed geodesics on spheres. In particular when M^2 is the 2-dimensional sphere we can find a 1–parameter family of curves starting and ending at a point curve in such a way that the induced map $F : \mathbf{S}^2 \to \mathbf{S}^2$ (see Fig. 1) has nonzero degree. Birkhoff's argument (or the min-max argument) allows us to conclude that M has a nontrivial closed geodesic of length less than or equal to the length of the longest curve in the 1-parameter family. A curve shortening argument gave that the geodesic obtained in this way is simple.

The difficulty in generalizing this method to get embedded minimal surfaces in 3–manifolds is three fold. The first problem is getting regularity of the min–max surface obtained. In Birkhoff's case (curves in surfaces) this was almost immediate. The second key difficulty is to show that the min–max surface is embedded. Using the technical tools of Geometric Measure Theory (mostly the theory of varifolds), these two problems are tackled at the same time. The third key difficulty is to get a good genus bound for the embedded minimal surface obtained.

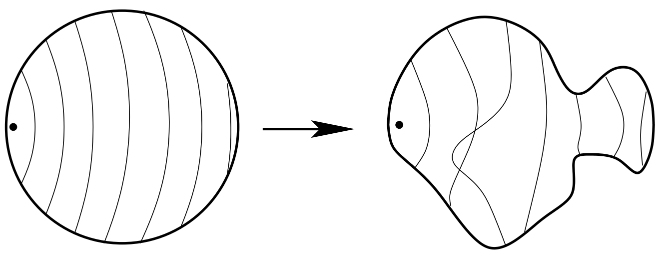

Figure 1. A 1–parameter family of curves on a 2–sphere which induces a map $F : \mathbf{S}^2 \to \mathbf{S}^2$ of degree 1.

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0.1. **The min–max construction in 3–manifolds.** In the following M denotes a closed 3–dimensional Riemannian manifold, Diff_0 is the identity component of the diffeomorphism group of M, and $\mathfrak{Is}$ is the set of smooth isotopies. Thus $\mathfrak{Is}$ is the set of maps $\psi \in C^\infty([0,1]\times M, M)$ such that $\psi(0,\cdot)$ is the identity and $\psi(t,\cdot)\in \mathrm{Diff}_0$ for every t.

We will first give a version of 1–parameter families of surfaces in 3-manifolds. The most direct way of doing this is to let $F:[0,1]\times\Sigma\to M$ be a smooth map such that $F(t,\cdot)$ is an embedding of the surface Σ for every $t\in[0,1]$. If we let $\Sigma_t = F(\{t\}\times\Sigma)$, then $\{\Sigma_t\}_{t\in[0,1]}$ is a smooth 1–parameter family of surfaces in M. This notion can be generalized in two directions. The first one is to relax the regularity required in the t–variable:

Definition 0.1. A family $\{\Sigma_t\}_{t\in[0,1]}$ of surfaces of M is said to be *continuous* if
(c1) $\mathcal{H}^2(\Sigma_t)$ is a continuous function of t;
(c2) $\Sigma_t\to\Sigma_{t_0}$ in the Hausdorff topology whenever $t\to t_0$.

A second generalization allows the family of surfaces to degenerate in finitely many points:

Definition 0.2. A family $\{\Sigma_t\}_{t\in[0,1]}$ of subsets of M is said to be a *generalized family* of surfaces if there are a finite subset T of $[0,1]$ and a finite set of points P in M such that

1. (c1) and (c2) hold;
2. Σ_t is a surface for every $t\notin T$;
3. For $t\in T$, Σ_t is a surface in $M\setminus P$.

Figure 1 gives (in one dimensions less) an example of a generalized 1–parameter family with $T=\{0,1\}$. To avoid confusion, families of surfaces will be denoted by $\{\Sigma_t\}$. Thus, when referring to a surface a subscript will denote a real parameter, whereas a superscript will denote an integer as in a sequence.

Given a generalized family $\{\Sigma_t\}$ we can generate new generalized families via the following procedure. Take an arbitrary map $\psi\in C^\infty([0,1]\times M,M)$ such that $\psi(t,\cdot)\in\mathrm{Diff}_0$ for each t and define $\{\Sigma'_t\}$ by $\Sigma'_t=\psi(t,\Sigma_t)$. We will say that a set Λ of generalized families is *saturated* if it is closed under this operation.

Remark 0.3. For technical reasons we will require that any of the saturated sets Λ that we consider has the additional property that there exists some $N=N(\Lambda)<\infty$ such that for any $\{\Sigma_t\}\subset\Lambda$, the set P in Definition 0.2 consists of at most N points. This additional property will play a crucial role in the proof of Theorem 0.5.

Given a family $\{\Sigma_t\}\subset\Lambda$ we denote by $\mathcal{F}(\{\Sigma_t\})$ the area of its maximal slice and by $m_0(\Lambda)$ the infimum of $\mathcal{F}$ taken over all families of Λ; that is,

$$\mathcal{F}(\{\Sigma_t\}) = \max_{t\in[0,1]}\mathcal{H}^2(\Sigma_t) \qquad \text{and} \qquad m_0(\Lambda)=\inf_{\Lambda}\mathcal{F} = \inf_{\{\Sigma_t\}\in\Lambda}\left[\max_{t\in[0,1]}\mathcal{H}^2(\Sigma_t)\right]. \tag{0.1}$$

If $\lim_n \mathcal{F}(\{\Sigma_t\}^n)=m_0(\Lambda)$, then we say that the sequence of generalized families of surfaces $\{\{\Sigma_t\}^n\}\subset\Lambda$ is a *minimizing sequence*. Assume $\{\{\Sigma_t\}^n\}$ is a minimizing sequence and let $\{t_n\}$ be a sequence of parameters. If the areas of the slices $\{\Sigma^n_{t_n}\}$ converge to m_0, i.e. if $\mathcal{H}^2(\Sigma^n_{t_n})\to m_0(\Lambda)$, then we say that $\{\Sigma^n_{t_n}\}$ is a *min–max sequence*.

An important point in the min–max construction is to find a saturated set Λ of generalized families of surfaces with $m_0(\Lambda)>0$. This can for instance be done by using the following elementary proposition proven in Appendix A; see Fig. 3:

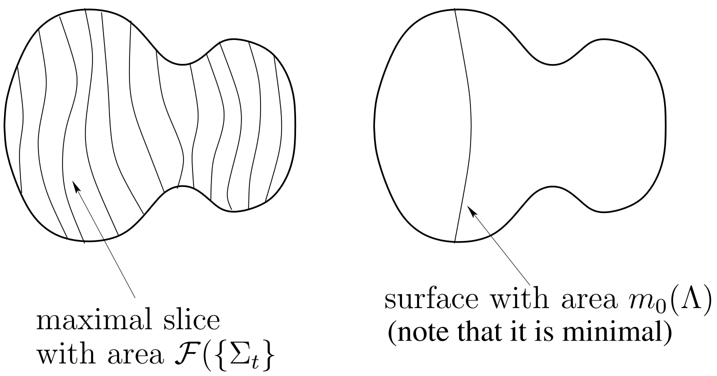

FIGURE 2. $\mathcal{F}(\{\Sigma_t\})$ and $m_0(\Lambda)$.

Proposition 0.4. Let M be a closed 3-manifold with a Riemannian metric and let $\{\Sigma_t\}$ be the level sets of a Morse function. The smallest saturated set Λ containing the family $\{\Sigma_t\}$ has $m_0(\Lambda) > 0$.

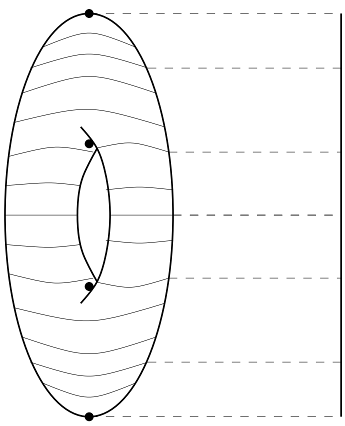

FIGURE 3. A sweep–out of the torus by level sets of a Morse function. In this case there are four degenerate slices in the 1–parameter family.

The following example of sweep–outs of the 3–sphere is a direct generalization of the families of curves on the 2–sphere considered by Birkhoff:

Example 0.2. Let x_4 be the coordinate function on the 3–sphere coming from its standard embedding into $\mathbf{R}^4$. By Proposition 0.4, for any fixed metric on $\mathbf{S}^3$ the level sets of x_4 generate a saturated set of generalized families of surfaces with $m_0 > 0$.

In this survey we will prove the following theorem:

Theorem 0.5. [Simon–Smith] Let M be a closed 3-manifold with a Riemannian metric. For any saturated set of generalized families of surfaces Λ, there is a min–max sequence obtained from Λ and converging in the sense of varifolds to a smooth embedded minimal surface with area $m_0(\Lambda)$ (multiplicity is allowed).

An easy corollary of Proposition 0.4 and Theorem 0.5 is the existence of a smooth embedded minimal surface in any closed Riemannian 3–manifold (Pitts proved that in any closed Riemannian manifold of dimension at most 7 there is a closed embedded minimal hypersurface; see theorem A and the final remark of the introduction of [P]).

For Λ as in Example 0.2 (where M is topologically a 3-sphere but could have an arbitrary metric and the sweep–outs are by 2-spheres) Simon and Smith proved that the min–max sequence given by Theorem 0.5 converges to a disjoint union of embedded minimal 2–spheres, possibly with multiplicity. The following generalization of this result was announced by Pitts and Rubinstein in [PR1] (in this theorem $\mathbf{g}(\Sigma)$ is the genus of the surface Σ):

Theorem 0.6. If $\{\Sigma^k_{t_k}\}$ is the min–max sequence of Theorem 0.5 and Σ^∞ its limit, then

$$\mathbf{g}(\Sigma^\infty) \leq \liminf_{k\to\infty} \mathbf{g}(\Sigma^k_{t_k})\,. \tag{0.3}$$

We plan to address the proof of Theorem 0.6 in a forthcoming paper.

Part 1. Overview of the proof

1. Preliminaries

1.1. **Notation.** We begin by fixing some notation which will be used throughout. When speaking of an isotopy ψ, that is, of a map $\psi : [0,1] \times M \to M$ such that $\psi(t,\cdot) \in \mathrm{Diff}_0$ for every t, then if not otherwise specified we assume that $\psi \in \mathfrak{Is}$. Recall that $\mathfrak{Is}$ is the set of smooth isotopies that start at the identity. Moreover, we say that ψ *is supported in* U if $\psi(t,x) = x$ for every $(t,x) \in [0,1] \times (M \setminus U)$.

Most places Γ and Σ will either denote smooth closed surfaces in M (multiplicity allowed) or smooth surfaces in some subset $U \subset M$ with $\overline{\Sigma} \setminus \Sigma \subset \partial U$. However, there are a few places where Σ and Γ denote surfaces which are smooth away from finitely many (singular) points.

Below is a list of our notation:

T_xM, TM	the tangent space of M at x and the tangent bundle of M.
$\mathrm{Inj}\,(M)$	the injectivity radius of M.
$\mathcal{H}^2$	the 2–d Hausdorff measure in the metric space (M,d).
$B_\rho(x)$, $\overline{B}_\rho(x)$, $\partial B_\rho(x)$	open ball, closed ball, and distance sphere of radius ρ in M.
$\mathrm{diam}(G)$	diameter of a subset $G \subset M$.
$d(G_1,G_2)$	the Hausdorff distance between the subsets G_1 and G_2 of M.
$\mathcal{D}$, $\mathcal{D}_\rho$	the unit disk and the disk of radius ρ in $\mathbf{R}^2$.
$\mathcal{B}$, $\mathcal{B}_\rho$	the unit ball and the ball of radius ρ in $\mathbf{R}^3$.
$\exp_x$	the exponential map in M at $x \in M$.
$\mathfrak{Is}(U)$	smooth isotopies supported in U.
$G^2(U)$ or $G(U)$	grassmannian of (unoriented) 2–planes on $U \subset M$.
$\mathrm{An}(x,\tau,t)$	the open annulus $B_t(x) \setminus \overline{B}_\tau(x)$.
$\mathcal{AN}_r(x)$	the set of annuli $\{\mathrm{An}(x,\tau,t)$ where $0<\tau<t<r\}$.
$C^\infty(X,Y)$	smooth maps from X to Y.
$C^\infty_c(X,Y)$	smooth maps with compact support from X to the vector space Y.

1.2. **Varifolds.** We will need to recall some basic facts from the theory of varifolds; see for instance chapter 4 and chapter 8 of [Si] for further information. Varifolds are a convenient way of generalizing surfaces to a category that has good compactness properties. Another advantage of varifolds, over other generalizations (like currents), is that they do not allow for cancellation of mass. This last property is fundamental for the min–max construction.

If U is an open subset of M, any finite nonnegative measure on the Grassmannian of unoriented 2–planes on U is said to be a 2*–varifold in* U. The Grassmannian of 2–planes will be denoted by $G^2(U)$ and the vector space of 2–varifolds is denoted by $\mathcal{V}^2(U)$. With the exception of Appendix C, throughout we will consider only 2–varifolds; thus we drop the 2.

We endow $\mathcal{V}(U)$ with the topology of the weak convergence in the sense of measures, thus we say that a sequence V^k of varifolds converge to a varifold V if for every function $\varphi \in C_c(G(U))$

$$\lim_{k\to\infty} \int \varphi(x,\pi)\, dV^k(x,\pi) \;=\; \int \varphi(x,\pi)\, dV(x,\pi)\,.$$

Here π denotes a 2–plane of T_xM. If $U' \subset U$ and $V \in \mathcal{V}(U)$, then we denote by $V \llcorner U'$ the restriction of the measure V to $G(U')$. Moreover, $\|V\|$ will be the unique measure on U satisfying

$$\int_U \varphi(x)\, d\|V\|(x) \;=\; \int_{G(U)} \varphi(x)\, dV(x,\pi) \qquad \forall \varphi \in C_c(U)\,.$$

The support of $\|V\|$, denoted by $\mathrm{supp}\,(\|V\|)$, is the smallest closed set outside which $\|V\|$ vanishes identically. The number $\|V\|(U)$ will be called the *mass of* V *in* U. When U is clear from the context, we say briefly the *mass of* V.

Recall also that a 2–dimensional rectifiable set is a countable union of closed subsets of C^1 surfaces (modulo sets of $\mathcal{H}^2$–measure 0). Thus, if $R \subset U$ is a 2–dimensional rectifiable set and $h : R \to \mathbf{R}^+$ is a Borel function, then we can define a varifold V by

$$\int_{G(U)} \varphi(x,\pi)\, dV(x,\pi) \;=\; \int_R h(x)\varphi(x, T_xR)\, d\mathcal{H}^2(x) \qquad \forall \varphi \in C_c(G(U))\,. \tag{1.1}$$

Here T_xR denotes the tangent plane to R in x. If h is integer–valued, then we say that V is an *integer rectifiable varifold.* If $\Sigma = \bigcup n_i\Sigma_i$, then by slight abuse of notation we use Σ for the varifold induced by Σ via (1.1).

