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Proof of a Conjecture by Lewandowski and Thiemann

by

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Abstract

It is proven that for compact, connected and semisimple structure groups every degenerate labelled web is strongly degenerate. This conjecture by Lewandowski and Thiemann implies that diffeomorphism invariant operators in the category of piecewise smooth immersive paths preserve the decomposition of the space of integrable functions w.r.t. the degeneracy and symmetry of the underlying labelled webs. This property is necessary for lifting these operators to well-defined operators on the space of diffeomorphism invariant states.

1 Introduction

One of the most striking features of general relativity is its invariance w.r.t. diffeomorphisms of the underlying space-time manifold. Its implementation into the Ashtekar formulation, however, is still not fully worked out. Here, one considers objects like generalized connections that are defined using finite graphs in the underlying space or space-time. For technical purposes, one assumed in the very beginning that these graphs are formed by piecewise analytic paths only. Namely, only in this case two finite graphs are always both contained in some third, bigger graph being again finite. This restriction has the drawback that only analyticity preserving diffeomorphisms can be implemented into that framework. In order to guarantee the inclusion of all diffeomorphisms, at least, piecewise smooth and immersive paths have to be considered as well. For the first time, this has been done by Baez and Sawin [5] introducing so-called webs. These are certain collections of paths that are independent enough to ensure the well-definedness of the generalized Ashtekar-Lewandowski measure μ_0. Applications to quantum geometry have then been studied first by Lewandowski and Thiemann [11]. For this purpose, they determined the set of possible parallel transports along webs and then discussed the diffeomorphism group averaging to generate diffeomorphism

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invariant states. Here it turned out that the extension of the spin-network formalism to the smooth-case spin-webs leads to degeneracies. These appear if some paths in a web share some full segment and the tensor product of their carried group representations includes the trivial representation. They impede the spin webs to form an orthonormal basis of μ_0-integrable functions – a striking contrast to the spin-networks in the analytic case. Moreover, the diffeomorphism averaging is defined only on those cylindrical functions that arise from nondegenerate spin-webs (having additionally finite symmetry group). To define now diffeomorphism invariant operators on diffeomorphism invariant states, these operators have to preserve the corresponding decomposition of integrable functions w.r.t. their degeneracy. In [11], Lewandowski and Thiemann showed that the images of non-degenerate spin-webs under such operators are at least still orthogonal to so-called strongly degenerate spin-webs. Now, they argued that these strongly degenerate spin-webs should be nothing but degenerate spin-webs, implying that diffeomorphism invariant operators respect the non-degeneracy of webs. In this article we are going to prove this conjecture.

The paper is organized as follows: After some preliminaries we recall the terms "richness" and "splitting" from [9]. They will be used to encode the relative position of (parts of) webs: do they coincide, are they in a certain sense independent? Next we study the decomposition of consistently parametrized paths into hyphs and list some properties of webs. In Section 6 we provide the technical details of the proof of the Lewandowski-Thiemann conjecture that will then be given in the subsequent section. In the final section of this paper we study the "canonical" example [4, 11] of a degenerate web.

2 Preliminaries

Let us briefly recall the basic facts and notations we need from the framework of generalized connections. General expositions can be found in [3, 2, 1] for the analytic framework. The smooth case is dealt with in [5, 4, 11]. The facts on hyphs and the conventions are due to [7, 8, 10].

Let $\mathbf{G}$ be some arbitrary Lie group (being compact from Section 6 on) and M be some manifold. Let $\mathcal{P}$ denote the set of all (finite) paths in M, i.e. the set of all piecewise smooth and immersive mappings from $[0,1]$ to M.[1] The set $\mathcal{P}$ is a groupoid (after imposing the standard equivalence relation, i.e., saying that reparametrizations and insertions/deletions of retracings are irrelevant). A hyph υ is some finite collection $(\gamma_1, \ldots, \gamma_n)$ of edges (i.e. non-selfintersecting paths) each having a "free" point. This means, for at least one direction none of the segments of γ_i starting in that point in this direction is a full segment of some of the γ_j with $j < i$. Graphs and webs are special hyphs. The subgroupoid generated by the paths in a hyph υ will be denoted by $\mathcal{P}_\upsilon$. Hyphs are ordered in the natural way. In particular, $\upsilon' \leq \upsilon''$ implies $\mathcal{P}_{\upsilon'} \subseteq \mathcal{P}_{\upsilon''}$. The set $\overline{\mathcal{A}}$ of generalized connections $\overline{A}$ is now defined by

$$\overline{\mathcal{A}} := \varprojlim_\upsilon \overline{\mathcal{A}}_\upsilon \cong \mathrm{Hom}(\mathcal{P}, \mathbf{G}),$$

with $\overline{\mathcal{A}}_{\boldsymbol{\gamma}} := \mathrm{Hom}(\mathcal{P}_{\boldsymbol{\gamma}}, \mathbf{G})$ given the topology induced by that of $\mathbf{G}$ for all finite tuples $\boldsymbol{\gamma}$ of paths. For those $\boldsymbol{\gamma}$ we define the (always continuous) map $\pi_{\boldsymbol{\gamma}} : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^{\#\boldsymbol{\gamma}}$ by $\pi_{\boldsymbol{\gamma}}(\overline{A}) := \overline{A}(\boldsymbol{\gamma})$. Note, that $\pi_{\boldsymbol{\gamma}}$ is surjective, if $\boldsymbol{\gamma}$ is a hyph. Finally, for compact $\mathbf{G}$, the Ashtekar-Lewandowski measure μ_0 is the unique regular Borel measure on $\overline{\mathcal{A}}$ whose push-forward $(\pi_\upsilon)_*\mu_0$ to $\overline{\mathcal{A}}_\upsilon \cong \mathbf{G}^{\#\upsilon}$ coincides with the Haar measure there for every hyph υ.

[1] Sometimes, for simplicity, we will speak about paths restricted to certain subintervals of $[0,1]$. By means of some affine map from that interval to $[0,1]$ we may regard these restrictions naturally as paths again.

3 Richness and Splittings

Let $n \in \mathbb{N}_+$ be some positive integer. We recall the notions "richness" and "splitting" from [9]. Proofs not presented in this section are either given in [9] or are obvious.

Definition 3.1 We define
- $\mathcal{V}_n$ to be the set of all n-tuples with entries equal to 0 or 1 only;
- $\mathbf{G}_v := \{(g^{v_1}, \ldots, g^{v_n}) \mid g \in \mathbf{G}\} \subseteq \mathbf{G}^n$ for every $v \in \mathcal{V}_n$; and
- $\mathbf{G}_V := \mathbf{G}_{v^1} \cdots \mathbf{G}_{v^k}$ for every ordered[2] subset $V = \{v^1, \ldots, v^k\} \subseteq \mathcal{V}_n$.

We have, e.g., $\mathbf{G}_{(1,0,1,0)} = \{(g, e_{\mathbf{G}}, g, e_{\mathbf{G}}) \mid g \in \mathbf{G}\}$.

3.1 Richness

Definition 3.2 An ordered subset $V \subseteq \mathcal{V}_n$ is called **rich** iff
1. for all $1 \leq i, j \leq n$ with $i \neq j$ there is an element $v \in V$ with $v_i \neq v_j$ and
2. for all $1 \leq i \leq n$ there is an element $w \in V$ with $w_i \neq 0$.

For instance, let $n = 4$. Then $V := \{(1,1,0,0),\ (1,0,1,0),\ (0,1,0,1),\ (0,0,1,1)\}$ is rich, but $\{(1,1,0,1),\ (1,0,1,1),\ (0,1,1,0)\}$ is not because it fails to fulfill the first condition for $i = 1$ and $j = 4$.

Next we quote the main theorem on rich ordered subsets from [9]. Note that every connected compact semisimple Lie group equals its commutator subgroup.

Theorem 3.1 Let $\mathbf{G}$ be a connected compact semisimple Lie group and n be some positive integer. Then there is a positive integer $q(n)$ such that $[\mathbf{G}_V]^{\bullet q(n)} = \mathbf{G}^n$ for any rich ordered subset V of $\mathcal{V}_n$.

Here, $[\mathbf{G}_V]^{\bullet q}$ denotes the q-fold multiplication $\mathbf{G}_V \cdots \mathbf{G}_V$ of $\mathbf{G}_V$. On the other hand, we use $\mathbf{G}^n$ as usual for the n-fold direct product $\mathbf{G} \times \cdots \times \mathbf{G}$ of $\mathbf{G}$. Note, moreover, that $q(n)$ in the theorem above does not depend on the ordering or the number of elements in V. Finally, we have $\mathbf{G}^n = [\mathbf{G}_V]^{\bullet q(n)} \subseteq [\mathbf{G}_V]^{\bullet q} \subseteq \mathbf{G}^n$ for all $q \geq q(n)$.