1.3. **Pushforward, first variation, monotonicity formula.** If V is a varifold induced by a surface $\Sigma \subset U$ and $\psi : U \to U'$ a diffeomorphism, then we let $\psi_\sharp V \in \mathcal{V}(U')$ be the varifold induced by the surface $\psi(\Sigma)$. The definition of $\psi_\sharp V$ can be naturally extended to *any* $V \in \mathcal{V}(U)$ by

$$\int \varphi(y,\sigma)\, d(\psi_\sharp V)(y,\sigma) \;=\; \int J\psi(x,\pi)\, \varphi(\psi(x), d\psi_x(\pi))\, dV(x,\pi)\,;$$

where $J\psi(x,\pi)$ denotes the Jacobian determinant (i.e. the area element) of the differential $d\psi_x$ restricted to the plane π; cf. equation (39.1) of [Si].

Given a smooth vector field χ, let ψ be the isotopy generated by χ, i.e. with $\frac{\partial\psi}{\partial t} = \chi(\psi)$. The first variation of V with respect to χ is defined as

$$[\delta V](\chi) \;=\; \frac{d}{dt}(\|\psi(t,\cdot)_\sharp V\|)\bigg|_{t=0}\;;$$

cf. sections 16 and 39 of [Si]. When Σ is a smooth surface we recover the classical definition of first variation of a surface:

$$[\delta\Sigma](\chi) \;=\; \int_\Sigma \mathrm{div}_\Sigma \chi \, d\mathcal{H}^2 \;=\; \frac{d}{dt}(\mathcal{H}^2(\psi(t,\Sigma)))\Big|_{t=0} \,.$$

If $[\delta V](\chi) = 0$ for every $\chi \in C^\infty_c(U, TU)$, then V is said to be *stationary in* U. Thus stationary varifolds are a natural generalization of minimal surfaces.

Stationary varifolds in Euclidean spaces satisfy the monotonicity formula (see sections 17 and 40 of [Si]):

$$\text{For every } x \text{ the function } f(\rho) = \frac{\|V\|(B_\rho(x))}{\pi\rho^2} \text{ is non–decreasing.} \tag{1.2}$$

When V is a stationary varifold in a Riemannian manifold a similar formula with an error term holds. Namely, there exists a constant $C(r) \geq 1$ such that

$$f(s) \;\leq\; C(r) f(\rho) \qquad \text{whenever } 0 < s < \rho < r. \tag{1.3}$$

Moreover, the constant $C(r)$ approaches 1 as $r \downarrow 0$. This property allows us to define the *density* of a stationary varifold V at x, by

$$\theta(x, V) \;=\; \lim_{r\downarrow 0} \frac{\|V\|(B_r(x))}{\pi r^2}.$$

Thus $\theta(x, V)$ corresponds to the upper density θ^{*2} of the measure $\|V\|$ as defined in section 3 of [Si]. The following theorem gives a useful condition for rectifiability in terms of density:

Theorem 1.1. (Theorem 42.4 of [Si]). If V is a stationary varifold with $\theta(V, x) > 0$ for $\|V\|$–a.e. x, then V is rectifiable.

1.4. **Tangent cones, Constancy Theorem.** Tangent varifolds are the natural generalization of tangent planes for smooth surfaces. In order to define tangent varifolds in a 3–dimensional manifold we need to recall what a dilation in a manifold is. If $x \in M$ and $\rho < \mathrm{Inj}\,(M)$, then the dilation around x with factor ρ is the map $T^x_\rho : B_\rho(x) \to \mathcal{B}_1$ given by $T^x_\rho(z) = (\exp^{-1}_x(z))/\rho$; thus if $M = \mathbf{R}^3$, then T^x_ρ is the usual dilation $y \to (y - x)/\rho$.

Definition 1.2. If $V \in \mathcal{V}(M)$, then we denote by V^x_ρ the dilated varifold in $\mathcal{V}(\mathcal{B}_1)$ given by $V^x_\rho = (T^x_\rho)_\sharp V$. Any limit $V' \in \mathcal{V}(\mathcal{B}_1)$ of a sequence $V^x_{s_n}$ of dilated varifolds, with $s_n \downarrow 0$, is said to be a *tangent varifold at* x. The set of all tangent varifolds to V at x is denoted by $T(x, V)$.

It is well known that if the varifold V is stationary, then any tangent varifold to V is a stationary *Euclidean* cone (see section 42 of [Si]); that is a stationary varifold in $\mathbf{R}^3$ which is invariant under the dilations $y \to y/\rho$. If V is also integer rectifiable and the support of V is contained in the union of a finite number of disjoint connected surfaces Σ_i, i.e. $\mathrm{supp}\,(\|V\|) \subset \bigcup \Sigma_i$, then the Constancy Theorem (see theorem 41.1 of [Si]) gives that $V = \bigcup m_i \Sigma^i$ for some natural numbers m_i.

1.5. **Curvature estimates for stable minimal surfaces.** In many of the proofs we will use Schoen's curvature estimate (see [Sc] or [CM2]) for stable minimal surfaces. Recall that this estimate asserts that if $U \subset\subset M$, then there exists a universal constant, $C(U)$, such that for every stable minimal surface $\Sigma \subset U$ with $\partial\Sigma \subset \partial U$ and second fundamental form A

$$|A|^2(x) \ \leq \ \frac{C(U)}{d^2(x, \partial U)} \qquad \forall x \in \Sigma\,. \tag{1.4}$$

In fact, what we will use is not the actual curvature estimate, rather it is the following consequence of it:

If $\{\Sigma^n\}$ is a sequence of stable minimal surfaces in U, then a subsequence converges, smoothly in U, to a (collection of) stable minimal surface Σ^∞. (1.5)

2. Overview of the proof of Theorem 0.5

In the following we fix a saturated set Λ of generalized 1-parameter families of surfaces and denote by $m_0 = m_0(\Lambda)$ the infimum of the areas of the maximal slices in Λ; cf. (0.1). The proof of Theorem 0.5, which we will outline in this section, follows by combining two results, Proposition 4.1 and Theorem 6.1. The proofs of these two results will involve all the material presented in Sections 3, 4, 5, and 6.

2.1. **Stationarity.** If $\{\{\Sigma_t\}^k\} \subset \Lambda$ is a minimizing sequence, then it is easy to show the existence of a min–max sequence which converge (after possibly passing to subsequences) to a stationary varifold. However, as Fig. 4 illustrate, a general minimizing sequence $\{\{\Sigma_t\}^k\}$ can have slices $\Sigma^k_{t_k}$ with area converging to m_0 but not "clustering" towards stationary varifolds.

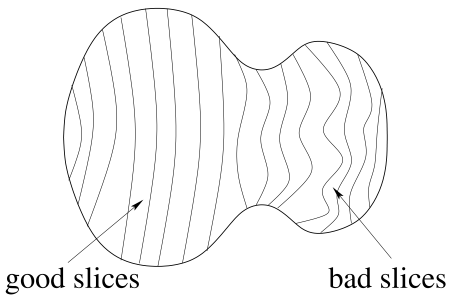

Figure 4. Slices with area close to m_0. The good ones are very near to a minimal surface of area m_0, whereas the bad ones are far from any stationary varifold.

In the language introduced above, this means that a given minimizing sequence $\{\{\Sigma_t\}^k\}$ can have min–max sequences which are not clustering to stationary varifolds. This is a source of some technical problems and forces us in Section 3 to show how to choose a "good" minimizing sequence $\{\{\Sigma_t\}^k\}$. This is the content of the following proposition:

Proposition 2.1. There exists a minimizing sequence $\{\{\Sigma_t\}^n\} \subset \Lambda$ such that every min–max sequence $\{\Sigma^n_{t_n}\}$ clusters to stationary varifolds.

A result similar to Proposition 2.1 appeared in [P] (see theorem 4.3 of [P]). The proof follows from ideas of [Alm] (cf. 12.5 there).

2.2. **Almost minimizing.** A stationary varifold can be quite far from an embedded minimal surface. The key point for getting regularity for varifolds produced by min–max sequences is the concept of "almost minimizing surfaces" or a.m. surfaces. Roughly speaking a surface Σ is almost minimizing if any path of surfaces $\{\Sigma_t\}_{t\in[0,1]}$ starting at Σ and such that Σ_1 has small area (compared to Σ) must necessarily pass through a surface with large area. That is, there must exist a $\tau \in]0,1[$ such that Σ_τ has large area compared with Σ; see Fig. 5.

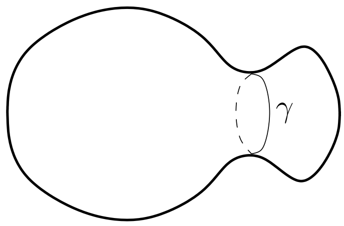

FIGURE 5. Curves near γ are ε–a.m.: It is impossible to deform any such curve isotopically to a much smaller curve without passing through a large curve.

The precise definition of a.m. surfaces is the following:

Definition 2.2. Given $\varepsilon > 0$, an open set $U \subset M^3$, and a surface Σ, we say that Σ is *ε–a.m. in U* if there DOES NOT exist any isotopy ψ supported in U such that

$$\mathcal{H}^2(\psi(t,N)) \leq \mathcal{H}^2(N) + \varepsilon/8 \text{ for all } t; \tag{2.1}$$

$$\mathcal{H}^2(\psi(1,N)) \leq \mathcal{H}^2(N) - \varepsilon. \tag{2.2}$$

A *sequence* $\{\Sigma^n\}$ is said to be *a.m. in U* if each Σ^n is ε_n–a.m. in U for some $\varepsilon_n \downarrow 0$.

This definition first appeared in Smith's dissertation, [Sm], and was inspired by a similar one of Pitts (see the definition of almost minimizing varifolds in 3.1 of [P]). In section 4 of his book, Pitts used combinatorial arguments (some of which were based on ideas of Almgren, [Alm]) to prove a general existence theorem for almost minimizing varifolds. The situation we deal with here is much simpler, due to the fact that we only consider 1–parameter families of surfaces and not general multi–parameter families. Using a version of the combinatorial arguments of Pitts, we will prove in Section 4 the following proposition:

Proposition 2.3. There exists a function $r : M \to \mathbf{R}^+$ and a min–max sequence $\{\Sigma^j\}$ such that:

- $\{\Sigma^j\}$ is a.m. in every annulus An centered at x and with outer radius at most $r(x)$;
- In any such annulus, Σ^j is smooth when j is sufficiently large;
- Σ^j converges to a stationary varifold V in M, as $j \uparrow \infty$.

The reason why we work with annuli is two fold. The first is that we allow the generalized families to have slices with point–singularities. The second is that even if any family of Λ were made of smooth surfaces, then the combinatorial proof of Proposition 2.3 would give a point $x \in M$ in which we are forced to work with annuli (cf. the proof of Proposition 4.1).

2.3. **Gluing replacements and regularity.** The task of the last sections is to prove that the stationary varifold V of Proposition 2.3 is a smooth surface. In Section 6 we will see that if An is an annulus in which $\{\Sigma^j\}$ is a.m., then there exists a stationary varifold V', referred to as a *replacement*, such that

$$V \text{ and } V' \text{ have the same mass and they coincide outside An.} \tag{2.3}$$

$$V' \text{ is a stable minimal surface inside An.} \tag{2.4}$$

In Lemma 5.4 we use this "replacement property" and (1.5) to show that the stationary varifold V of Proposition 2.3 is integer rectifiable. The properties of (smooth) minimal surfaces would naturally lead to the following unique continuation–type conjecture:

Conjecture 2.5. Let V and V' be stationary integer rectifiable and let $\rho > 0$ be smaller than the convexity radius. If $V = V'$ on $M \setminus B_\rho(x)$ and V' is a stable minimal surface in U, then $V = V'$.

An affirmative answer to Conjecture 2.5 would immediately yield the regularity of the stationary varifold V of Proposition 2.3 in sufficiently small annuli. By letting the inner radius of such annuli go to zero, we would be able to conclude that V is a stable minimal surface in $B_{\rho(x)}(x) \setminus \{x\}$, provided that $\rho(x)$ is sufficiently small. Hence, after showing that x is a removable singularity we would get that V is an embedded minimal surface.

Unluckily we are not able to argue in this way. In fact, in Appendix C we give an example of two distinct integer rectifiable 1–varifolds V_1 and V_2 in $\mathbf{R}^2$ which have the same mass and coincide outside a disk. This example does not disprove Conjecture 2.5; because, besides the dimensional difference, in the disk where $V_1 \neq V_2$, both the varifolds are singular. It does however show that a proof of Conjecture 2.5 could be rather delicate.

In [Sm] this problem of unique continuation was overcome by showing that for V as in Proposition 2.3, one can construct "secondary" replacements V'' also for the replacements V'. This idea goes back to [P]. In Section 5 we follow [Sm] and show that if we can replace sufficiently many times, then V is regular (cf. Definition 5.2 and Proposition 5.3 for the precise statement).

2.4. **Replacements.** As discussed in the previous subsection, to prove the regularity of V, we need to construct (sufficiently many) replacements. This task is accomplished in two steps in Section 6.

Step 1: Fix an annulus An in M in which Σ^k is ε_k–a.m. In this annulus we deform Σ^k into a further sequence of surfaces $\{\Sigma^{k,l}\}^l$ with the following properties:

- $\Sigma^{k,l}$ is the image of Σ^k under some isotopy ψ which satisfies (2.1) (with $\varepsilon = \varepsilon_k$ and $U = \text{An}$);
- If we denote by $\mathcal{S}^k$ the family of all such isotopies, then

$$\lim_{l\to\infty} \mathcal{H}^2(\Sigma^{k,l}) = \inf_{\psi\in\mathcal{S}^k} \mathcal{H}^2(\psi(1,\Sigma^k))\,. \tag{2.6}$$

After possibly passing to a subsequence, then $\Sigma^{k,l} \to V^k$ and $V^k \to V'$, where V^k is a varifold which is stationary *in* An. By the a.m. property of V, it follows that V' is stationary *in all of* M and satisfies (2.3).

The second step is to prove that V^k is a (smooth) stable minimal surface in An. Thus, (1.5) will give that also V' is a stable minimal surface in An. After checking some details we show that V meets the technical requirements of Proposition 5.3.

Step 2: It remains to prove that V^k is a stable minimal surface. Stability is a trivial consequence of (2.6). For the regularity we use again Proposition 5.3. The key is proving the following property:

(P) If $B \subset$ An is a sufficiently small ball and l is a sufficiently large number, then *any* $\psi \in \mathfrak{Is}(B)$ with $\mathcal{H}^2(\psi(1, \Sigma^{k,l})) \leq \mathcal{H}^2(1, \Sigma^k)$ can be replaced by a $\Psi \in \mathfrak{Is}(\mathrm{An})$ with

$$\Psi(1, \cdot) \;=\; \psi(1, \cdot) \qquad \text{and} \qquad \mathcal{H}^2(\Psi(t, \Sigma^{k,l})) \;\leq\; \mathcal{H}^2(\Sigma^{k,l}) + \varepsilon_k/8 \quad \text{for all } t. \tag{2.7}$$

We will now discuss how (P) gives the regularity of V^k.

Fix a sufficiently small ball B and a large number l so that the property (P) above holds. Take a sequence of surfaces $\Gamma^j = \Sigma^{k,l,j}$ which are isotopic to $\Sigma^{k,l}$ in B and such that $\mathcal{H}^2(\Gamma^j)$ converges to

$$\inf_{\psi \in \mathfrak{Is}(B)} \mathcal{H}^2(1, \psi(\Sigma^{k,l}))\,.$$

By a result of Meeks–Simon–Yau, [MSY], Γ^j converges to a varifold $V^{k,l}$ which is a stable minimal surface in B. Thus, by (1.5), the sequence of varifolds $\{V^{k,l}\}^l$ converges to a varifold W^k which is a stable minimal surface in B. The property (P) is used to show that, for j and l sufficiently large, $\Sigma^{k,l,j}$ is a good competitor with respect to the ε_k–a.m. property of Σ^k. This is then used to show that W^k is a replacement for V^k in B. Again it is only a technical step to check that we can apply Proposition 5.3, and hence get that V^k is a stable minimal surface in An.

Part 2. Proof of Theorem 0.5

3. Limits of suitable min–max sequences are stationary

This section is devoted to the proof of Proposition 2.1. For simplicity we metrize the weak topology on the space of varifolds and restate Proposition 2.1 using this metric.

Denote by X the set of varifolds $V \in \mathcal{V}(M)$ with mass bounded by $4\,m_0$, i.e., with $\|V\|(M) \leq 4m_0$. Endow X with the weak* topology and let $\mathcal{V}_\infty$ be the set of stationary varifolds contained in X. Clearly, $\mathcal{V}_\infty$ is a closed subset of X. Moreover, by standard general topology theorems, X is compact and metrizable. Fix one such metric and denote it by $\mathfrak{d}$. The ball of radius r and center V in this metric will be denoted by $U_r(V)$.

Proposition 3.1. There exists a minimizing sequence $\{\{\Sigma_t\}^n\} \subset \Lambda$ such that, if $\{\Sigma^n_{t_n}\}$ is a min–max sequence, then $\mathfrak{d}\,(\Sigma^n_{t_n}, \mathcal{V}_\infty) \to 0$.

Proof. The key idea of the proof is building a continuous map $\Psi : X \to \mathfrak{Is}$ such that :

- If V is stationary, then Ψ_V is the trivial isotopy;
- If V is not stationary, then Ψ_V decreases the mass of V.

Since each Ψ_V is an isotopy, and thus is itself a map from $[0, 1] \times M \to M$, to avoid confusion we use the subscript V to denote the dependence on the varifold V. The map Ψ will be used to deform a minimizing sequence $\{\{\Sigma_t\}^n\} \subset \Lambda$ into another minimizing sequence $\{\{\Gamma_t\}^n\}$ such that :

For every $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\text{if} \quad \left\{ \begin{array}{cc} & n > N \\ \text{and} & \mathcal{H}^2(\Gamma^n_{t_n}) > m_0 - \delta \end{array} \right\}, \quad \text{then} \quad \mathfrak{d}\,(\Gamma^n_{t_n}, \mathcal{V}_\infty) < \varepsilon. \tag{3.1}$$

Such a $\{\{\Gamma_t\}^n\}$ would satisfy the requirement of the proposition.

The map Ψ_V should be thought of as a natural "shortening process" of varifolds which are not stationary. If the mass (considered as a functional on the space of varifolds) were smoother, then a gradient flow would provide a natural shortening process like Ψ_V. However, this is not the case; even if we start with smooth initial datum, in very short time the motion by mean curvature, i.e. the gradient flow of the area functional on smooth submanifolds, gives surfaces which are not isotopic to the initial one.

Step 1: A map from X to the space of vector fields.

The isotopies Ψ_V will be generated as 1–parameter families of diffeomorphisms satisfying certain ODE's. In this step we associate to any V a suitable vector field, which in Step 2 will be used to construct Ψ_V.

For $k \in \mathbb{Z}$ define the annular neighborhood of $\mathcal{V}_\infty$

$$\mathcal{V}_k \;=\; \left\{ V \in X | 2^{-k+1} \geq \mathfrak{d}\,(V, \mathcal{V}_\infty) \geq 2^{-k-1} \right\}.$$

There exists a positive constant $c(k)$ depending on k such that to every $V \in \mathcal{V}_k$ we can associate a smooth vector field χ_V with

$$\|\chi_V\|_\infty \leq 1 \qquad \text{and} \qquad \delta V(\chi_V) \leq -c(k).$$

Our next task is choosing χ_V with continuous dependence on V. Note that for every V there is some radius r such that $\delta W(\chi_V) \leq -c(k)/2$ for every $W \in U_r(V)$. Hence, for any k we can find balls $\{U^k_i\}_{i=1,\ldots,N(k)}$ and vector fields χ^k_i such that :

$$\text{The balls } \tilde{U}^k_i \text{ concentric to } U^k_i \text{ and with half the radii cover } \mathcal{V}_k; \tag{3.2}$$

$$\text{If } W \in U^k_i, \text{ then } \delta W(\chi) \leq -c(k)/2; \tag{3.3}$$

$$\text{The balls } U^k_i \text{ are disjoint from } \mathcal{V}_j \text{ if } |j-k| \geq 2. \tag{3.4}$$

Hence, $\{U^k_i\}_{k,i}$ is a locally finite covering of $X \setminus \mathcal{V}_\infty$. To this family we can subordinate a continuous partition of unit φ^k_i. Thus we set $H_V = \sum_{i,k} \varphi^k_i(V)\chi^k_i$. The map $H : X \to C^\infty(M, TM)$ which to every V associates H_V is continuous. Moreover, $\|H_V\|_\infty \leq 1$ for every V.

Step 2: A map from X to the space of isotopies.

For $V \in \mathcal{V}_k$ we let $r(V)$ be the radius of the smaller ball $\tilde{U}^j_i$ which contains it. We find that $r(V) > r(k) > 0$, where $r(k)$ only depends on k. Moreover, by (3.3) and (3.4), for every W contained in the ball $U_{r(V)}(V)$ we have that

$$\delta W(H_V) \;\leq\; -\frac{1}{2}\min\{c(k-1), c(k), c(k+1)\}.$$

Summarizing there are two continuous functions $g : \mathbf{R}^+ \to \mathbf{R}^+$ and $r : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\delta W(H_V) \;\leq\; -g(\mathfrak{d}\,(V, \mathcal{V}_\infty)) \qquad \text{if} \qquad \mathfrak{d}\,(W, V) \leq r(\mathfrak{d}\,(V, \mathcal{V}_\infty)). \tag{3.5}$$

Now for every V construct the 1–parameter family of diffeomorphisms

$$\Phi_V : [0,+\infty)\times M \to M \qquad \text{with} \qquad \frac{\partial \Phi_V(t,x)}{\partial t} \;=\; H_V(\Phi_V(t,x)).$$

For each t and V, we denote by $\Phi_V(t,\cdot)$ the corresponding diffeomorphism of M. We claim that there are continuous functions $T:\mathbf{R}^+\to[0,1]$ and $G:\mathbf{R}^+\to\mathbf{R}^+$ such that

- If $\gamma=\mathfrak{d}\,(V,\mathcal{V}_\infty)>0$ and we transform V into V' via the diffeomorphism $\Phi_V(T(\gamma),\cdot)$, then $\|V'\|(M)\leq\|V\|(M)-G(\delta)$;
- $G(s)$ and $T(s)$ both converge to 0 as $s\downarrow 0$.

Indeed fix V. For every $r>0$ there is a $T>0$ such that the curve of varifolds

$$\{V(t)=(\Phi_V(t,\cdot))_\sharp V\,,\quad t\in[0,T]\}$$

stays in $U_r(V)$. Thus

$$\|V(T)\|(M)-\|V\|(M) \;=\; \|V(T)\|(M)-\|V(0)\|(M) \;\leq\; \int_0^T[\delta V(t)](H_V)\,dt\,,$$

and therefore if we choose $r=r(\mathfrak{d}\,(V,\mathcal{V}_\infty))$ as in (3.5), then we get the bound

$$\|\Gamma(T)\|(M)-\|V\|(M) \;\leq\; -Tg(\mathfrak{d}\,(V,\mathcal{V}_\infty)).$$

Using a procedure similar to that of Step 1 we can choose T depending continuously on V. It is then trivial to see that we can in fact choose T so that at the same time it is continuous and depends only on $\mathfrak{d}\,(V,\mathcal{V}_\infty)$.

Step 3: Constructing the competitor and the conclusion.
For each V, set $\gamma=\mathfrak{d}\,(V,\mathcal{V}_\infty)$ and

$$\Psi_V(t,\cdot) \;=\; \Phi_V([T(\gamma)]\,t,\cdot) \qquad \text{for } t\in[0,1].$$

Ψ_V is a "normalization" of Φ_V. From Step 2 we know that there is a continuous function $L:\mathbf{R}\to\mathbf{R}$ such that

- L is strictly increasing and $L(0)=0$;
- $\Psi_V(1,\cdot)$ deforms V into a varifold V' with $\|V'\|\leq\|V\|-L(\gamma)$.

Choose a sequence of families $\{\{\Sigma_t\}^n\}\subset\Lambda$ with $\mathcal{F}(\{\Sigma_t\}^n)\leq m_0+1/n$ and define $\{\Gamma_t\}^n$ by

$$\Gamma^n_t=\Psi_{\Sigma^n_t}(1,\Sigma^n_t) \qquad \text{for all } t\in[0,1] \text{ and all } n\in\mathbb{N} \tag{3.6}$$

Thus

$$\mathcal{H}^2(\Gamma^n_t)\leq\mathcal{H}^2(\Sigma^n_t)-L(\mathfrak{d}\,(\Sigma^n_t,\mathcal{V}_\infty)). \tag{3.7}$$

Note that $\{\Gamma_t\}^n$ does not necessarily belong to Λ, since the families of diffeomorphisms $\psi_t(\cdot)=\Psi_{\Sigma^n_t}(1,\cdot)$ may not depend smoothly on t. In order to overcome this technical obstruction fix n and note that $\Psi_t=\Psi_{\Sigma^n_t}$ is the 1–parameter family of isotopies generated by the 1–parameter family of vector fields $h_t=T(\Sigma^n_t)H_{\Sigma^n_t}$. Think of h as a continuous map

$$h:[0,1]\to C^\infty(M,TM) \qquad \text{endowed with the topology of } C^k \text{ seminorms.}$$

Thus h can be approximated by a *smooth* map $\tilde h:[0,1]\to C^\infty(M,TM)$. Consider the *smooth* 1–parameter family of isotopies $\tilde\Psi_t$ generated by the vector fields $\tilde h_t$ and the family of surfaces $\{\Gamma_t\}^n$ given by $\Gamma^n_t = \tilde{\Psi}_t(1, \Sigma^n_t)$. If $\sup_t \|h_t - \tilde{h}_t\|_{C^1}$ is sufficiently small, then we easily get (by the same calculations of the previous steps)

$$\mathcal{H}^2(\Gamma^n_t) \leq \mathcal{H}^2(\Sigma^n_t) - L(\mathfrak{d}\,(\Sigma^n_t, \mathcal{V}_\infty))/2. \tag{3.8}$$

Moreover, since $\tilde{\Psi}_t(1, \cdot)$ is a smooth map, this new family belongs to Λ.

Clearly $\{\{\Gamma_t\}^n\}$ is a minimizing sequence. We next show that $\{\{\Gamma_t\}^n\}$ satisfies (3.1). Note first that the construction yields a continuous and increasing function $\lambda : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\lambda(0) = 0 \qquad \text{and} \qquad \mathfrak{d}\,(\Sigma^n_t, \mathcal{V}_\infty) \geq \lambda(\mathfrak{d}\,(\Gamma^n_t, \mathcal{V}_\infty))\,. \tag{3.9}$$

Fix $\varepsilon > 0$ and choose $\delta > 0$, $N \in \mathbb{N}$ such that $L(\lambda(\varepsilon))/2 - \delta > 1/N$. We claim that (3.1) is satisfied with this choice. Suppose not; then there are $n > N$ and t such that $\mathcal{H}^2(\Gamma^n_t) > m_0 - \delta$ and $\mathfrak{d}\,(\Gamma^n_t, \mathcal{V}_\infty) > \varepsilon$. Hence, from (3.8) and (3.9) we get

$$\mathcal{H}^2(\Sigma^n_t) \;\geq\; \mathcal{H}^2(\Gamma^n_t) + \frac{L(\lambda(\varepsilon))}{2} - \delta \;>\; m_0 + \frac{1}{N} \geq m_0 + \frac{1}{n}\,.$$

This contradicts the assumption that $\mathcal{F}(\{\Sigma_t\}^n) \leq m_0 + 1/n$. Thus (3.1) holds and the proof is completed. □

4. Almost minimizing min–max sequences

As above, Λ is a fixed saturated set of 1–parameter families $\{\Sigma_t\}$ in M. In the previous section we showed that there exists a family $\{\Sigma_t\}$ such that every min–max sequence is clustering towards stationary varifolds. We will now prove that one of these min–max sequences is a.m. in sufficiently many annuli.

Proposition 4.1. There exists a function $r : M \to \mathbf{R}^+$ and a min–max sequence $\{\Sigma^j\}$ such that:

$$\{\Sigma^j\} \text{ is a.m. in every An} \in \mathcal{AN}_{r(x)}(x), \text{ for all } x \in M\,. \tag{4.1}$$

$$\text{In every such An, } \Sigma^j \text{ is a smooth surface when } j \text{ is sufficiently large}\,. \tag{4.2}$$

$$\Sigma^j \text{ converges to a stationary varifold } V \text{ as } j \uparrow \infty. \tag{4.3}$$

We first fix some notation.

Definition 4.2. Given a pair of open sets (U^1, U^2) we say that a surface Σ is *ε–a.m. in* (U^1, U^2) if it is ε–a.m. in at least one of the two open sets. We denote by $\mathcal{CO}$ the set of pairs (U^1, U^2) of open sets with

$$\mathrm{d}\,(U^1, U^2) \geq 2 \min\{\mathrm{diam}(U^1), \mathrm{diam}(U^2)\}\,.$$

Proposition 4.1 will be an easy corollary of the following:

Proposition 4.3. There exists a min–max sequence $\{\Sigma^L\} = \{\Sigma^{n(L)}_{t_{n(L)}}\}$ which converges to a stationary varifold and such that

$$\text{each } \Sigma^L \text{ is } 1/L\text{–a.m. in every } (U^1, U^2) \in \mathcal{CO}\,. \tag{4.4}$$

Note that the Σ^L's in the previous proposition may be degenerate slices (that is, they may have a finite number of singular points). The key point for proving Proposition 4.3 is the following obvious lemma:

Lemma 4.4. If (U^1, U^2) and $(V^1, V^2) \in \mathcal{CO}$, then there are $i, j \in \{1,2\}$ with $\mathrm{d}\,(U^i, V^j) > 0$.