3.2 Splittings

Definition 3.3
- A subset $V \subseteq \mathcal{V}_n$ is called ***n*-splitting** iff
  1. $\sum_{v \in V} v = (1, \ldots, 1)$ and
  2. $(0, \ldots, 0) \notin V$.
- Let V and V' be n-splittings. V' is called **refinement** of V (shortly: $V' \geq V$) iff every $v \in V$ can be written as a sum of elements in V'.

Directly from the definition we get

Lemma 3.2
- We have $V \leq V_{\max}$ for all n-splittings V, where $V_{\max}$ contains precisely the elements of $\mathcal{V}_n$ having precisely one component equal 1.
- An n-splitting V is rich iff $V = V_{\max}$.

[2] By an ordered subset of X we mean an arbitrary tuple of elements in X where every element in X occurs at most once as a component of that tuple. However, we will use the standard terminology of sets if misunderstandings seem to be impossible.

Definition 3.4 For all n-splittings V we define

$$\begin{array}{rccc} \pi_V : & \mathbf{G}^n & \longrightarrow & \mathbf{G}^n, \\ & (g_1,\ldots,g_n) & \longmapsto & (g_{s_V(1)},\ldots,g_{s_V(n)}) \end{array}$$

where $s_V(i)$ is given by

$$s_V(i) := \min\{j \in [1,n] \mid \text{there is a } v \in V \text{ with } v_j = 1 = v_i\}.$$

Lemma 3.3 We have for all n-splittings V and V' with $V \leq V'$:

1. $s_{V'} \circ s_V = s_V$,
2. $\pi_{V'} \circ \pi_V = \pi_V$,
3. π_V is a $*$-homomorphism and
4. $\pi_{V_{\max}}$ is the identity.

Proof
1. Let $1 \leq i \leq n$ be given. Choose $v \in V$ and $v' \in V'$, such that $v'_{s_V(i)} = 1 = v_{s_V(i)}$. By definition, we have $v'_{s_{V'}(s_V(i))} = 1 = v'_{s_V(i)}$. Due to $V \leq V'$, this implies $v_{s_{V'}(s_V(i))} = 1 = v_{s_V(i)}$, hence $v_{s_{V'}(s_V(i))} = 1 = v_i$ by definition of s_V. Again, by the minimum requirement in the definition of s_V we have $s_V(i) \leq s_{V'}(s_V(i))$. On the other hand, $s_{V'}$ is obviously non-increasing, hence $s_V(i) = s_{V'}(s_V(i))$.
2. Follows immediately from $s_{V'} \circ s_V = s_V$.
3. Clear by the properties of n-splittings.
4. Trivial. **qed**

Lemma 3.4 For every n-splitting V we have $\mathbf{G}_V = \prod_{v\in V} \mathbf{G}_v = \pi_V(\mathbf{G}^n)$ independently of the ordering in V. Moreover, $\mathbf{G}_V$ is a Lie subgroup of $\mathbf{G}^n$.

Definition 3.5 Let $n \in \mathbb{N}_+$ be some positive integer, S be some set and $\vec{s}$ be some n-tuple of elements of S. Then the **splitting** $V(\vec{s})$ **for** $\vec{s}$ is given by

$$V(\vec{s}) := \{v \in \mathcal{V}_n \mid v_i = 1 = v_j \Longleftrightarrow s_i = s_j\} \setminus \{(0,\ldots,0)\}.$$

For example, the splitting for $\vec{s} = (s_1, s_2, s_3, s_2)$ is $V(\vec{s}) = \{(1,0,0,0),(0,1,0,1),(0,0,1,0)\}$.

Lemma 3.5 For every n, S and $\vec{s}$ as given in Definition 3.5, $V(\vec{s})$ is an n-splitting.

4 Consistent Parametrization

In this short section, consistently parametrized paths [5] are studied. These are paths whose parameters coincide if their images in the manifold M coincide. We will prove that those paths can always be decomposed at finitely many parameter values such that the subpaths generated this way are graph-theoretically (hence [7] measure-theoretically) independent, unless they are equal.

Definition 4.1 Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ be some n-tuple of edges.

- $\boldsymbol{\gamma}$ is called **nice** iff its **reduction** $\mathcal{R}(\boldsymbol{\gamma}) := \{\gamma_1, \ldots, \gamma_n\}$ is a hyph.[3]
- $\boldsymbol{\gamma}$ is called **consistently parametrized** iff for all $i, j = 1, \ldots, n$ we have
$$\gamma_i(t') = \gamma_j(t'') \implies t' = t''.$$

For example, we have for $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ with $\gamma_2 = \gamma_4$

$$\begin{array}{rcl} \mathcal{R}(\boldsymbol{\gamma}) & = & \{\gamma_1, \gamma_2, \gamma_3\}, \\ V(\boldsymbol{\gamma}) & = & \{(1,0,0,0),(0,1,0,1),(0,0,1,0)\}. \end{array}$$

[3] Observe that $\{\gamma_1, \ldots, \gamma_n\}$ denotes the (if necessary, ordered) *set* of all components of the *tuple* $\boldsymbol{\gamma}$.

Proposition 4.1 Let $\boldsymbol{\gamma}$ be a consistently parametrized n-tuple of edges and $I \subseteq [0,1]$ be some nontrivial interval (i.e. I consists of at least two points).
Then there is some $N \in \mathbb{N}_+$ and a sequence
$$\min I = \tau_0 < \tau_1 < \ldots < \tau_N = \max I,$$
such that $\bigcup_{i=1}^N \mathcal{R}(\boldsymbol{\gamma}|_{[\tau_{i-1},\tau_i]})$ is a disjoint union and a hyph, i.e., in particular, each $\boldsymbol{\gamma}|_{[\tau_{i-1},\tau_i]}$ is nice.

Proof • Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$. Define for every $\tau \in I$ and every $i = 1, \ldots, n$ the sets
$$I_{\tau,+,j,k} := \{\tau' \in [\tau,1] \mid \gamma_j|_{[\tau,\tau']} = \gamma_k|_{[\tau,\tau']}\} \cap I$$
and
$$I_{\tau,-,j,k} := \{\tau' \in [0,\tau] \mid \gamma_j|_{[\tau',\tau]} = \gamma_k|_{[\tau',\tau]}\} \cap I.$$
Observe first, that $I_{\tau,\pm,j,k}$ is always closed, since edges are continuous mappings from $[0,1]$ to M and $\boldsymbol{\gamma}$ is consistently parametrized. Moreover, it is always connected and contains τ unless it is empty. Consequently,
$$I_{\tau,\pm} := \bigcap_{\substack{j,k=1,\ldots,n \\ j \neq k \text{ and } I_{\tau,\pm,j,k} \setminus \{\tau\} \neq \varnothing}} I_{\tau,\pm,j,k} \tag{1}$$
is always a closed and connected, but possibly empty subset of I. (The sets $I_{\tau,\pm}$ are assumed empty, if $I_{\tau,\pm,j,k} \setminus \{\tau\} = \varnothing$ for all $j \neq k$.) More precisely, we have two cases. Excluding the exception $\tau = \max I$, we have:

– If $I_{\tau,+}$ is non-empty, then $I_{\tau,+}$ is a nontrivial interval (i.e. not a single point) because the intersection in (1) is finite.[4]

– If $I_{\tau,+}$ is empty, then again by that finiteness we have $I_{\tau,\pm,j,k} \setminus \{\tau\} = \varnothing$, hence $\gamma_j|_{[\tau,1]} \not\upuparrows \gamma_k|_{[\tau,1]}$ for all $j \neq k$.

Similar results are true for $I_{\tau,-}$.

• Assume first that there is some $\tau \in I$ such that $I_{\tau,+}$ (for $\tau \neq \max I$) or $I_{\tau,-}$ (for $\tau \neq \min I$) is empty. Then we have in the first case $\gamma_j|_{[\tau,\max I]} \not\upuparrows \gamma_k|_{[\tau,\max I]}$ for all $j \neq k$, hence, $\boldsymbol{\gamma}|_I \equiv \mathcal{R}(\boldsymbol{\gamma}|_I)$ is a hyph. Defining $\tau_0 := \min I$ and $\tau_1 := \max I$, we get the assertion. The second case is completely analogous.

• Assume now that there is no $\tau \in I$ such that $I_{\tau,+}$ (for $\tau \neq \max I$) or $I_{\tau,-}$ (for $\tau \neq \min I$) is empty.

– Construction of the sequence (τ_i) in I
Set $\tau_0 := \min I$. Then proceed successively, until $\tau_j = \max I$ for some j:
1. $\tau_{2i+1} := \max I_{\tau_{2i},+}$.
2. $\tau_{2i+2} := \max\{\tau \in [\tau_{2i+1}, \max I] \mid I_{\tau_{2i},+} \cap I_{\tau,-} \neq \varnothing\}$.

– Well-definedness of the construction
1. τ_{2i+1} exists, since $I_{\tau_{2i},+}$ is always a closed interval. Moreover, $\tau_{2i+1} > \tau_{2i}$, since by assumption $\tau_{2i} \neq \max I$ and $I_{\tau_{2i},+}$ is nonempty, hence a nontrivial interval starting at τ_{2i}.
2. τ_{2i+2} exists. In fact, since by construction $\min I \leq \tau_{2i} < \tau_{2i+1} < \max I$ and so neither $I_{\tau_{2i},+}$ nor $I_{\tau_{2i+1},-}$ are empty, we get $\tau_{2i+1} \in I_{\tau_{2i},+} \cap I_{\tau_{2i+1},-}$. Hence, the set J which τ_{2i+2} is supposed to be the maximum of, is non-empty. It remains the question whether J has indeed a maximum. For this, set $\sigma := \sup J$ and assume $\sigma_l \uparrow \sigma$ strictly increasing with non-empty $I_{\tau_{2i},+} \cap I_{\sigma_l,-}$ for all $l \in \mathbb{N}$. Fix $j \neq k$. There are two cases:
   · Let there exist some l' such that $\gamma_j|_{[\tau_{2i+1},\sigma_l]} \neq \gamma_k|_{[\tau_{2i+1},\sigma_l]}$ for all $l \geq l'$. Then $I_{\sigma,-,j,k} \setminus \{\sigma\}$ is empty: Otherwise, there would be some $l_0 \geq l'$ such that $\sigma_{l_0} \in I_{\sigma,-,j,k}$, and then $\sigma_{l_0} \in I_{\sigma_{l_0+1},-,j,k} \supseteq I_{\sigma_{l_0+1},-} \ni \tau_{2i+1}$

[4] A finite intersection of intervals containing τ and some other point larger than τ is again such an interval.

which would imply that $\gamma_j|_{[\tau_{2i+1},\sigma_{l_0}]} = \gamma_k|_{[\tau_{2i+1},\sigma_{l_0}]}$. Contradiction.

· Let there exist no l' such that $\gamma_j|_{[\tau_{2i+1},\sigma_l]} \neq \gamma_k|_{[\tau_{2i+1},\sigma_l]}$ for all $l \geq l'$. Then there is an infinite subsequence (σ_{l_q}) of (σ_l), such that we have $\gamma_j|_{[\tau_{2i+1},\sigma_{l_q}]} = \gamma_k|_{[\tau_{2i+1},\sigma_{l_q}]}$ for all q. Hence, $\gamma_j|_{[\tau_{2i+1},\sigma]} = \gamma_k|_{[\tau_{2i+1},\sigma]}$, i.e. $\tau_{2i+1} \in I_{\sigma,-,j,k}$.

Altogether, since $I_{\sigma,-} \neq \varnothing$ by assumption, we have $\tau_{2i+1} \in I_{\sigma,-}$, and so $\sigma \in J$, since $\tau_{2i+1} \in I_{\tau_{2i},+}$. Obviously, $\tau_{2i+2} \geq \tau_{2i+1}$.

– Stopping of the Construction
Suppose, there were no $N \in \mathbb{N}$ such that $\tau_N = \max I$. Then $(\tau_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence in I having some limit $\tau \in I$ with $\tau_i < \tau$ for all i. Of course, $\tau > \min I$.
Let $\tau' \in I_{\tau,-}$ with $\tau' < \tau$. (Remember that $I_{\tau,-}$ is nonempty.) Then there is some $i_0 \in \mathbb{N}$ with $\tau' \leq \tau_{2i_0+1} < \tau$. Consequently, $I_{\tau_{2i_0},+} \cap I_{\tau,-}$ contains τ_{2i_0+1}. This implies by the second step of the construction above, that $\tau \leq \tau_{2i_0+2}$. This, however, is a contradiction to $\tau > \tau_i$ for all i.

– Final adjustment
Drop now all τ_{2i+2} from that sequence with $\tau_{2i+1} = \tau_{2i+2}$, and denote the resulting finite subsequence again by $(\tau_0, \ldots, \tau_N)$.

This sequence fulfills the requirements of the proposition:

1. $\mathcal{R}(\boldsymbol{\gamma}|_{[\tau_{i-1},\tau_i]})$ is a hyph.
Let first i correspond to some "originally" odd i. Choose some path in $\mathcal{R}(\boldsymbol{\gamma}|_{[\tau_{i-1},\tau_i]})$, say $\gamma_j|_{[\tau_{i-1},\tau_i]}$. If $\gamma_j|_{[\tau_{i-1},\tau_i]} \uparrow\uparrow \gamma_k|_{[\tau_{i-1},\tau_i]}$, then there is some $\sigma \in (\tau_{i-1}, \tau_i]$ with $\gamma_j|_{[\tau_{i-1},\sigma]} = \gamma_k|_{[\tau_{i-1},\sigma]}$, hence $[\tau_{i-1}, \sigma] \subseteq I_{\tau_{i-1},+,j,k}$. By construction, we have $I_{\tau_{i-1},+,j,k} \supseteq I_{\tau_{i-1},+} = [\tau_{i-1}, \tau_i]$ and thus $\gamma_j|_{[\tau_{i-1},\tau_i]} = \gamma_k|_{[\tau_{i-1},\tau_i]}$. This means, they define the same element in $\mathcal{R}(\boldsymbol{\gamma}|_{[\tau_{i-1},\tau_i]})$. Therefore, $\gamma_j|_{[\tau_{i-1},\tau_i]} \not\uparrow\not\uparrow \gamma_k|_{[\tau_{i-1},\tau_i]}$ for different elements. By the consistent parametrization, $\mathcal{R}(\boldsymbol{\gamma}|_{[\tau_{i-1},\tau_i]})$ is a hyph.
The case of "even" i goes analogously.
2. $\bigcup_i \mathcal{R}(\boldsymbol{\gamma}|_{[\tau_{i-1},\tau_i]})$ is a hyph and a disjoint union.
The consistent parametrization of $\boldsymbol{\gamma}$ implies that $\gamma_j|_{[\tau_{i-1},\tau_i]} \uparrow\uparrow \gamma_{j'}|_{[\tau_{i'-1},\tau_{i'}]}$. (or any other relation $\downarrow\uparrow$, $\uparrow\downarrow$ or $\downarrow\downarrow$) is possible for $i = i'$ only. Together with the previous step we get the assertion. **qed**

5 Basic Facts about Webs

Let us start with some definitions. Note that the definition of the $\boldsymbol{\gamma}$-type of a point is slightly different from that in [5].

Definition 5.1 Let $\boldsymbol{\gamma}$ be some n-tuple of paths.

- A point $x \in M$ is called **$\boldsymbol{\gamma}$-regular** iff x is not an endpoint or nondifferentiable point of one of the paths in $\boldsymbol{\gamma}$ and there is a neighbourhood of x whose intersection with im $\boldsymbol{\gamma}$ is an embedded interval. [5]
- $\tau \in [0, 1]$ is called $\boldsymbol{\gamma}$-regular iff $\gamma(\tau)$ is $\boldsymbol{\gamma}$-regular for all $\gamma \in \boldsymbol{\gamma}$.

Definition 5.2 Let $\boldsymbol{\gamma}$ be some n-tuple of paths.

- For every $x \in M$ we define the **$\boldsymbol{\gamma}$-type** $v(x) \in \mathcal{V}_n$ of x by
$$v(x)_i := \begin{cases} 1 & \text{if } x \in \text{im } \gamma_i \\ 0 & \text{if } x \notin \text{im } \gamma_i \end{cases}.$$

- For every consistently parametrized $\boldsymbol{\gamma}$ we define
$$V_{\boldsymbol{\gamma}} := \bigcup_{\tau \,\in\, [0,1],\ \tau\ \boldsymbol{\gamma}\text{-regular}} V(\boldsymbol{\gamma}(\tau)).$$

For consistently parametrized $\boldsymbol{\gamma}$, obviously, $V(\boldsymbol{\gamma}(\tau))$ is the set of all $\boldsymbol{\gamma}$-types of points in $\boldsymbol{\gamma}(\tau)$. Note, moreover, that in general the set $V_{\boldsymbol{\gamma}}$ of types in $\boldsymbol{\gamma}$ and the splitting $V(\boldsymbol{\gamma})$ for $\boldsymbol{\gamma}$ do not coincide. For instance, we have in the case of Figure 1 (see page 17 with $\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$)

$$\begin{aligned} V_{\boldsymbol{\gamma}} &= \{(1,1,0,0),(1,0,1,0),(0,1,0,1),(0,0,1,1)\}, \\ V(\boldsymbol{\gamma}) &= \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}. \end{aligned}$$

In a certain sense, $V_{\boldsymbol{\gamma}}$ is finer. $V(\boldsymbol{\gamma})$ only looks whether two whole paths are equal or not. $V_{\boldsymbol{\gamma}}$ looks closer at the image points of $\boldsymbol{\gamma}$.