Before giving a rigorous proof of Proposition 4.3 we will explain the ideas behind it.

4.1. **Outline of the proof of Proposition 4.3.** First of all note that if a slice $\Sigma^n_{t_0}$ is not ε–a.m. in a given open set U, then we can decrease its area by an isotopy ψ satisfying (2.1) and (2.2). Now fix an open interval I around t_0 and choose a smooth bump function $\varphi \in C^\infty_c(I, [0,1])$ with $\varphi(t_0) = 1$. Define $\{\Gamma_t\}^n$ by

$$\Gamma^n_t \;=\; \psi(\varphi(t), \Sigma^n_t)\,.$$

If the interval I is sufficiently small, then by (2.1), for any $t \in I$, the area of Γ^n_t will not be much larger than the area of Σ^n_t. Moreover, for t very close to t_0 (say, in a smaller interval $J \subset I$) the area of Γ^n_t will be much less than the area of Σ^n_t.

We will show Proposition 4.3 by arguing by contradiction. So suppose that the proposition fails; we will construct a better competitor $\{\{\Gamma_t\}^n\}$. Here the pairs $\mathcal{CO}$ will play a crucial role. Indeed when the area of Σ^n_t is sufficiently large (i.e. close to m_0), we can find *two* disjoint open sets U_1 and U_2 in which Σ^n_t is not almost minimizing. Consider the set $K_n \subset [0,1]$ of slices with sufficiently large area. Using Lemma 4.4 (and some elementary considerations), we find a finite family of intervals I_j, open sets U_j, and isotopies $\psi_j : I_j \times M \to M$ satisfying the following conditions; see Fig. 6:

ψ_j is supported in U_j and is the identity at the endpoints of I_j. (4.5)

If $I_j \cap I_k \neq \emptyset$ then $U_j \cap U_k = \emptyset$. (4.6)

No point of $[0,1]$ belong to more than two I_j's. (4.7)

$\mathcal{H}^2(\psi_j(t, \Sigma^n_t))$ is never much larger than $\mathcal{H}^2(\Sigma^n_t)$. (4.8)

For every $t \in K_n$ there is j s.t. $\mathcal{H}^2(\psi_j(t, \Sigma^n_t))$ is much smaller than $\mathcal{H}^2(\Sigma^n_t)$. (4.9)

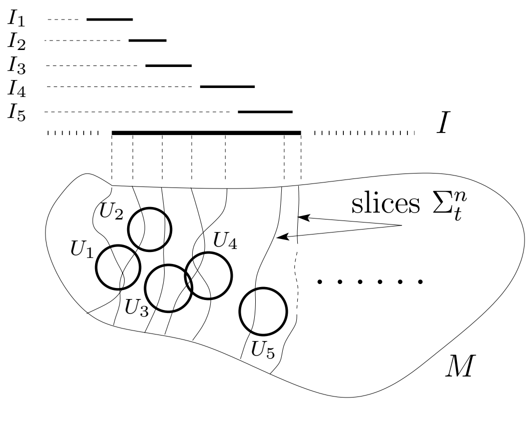

FIGURE 6. The covering I_j and the sets U_j. No point of I is contained in more than two I_j's. The intersection $U_j \cap U_k = \emptyset$ if I_j and I_k overlap.

Conditions (4.5) and (4.6) allow us to "glue" the ψ_j's in a unique $\psi \in \mathfrak{Is}$ such that $\psi = \psi_j$ on $I_j \times U_j$. The family $\{\Gamma_t\}^n$ given by $\Gamma^n_t = \psi(t, \Sigma^n_t)$ is our competitor. Indeed for every t, there are at most two ψ_j's which change Σ^n_t. If $t \notin K_n$, then none of them increases the area of Σ^n_t too much. Whereas, if $t \in K_n$, then one ψ_j decreases the area of Σ^n_t a definite amount, and the other increases the area of Σ_t^n a small amount. Thus, the area of the "small–area" slices are not increased much and the area of "large–area" slices are decreased. This yields that $\mathcal{F}(\{\Gamma_t\}^n) - \mathcal{F}(\{\Sigma_t\}^n) < 0$. We will now give a rigorous bound for this (negative) difference.

4.2. **Proof of Proposition 4.3.**

Proof of Proposition 4.3. We choose $\{\{\Sigma_t\}^n\} \subset \Lambda$ such that $\mathcal{F}(\{\Sigma_t\}^n) < m_0 + 1/n$ and it satisfies the requirements of Proposition 3.1. Fix $L \in \mathbb{N}$. To prove the proposition we claim there exist $n > L$ and $t_n \in [0,1]$ such that $\Sigma^n = \Sigma^n_{t_n}$ satisfies (4.4) and $\mathcal{H}^2(\Sigma^n) \geq m_0 - 1/L$. We define the sets

$$K_n = \left\{ t \in [0,1] : \mathcal{H}^2(\Sigma_t^n) \geq m_0 - \frac{1}{L} \right\}$$

and argue by contradiction. Suppose not; then for every $t \in K_n$ there exists a pair of open subsets (U_t^1, U_t^2) such that Σ_t^n is not $1/L$–a.m. in either of them. So for every $t \in K_n$ there exists isotopies ψ_t^i such that

(1) ψ_t^i is supported on U_t^i;
(2) $\mathcal{H}^2(\psi_t^i(1,\Sigma_t^n)) \leq \mathcal{H}^2(\Sigma_t^n) - 1/L$;
(3) $\mathcal{H}^2(\psi_t^i(\tau,\Sigma_t^n)) \leq \mathcal{H}^2(\Sigma_t^n) + 1/(8L)$ for every $\tau \in [0,1]$.

In the following we fix n and drop the subscript from K_n. Since $\{\Sigma_t^n\}$ is continuous in t, if $t \in K$ and $|s-t|$ is sufficiently small, then

(2') $\mathcal{H}^2(\psi_t^i(1,\Sigma_s^n)) \leq \mathcal{H}^2(\Sigma_s^n) - 1/(2L)$;
(3') $\mathcal{H}^2(\psi_t^i(\tau,\Sigma_s^n)) \leq \mathcal{H}^2(\Sigma_s^n) + 1/(4L)$ for every $\tau \in [0,1]$.

By compactness we can cover K with a finite number of intervals satisfying (2') and (3'). This covering $\{I_k\}$ can be chosen so that I_k overlaps only with I_{k-1} and I_{k-2}. Summarizing we can find

closed intervals	$I_1, \ldots I_r$
pairs of open sets	$(U_1^1, U_1^2), \ldots, (U_r^1, U_r^2) \in \mathcal{CO}$
and pairs of isotopies	$(\psi_1^1, \psi_1^2), \ldots, (\psi_r^1, \psi_r^2)$

such that

(A) the interiors of I_j cover K and $I_j \cap I_k = \emptyset$ if $|k-j| \geq 2$;
(B) ψ_j^i is supported in U_j^i;
(C) $\mathcal{H}^2(\psi_j^i(1,\Sigma_s^n)) \leq \mathcal{H}^2(\Sigma_s^n) - 1/(2L)$ $\forall s \in I_j$;
(D) $\mathcal{H}^2(\psi_j^i(\tau,\Sigma_s^n)) \leq \mathcal{H}^2(\Sigma_s^n) + 1/(4L)$ $\forall s \in I_j$ and $\tau \in [0,1]$.

In Step 1 we refine this covering. In Step 2 we use the refined covering to construct a competitor $\{\Gamma_t\}^n \in \Lambda$ with

$$\mathcal{F}(\{\Gamma_t\}^n) \leq \mathcal{F}(\{\Sigma_t\}^n) - 1/(2L)\,. \tag{4.10}$$

The arbitrariness of n will give that $\liminf_n \mathcal{F}(\{\Gamma_t\}^n) < m_0$. This is the desired contradiction which yields the proposition.

Step 1: Refinement of the covering.

First we want to find

$$\begin{array}{rl} \text{a covering } \{J_1, \ldots, J_R\} & \text{which is a refinement of } \{I_1, \ldots I_r\}\,, \\ \text{open sets } V_1, \ldots V_R & \text{among } \{U^i_j\}\,, \\ \text{and isotopies } \varphi_1, \ldots, \varphi_R & \text{among } \{\psi^i_j\}\,, \end{array}$$

such that:

(A1) The interiors of J_i cover K and $J_i \cap J_k = \emptyset$ for $|k - i| \geq 2$;
(A2) If $J_i \cap J_k \neq \emptyset$, then $d(V_i, V_k) > 0$;
(B') φ_i is supported in V_i;
(C') $\mathcal{H}^2(\varphi_i(1, \Sigma^n_s)) \leq \mathcal{H}^2(\Sigma^n_s) - 1/(2L)$ $\forall s \in J_i$;
(D') $\mathcal{H}^2(\varphi_i(\tau, \Sigma^n_s)) \leq \mathcal{H}^2(\Sigma^n_s) + 1/(4L)$ $\forall s \in J_i$ and $\tau \in [0, 1]$.

We start by setting $J_1 = I_1$ and we distinguish two cases.

- **Case a1**: $I_1 \cap I_2 = \emptyset$; we set $V_1 = U^1_1$, and $\varphi_1 = \psi^1_1$.
- **Case a2**: $I_1 \cap I_2 \neq \emptyset$; by Lemma 4.4 we can choose $i, k \in \{1, 2\}$ such that $\mathrm{d}\,(U^i_1, U^k_2) > 0$ and we set $V_1 = U^i_1$, $\varphi_1 = \psi^i_1$.

We now come to the choice of J_3. If we come from case a1 then:

- **Case b1:** We make our choice as above replacing I_1 and I_2 with I_2 and I_3;

If we come from case a2, then we let i and k be as above and we further distinguish two cases.

- **Case b21**: $I_2 \cap I_3 = \emptyset$; we define $J_2 = I_2$, $V_2 = U^k_2$, $\varphi_2 = \psi^k_2$.
- **Case b22**: $I_2 \cap I_3 \neq \emptyset$; by Lemma 4.4 there exist $l, m \in \{1, 2\}$ such that $d(U^l_2, U^m_3) > 0$. If $l = k$, then we define $J_2 = I_2$, $V_2 = U^k_2$, $\varphi_2 = \psi^k_2$. Otherwise we choose two closed intervals $J_2, J_3 \subset I_2$ such that
  – their interiors cover the interior of I_2,
  – J_2 does not overlap with any I_h for $h \neq 1, 2$,
  – J_3 does not overlap with any I_h for $h \neq 2, 3$.

  Thus we set $V_2 = U^k_2$, $\varphi_2 = \psi^k_2$, and $V_3 = U^l_2$, $\varphi_3 = \psi^l_2$.

An inductive argument using this procedure gives the desired covering. Note that the cardinality of $\{J_1, \ldots, J_R\}$ is at most $2r - 1$.

Step 2: Construction.

Choose C^∞ functions η_i on $\mathbf{R}$ which take values in $[0, 1]$, are supported in J_i and such that for every $s \in K$ there exists η_i with $\eta_i(s) = 1$. Fix $t \in [0, 1]$ and consider the set $\mathrm{Ind}_t \subset \mathbb{N}$ of all i containing t; thus Ind_t consists of at most two elements. Define subsets of M by

$$\Gamma^n_t \;=\; \begin{cases} \varphi_i(\eta_i(t), \Sigma^n_t) & \text{in the open sets } V^i,\ i \in \mathrm{Ind}_t\;, \\ \Sigma^n_t & \text{outside.} \end{cases} \tag{4.11}$$

In view of (A1), (A2) and (B'), then $\{\Gamma_t\}^n$ is well defined and belongs to Λ.

Step 3: The contradiction.

We now want to bound the energy $\mathcal{F}(\{\Gamma_t\}^n)$ and hence we have to estimate $\mathcal{H}^2(\Gamma^n_t)$. Note that by (A1) every Ind_t consists of at most two integers. Assume for the sake of argument that Ind_t consists of *exactly* two integers. From the construction, there exist $s_i, s_k \in [0, 1]$

such that Γ^n_t is obtained from Σ^n_t via the diffeomorphisms $\varphi_i(s_i,\cdot)$, $\varphi_k(s_k,\cdot)$. By (A2) these diffeomorphisms are supported on disjoint sets. Thus if $t\notin K$, then (D') gives

$$\mathcal{H}^2(\Gamma^n_t)\ \le\ \mathcal{H}^2(\Sigma^n_t)+\frac{2}{4L}\ \le\ m_0-\frac{1}{2L}.$$

If $t\in K$, then at least one of s_i, s_k is equal to 1. Hence (C) and (D) give

$$\mathcal{H}^2(\Gamma^n_t)\ \le\ \mathcal{H}^2(\Sigma^n_t)-\frac{1}{L}+\frac{1}{4L}\ \le\ \mathcal{F}(\{\Sigma^n_t\})-\frac{3}{4L}\,.$$

Therefore $\mathcal{F}(\{\Gamma_t\}^n)\le\mathcal{F}(\{\Sigma_t\}^n)-1/(2L)$. This is the desired bound (4.10). □

We now come to Proposition 4.1.

Proof of Proposition 4.1. We claim that a subsequence of the Σ^k's of Proposition 4.3 satisfies the requirements of the proposition we are proving. Indeed fix $k\in\mathbb{N}$ and r such that $\mathrm{Inj}\,(M)>4r>0$. Since $(B_r(x), M\setminus B_{4r}(x))\in\mathcal{CO}$ we then know that Σ^N is $1/k$–a.m. in $M\setminus B_{4r}(x)$. Thus we have that

$$\text{either } \Sigma^k \text{ is } 1/k\text{–a.m. on } B_r(y) \text{ for every } y \tag{4.12}$$

$$\text{or there is } x^k_r\in M \text{ such that } \Sigma^k \text{ is } 1/k\text{–a.m. on } M\setminus B_{4r}(x^k_r). \tag{4.13}$$

If for some $r>0$ there exists a subsequence $\{\Sigma^{k(n)}\}$ satisfying (4.12), then we are done. Otherwise we may assume that there are two sequences of natural numbers $n\uparrow\infty, j\uparrow\infty$ and points x^n_j such that

- For every j, and for n large enough, Σ^n is $1/n$–a.m. in $M\setminus B_{1/j}(x^n_j)$.
- $x^n_j\to x_j$ for $n\uparrow\infty$ and $x_j\to x$ for $j\uparrow\infty$.

Thus for every j, the sequence $\{\Sigma^n\}$ is a.m. in $M\setminus B_{2/j}(x)$. Of course if $U\subset V$ and N is ε–a.m. in V, then N is ε–a.m. in U. This proves that there exists a subsequence $\{\Sigma^j\}$ which satisfies conditions (4.1) and (4.3) for some positive function $r: M\to\mathbf{R}^+$.

It remains to show that an appropriate further subsequence satisfies (4.2). Each Σ^j is smooth except for finitely many points. We denote by P_j the set of singular points of Σ^j. After extracting another subsequence we can assume that P_j is converging, in the Hausdorff topology, to a finite set P. If $x\in P$ and An is any annulus centered at x, then $P_j\cap\mathrm{An}=\emptyset$ for j large enough. If $x\notin P$ and An is any annulus centered at x with outer radius less than $\mathrm{d}\,(x,P)$, then $P_j\cap\mathrm{An}=\emptyset$ for j large enough. Thus, after possibly modifying the function r above, the sequence $\{\Sigma^j\}$ satisfies (4.1), (4.2), and (4.3). □

5. Regularity for the replacements

We will now define a notion of a "good replacement" for stationary varifolds and prove that the existence of (sufficiently many) replacements for a stationary varifold implies that it is a smooth minimal surface; see Proposition 5.3. In section 6 we will show that the varifold V of Proposition 4.1 satisfies the hypotheses of Proposition 5.3 and thus is smooth.

Definition 5.1. Let $V\in\mathcal{V}(M)$ be stationary and $U\subset M$ be an open subset. A stationary varifold $V'\in\mathcal{V}(M)$ is said to be a *replacement for* V *in* U if (5.1) and (5.2) below hold.

$$V'=V \text{ on } G(M\setminus\overline{U}) \text{ and } \|V'\|(M)=\|V\|(M). \tag{5.1}$$

$$V\llcorner U \text{ is a stable minimal surface } \Sigma \text{ with } \overline{\Sigma}\setminus\Sigma\subset\partial U. \tag{5.2}$$

Definition 5.2. Let V be a stationary varifold and $U \subset M$ be an open subset. We say that V has the *good replacement property* in U if (a), (b), and (c) below hold.

(a) There is a positive function $r : U \to \mathbf{R}$ such that for every annulus $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$ there is a replacement for V' in An.

(b) The replacement V' has a replacement V'' in any $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$ and in any $\mathrm{An} \in \mathcal{AN}_{r'(y)}(y)$ (where r' is positive).

(c) The replacement V'' has a replacement V''' in any $\mathrm{An} \in \mathcal{AN}_{r''(y)}(y)$ (where $r'' > 0$).

If V and V' are as above, then we will say that V' is a *good replacement* and V'' a *good further replacement*.

This section is devoted to prove the following:

Proposition 5.3. Let G be an open subset of M. If V has the good replacement property in G, then V is a (smooth) minimal surface in G.

In the proof Proposition 5.3 we need the two technical Lemmas B.1 and B.2, stated and proved in Appendix B. Note that Lemma B.1 is just a weak version (in the framework of varifolds) of the classical maximum principle for minimal surfaces. As a first step towards the proof of Proposition 5.3 we have the following:

Lemma 5.4. Let U be an open subset of M and V a stationary varifold in U. If there exists a positive function r on M such that V has a replacement in any annulus $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$, then V is integer rectifiable. Moreover, $\theta(x, V) \geq 1$ for any $x \in U$ and any tangent cone to V in x is an integer multiple of a plane.

Proof. Since V is stationary, the monotonicity formula (1.3) gives a constant C_M such that

$$\frac{\|V\|(B_\sigma(x))}{\sigma^2} \leq C_M \frac{\|V\|(B_\rho(x))}{\rho^2} \qquad \forall \sigma < \rho < \mathrm{Inj}\,(M) \text{ and } \forall x \in M\,. \tag{5.3}$$

Fix $x \in \mathrm{supp}\,(\|V\|)$ and $r < r(x)$ so that $4r$ is smaller than the convexity radius. Replace V with V' in $\mathrm{An}(x, r, 2r)$. We claim that $\|V'\|$ cannot be identically 0 on $\mathcal{AN}(x, r, 2r)$. Assume it was; since $x \in \mathrm{supp}\,(\|V'\|)$, there would be a $\rho \leq r$ such that V' "touches" ∂B_ρ from the interior. More precisely, there would exist ρ and ε such that $\mathrm{supp}\,\|V'\| \cap \partial B_\rho(x) \neq \emptyset$ and $\mathrm{supp}\,\|V'\| \cap \mathcal{AN}(x, \rho, \rho + \varepsilon) = \emptyset$. Since $B_\rho(x)$ is convex this would contradict Lemma B.1. Thus $V' \llcorner \mathrm{An}(x, r, 2r)$ is a non–empty smooth surface and so there is $y \in \mathrm{An}(x, r, 2r)$ with $\theta(V', y) \geq 1$. Using (5.3) we get

$$\frac{\|V\|(B_{4r}(x))}{16r^2} = \frac{\|V'\|(B_{4r}(x))}{16r^2} \geq \frac{C_M \|V'\|(B_{2r}(y))}{16r^2} \overset{(5.3)}{\geq} \frac{\pi C_M}{4}. \tag{5.4}$$

Hence, $\theta(x, V)$ is bounded uniformly from below on $\mathrm{supp}\,(\|V\|)$ and applying Theorem 1.1 we conclude that V is rectifiable.

We next prove that V is *integer* rectifiable. We use the notation of Definition 1.2. Fix $x \in \mathrm{supp}\,(\|V\|)$, a stationary cone $C \in TV(x, V)$, and a sequence $\rho_n \downarrow 0$ such that $V^x_{\rho_n} \to C$. Replace V by V'_n in $\mathrm{An}(x, \rho_n/4, 3\rho_n/4)$ and set $W'_n = (T^x_{\rho_n})_\sharp V'_n$. After possibly passing to a subsequence, we can assume that $W'_n \to C'$, where C' is a stationary varifold. The following properties of C' are trivial consequences of the definition of replacements:

$$C' = C \text{ in } \mathcal{B}_{1/4}(x) \cup \mathrm{An}(x, 3/4, 1), \tag{5.5}$$

$$\|C'\|(\mathcal{B}_\rho) = \|C\|(\mathcal{B}_\rho) \text{ if } \rho \in]0, 1/4[\cup]3/4, 1[. \tag{5.6}$$

Since C is a cone, in view of (5.6) we have

$$\frac{\|C'\|(\mathcal{B}_\sigma)}{\sigma^2} = \frac{\|C'\|(\mathcal{B}_\rho)}{\rho^2} \qquad \forall \sigma, \rho \in]0, 1/4[\cup]3/4, 1[. \tag{5.7}$$

Hence, the stationarity of C' and the monotonicity formula imply that C' is a cone. By (1.5), W'_n converge to a stable embedded minimal surface in $\mathrm{An}(x, 1/4, 3/4)$. This means that $C' \llcorner \mathrm{An}(x, 1/4, 3/4)$ is an embedded minimal cone in the classical sense and hence it is supported on a disk containing the origin. This forces C' and C to coincide and be an integer multiple of the same plane. □

Proof of Proposition 5.3. The strategy of the proof is as follows. Fix $x \in M$, a good replacement V' for V in $\mathrm{An}(x, \rho, 2\rho)$, and let Σ' be the stable minimal surface given by V' in $\mathrm{An}(x, \rho, 2\rho)$. Consider $t \in]\rho, 2\rho[$, $s < \rho$ and the replacement V'' of V' in $\mathrm{An}(x, s, t)$, which in this annulus coincides with a smooth minimal surface Σ''. In step 2 we will prove that, for ρ sufficiently small and for an appropriate choice of t, then $\Sigma'' \cup \Sigma'$ is a smooth surface. Letting $s \downarrow 0$ we get a minimal surface $\Sigma \subset B_\rho(x) \setminus \{x\}$ such that every Σ'' constructed as above is a subset of Σ. Loosely speaking, any replacement of V' will coincide with Σ in the annular region where it is smooth.

Now, fix a z which belongs to $\mathrm{supp}\,(\|V\|)$ and such that V intersects $\partial B_s(x)$ "transversally" in z. If we consider a replacement V'' of V' in $\mathrm{An}(x, s, \rho)$, then z will belong to the closure of the minimal surface $\Sigma'' = V'' \llcorner \mathrm{An}(x, s, \rho)$. The discussion above gives that $z \in \Sigma$. Lemma B.2 implies that "transversality" to the spheres centered at x in a dense subset of $(\mathrm{supp}\,(\|V\|)) \cap B_\rho(x)$. Thus in step 3 we conclude that

$$(\mathrm{supp}\,(\|V\|)) \cap B_\rho(x) \setminus \{x\} \subset \Sigma.$$

Since $\mathcal{H}^2(\Sigma \cap B_\rho(x)) = \|V\|(B_\rho(x))$, then $V = \Sigma$ in $B_\rho(x)$. Step 4 concludes the proof by showing that x is a removable singularity for Σ.

The key fact that Σ'' and Σ' can be "glued" smoothly together is a consequence of the curvature estimates for stable minimal surfaces combined with the characterization of the tangent cones given in Lemma 5.4. These two ingredients will be used to prove that Σ'' is (locally) a Lipschitz graph nearby $\partial B_t(x)$; thus allowing us to apply standard theory of Elliptic PDE.

Step 1: The set up.

Fix x, V, V', V'', Σ', Σ'', ρ, s, and t as above. We require that 2ρ is less than the convexity radius of M and that Σ' intersects $\partial B_t(x)$ transversally. Fix a point $y \in \Sigma' \cap \partial B_t(x)$ and a sufficiently small radius r, so that $\Sigma' \cap B_r(y)$ is a disk and $\gamma = \Sigma' \cap \partial B_t(x) \cap B_r(y)$ is a smooth arc.

Let $\zeta : B_r(y) \to \mathcal{B}_1$ be a diffeomorphism such that

$$\zeta(\partial B_t(x)) \subset \{z_1 = 0\} \qquad \text{and} \qquad \zeta(\Sigma'') \subset \{z_1 > 0\},$$

where z_1, z_2, z_3 are orthonormal coordinates on $\mathcal{B}_1$; see Fig. 7. We will also assume that $\zeta(\gamma) = \{(0, z_2, g'(0, z_2))\}$ and $\zeta(\Sigma') \cap \{z_1 \leq 0\} = \{(z_1, z_2, g'(z_1, z_2))\}$ where g' is smooth.

Note the following elementary facts:

- Any kind of estimates (like curvature or area bounds or monotonicity) for a minimal surface $\Sigma \subset B_r(y)$ translates into similar estimates for the surface $\zeta(\Sigma)$.

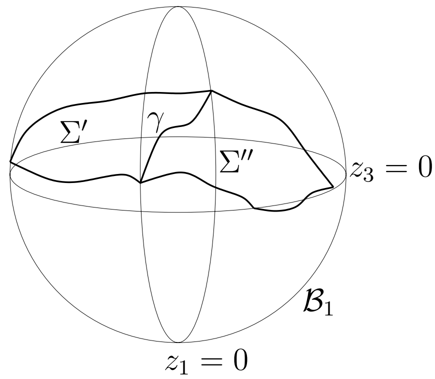

FIGURE 7. The surfaces Σ' and Σ'' and the curve γ in $\mathcal{B}_1$.

- Varifolds in $B_r(y)$ are push–forwarded to varifolds in $\mathcal{B}_1$ and there is a natural correspondence between tangent cones to V in ξ and tangent cones to $\zeta_\sharp V$ in $\zeta(\xi)$.

By slight abuse of notation, we use the same symbols (e.g. γ, V', Σ') for both the objects of $B_r(y)$ and their images under ζ.

Step 2: Graphicality; gluing Σ' and Σ'' smoothly together.

The varifold V'' consists of $\Sigma'' \cup \Sigma'$ in $B_r(y)$. Moreover, Lemma 5.4 applied to V'' gives that $TV(z, V'')$ is a family of (multiples of) 2–planes. Fix $\overline{z} \in \gamma$. Since Σ' is regular and transversal to $\{z_1 = 0\}$ in $\overline{z}$, each plane $P \in TV(\overline{z}, V'')$ coincides with the half plane $T_{\overline{z}}\Sigma'$ in $\{z_1 < 0\}$. Hence $TV(\overline{z}, V'') = \{T_{\overline{z}}\Sigma'\}$. Let $\tau(\overline{z})$ be the unit normal to the graph of g'

$$\tau(\overline{z}) \;=\; \frac{(-\partial_1 g'(0,\overline{z}_2), -\partial_2 g'(0,\overline{z}_2), 1)}{\sqrt{1+|\nabla g' 0,\overline{z}_2)|^2}}$$

and let $R^{\overline{z}}_r : \mathbf{R}^3 \to \mathbf{R}^3$ be the dilatation of 3-space defined by

$$R^{\overline{z}}_r(z) \;=\; \frac{z-\overline{z}}{r}\,.$$

Since $TV(\overline{z}, V'') = \{T_{\overline{z}}\Sigma'\}$, the surfaces $\Sigma_r = R^{\overline{z}}_r(\Sigma'')$ converge to the half plane $HP = \{\tau(\overline{z}) \cdot v = 0, v_1 > 0\}$ — half of the plane $\{\tau(\overline{z}) \cdot v = 0\}$. This convergence implies that

$$\lim_{z\to\overline{z}, z\in\Sigma''} \frac{|(z-\overline{z})\cdot\tau(\overline{z})|}{|\overline{z}-z|} = 0\,. \tag{5.8}$$

Indeed assume that (5.8) fails; then there is a sequence $\{z_n\} \subset \Sigma''$ such that $z_n \to \overline{z}$ and $|(z_n - \overline{z}) \cdot \tau(\overline{z})| \geq k|z_n - \overline{z}|$ for some $k > 0$. Set $r_n = |z_n - \overline{z}|$. There exists a constant k_2 such that $\mathcal{B}_{2k_2 r_n}(z_n) \cap HP = \emptyset$. Thus $\mathrm{dist}(HP, \mathcal{B}_{k_2 r_n}(z_n)) \geq k_2 r_n$. Since Σ'' is regular in z_n we get by the monotonicity formula that

$$\|V''\|(\mathcal{B}_{k_2 r_n}(z_n)) \geq C k_2^2 r_n^2 \qquad \text{where } C \text{ depends on } \zeta.$$

This would contradict the fact that HP is the only element of $TV(\overline{z}, V'')$. Note also that the convergence of (5.8) is uniform for $\overline{z}$ in compact subsets of γ. The argument is explained in Fig. 8.

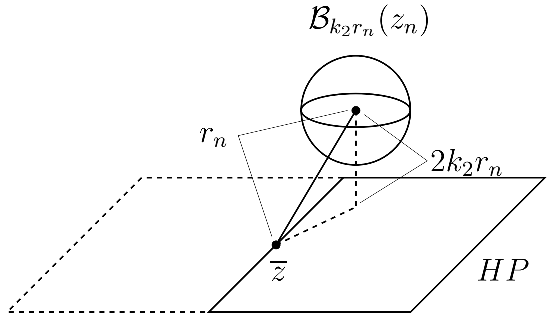

FIGURE 8. If $z_n \in \Sigma''$ is far from the plane HP, the monotonicity formula gives a "good amount" of the varifold V'' which lives far from HP.