We now recall the definition of tassels and webs owing to Baez and Sawin [5, 4].

Definition 5.3
- A finite ordered set $T = \{c_1, \ldots, c_n\}$ of paths is called **tassel based on** $p \in \operatorname{im} T$ iff the following conditions are met:
  1. $\operatorname{im} T$ lies in a contractible open subset of M.
  2. T can be consistently parametrized in such a way that $c_i(0) = p$ is the left endpoint of every path c_i.
  3. Two paths in T that intersect at a point other than p intersect at a point other than p in every neighborhood of p.
  4. For every neighbourhood U of p, any T-type which occurs at some regular point in $\operatorname{im} T$ occurs at some regular point in $U \cap \operatorname{im} T$.
  5. No two paths in T have the same image.
- A finite collection $w = w^1 \cup \cdots \cup w^k$ of tassels is called **web** iff for all $i \neq j$ the following conditions are met:
  1. Any path in the tassel w^i intersects any path in w^j, if at all, only at their endpoints.
  2. There is a neighborhood of each such intersection point whose intersection with $\operatorname{im}\,(w^i \cup w^j)$ is an embedded interval.
  3. $\operatorname{im} w^i$ does not contain the base of w^j.

Next, we list some important properties of webs that can be derived immediately from statements in [5]. The proofs are given in [9].

Proposition 5.1 For every web w the set $[0,1]_{\mathrm{reg}}$ of w-regular parameter values is open and dense in $[0,1]$. Moreover, the function $V(w(\cdot)) : [0,1]_{\mathrm{reg}} \longrightarrow \mathcal{V}_{\#w}$, assigning to every w-regular τ its splitting, is locally constant.

Lemma 5.2 For every web w the set V_w of w-types occurring in w is rich.

Let us define the set $\mathcal{V}(w) := \bigcap_{\tau \in (0,1]} \bigcup_{\sigma \in [0,\tau]_{\mathrm{reg}}} \{V(w(\sigma))\}$ of all those splittings $V(w(\sigma))$ that appear in every neighbourhood of 0. Here, I_{reg} denotes the set of w-regular elements in an arbitrary interval $I \subseteq [0,1]$.

Lemma 5.3 Let w be a web. Then for all $v \in V_w$ there is some $V \in \mathcal{V}(w)$ with $v \in V$. In particular, $\mathcal{V}(w)$ is nonempty (if w is nonempty).

Corollary 5.4 $\bigcup_{V \in \mathcal{V}(w)} V$ equals V_w for every web w and is rich.

6 Operator-Valued Integrals

Let now $\mathbf{G}$ be *compact*. Let us fix some positive integer n and some n-tuple $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n)$ of irreducible (unitary) representations of $\mathbf{G}$ with X_i being the representation space of φ_i. Set $\boldsymbol{\varphi} := \bigotimes_k \varphi_k$ to be the tensor product representation of $\mathbf{G}^n$ on $X := \bigotimes_k X_k$ corresponding to $\vec{\varphi}$. Moreover, let $Y := \overline{X} \otimes X$. Y is now the representation space for the $\mathbf{G}^n$-representation $\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi} = \bigotimes_k \overline{\varphi_k} \otimes \bigotimes_k \varphi_k$. Finally, we equip $\operatorname{End} X$ and $\operatorname{End} Y$ with the standard operator norm.

Definition 6.1 For every continuous function $D : \mathbf{G}^n \longrightarrow \operatorname{End} X$ we define

- the (integrated and normalized Frobenius) norm[5] of D by
$$\|D\|_F^2 := \frac{1}{\dim X} \int_{\mathbf{G}^n} \operatorname{tr}(D^* D) \, \mathrm{d}\mu_{\text{Haar}}$$
and
- the operator $Q_D \in \operatorname{End} Y$ by
$$Q_D := \int_{\mathbf{G}^n} \overline{D} \otimes D \, \mathrm{d}\mu_{\text{Haar}}.$$

Lemma 6.1 Let $D, E : \mathbf{G}^n \longrightarrow \operatorname{End} X$ be continuous functions.
If E is unitary, then $\|E\|_F = 1$ and $\|E^* D E\|_F = \|D\|_F$.

Proof Trivial. **qed**

Lemma 6.2 Let $D : \mathbf{G}^n \longrightarrow \operatorname{End} X$ be a $*$-homomorphism.
Then $Q_D : Y \longrightarrow Y$ is an orthogonal projector.

Proof By $Q_D^* = Q_{D^*}$ and $D^*(\vec{g}) = D(\vec{g}^*)$, the homomorphy property of D implies

$$\begin{aligned} Q_D^* Q_D &= \left(\int_{\mathbf{G}^n} (\overline{D}^* \otimes D^*)(\vec{g}) \, \mathrm{d}\mu_{\text{Haar}} \right) \left(\int_{\mathbf{G}^n} (\overline{D} \otimes D)(\vec{g}') \, \mathrm{d}\mu_{\text{Haar}} \right) \\ &= \int_{\mathbf{G}^n} \int_{\mathbf{G}^n} (\overline{D} \otimes D)(\vec{g}^* \vec{g}') \, \mathrm{d}\mu_{\text{Haar}} \, \mathrm{d}\mu_{\text{Haar}} \;=\; \int_{\mathbf{G}^n} (\overline{D} \otimes D)(\vec{g}) \, \mathrm{d}\mu_{\text{Haar}} \\ &= Q_D \end{aligned}$$

using the translation invariance and normalization of the Haar measure. Hence, we get $Q_D^* = (Q_D^* Q_D)^* = Q_D^* Q_D = Q_D$ and $Q_D = Q_D Q_D$. **qed**

6.1 Scalar-Product Projectors

Definition 6.2 We define for all n-splittings V
$$P_V := Q_{\boldsymbol{\varphi} \circ \pi_V} = \int_{\mathbf{G}^n} (\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi}) \circ \pi_V \, \mathrm{d}\mu_{\text{Haar}} \in \operatorname{End} Y$$
and set $P_0 := P_{V_{\max}}$.

Lemma 6.3 We have for all n-splittings V and V':

1. $P_{V'} P_V = P_{V'} = P_V P_{V'}$ if $V \leq V'$.
2. P_V is an orthogonal projection on Y.
3. $P_0 P_V = P_0 = P_V P_0$.
4. $\|P_V\| = 1$.
5. $P_V Y = \{y \in Y \mid y \text{ is } (\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi})(\mathbf{G}_V)\text{-invariant}\}$.

[5]Note, that $\|\cdot\|_F$ is, in general, not a matrix norm due to the normalization.

Proof 1. Using Lemma 3.3 and the homomorphy property of $\boldsymbol{\varphi}$, we have

$$\boldsymbol{\varphi}(\pi_V(\vec{g}))\,\boldsymbol{\varphi}(\pi_{V'}(\vec{g}')) = \boldsymbol{\varphi}(\pi_{V'}(\pi_V(\vec{g})))\,\boldsymbol{\varphi}(\pi_{V'}(\vec{g}')) = \boldsymbol{\varphi}(\pi_{V'}(\pi_V(\vec{g})\,\vec{g}')).$$

Consequently,

$$\begin{aligned} P_V P_{V'} &= \Big(\int_{\mathbf{G}^n} (\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi})(\pi_V(\vec{g}))\,\mathrm{d}\mu_{\mathrm{Haar}}\Big)\Big(\int_{\mathbf{G}^n} (\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi})(\pi_{V'}(\vec{g}'))\,\mathrm{d}\mu_{\mathrm{Haar}}\Big) \\ &= \int_{\mathbf{G}^n}\int_{\mathbf{G}^n} (\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi})(\pi_{V'}(\pi_V(\vec{g})\,\vec{g}'))\,\mathrm{d}\mu_{\mathrm{Haar}}\,\mathrm{d}\mu_{\mathrm{Haar}} \\ &= \int_{\mathbf{G}^n} (\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi})(\pi_{V'}(\vec{g}'))\,\mathrm{d}\mu_{\mathrm{Haar}} \\ &= P_{V'}, \end{aligned}$$

where we used in the third step that the Haar measure is normalized and invariant w.r.t. $\vec{g}' \longmapsto \pi_V(\vec{g})^{-1}\vec{g}'$. $P_{V'}P_V = P_V$ follows precisely the same way.

2. Follows from Lemma 6.2 since each φ_k is unitary and π_V is a $*$-homomorphism.
3. Follows from $V \leq V_{\max}$ for all V and the statements above.
4. Being a projection, $\|P_V\| = 1$ unless P_V is zero. Since $P_V P_0 = P_0$ and $P_0 \neq 0$ (for an explicit computation of its matrix elements see the proof of Lemma 6.5), $P_V = 0$ is impossible.
5. Let $\phi_V : Y \longrightarrow \bigoplus_l W_l$ be a unitary map decomposing the $\mathbf{G}^n$-representation $(\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi}) \circ \pi_V$ into a direct sum of irreducible representations ρ_l on W_l. Then we have

$$\phi_V(P_V y) \;=\; (\phi_V \circ P_V \circ \phi_V^{-1})(\phi_V(y)) \;=\; \big(\textstyle\int_{\mathbf{G}^n} \bigoplus_l \rho_l\,\mathrm{d}\mu_{\mathrm{Haar}}\big)(\phi_V(y)).$$

Since $\int_{\mathbf{G}^n} \rho_l\,\mathrm{d}\mu_{\mathrm{Haar}}$ equals 0 if ρ_l is non-trivial and equals 1 if ρ_l is trivial, we have

$$\begin{aligned} & P_V y = y \\ \Longleftrightarrow\;& \phi_V(P_V y) = \phi_V(y) \\ \Longleftrightarrow\;& \phi_V(y) \text{ is contained in } \textstyle\bigoplus_{\rho_l = 0} W_l \\ \Longleftrightarrow\;& \phi_V(y) \text{ is invariant w.r.t. } \textstyle\bigoplus_l \rho_l(\mathbf{G}^n) \\ \Longleftrightarrow\;& y \text{ is invariant w.r.t. } \big((\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi}) \circ \pi_V\big)(\mathbf{G}^n) = \phi_V^{-1}\big(\textstyle\bigoplus_l \rho_l(\mathbf{G}^n)\big)\phi_V. \end{aligned}$$

$\pi_V(\mathbf{G}^n) = \mathbf{G}_V$ gives the assertion. **qed**

Lemma 6.4 Let $\mathcal{V} \subseteq \mathcal{V}_n$ be some subset and define $V_{\mathcal{V}} := \bigcup_{V \in \mathcal{V}} V$. Assume, moreover, that there is some $q \in \mathbb{N}_+$ with $[\mathbf{G}_{V_{\mathcal{V}}}]^{\bullet q} = \mathbf{G}^n$. Then we have $\bigcap_{V \in \mathcal{V}} P_V Y = P_0 Y$.

Proof "$\supseteq$" Since $P_V P_0 = P_0$, we have $P_0 Y \subseteq P_V Y$ for all $V \in \mathcal{V}_n \supseteq \mathcal{V}$.

"$\subseteq$" Let now $y \in P_V Y$ for all $V \in \mathcal{V}$. By Lemma 6.3, y is invariant under each corresponding $(\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi})(\mathbf{G}_V)$, hence w.r.t. $(\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi})(\mathbf{G}_v)$ for all $v \in V_{\mathcal{V}}$. By assumption, every element in $\mathbf{G}^n$ can be written as some finite product of elements in $\mathbf{G}_{V_{\mathcal{V}}}$, hence in $\bigcup_{v \in V_{\mathcal{V}}} \mathbf{G}_v$ as well. By the homomorphy property of $\boldsymbol{\varphi}$, we get the invariance of y w.r.t. $(\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi})(\mathbf{G}^n)$, hence $y \in P_0 Y$. **qed**

6.2 More General Operators

Lemma 6.5 For every continuous $D : \mathbf{G}^n \longrightarrow \mathrm{End}\,X$ we have $P_0 Q_D P_0 = \|D\|_F^2\, P_0$.

Proof Introducing some bases on the X_i and then forming multi-indices we have

$$\begin{aligned} (P_0)^{\boldsymbol{im}}_{\boldsymbol{jn}} &= \textstyle\int_{\mathbf{G}^n} (\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi})^{\boldsymbol{im}}_{\boldsymbol{jn}}\,\mathrm{d}\mu_{\mathrm{Haar}} &&= \textstyle\prod_k \int_{\mathbf{G}} (\overline{\varphi_k})^{i_k}_{j_k}(\varphi_k)^{m_k}_{n_k}\,\mathrm{d}\mu_{\mathrm{Haar}} \\ &= \textstyle\prod_k \frac{1}{\dim \varphi_k}\,\delta^{i_k m_k}\delta_{j_k n_k} &&= \frac{1}{\dim X}\,\delta^{\boldsymbol{im}}\delta_{\boldsymbol{jn}} \end{aligned}$$

and hence

$$\begin{aligned}(P_0 Q_D P_0)^{\boldsymbol{im}}_{\boldsymbol{jn}} &= (P_0)^{\boldsymbol{im}}_{\boldsymbol{pr}} (Q_D)^{\boldsymbol{pr}}_{\boldsymbol{qs}} (P_0)^{\boldsymbol{qs}}_{\boldsymbol{jn}} = \frac{1}{\dim X}\,\delta^{\boldsymbol{im}}\delta_{\boldsymbol{pr}}\,(Q_D)^{\boldsymbol{pr}}_{\boldsymbol{qs}}\,\frac{1}{\dim X}\,\delta^{\boldsymbol{qs}}\delta_{\boldsymbol{jn}} \\ &= \frac{1}{\dim X}\,\delta^{\boldsymbol{im}}\delta_{\boldsymbol{jn}}\,\frac{1}{\dim X}\,(Q_D)^{\boldsymbol{pp}}_{\boldsymbol{qq}} = (P_0)^{\boldsymbol{im}}_{\boldsymbol{jn}}\,\frac{1}{\dim X}\,(Q_D)^{\boldsymbol{pp}}_{\boldsymbol{qq}}.\end{aligned}$$

Using

$$(Q_D)^{\boldsymbol{pp}}_{\boldsymbol{qq}} = \int_{\mathbf{G}^n} \overline{D^{\boldsymbol{p}}_{\boldsymbol{q}}} D^{\boldsymbol{p}}_{\boldsymbol{q}}\ \mathrm{d}\mu_{\mathrm{Haar}} = \int_{\mathbf{G}^n} (D^*)^{\boldsymbol{q}}_{\boldsymbol{p}} D^{\boldsymbol{p}}_{\boldsymbol{q}}\ \mathrm{d}\mu_{\mathrm{Haar}} = (\dim X)\,\|D\|_F^2,$$

we have $P_0 Q_D P_0 = \|D\|_F^2\, P_0$. **qed**

Definition 6.3 Let V be some n-splitting and let $\bigoplus_{l_k} \rho_{k,l_k}$ for each $k = 1,\ldots,n$ be the decomposition of $\bigotimes_{i:s_V(i)=k} \varphi_i$ into irreducible $\mathbf{G}$-representations. Furthermore, denote the representation space of each ρ_{k,l_k} by W_{k,l_k}, and let $\phi : X \longrightarrow \bigotimes_{k=s_V(k)} \bigoplus_{l_k} W_{k,l_k}$ be the corresponding unitary intertwiner. Define $D_{V,q} : \mathbf{G}^n \longrightarrow \mathrm{End}\, X$ for all $1 \leq q \leq n$ via

$$\phi\, D_{V,q}(g_1,\ldots,g_n)\,\phi^{-1} := \bigotimes_{\substack{k \\ k=s_V(k)}} \begin{cases} \bigoplus_{l_k} \rho_{k,l_k}(g_k) & \text{for } s_V(k) \neq s_V(q) \\ \bigoplus_{l_k \neq 0} \rho_{k,l_k}(g_k) & \text{for } s_V(k) = s_V(q) \end{cases}$$

and set $Q_{V,q} := Q_{D_{V,q}}$.

In other words, $D_{V,q}$ just projects to the subspace of X which is orthogonal to the subspace that carries the trivial representation after tensoring all φ_i where i is "equivalent" to q, i.e. where i is running over all components in v being 1 where v is just the element in V whose q-component is 1. Note, furthermore, that $D_{V,q} = D_{V,q} \circ \pi_V$.

Lemma 6.6 For every n-splitting V and every $1 \leq q \leq n$ we have

$$\|D_{V,q}\|_F^2 = 1 - \frac{d_q^0}{d_q}$$

where d_q is the dimension of the representation $\bigoplus_l \rho_{s_V(q),l}$ and d_q^0 the number of trivial $\rho_{s_V(q),l}$ in this direct sum.