Let ν denote the smooth unit vector field to Σ'' such that $\nu \cdot (0,0,1) \geq 0$. We next use the stability of Σ'' to show that

$$\lim_{z \to \overline{z}, z \in \Sigma''} \nu(z) \;=\; \tau(\overline{z})\,. \tag{5.9}$$

Indeed let σ be the plane $\{(0,\alpha,\beta),\, \alpha,\beta \in \mathbf{R}\}$, assume that $z_n \to \overline{z}$ and set $r_n = \mathrm{dist}(z_n,\sigma)$. Define the rescaled surfaces $\Sigma^n = R^{z_n}_{r_n}(\Sigma''_n \cap \mathcal{B}_{r_n}(z_n))$. Each Σ^n is a stable minimal surface in $\mathcal{B}_1$, and hence, after possibly passing to a subsequence, Σ^n converges smoothly in $\mathcal{B}_{1/2}$ to a minimal surface Σ^∞ (by (1.5)). By (5.8), we have that Σ^∞ is the disk $T_{\overline{z}}\Sigma' \cap \mathcal{B}_{1/2}$. Thus the normals to Σ^n in 0, which are given by $\nu(z_n)$, converge to $\tau(\overline{z})$; see Fig. 9. It is easy to see that the convergence in (5.9) is uniform on compact subsets of γ.

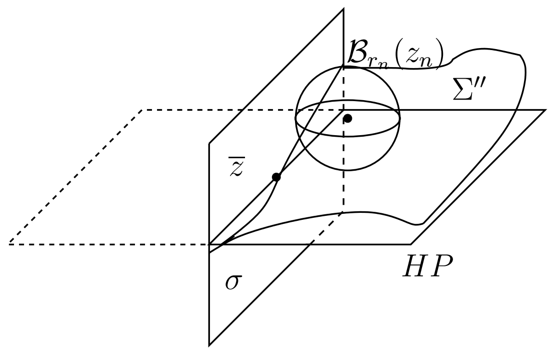

FIGURE 9. If we rescale $\mathcal{B}_{r_n}(z_n)$, then we find a sequence of stable minimal surfaces Σ^n which converge to the half–plane HP.

Hence, for each $\overline{z} \in \gamma$, there exists $r > 0$ and a function $g'' \in C^1(\{z_1 \geq 0\})$ such that

$$\Sigma'' \cap B_r(\overline{z}) \;=\; \{(z_1, z_2, g''(z_1,z_2)), z_1 > 0\}\,,$$
$$g''(0,z_2) = g'(0,z_2)\,, \qquad \text{and} \qquad \nabla g''(0,z_2) = \nabla g'(0,z_2)\,.$$

In the coordinates z_1, z_2, z_3, the minimal surface equation yields a second order uniformly elliptic equation for g' and g''. Thus the classical theory of elliptic PDE gives that g' and g'' are restrictions of a unique smooth function g.

Step 3: Regularity of V in the punctured ball.

Let Σ' and Σ'' be as in the previous step. We will now show that :

$$\text{If } \Gamma \text{ is a connected component of } \Sigma'', \text{ then } \overline{\Gamma} \cap \Sigma' \cap \partial B_t(y) \neq \emptyset. \tag{5.10}$$

Indeed assume that for some Γ equation (5.10) fails. Since t is assumed to be less than the convexity radius we have by the maximum principle that $\overline{\Gamma}\cap\partial B_t(x) \neq \emptyset$. Fix z in $\overline{\Gamma}\cap\partial B_t(x)$. If (5.10) were false, then the varifold V'' would "touch" $\partial B_t(x)$ in z from the interior. More precisely, there would be an $r > 0$ such that

$$z \in \operatorname{supp}(\|V''\|) \qquad \text{and} \qquad (B_r(y) \cap \operatorname{supp}(\|V''\|)) \subset \overline{B}_t(x)\,.$$

This contradicts Lemma B.1; thus (5.10) holds.

Let t, ρ be as in the first paragraph of Step 1. Step 2 and (5.10) imply the following:

$$\text{if } s < \rho, \text{ then } \Sigma' \text{ can be extended to a surface } \Sigma_s \text{ in } \mathrm{An}(x, s, 2\rho) \tag{5.11}$$

$$\text{if } s_1 < s_2 < \rho, \text{ then } \Sigma_{s_1} = \Sigma_{s_2} \text{ in } \mathrm{An}(x, s_2, 2\rho). \tag{5.12}$$

Thus $\Sigma = \bigcup_s \Sigma_s$ is a stable minimal surface Σ with $\overline{\Sigma} \setminus \Sigma \subset (\partial B_{2\rho}(x) \cup \{x\})$, i.e. Σ' can be continued up to x (which, in principle, could be a singular point).

We will next show that V coincides with Σ in $B_\rho(x) \setminus \{x\}$. Recall that $V = V'$ in $B_\rho(x)$. Fix

$$y \in (\operatorname{supp}(\|V\|)) \cap B_\rho(x) \setminus \{x\} \qquad \text{and set } s = d(y, x).$$

We first prove that if $TV(y, V)$ consists of a (multiple of a) plane π transversal to $\partial B_s(x)$, then y belongs to Σ. Consider the replacement V'' of V' in $\mathrm{An}(x, s, t)$ and split V'' into the three varifolds

$$\begin{aligned} V_1 &= V'' \llcorner B_s(x) &&= V \llcorner B_s(x)\,, \\ V_2 &= V'' \llcorner \mathrm{An}(x, s, 2\rho) &&= \Sigma \cap \mathrm{An}(x, s, 2\rho)\,, \\ V_3 &= V'' - V_1 - V_2\,. \end{aligned}$$

By Lemma 5.4, the set $TV(y, V'')$ consists of planes and since $V_1 = V \llcorner B_s(x)$, all these planes have to be multiples of π. Thus y is in the closure of $(\operatorname{supp}(\|V''\|)) \setminus \overline{B}_s(x)$, which implies $y \in \overline{\Sigma}_t \subset \Sigma$.

Let T be the set of points $y \in B_\rho(x)$ such that $TV(y, V)$ consists of a (multiple of) a plane transversal to $\partial B_{d(y,x)}(x)$. Lemma B.2 gives that T is dense in $\operatorname{supp}(\|V\|)$. Thus

$$(\operatorname{supp}(\|V\|)) \cap B_\rho(x) \setminus \{x\} \;\subset\; \Sigma\,.$$

Property (5.1) of replacements implies that $\mathcal{H}^2(\Sigma \cap B_\rho(x)) = \|V\|(B_\rho(x))$. Hence $V = \Sigma$ on $B_\rho(x) \setminus \{x\}$.

Step 4: Regularity in x.

We will next show that Σ is smooth also in x, i.e. that x is a removable singularity for Σ. If $x \notin \operatorname{supp}(\|V\|)$, then we are done. So assume that $x \in \operatorname{supp}(\|V\|)$. In the following we will use that, by Lemma 5.4, every $C \in TV(x, V)$ is a multiple of a plane.

Map $B_t(x)$ into $\mathcal{B}_t(0)$ by the exponential map, use the notation of Step 1, and set $\Sigma_r = R^x_r(\Sigma)$. Every convergent subsequence $\{\Sigma_{r_n}\}$ converges to a plane in the sense of varifolds. The curvature estimates for stable minimal surfaces (see (1.5)) gives that this convergence is actually smooth in $\mathcal{B}_1 \setminus \mathcal{B}_{1/2}$. Thus, for r sufficiently small, there exist natural numbers $N(\rho)$ and $m_i(\rho)$ such that

$$\Sigma \cap \mathrm{An}(x, \rho/2, \rho) \;=\; \bigcup_{i=1}^{N(\rho)} m_i(\rho)\Sigma^i_\rho\,,$$

where each Σ^i_ρ is a Lipschitz graph over a (planar) annulus. Note also that the Lipschitz constants are uniformly bounded, independently of ρ.

By continuity, the numbers $N(r)$ and $m_i(r)$ do not depend on r. Moreover, if $s \in]\rho/2, \rho[$, then each Σ^i_ρ can be continued through $\mathrm{An}(s/2, \rho/2, x)$ by a Σ^j_s. Repeating this argument a countable number of times, we get N minimal punctured disks Σ^i with

$$\Sigma \cap B_\rho(x) \setminus \{x\} \;=\; \bigcup_{i=1}^{N} m_i \Sigma^i\,.$$

Note that x is a removable singularity for each Σ^i. Indeed, Σ^i is a stationary varifold in $B_\rho(x)$ and $TV(x, \Sigma^i)$ consists of planes with multiplicity one. This means that

$$\lim_{r \downarrow 0} \frac{\|V\|(B_r(x))}{\pi r^2} \;=\; 1\,.$$

Hence we can apply Allard's regularity theorem (see section 8 of [All]) to conclude that Σ^i is a graph in a sufficiently small ball around x. Standard elliptic PDE theory gives that x is a removable singularity.

Finally, the maximum principle for minimal surfaces implies that N must be 1. This completes the proof. □

Remark 5.5. In the case at hand, there are other ways of proving that x is a removable singularity. For example one could use the existence of a conformal parameterization $u : \mathbf{C} \setminus \{0\} \to \Sigma^i \cap B_\rho(x) \setminus \{x\}$. The minimality of Σ^i gives that u is an harmonic map. Since the energy of u is finite, we can use theorem 3.6 in [SU] to conclude that u is smooth in 0.

6. Construction of the replacements

In this section we conclude the proof of Theorem 0.5 by showing that the varifold V of Proposition 4.1 is a smooth minimal surface.

Theorem 6.1. Let $\{\Sigma^j\}$ be a sequence of compact surfaces in M which converge to a stationary varifold V. If there exists a function $r : M \to \mathbf{R}^+$ such that

- in every annulus of $\mathcal{AN}_{r(x)}(x)$ and for j large enough Σ^j is a $1/j$–a.m. smooth surface in An,

then V is a smooth minimal surface.

To prove this theorem, we will show that V satisfies the requirements of Proposition 5.3. Thus we need to construct good replacements for V, using the strategy outlined in Section 2. In subsection 6.1 we fix some notation and recall a theorem of Meeks–Simon–Yau. In subsection 6.2 we show how to construct the varifolds V^* which are our candidates for replacements. Subsections 6.3 and 6.4 prove the regularity of the V^*'s constructed in subsection 6.2. Finally, in subsection 6.5 we prove the last details needed to show that V meets the requirements of Proposition 5.3.

6.1. **The result of Meeks–Simon–Yau.**

Definition 6.2. Let $\mathcal{I}$ be a class of isotopies of M and $\Sigma \subset M$ a smooth embedded surface. If $\{\varphi^k\} \subset \mathcal{I}$ and

$$\lim_{k\to\infty} \mathcal{H}^2(\varphi^k(1,\Sigma)) \;=\; \inf_{\psi\in\mathcal{I}} \mathcal{H}^2(\psi(1,\Sigma))\,,$$

then we say that $\varphi^k(1,\Sigma)$ is a *minimizing sequence for Problem* $(\Sigma,\mathcal{I})$.

Theorem 6.3. [Meeks–Simon–Yau [MSY]] If $\{\Sigma^k\}$ is a minimizing sequence for Problem $(\Sigma, \mathfrak{Is}(U))$ which converges to a varifold V, then there exists a stable minimal surface Γ with $\overline{\Gamma}\setminus\Gamma \subset \partial U$ and $V = \Gamma$ in U.

In [MSY] Theorem 6.3 is proved for $U = M$. However, the theory developed there is local and can be extended in a straightforward way to cover the case at hand.

6.2. **Construction of replacements.** Let V be as in Theorem 6.1 and fix an annulus $\mathrm{An} \in \mathcal{AN}_{r(x)}(x)$. Set

$$\begin{aligned}
\mathfrak{Is}_j(\mathrm{An}) &= \left\{\psi\in\mathfrak{Is}(\mathrm{An})\middle|\, \mathcal{H}^2(\psi_j(\tau,\Sigma^j)) \le \mathcal{H}^2(\Sigma_j) + 1/(8j) \quad \text{for all } \tau\in[0,1]\right\},\\
m_j &= \inf_{\psi\in\mathfrak{Is}_j} \mathcal{H}^2(\psi(1,\Sigma^j))\,.
\end{aligned}$$

Fix j. The following lemma implies that we can deform Σ^j into a sequence $\Sigma^{j,k}$ which is minimizing for Problem $(\Sigma^j, \mathfrak{Is}_j(\mathrm{An}))$ and converges, as $k\to\infty$, to a stable minimal surface in An.

Lemma 6.4. If a sequence $\{\Sigma^{j,k}\}^k$ is minimizing for Problem $(\Sigma^j, \mathfrak{Is}_j(\mathrm{An}))$ and converges to a varifold V^j, then V^j is a stable minimal surface in An.

Lemma 6.4 will be proved in the next subsection. Here we use it for constructing a replacement for V in An.

Proposition 6.5. Let V^j be the varifold of Lemma 6.4. Any V^* which is the limit of a subsequence of $\{V^j\}$ is a replacement for V.

Proof. Without loss of generality, we can assume that the sequence $\{V^j\}$ converges to V. Note that every V^j coincides with V in $M\setminus\overline{\mathrm{An}}$; thus the same is true for V^*. Moreover, $\|V^j\|(M) \ge \mathcal{H}^2(\Sigma^j) - 1/j$ since Σ^j is a.m. This gives that $\|V^*\|(M) = \|V\|(M)$. By Lemma 6.4 and (1.5) we have that V^* is a stable minimal surface in An.