Proof Since $D_{V,q} = D_{V,s_V(q)}$, we may assume $q = s_V(q)$. Then, using the unitarity of ϕ and φ_k and the fact that tensor products for terms depending on different g_k contribute to the norm as separate factors, we have

$$\|D_{V,q}\|_F^2 = \frac{\int_{\mathbf{G}} \mathrm{tr}\big(\bigoplus_{l \text{ with } \rho_{q,l}\neq 0} \rho_{q,l}^* \rho_{q,l}\big)\,\mathrm{d}\mu_{\mathrm{Haar}}}{\prod_{i:s_V(i)=q} \dim \varphi_i} = \frac{\sum_{l \text{ with } \rho_{q,l}\neq 0} \dim \rho_{q,l}}{\sum_l \dim \rho_{q,l}} = 1 - \frac{d_q^0}{d_q}.$$

qed

Lemma 6.7 For every n-splitting V and every $1 \leq q \leq n$ we have $\|Q_{V,q}\| \leq 1$.

Proof By construction, $D_{V,q}$ is a $*$-homomorphism. Now, Lemma 6.2 gives the assertion. **qed**

6.3 Application to Nice Sets of Paths

Lemma 6.8 For every nice n-tuple $\boldsymbol{\gamma}$ of edges and every continuous $f : \mathbf{G}^n \longrightarrow \mathbb{C}$ we have

$$\int_{\overline{\mathcal{A}}} f \circ \pi_{\boldsymbol{\gamma}}\, \mathrm{d}\mu_0 = \int_{\mathbf{G}^n} f \circ \pi_{V(\boldsymbol{\gamma})}\, \mathrm{d}\mu^n_{\mathrm{Haar}}.$$

Proof Assume $\boldsymbol{\gamma}$ nice and, w.l.o.g., $\mathcal{R}(\boldsymbol{\gamma}) = \{\gamma_1,\ldots,\gamma_k\}$. Then

$$\begin{aligned}\pi_{\boldsymbol{\gamma}}(\overline{A}) &= \big(h_{\overline{A}}(\gamma_1),\ldots,h_{\overline{A}}(\gamma_k),h_{\overline{A}}(\gamma_{k+1}),\ldots,h_{\overline{A}}(\gamma_n)\big) \\ &= \pi_{V(\boldsymbol{\gamma})}\big(h_{\overline{A}}(\gamma_1),\ldots,h_{\overline{A}}(\gamma_k),h_{\overline{A}}(\gamma_{k+1}),\ldots,h_{\overline{A}}(\gamma_n)\big) \\ &= \pi_{V(\boldsymbol{\gamma})}\big(h_{\overline{A}}(\gamma_1),\ldots,h_{\overline{A}}(\gamma_k),e_{\mathbf{G}},\ldots,e_{\mathbf{G}}\big) \\ &= \pi_{V(\boldsymbol{\gamma})}(\pi_{\mathcal{R}(\boldsymbol{\gamma})} \times 1_{n-k})(\overline{A}).\end{aligned}$$

Since, by assumption, $\mathcal{R}(\boldsymbol{\gamma})$ is a hyph and μ_{Haar} is normalized, we get the assertion from $(\pi_{\mathcal{R}(\boldsymbol{\gamma})})_* \mu_0 = \mu_{\text{Haar}}^k$. Since the Haar measure is permutation invariant, we get the proof for arbitrary $\mathcal{R}(\boldsymbol{\gamma})$. **qed**

Corollary 6.9 For every nice n-tuple $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ of edges we have

$$P_{V(\boldsymbol{\gamma})} = \int_{\overline{\mathcal{A}}} (\overline{\boldsymbol{\varphi}} \otimes \boldsymbol{\varphi}) \circ \pi_{\boldsymbol{\gamma}} \, \mathrm{d}\mu_0.$$

7 Conjecture of Lewandowski and Thiemann

First we recall very briefly the definition of spin webs and then the two different notions of degeneracy [11]. The conjecture of Lewandowski and Thiemann will say that both are equivalent. Throughout the whole section, let $\mathbf{G}$ be compact.

Definition 7.1
- A **spin web** $(w, \vec{\varphi})$ consists of a web w and some $\#w$-tuple $\vec{\varphi}$ of (equivalence classes of) irreducible representations of $\mathbf{G}$.
- The **spin web state** $(T_{w,\vec{\varphi}})^{\boldsymbol{i}}_{\boldsymbol{j}}$ to a spin web $(w, \vec{\varphi})$ is defined by
$$(T_{w,\vec{\varphi}})^{\boldsymbol{i}}_{\boldsymbol{j}} := \boldsymbol{\varphi}^{\boldsymbol{i}}_{\boldsymbol{j}} \circ \pi_w : \overline{\mathcal{A}} \longrightarrow \mathbb{C}$$
with the tensor-matrix functions
$$\begin{array}{rccc} \boldsymbol{\varphi}^{\boldsymbol{i}}_{\boldsymbol{j}} = \bigotimes_k (\varphi_k)^{i_k}_{j_k} : & \mathbf{G}^{\#\vec{\varphi}} & \longrightarrow & \mathbb{C}. \\ & \vec{g} & \longmapsto & \prod_k \varphi_k(g_k)^{i_k}_{j_k} \end{array}$$
- The **spin web space** $\mathcal{H}_{w,\vec{\varphi}}$ for the spin web $(w, \vec{\varphi})$ is the $\mathbb{C}$-linear span of all spin web states for $(w, \vec{\varphi})$. The web space $\mathcal{H}_w$ is defined to be the closure of the $\mathbb{C}$-span of all possible spin web states to the web w.

We remark that the definition above can be extended directly from webs to hyphs.

Before we come to the definition of degeneracy, we still have to define for every edge s the projection $p_s : \mathcal{H} \longrightarrow \mathcal{H}$ as follows [11]: Let first e be an edge and $\Psi \in \mathcal{H}_e$. Then, $p_s\Psi := \Psi$ if e and s are disjoint (maybe up to their endpoints), and $p_s\Psi := (T_{e,0}, \Psi)T_{e,0} \equiv (1, \Psi)1$ if e is a nontrivial subpath of s. This means, p_s projects onto the part in $\mathcal{H}_s$ carrying the trivial representation. For the general case, let $\upsilon = \{\gamma_1, \ldots, \gamma_n\}$ be some hyph with $\upsilon \geq \{s\}$ and let $\Psi = \bigotimes \Psi_k \in \bigotimes \mathcal{H}_{\gamma_k}$, then $p_s\Psi := \bigotimes p_s\Psi_k$. One immediately checks that p_s is well defined. Thus, we may extend this definition by linearity and continuity.

Definition 7.2 A splitting V is called **$\vec{\varphi}$-degenerate** iff there is some $v \in V$ such that the decomposition of $\bigotimes_{k:v_k=1} \varphi_k$ into irreducible $\mathbf{G}$-representations contains the trivial representation.

For example, let $\mathbf{G} = SU(2)$ whose irreducible representations are labelled by half-integers. Then $\{(1,1,0,0),(0,0,1,1)\}$ is $(\frac{1}{2}, \frac{1}{2}, 3, \frac{5}{2})$-degenerate, since $\frac{1}{2} \otimes \frac{1}{2} \cong 1 \oplus 0$.

Definition 7.3
- A spin web $(w, \vec{\varphi})$ is called **(weakly) degenerate** iff there is some w-regular $\tau \in [0,1]$ such that $V(w(\tau))$ is $\vec{\varphi}$-degenerate, i.e. there is some w-regular point $x \in \operatorname{im} w$ such that the trivial representation is contained in the decomposition of $\bigotimes_{j:x\in \operatorname{im} w_j} \varphi_j$ into irreducible representations.
- A spin web $(w, \vec{\varphi})$ is called **strongly degenerate** iff there is a sequence $(s_l)_{l\in\mathbb{N}}$ of disjoint w-regular segments in w such that
$$\lim_{l\to\infty} (1 - p_{s_0}) \cdots (1 - p_{s_l})\Psi = 0$$
for all $\Psi \in \mathcal{H}_{w,\vec{\varphi}}$.

Here, a w-regular segment equals $w_q|_I$ for some $w_q \in w$ and some interval $I \subseteq [0,1]_{\mathrm{reg}}$.
Let us now state

Theorem 7.1 Lewandowski-Thiemann Conjecture
Let $\mathbf{G}$ be compact, connected and semisimple.
Then a spin web is weakly degenerate iff it is strongly degenerate.

Lemma 7.2 Let $\mathbf{G}$ be compact, connected and semisimple.
Then we have $\bigcap_{V \in \mathcal{V}(w)} P_V Y = P_0 Y$ for every web w.

Proof Since $V_w = \bigcup_{V \in \mathcal{V}(w)} V$ is rich by Corollary 5.4, and since $\mathbf{G}$ is compact, connected and semisimple, Theorem 3.1 guarantees that $[\mathbf{G}_{V_w}]^{\bullet q} = \mathbf{G}^n$ for some $q \in \mathbb{N}_+$. Now, Lemma 6.4 gives the proof. **qed**

Proof Theorem 7.1
Let first $(w, \vec{\varphi})$ be some spin web that is not weakly degenerate. Then $p_s \Psi = 0$ for all $\Psi \in \mathcal{H}_{w,\vec{\varphi}}$ and all w-regular segments s in w. Consequently,
$$\lim_{l \to \infty} (1 - p_{s_0}) \cdots (1 - p_{s_l}) \Psi = \Psi,$$
whence $(w, \vec{\varphi})$ is not strongly degenerate.
Let now $(w, \vec{\varphi})$ be some weakly degenerate spin web. Since the proof of its strong degeneracy is much more technical, we proceed in several steps.

1. Notations
   We denote the elements of $\mathcal{V}(w)$ by $V_1, \ldots, V_N$. Since $(w, \vec{\varphi})$ is weakly degenerate, there is some $v \in V_w$, such that $\bigotimes_{k: v_k = 1} \varphi_k$ contains the trivial representation. By Lemma 5.3, there is some $W \in \mathcal{V}(w)$ with $v \in W$. Finally, let $1 \leq q \leq n$ be some number with $v_q = 1$, where n as usual is the number of paths in w.
2. Decomposition of w
   Let us construct a sequence (τ_i) in $[0,1]$ that will be used for the decomposition of w. For this, we first define inductively a strictly decreasing sequence $(\sigma_{i,j})_{i \in \mathbb{N}, 0 \leq j \leq N}$ as follows ($\sigma_{-1,N} := 1$):
   a) $\sigma_{i+1,0}$ is some w-regular element in $[0, \sigma_{i,N})$, such that $V(w(\sigma_{i+1,0})) = W$;
   b) $\sigma_{i,j+1}$ is some w-regular element in $[0, \sigma_{i,j})$, such that $V(w(\sigma_{i,j+1})) = V_{j+1}$.
   By construction, such $\sigma_{i,j}$ always exist and $\sigma_{i,j} > 0$ for all i, j.
   Since $\sigma_{i,j}$ is always regular and the splitting function $[0,1]_{\mathrm{reg}} \ni \tau \longmapsto V(w(\tau))$ is locally constant, there are regular $\sigma^{\pm}_{i,j}$ such that
   - the splitting function on $[\sigma^-_{i,j}, \sigma^+_{i,j}] \ni \sigma_{i,j}$ is constant (i.e. equal to $V(w(\sigma_{i,j}))$);
   - $\sigma^+_{i,0} > \sigma^-_{i,0} > \sigma^+_{i,1} > \sigma^-_{i,1} > \sigma^+_{i,2} > \ldots > \sigma^-_{i,N-1} > \sigma^+_{i,N} > \sigma^-_{i,N} > \sigma^+_{i+1,0}$ for all i.

   Now we decompose, according to Proposition 4.1, those intervals in $[0,1]$ that remain after removing all the intervals $[\sigma^-_{i,j}, \sigma^+_{i,j}]$. More precisely, there are $N_{i,j} \in \mathbb{N}_+$ and $\tau_{i,j,k} \in [0,1]$ for $i, j, k \in \mathbb{N}$ with $0 \leq j \leq N$ and $0 \leq k \leq N_{i,j}$, such that for all i, j
   - $\sigma^-_{i,j-1} = \tau_{i,j,0} > \tau_{i,j,1} > \ldots > \tau_{i,j,N_{i,j}} = \sigma^+_{i,j}$;
   - $\mathcal{R}(w|_{[\tau_{i,j,k+1}, \tau_{i,j,k}]})$ is a hyph for $k = 0, \ldots, N_{i,j} - 1$.

   Here, we have been quite sloppy with the notation in the case that i or j are getting out of range. In these cases, we extended our definitions naturally, i.e., $\sigma^-_{i,-1} := \sigma^-_{i-1,N}$ and $\sigma_{0,-1} := 1$.
   To simplify the notation we denote the members of the sequence
   $$(\tau_{0,0,0},\ \tau_{0,0,1},\ \ldots,\ \tau_{0,0,N_{0,0}},\ \tau_{0,1,0},\ \ldots,\ \ldots,\ \tau_{0,N,N_{0,N}},\ \tau_{1,0,0},\ \ldots)$$

by $(\tau_0, \tau_1, \tau_2, \ldots)$. Additionally, we define $a_i \in \mathbb{N}$ for every i by $\tau_{a_i} = \tau_{i,0,N_{i,0}}$. This is precisely the endpoint of the $(2i+1)$-st[6] interval (i.e., $[\tau_{a_i+1}, \tau_{a_i}]$) in our construction having splitting W. Finally, we define

$$I_i \;:=\; [\tau_{i+1}, \tau_i] \quad\text{and}\quad J_i \;:=\; [0, \tau_i],$$

and set

$$V(i) \;:=\; V(w|_{I_i}).$$

3. Properties of the decomposition
   We have for all $i, i' \in \mathbb{N}$:
   a) $\mathcal{R}(w|_{I_i})$ is a hyph.
      If I_i corresponds to some interval $[\tau_{s,j,k+1}, \tau_{s,j,k}]$ with $k \neq N_{s,j}$, this follows directly from the construction. Otherwise, i.e. for $I_i = [\sigma^-_{s,j}, \sigma^+_{s,j}]$, the assertion follows because I_i then contains w-regular elements only and the splitting function is constant on I_i. Therefore, by the consistent parametrization, the paths in $w|_{I_i}$ are disjoint or equal, proving the hyph property.
   b) $\mathcal{R}(w|_{J_i}) = w|_{J_i}$ is a web, hence a hyph as well.
      To see this, use that $\widehat{w}|_{[0,\tau]}$ is a web again for all webs $\widehat{w}$ and all $\tau > 0$.
   c) $w|_{I_i} \cap w|_{I_{i'}} \neq \varnothing$ iff $i = i'$.
      This is a consequence of the consistent parametrization of w.
   d) $w|_{I_i} \cap w|_{J_{i'}} = \varnothing$ for $i < i'$.
      This comes from the consistent parametrization again.
   e) Performing the multiplication with decreasing indices, we have
      $$w \;=\; w|_{J_{i+1}} \prod_{i'=i}^{0} w|_{I_{i'}} \;\equiv\; w|_{J_{i+1}} \circ w|_{I_i} \circ \cdots \circ w|_{I_0}$$
      directly from the definitions above.
   f) $\mathcal{R}(w|_{J_{i+1}}) \cup \bigcup_{i'=0}^{i} \mathcal{R}(w|_{I_{i'}})$ is a hyph.
      Since each reduction involved is a hyph itself, this comes from the consistent parametrization.
4. Estimation of products of projections
   Let ε be given. Consider the set $\mathcal{V} := \bigcup_i \{V(i)\}$ of all splittings occurring in the above decomposition. Of course, $\mathcal{V}$ is finite, because there are only finitely many n-splittings at all. Moreover, $\mathcal{V}(w) \subseteq \mathcal{V}$, and every $V_l \in \mathcal{V}(w)$ occurs infinitely often in $(I_0, I_1, \ldots)$. Since every P_V is a projection (Lemma 6.3) and since $\bigcap_{V \in \mathcal{V}(w)} P_V Y = P_0 Y$ (Lemma 7.2), Proposition A.1 guarantees that for every $i \in \mathbb{N}$ there is some integer $K(i, \varepsilon) > i$, such that
   $$\bigl\| \prod_{i'=K(i,\varepsilon)}^{i+1} P_{V(i')} - P_0 \bigr\| < \varepsilon.$$
   Since $P_V P_0 = P_0 = P_0 P_V$ and $\|P_V\| = 1$ for all V, we get
   $$\|P_{V(i_-)} \cdots P_{V(i_+)} - P_0\| < \varepsilon$$
   for all $i_\pm$ with $i_- \geq K(i, \varepsilon) \geq i + 1 \geq i_+$. Choose now a strictly increasing sequence $(l^\varepsilon_0, l^\varepsilon_1, \ldots)$ in $\mathbb{N}$ fulfilling
   $$a_{l^\varepsilon_{\nu+1}} > K(a_{l^\varepsilon_\nu}, \varepsilon_\nu) \quad\text{with}\quad \varepsilon_\nu := (1+\varepsilon)^{1/2^{\nu+2}} - 1$$
   for all $\nu \in \mathbb{N}$. For starting, we set $l^\varepsilon_{-1} := -1$ and $a_{-1} := -1$.
   Since $K(l, \cdot) > l$ for all l, we have $a_{l^\varepsilon_{\nu+1}} > a_{l^\varepsilon_\nu}$, i.e. indeed a strictly increasing sequence (l^ε_ν). Moreover, we have $a_{l^\varepsilon_{\nu+1}} - 1 \geq K(a_{l^\varepsilon_\nu}, \varepsilon_\nu) \geq a_{l^\varepsilon_\nu} + 1$. Consequently, by $\|P_0\| = 1$, $\|Q_{W,q}\| \leq 1$ and Proposition A.2, we have for all $L \in \mathbb{N}$
   $$\bigl\| P_0 \prod_{\nu=L}^{0} \bigl(Q_{W,q} P_{V(a_{l^\varepsilon_\nu} - 1)} \cdots P_{V(a_{l^\varepsilon_{\nu-1}} + 1)}\bigr) - P_0 \prod_{\nu=L}^{0} \bigl(Q_{W,q} P_0\bigr) \bigr\| \;<\; \varepsilon.$$

[6] Note that, since W is also a member of the sequence $V_1, \ldots, V_N$, it has two tasks and occurs therefore roughly twice as often as the other V_is. In fact, first it will be used to pick up the degeneracy and second it will be used to make the sequence $V_1, \ldots, V_N$ rich. Hence, we will need W partially in the terms below that are affected by p_s and partially in those that are not.