To complete the proof we need to show that V^* is stationary. Since $V = V^*$ in $M\setminus\overline{\mathrm{An}}$, then V^* is stationary in this open set. Hence it suffices to prove that V^* is stationary in an open annulus $\mathrm{An}' \in \mathcal{AN}_r$ containing $\overline{\mathrm{An}}$. Choose such an An′ and suppose that V^* is not stationary in An′; we will show that this contradict that $\{\Sigma^j\}$ is a.m. in An′. Namely, suppose that for some vector field χ supported in An′ we have $\delta V^*(\chi) \le -C < 0$. Let ψ be the isotopy given by that $\frac{\partial\psi(t,x)}{\partial t} = \chi(\psi(t,x))$ and set

$$\begin{aligned}
V^*(t) &= \psi(t)_\sharp V^*\,,\\
V^j(t) &= \psi(t)_\sharp V^j\,,\\
\Sigma^{j,k}(t) &= \psi(t,\Sigma^{j,k})\,.
\end{aligned}$$

For ε sufficiently small, we have that

$$[\delta V^*(t)](\chi) \;\leq\; -\frac{C}{2} \qquad \text{for all } t<\varepsilon.$$

Since $V^j(t) \to V^*(t)$, there exists J such that

$$[\delta V^j(t)](\chi) \leq -\frac{C}{4} \qquad \text{for every } j>J \text{ and every } t<\varepsilon.$$

Moreover, since $\Sigma^{j,k}(t) \to V^j(t)$, for each $j>J$ there exists $K(j)$ with

$$[\delta\Sigma^{j,k}(t)](\chi) \;\leq\; -\frac{C}{8} \qquad \text{for all } t<\varepsilon \text{ and all } k>K(j). \tag{6.1}$$

Integrating both sides of (6.1) we get

$$\mathcal{H}^2(\psi(t,\Sigma^{j,k})) - \mathcal{H}^2(\Sigma^{j,k}) \;\leq\; -\frac{tC}{8} \tag{6.2}$$

Choose j and k sufficiently large so that $\varepsilon C/8 > 1/j$ and (6.2) holds. Each $\Sigma^{j,k}$ is isotopic to Σ^j via an isotopy $\varphi^{j,k} \in \mathfrak{Is}_j(\mathrm{An})$. By gluing $\varphi^{j,k}$ and ψ smoothly together, we find a smooth isotopy $\Phi : [0,1+\varepsilon]\times M \to M$ supported on An'. In view of (6.2), Φ satisfies

$$\begin{aligned} \mathcal{H}^2(\Phi(t,\Sigma^j)) &\;\leq\; \mathcal{H}^2(\Sigma^j) + 1/(8j) \qquad \forall t\in[0,1+\varepsilon]\,, \\ \mathcal{H}^2(\Phi(1+\varepsilon,\Sigma^j) &\;<\; \mathcal{H}^2(\Sigma^j) - 1/j\,, \end{aligned}$$

which give the desired contradiction and prove the proposition. □

6.3. **Proof of Lemma 6.4.** Without loss of generality we may assume that $j=1$ and use V', Σ and Σ^k in place of V^j, $\Sigma^{j,k}$ and Σ^j. Clearly V' is stationary and stable in An, by its minimizing property. Thus we need only prove that V' is regular. The proof of this uses Theorem 6.3 and the following:

Lemma 6.6. Let $x\in\mathrm{An}$ and assume that $\{\Sigma^k\}$ is minimizing for Problem $(\Sigma,\mathfrak{Is}_1(\mathrm{An}))$. There exists $\varepsilon>0$ such that, for k sufficiently large, the following holds:

(Cl) For any $\varphi\in\mathfrak{Is}(B_\varepsilon(x))$ with $\mathcal{H}^2(\varphi(1,\Sigma^k)) \leq \mathcal{H}^2(\Sigma^k)$, there exists an isotopy $\Phi\in\mathfrak{Is}(B_\varepsilon)$ such that

$$\Phi(1,\cdot) \;=\; \varphi(1,\cdot)\,, \tag{6.3}$$

$$\mathcal{H}^2(\Phi(t,\Sigma^k)) \;\leq\; \mathcal{H}^2(\Sigma^k) + 1/8. \tag{6.4}$$

Moreover, $\varepsilon>0$ can be chosen so that (Cl) holds for *any* sequence $\{\tilde\Sigma^k\}$ which is minimizing for Problem $(\Sigma,\mathfrak{Is}_1(\mathrm{An}))$ and with $\Sigma^j=\tilde\Sigma^j$ on $M\setminus\overline{B}_\varepsilon(x)$.

Lemma 6.6 will be proved in the next subsection. We now return to the proof of Lemma 6.4. We will use Proposition 5.3. Hence, once again we need to construct replacements for a varifold, which this time is V'. We divide the proof into two steps. The first one is the basic construction of replacements for V'. The second shows that the replacements satisfy (b) and (c) in Definition 5.2

Step 1 Fix $x\in\mathrm{An}$ and $\varepsilon>0$ such that Lemma 6.6 holds. Fix any annulus

$$\mathrm{An}^* \;=\; \mathrm{An}(x,\tau,t) \;\subset\; B_\varepsilon(x) \;\subset\; \mathrm{An}$$

and consider a minimizing sequence $\{\overline{\Sigma}^{k,l}\}^l$ for Problem $(\Sigma^k, \mathfrak{Is}(\mathrm{An}^*))$. Lemma 6.6 implies that, for k sufficiently large, $\overline{\Sigma}^{k,l}$ can be constructed from Σ via an isotopy of $\mathfrak{Is}_1(\mathrm{An})$. Thus if we let W^k be the varifold limit of $\overline{\Sigma}^{k,l}$ and W the limit of W^k, then we have that $\|W\|(M) = \|V'\|(M)$.

Let $\{\overline{\Sigma}^{k,l(k)}\}$ be a subsequence which converges to W. By the discussion above the subsequence is a minimizing sequence for Problem $(\Sigma, \mathfrak{Is}_1(\mathrm{An}))$. Hence W is stationary in An. Moreover, by Theorem 6.3, every W^k is a stable minimal surface in the annulus An^*: the curvature estimates (see (1.5)) give that W is a stable minimal surface in An^*. Hence W is a replacement for V'.

Step 2 Summarizing we have proven in Step 1 that for any $y \in \mathrm{An}$ there exists $r(y) > 0$ such that in the class of annuli $\mathcal{AN}_{r(y)}(y)$ we can construct replacements. In order to complete the proof we have to check that the replacements so constructed satisfy all the technical requirements of Proposition 5.3. Thus define W as in Step 1. Since $\{\overline{\Sigma}^{k,l(k)}\}$ is a minimizing sequence for Problem $(\Sigma, \mathfrak{Is}_1(\mathrm{An}))$, in all the arguments of Step 1 we can use W in place of V'. Thus for every $y \in \mathrm{An}$, W has the replacement property for a class of annuli centered at y. This shows the second part of (b) in Definition 5.2

We still have to settle the first part of (b) in Definition 5.2, i.e. that if the W constructed in Step 1 replaces V' in an annulus of $\mathcal{AN}_{r(x)}(x)$, then W has the replacement property on the *whole* collection of annuli $\mathcal{AN}_{r(x)}(x)$. Note that our $r(x) = \varepsilon$, where ε is given by Lemma 6.6. Every $\overline{\Sigma}^{k,l(k)}$ coincides with Σ^k in $M \setminus B_\varepsilon(x)$ and $\{\overline{\Sigma}^{k,l(k)}\}$ is minimizing for Problem $(\Sigma, \mathfrak{Is}_1(\mathrm{An}))$. Thus the last line of Lemma 6.6 applies and again we can argue as in Step 1 with W in place of V'. We conclude that also W has a replacement in every annulus of $\mathcal{AN}_\varepsilon(x)$.

Condition (c) in Definition 5.2 follows from similar arguments. Summarizing, V' and all its replacements just constructed satisfy the requirements of Proposition 5.3. Hence V' is a smooth surface in An. □

6.4. **Proof of Lemma 6.6. Step 1: Small area slices.** Let x, ε, An, Σ^k and φ be as in the statement of the lemma. We know that Σ^k converges to a varifold V' which is stationary in An and has mass m_0. Thus the monotonicity formula gives that

$$\|V'\|(\mathrm{An}(x, \varepsilon, 2\varepsilon)) \;<\; 4m_0\varepsilon^2 \,.$$

Therefore, if k is sufficiently large we get that $\mathcal{H}^2(\Sigma^k \cap B_{2\varepsilon}(\delta)) < 5m_0\varepsilon^2$. Applying the coarea formula we have that, for every such k, there exists an interval $I_k \subset]\varepsilon, 2\varepsilon[$ such that

$$\mathcal{L}^1(I_k) > 0 \qquad \text{and} \qquad \mathcal{H}^1(\Sigma^k \cap \partial B_\tau(x)) \;\leq\; 10m_0\varepsilon \quad \text{for all } \tau \in I_k\,. \tag{6.5}$$

Thus, applying Sard's Theorem to the function $\mathrm{d}\,(\cdot, x)$ on Σ^k, we can find $\tau_k \in]\varepsilon, 2\varepsilon[$ such that

$$\mathcal{H}^1(\Sigma^k \cap \partial B_{\tau_k}(x)) \;<\; 10m_0\varepsilon \qquad \text{and} \qquad \Sigma^k \text{ is transversal to } \partial B_{\tau_k}(x)\,. \tag{6.6}$$

Moreover, the smoothness of Σ^k implies that we can choose a small interval $]\sigma_k, s_k[$ with $\varepsilon < \sigma_k$ and so that (6.6) holds for every $\tau_k \in]\sigma_k, s_k[$.

Step 2: Radial deformations.

For $\eta > 0$ we denote by O_η the usual radial deformation of Euclidean 3–space given by $O_\eta(x) = \eta x$. If both r and $r\eta$ are less than $\mathrm{Inj}\,(M)/2$, then we define the diffeomorphism

$$I_\eta : B_r(x) \to B_{\eta r}(x) \qquad \text{by} \quad \exp \circ O_\eta \circ \exp^{-1}.$$

By the smoothness of M there exists $\mu > 0$ such that, for any surface $\Gamma \subset M$,

$$\mathcal{H}^2(I_\eta(\Gamma) \cap B_{\eta r}(x)) \leq \mu\,\eta^2\,\mathcal{H}^2(\Gamma \cap B_r(x))\,, \tag{6.7}$$

$$\mathcal{H}^1(I_\eta(\Gamma) \cap \partial B_{\eta r}(x)) \leq \mu\,\eta\,\mathcal{H}^1(\Gamma \cap \partial B_r(x)) \quad \text{if } \Gamma \text{ is transversal to } \partial B_r(x). \tag{6.8}$$

$$\mathcal{H}^2(\Gamma \cap B_r(x)) \leq \mu \int_0^r \mathcal{H}^1(\Gamma \cap \partial B_\rho(x))\, d\rho. \tag{6.9}$$

Fix k and choose ε, σ_k and s_k as in the previous step. In the current step we use I_η to construct a smooth $\psi : [0,1] \times M \to M$ such that

- For every $\delta > 0$, $\psi|_{[0,1-\delta]\times M}$ is a smooth isotopy supported in $\mathfrak{Is}(B_{s_k}(x))$;
- $\psi|_{\{1\}\times M}$ "squeezes" the ball $B_{\sigma_k}(x)$ to the point $\{x\}$ and "stretches" the annulus $\mathrm{An}(x, \sigma_k, s_k)$ to the ball $B_{s_k}(x)$;
- For some constant C depending on μ we have (see Fig. 10)

$$\mathcal{H}^2(\psi(t, \Sigma^k)) \leq \mathcal{H}^2(\Sigma^k) + C\varepsilon^2\,. \tag{6.10}$$

We construct ψ explicitly. We first choose a nondecreasing smooth $f : [0,1] \times [0,1] \to [0,1]$ such that

$$f(t,r) = \begin{cases} (1-t) & \text{if } r \in [0, \sigma_k], \\ 1 & \text{if } r \in [s_k, 1], \end{cases}$$

and then set

$$\psi(t,y) = \begin{cases} y & \text{if } \mathrm{d}\,(y,x) \geq s_k\,, \\ I_{f(t,\mathrm{d}\,(x,y))}(y) & \text{otherwise.} \end{cases}$$

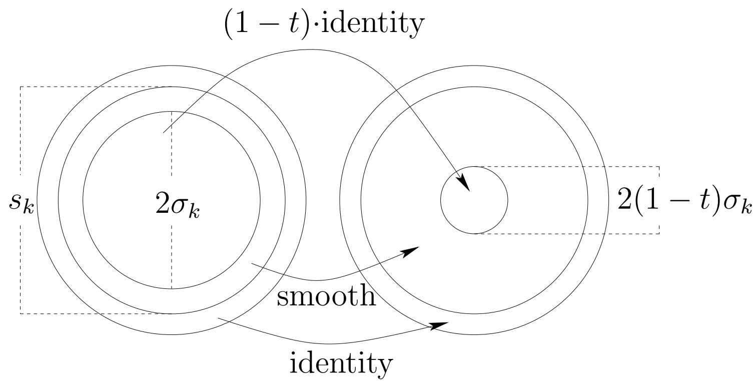

FIGURE 10. The map $\psi(t,\cdot) : M \to M$

We will only prove (6.10), since the other properties are easy to check. First of all, since $\psi(t,y) = y$ on $M \setminus B_{\sigma_k}(x)$, we have

$$\mathcal{H}^2(\psi(t, \Sigma^k) \cap (M \setminus B_{2\varepsilon}(x)) = \mathcal{H}^2(\Sigma^k \cap (M \setminus B_{2\varepsilon}(x)). \tag{6.11}$$

By (6.7)

$$\mathcal{H}^2(\psi(t, \Sigma^k) \cap B_{(1-t)\sigma_k}(x)) \leq \mu(1-t)^2 \mathcal{H}^2(\Sigma^k \cap B_{\sigma_k}) \leq 10\mu m_0 \varepsilon\,. \tag{6.12}$$

To estimate the remaining portion of $\psi(t,\Sigma^k)$ we use (6.9) and get

$$\mathcal{H}^2(\psi(t,\Sigma^k)\cap \mathrm{An}(x,(1-t)\sigma_k,s_k)) \;\leq\; \mu\int_{(1-t)\sigma_k}^{s_k} \mathcal{H}^1(\psi(t,\Sigma^k)\cap\partial B_\rho(x))d\rho. \tag{6.13}$$

Note that for $\rho\in((1-t)\sigma_k,s_k)$ there exists $\tau\in(\sigma_k,s_k)$ and $\eta\leq 1$ such that

$$\psi(t,\Sigma^k)\cap\partial B_\rho \;=\; I_\eta(\Sigma^k\cap\partial B_\tau(x))\,.$$

Thus, by (6.8) and (6.6) we have

$$\mathcal{H}^1(\psi(t,\Sigma^k)\cap\partial B_\rho) \;\leq\; \mu\mathcal{H}^1(\Sigma^k\cap\partial B_\tau(x)) \;\leq\; 10\mu m_0\varepsilon\,. \tag{6.14}$$

Inserting (6.14) in (6.13) we find that

$$\mathcal{H}^2(\psi(t,\Sigma^k)\cap \mathrm{An}(x,(1-t)\sigma_k,s_k)) \;\leq\; 20\mu^2 m_0\varepsilon^2. \tag{6.15}$$

Equations (6.11), (6.12), and (6.15) yield

$$\mathcal{H}^2(\psi(t,\Sigma^k)) \;\leq\; \mathcal{H}^2(\Sigma^k)+C\varepsilon^2\,. \tag{6.16}$$

Step 3: The conclusion.
Choose ε such that $\mu C\varepsilon^2<1/32$ and K such that :

- We can construct the ψ of the previous step with (6.16) valid for any $k>K$;
- $\mathcal{H}^2(\Sigma^k)\leq\mathcal{H}^2(\Sigma)+1/32$ for any $k>K$.

We want to prove that ε satisfies the requirement of the lemma. Indeed choose any smooth isotopy φ which is supported on $B_\varepsilon(x)$ and such that $\mathcal{H}^2(\varphi(1,\Sigma^k))\leq\mathcal{H}^2(\Sigma^k)$. Set

$$K \;=\; \sup_t \mathcal{H}^2(\varphi(t,\Sigma^k)\cap B_{\sigma_k}(x))$$

and choose t such that $\mu(1-t)^2K\leq 1/32$.

Define isotopies ψ^- and $\tilde\varphi$ by

$$\psi^-(s,\cdot) \;=\; \psi(1-s,\cdot) \qquad\text{and}\qquad \tilde\varphi \;=\; I_{(1-t)}\circ\varphi\circ I_{(1-t)^{-1}}\,,$$

and note that ψ^- is the "backward" of ψ and hence instead of "squeezing", it magnifies, whereas $\tilde\varphi$ is the "$(1-t)$–shrunk" version of φ. We now define a (piecewise) smooth isotopy $\Psi: I_1\cup I_2\cup I_3\times M\to M$ by gluing ψ, $\tilde\phi$, and ψ^- together. Namely, set

- $\Psi(s,\cdot)=\psi(s,\cdot)$ for $s\in I_1=[0,1-t]$;
- $\Psi(s,\cdot)=\tilde\varphi(s-(1-t))$ for $s\in I_2=[1-t,2-t]$;
- $\Psi(s,\cdot)=\psi^-((s-(2-t))+t,\tilde\varphi(1,\cdot))$ for $s\in I_3=[2-t,3-2t]$.

Loosely speaking, the isotopy Ψ performs the following operations (see Fig. 11):

First it "shrinks" the ball $B_{s_k}(x)$ to the ball $B_{(1-t)s_k}(x)$;
Then it applies the "$(1-t)$–shrunk" version of φ to the ball $B_{(1-t)s_k}(x)$;
Finally it magnifies back $B_{(1-t)s_k}(x)$ to the ball $B_{s_k}(x)$.

By changing parameter we can assume that Ψ is smooth and that $I_i=[(i-1)/3,i/3]$. Note that (6.10) implies that

$$\mathcal{H}^2(\Psi(s,\Sigma^k)) \;\leq\; \mathcal{H}^2(\Sigma^k)+1/32 \qquad\text{for } s\in[0,1/3]. \tag{6.17}$$

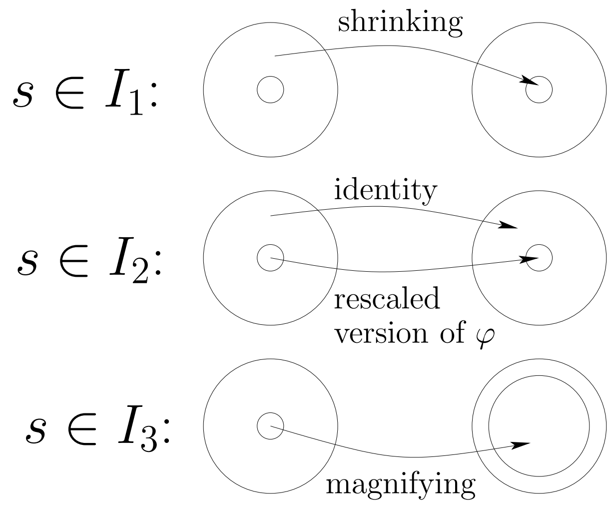

FIGURE 11. The isotopy Ψ.

Since $\mu(1-t)^2K \leq 1/32$ we have

$$\mathcal{H}^2(\Psi(s,\Sigma^k)) \;\leq\; \mathcal{H}^2(\Sigma^k)+\mu(1-t)^2K \;\leq\; \frac{1}{32} \qquad \text{for } s\in[1/3,2/3]. \tag{6.18}$$

Again by (6.10), and since $\mathcal{H}^2(\varphi(1,\Sigma^k))\leq\mathcal{H}^2(\Sigma^k)$, we have

$$\begin{aligned}\mathcal{H}^2(\Psi(s,\Sigma^k)) &\leq \mathcal{H}^2(\Sigma^k)+1/32+\mu\mathcal{H}^2(\varphi(1,\Sigma^k)\cap B_{\sigma_k}(x))\\ &\leq \mathcal{H}^2(\Sigma^k)+1/32+\mu C\varepsilon^2\\ &\leq \mathcal{H}^2(\Sigma^k)+1/16 \qquad\qquad \text{for } s\in[2/3,1].\end{aligned} \tag{6.19}$$

Thus for every s

$$\mathcal{H}^2(\Psi(s,\Sigma^k)) \;\leq\; \mathcal{H}^2(\Sigma^k)+1/16 \;\leq\; \mathcal{H}^2(\Sigma)+3/32. \tag{6.20}$$

Note also that $\Psi(1,\Sigma^k)=\varphi(1,\Sigma^k)$. Finally, recall that Σ^k was obtained via an isotopy φ^k such that

$$\mathcal{H}^2(\varphi^k(t,\Sigma)) \;\leq\; \mathcal{H}^2(\Sigma)+1/8.$$

Gluing together φ_k and Ψ we easily obtain a Φ which satisfies both (6.3) and (6.4). Clearly the ε found in this proof satisfies the last requirement of the statement of the lemma. □

6.5. **Proof of Theorem 6.1.** We will apply Proposition 5.3. From Proposition 6.5 we know that in every annulus $\mathrm{An}\in\mathcal{AN}_{r(x)}(x)$ there is a replacement V^* for V. We still need to show that V satisfies (a), (b), and (c) in Definition 5.2. Consider the family of surfaces $\Sigma^{j,k}$ of Lemma 6.4. By a diagonal argument we can extract a subsequence $\Sigma^{j,k(j)}$ converging to V^*. Note the following consequence of the way we constructed $\{\Sigma^{j,k(j)}\}^j$. If U is open and

- either $U\cup\mathrm{An}$ is contained in some annulus $\mathcal{AN}_{r(x)}(x)$
- or $U\cap\mathrm{An}=\emptyset$ and U is contained in some annulus of $\mathcal{AN}_{r(y)}(y)$ with $y\neq x$,

then $\Sigma^{j,k(j)}$ is a.m. in U. Thus $\{\Sigma^{j,k(j)}\}$ is still a.m. in

- every annulus of $\mathcal{AN}_{r(x)}(x)$;
- every annulus of $\mathcal{AN}_{\rho(y)}(y)$ for $y\neq x$, provided $\rho(y)$ is sufficiently small.

This shows that (b) in Definition 5.2 holds for V. Similarly, we can show that also condition (c) of that Definition holds. Hence Proposition 5.3 applies and we conclude that V is a smooth surface. □

Appendix A. Proof of Proposition 0.4

Let M^3 be a closed Riemannian 3–manifold with a Morse function $f : M \to [0,1]$. Denote by Σ_t the level set $f^{-1}(\{t\})$ and let Λ be the saturated set of families

$$\left\{\{\Gamma_t\}\,\middle|\, \Gamma_t = \psi(t,\Sigma_t) \text{ for some } \psi \in C^\infty([0,1]\times M, M) \text{ with } \psi_t \in \mathrm{Diff}_0 \text{ for every } t\right\}.$$

To prove Proposition 0.4 we need to show that $m_0(\Lambda) > 0$. To do that set $U_t = f^{-1}([0,t[)$ and $V_t = \psi(t,U_t)$. Clearly $\Gamma_t = \partial V_t$ and if we let Vol denote the volume on M, then $\mathrm{Vol}(U_t)$ is a continuous function of t. Since V_0 is a finite set of points and $V_1 = M$, then there exists an s such that $\mathrm{Vol}(V_s) = \mathrm{Vol}(M)/2$. By the isoperimetric inequality there exists a constant $c(M)$ such that

$$\frac{\mathrm{Vol}(M)}{2} \;=\; \mathrm{Vol}(V_s) \;\leq\; c(M)\mathcal{H}^2(\Gamma_s)^{3/2}\,.$$

Hence,

$$\mathcal{F}(\{\Gamma_t\}) \;=\; \max_{t\in[0,1]} \mathcal{H}^2(\Gamma_t) \;\geq\; \left(\frac{\mathrm{Vol}(M)}{2c(M)}\right)^{\frac{2}{3}} > 0\,, \tag{A.1}$$

and the proposition follows.

Appendix B. Two lemmas about varifolds

Lemma B.1. Let U be an open subset of a 3–manifold M and W a stationary 2–varifold in $\mathcal{V}(U)$. If $K \subset\subset U$ is a smooth strictly convex set and $x \in (\mathrm{supp}\,(\|W\|)) \cap \partial K$, then

$$(B_r(x)\setminus \overline{K}) \cap \mathrm{supp}\,(\|W\|) \;\neq\; \emptyset \qquad \text{for every } r>0.$$

Proof. For simplicity assume that $M = \mathbf{R}^3$. The proof can be easily adapted to the general case. Let us argue by contradiction; so assume that there are $x \in \mathrm{supp}\,(\|W\|)$ and $B_r(x)$ such that $(B_r(x)\setminus \overline{K}) \cap \mathrm{supp}\,(\|W\|) = \emptyset$. Given a vector field $\chi \in C^\infty_c(U,\mathbf{R}^3)$ and a 2–plane π we set

$$\mathrm{Tr}\,(D\chi(x),\pi) \;=\; D_{v_1}\chi(x)\cdot v_1 + D_{v_2}\chi(x)\cdot v_2$$

where $\{v_1, v_2\}$ is an orthonormal base for π. Recall that the first variation of W is given by

$$\delta W(\chi) \;=\; \int_{G(U)} \mathrm{Tr}\,(D\chi(x),\pi)\, dW(x,\pi)\,.$$

Take an increasing function $\eta \in C^\infty([0,1])$ which vanishes on $[3/4,1]$ and is identically 1 on $[0,1/4]$. Denote by φ the function given by $\varphi(x) = \eta(|y-x|/r)$ for $y \in B_r(x)$. Take the interior unit normal ν to ∂K in x, and let z_t be the point $x + t\nu$. If we define vector fields ψ_t and χ_t by

$$\psi_t(y) \;=\; -\frac{y-z_t}{|y-z_t|} \qquad \text{and} \qquad \chi_t = \varphi\psi_t\,,$$

then χ_t is supported in $B_r(x)$ and $D\chi_t = \varphi D\psi_t + \nabla\varphi \otimes \psi_t$. Moreover, by the strict convexity of the subset K,

$$\nabla\varphi(y)\cdot\nu > 0 \qquad \text{if } y \in \overline{K}\cap B_r(x) \text{ and } \nabla\varphi(y)\neq 0.$$

Note that ψ_t converges to ν uniformly in $B_r(x)$, as $t\uparrow\infty$. Thus, $\psi_T(y)\cdot\nabla\varphi(y)\geq 0$ for every $y\in\overline{K}\cap B_r(x)$, provided T is sufficiently large. This yields that

$$\mathrm{Tr}\,(\nabla\varphi(y)\otimes\psi_T(y),\pi)\geq 0 \qquad \text{for all } (y,\pi)\in G(B_r(x)\cap\overline{K})\,. \tag{B.1}$$

Note that $\mathrm{Tr}\,(D\psi_t(y),\pi)>0$ for all $(y,\pi)\in G(B_r(x))$ and all $t>0$. Thus

$$\begin{aligned}\delta W(\chi_T) &= \int_{G(B_r(x)\cap\overline{K})}\mathrm{Tr}\,(D\chi_T(y),\pi)\,dW(y,\pi)\\ &\overset{(B.1)}{\geq} \int_{G(B_r(x)\cap\overline{K})}\mathrm{Tr}\,(\varphi(y)D\psi_T(y),\pi)\,dW(y,\pi)\\ &\geq \int_{G(B_{r/4}(x)\cap\overline{K})}\mathrm{Tr}\,(D\psi_T(y),\pi)\,dW(y,\pi) \;>\; 0.\end{aligned}$$

This contradicts that W is stationary and completes the proof. □

Lemma B.2. Let $x\in M$ and V be a stationary integer rectifiable varifold in M. Assume T is the subset of the support of $\|V\|$ given by

$$T \;=\; \{T(y,V) \text{ consists of a plane transversal to } \partial B_{d(x,y)}(x)\}\,.$$

If $\rho<\mathrm{Inj}\,(M)$, then T is dense in $(\mathrm{supp}\,(\|V\|))\cap B_\rho(x)$.

Proof. Since V is integer rectifiable, then V is supported on a rectifiable 2–dimensional set R and there exists a Borel function $h: R\to\mathbb{N}$ such that $V=hR$. Assume the lemma is false; then there exists $y\in B_\rho(x)\cap\mathrm{supp}\,(\|V\|)$ and $t>0$ such that

$$\text{the tangent plane to } R \text{ in } z \text{ is tangent to } \partial B_{d(z,x)}(x), \text{ for any } z\in B_t(y)\,.$$

We choose t so that $B_t(y)\subset B_\rho(x)$. Take polar coordinates (r,θ,φ) in $B_\rho(x)$ and let f be a smooth nonnegative function in $C^\infty_c(B_t(y))$ with $f=1$ on $B_{t/2}(y)$. Denote by χ the vector field $\chi(\theta,\varphi,r)=f(\theta,\varphi,r)\frac{\partial}{\partial r}$. We use the notation of the proof of Lemma B.1. For every $z\in R\cap B_t(x)$, the plane π tangent to R in z is also tangent to the sphere $\partial B_{d(z,x)}(x)$. Hence, an easy computation yields that $\mathrm{Tr}\,(\chi,\pi)(z)=2\psi(z)/d(z,x)$. This gives

$$[\delta V](\chi)=\int_{R\cap B_t(y)}\frac{2h(z)\psi(z)}{\mathrm{d}\,(z,x)}d\mathcal{H}^2(z) \;>\; C\|V\|(B_{t/2}(y))\,,$$

for some positive constant C. Since $y\in\mathrm{supp}\,(\|V\|)$, we have $\|V\|(B_{t/2}(y))>0$. This contradicts that V is stationary. □

Appendix C. An example

Let $V_1\in\mathcal{V}^1(\mathcal{D}_2)$ be the 1–dimensional varifold given by three straight lines ℓ_1,ℓ_2,ℓ_3 which meet in the origin at angles of 60 degrees and let V_2 be the 1–dimensional varifold given by (see Fig. 12):

- $V_2=V_1$ in $\mathcal{D}_2\setminus\mathcal{D}_1$;

- In $\mathcal{D}_1$, V_2 is given by the regular hexagon Hex with sides of Length 1 and vertices lying on the l_i's.

Note that both V_1 and V_2 are stationary in $\mathcal{D}_2$, they have the same mass, and they coincide in $\mathcal{D}_2 \setminus \mathcal{D}_1$.

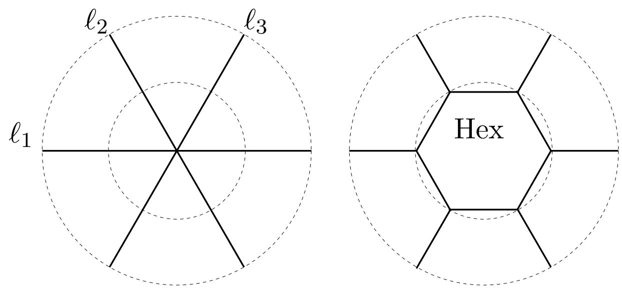

FIGURE 12. The varifolds V_1 and V_2.

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Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012

Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22 - 26, 04103 Leipzig / Germany

E-mail address: colding@cims.nyu.edu and delellis@mis.mpg.de