Let us consider the second product. By Lemma 6.5 we have

$$\begin{aligned}\|P_0(Q_{W,q}P_0)^{L+1}\| &= \|P_0(Q_{W,q}P_0P_0)^{L+1}\| = \|(P_0Q_{W,q}P_0)^{L+1}\,P_0\| \\ &\leq \|P_0Q_{W,q}P_0\|^{L+1} = \|D_{W,q}\|_F^{2(L+1)}.\end{aligned}$$

Due to the choice of W and q, we have $\|D_{W,q}\|_F < 1$ by Lemma 6.6. Thus, there is some $L(\varepsilon) \in \mathbb{N}$, such that

$$\left\|P_0\,(Q_{W,q}P_0)^{L(\varepsilon)+1}\right\| < \varepsilon.$$

Consequently,

$$\left\|P_0 \prod_{\nu=L(\varepsilon)}^{0} \Big(Q_{W,q}P_{V(a_{l_\nu^\varepsilon}-1)} \cdots P_{V(a_{l_{\nu-1}^\varepsilon}+1)}\Big)\right\| < 2\varepsilon. \tag{2}$$

5. Application to the spin web $(w, \vec{\varphi})$
We have for all $i' \in \mathbb{N}$

$$T_{w,\vec{\varphi}} = \boldsymbol{\varphi} \circ \pi_w = \big(\boldsymbol{\varphi} \circ \pi_{w|_{J_{i'+1}}}\big) \cdot \textstyle\prod_{i=i'}^{0} \boldsymbol{\varphi} \circ \pi_{w|_{I_i}}.$$

Set now $s_i := w_q|_{I_{a_i}}$, i.e., s_i is the restriction of w_q to $[\sigma^-_{i,0}, \sigma^+_{i,0}]$ which is just the $(2i+1)$-st interval in our originally chosen sequence whose corresponding splitting is W. Extending the action of p_s naturally from the spin web states $(T_{w,\vec{\varphi}})^{\boldsymbol{i}}_{\boldsymbol{j}}$ to the corresponding operators $T_{w,\vec{\varphi}}$, we get

$$\begin{aligned} &(1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})T_{w,\vec{\varphi}} \\ =\ & \big(\boldsymbol{\varphi} \circ \pi_{w|_{J_{a_{l_L}+1}}}\big) \cdot \prod_{i=a_{l_L}}^{0} (1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})(\boldsymbol{\varphi} \circ \pi_{w|_{I_i}}) \\ =\ & \big(\boldsymbol{\varphi} \circ \pi_{w|_{J_{a_{l_L}+1}}}\big) \cdot \prod_{\nu=L}^{0} \Big((1-p_{s_{l_\nu}})(\boldsymbol{\varphi} \circ \pi_{w|_{I_{a_{l_\nu}}}}) \cdot \prod_{i=a_{l_\nu}-1}^{a_{l_{\nu-1}}+1} (\boldsymbol{\varphi} \circ \pi_{w|_{I_i}})\Big)\end{aligned}$$

for all strictly increasing (finite) sequences $(l_0, \ldots, l_L)$, where w.l.o.g. $l_{-1} = -1$. Thus, we get

$$\begin{aligned} & \int_{\overline{\mathcal{A}}} \overline{(1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})T_{w,\vec{\varphi}}} \otimes (1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})T_{w,\vec{\varphi}}\ \mathrm{d}\mu_0 \\ =\ & \int_{\overline{\mathcal{A}}} \overline{(\boldsymbol{\varphi} \circ \pi_{w|_{J_{a_{l_L}+1}}})} \otimes (\boldsymbol{\varphi} \circ \pi_{w|_{J_{a_{l_L}+1}}})\ \mathrm{d}\mu_0\ \cdot \\ & \cdot \prod_{\nu=L}^{0} \Big(\int_{\overline{\mathcal{A}}} \overline{(1-p_{s_{l_\nu}})(\boldsymbol{\varphi} \circ \pi_{w|_{I_{a_{l_\nu}}}})} \otimes (1-p_{s_{l_\nu}})(\boldsymbol{\varphi} \circ \pi_{w|_{I_{a_{l_\nu}}}})\ \mathrm{d}\mu_0\ \cdot \\ & \cdot \prod_{i=a_{l_\nu}-1}^{a_{l_{\nu-1}}+1} \int_{\overline{\mathcal{A}}} \overline{(\boldsymbol{\varphi} \circ \pi_{w|_{I_i}})} \otimes (\boldsymbol{\varphi} \circ \pi_{w|_{I_i}})\ \mathrm{d}\mu_0\Big) \qquad \text{(Corollary B.2)} \\ =\ & P_0 \cdot \prod_{\nu=L}^{0} \Big(Q_{W,q} \cdot \prod_{i=a_{l_\nu}-1}^{a_{l_{\nu-1}}+1} P_{V(i)}\Big). \qquad \text{(Lemma 6.8 and Corollary 6.9)}\end{aligned}$$

Here we used that $V(w|_{I_i}) = V(i)$ and $V(a_{l_\nu}) = W$. Moreover, we exploited the definitions of $Q_{W,q}$ (Definition 6.3) and p_s (page 11) to replace the $(1-p_{s_l})$-terms by $Q_{W,q}$. Finally, note that $w|_{J_i}$ is always a web, hence $V(w|_{J_i}) = V_0$.
Let now $(T_{w,\vec{\varphi}})^{\boldsymbol{i}}_{\boldsymbol{j}}$ be some spin web state for $(w, \vec{\varphi})$. Then

$$
\begin{aligned}
&\|(1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})(T_{w,\vec\varphi})^{\boldsymbol{i}}_{\boldsymbol{j}}\|^2\\
=\;&\langle(1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})(T_{w,\vec\varphi})^{\boldsymbol{i}}_{\boldsymbol{j}},(1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})(T_{w,\vec\varphi})^{\boldsymbol{i}}_{\boldsymbol{j}}\rangle\\
=\;&\Bigl(\int_{\overline{\mathcal{A}}}\overline{(1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})T_{w,\vec\varphi}}\otimes(1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})T_{w,\vec\varphi}\;\mathrm{d}\mu_0\Bigr)^{\boldsymbol{ii}}_{\boldsymbol{jj}}
\end{aligned}
$$

is just some matrix element of the above operator on Y. Since Y is a finite-dimensional Hilbert space, all norms are equivalent, hence there is some constant $C \in \mathbb{R}$ (depending only on Y and the norms fixed from the beginning), such that

$$
\begin{aligned}
&\|(1-p_{s_{l_0}})\cdots(1-p_{s_{l_L}})(T_{w,\vec\varphi})^{\boldsymbol{i}}_{\boldsymbol{j}}\|^2\\
&\qquad\le\; C\,\Bigl\|P_0\cdot\prod_{\nu=L}^{0}\Bigl(Q_{W,q}\cdot\prod_{i=a_{l_\nu}-1}^{a_{l_{\nu-1}}+1}P_{V(i)}\Bigr)\Bigr\|.
\end{aligned}
$$

6. Final step: Proof of the Lewandowski-Thiemann conjecture
Let $\varepsilon > 0$ be given. Choose $(l_0^\varepsilon, l_1^\varepsilon, \ldots)$ as above. Then there is some $L(\varepsilon)$, such that (2) is fulfilled. Consequently, setting $N(\varepsilon) := l^\varepsilon_{L(\varepsilon)}$ we have

$$
\begin{aligned}
\|(1-p_{s_0})\cdots(1-p_{s_{N(\varepsilon)}})(T_{w,\vec\varphi})^{\boldsymbol{i}}_{\boldsymbol{j}}\|^2 &\le \|(1-p_{s_{l_0^\varepsilon}})\cdots(1-p_{s_{l^\varepsilon_{L(\varepsilon)}}})(T_{w,\vec\varphi})^{\boldsymbol{i}}_{\boldsymbol{j}}\|^2\\
&< 2C\varepsilon
\end{aligned}
$$

because $(1-p_s)$ is a projection. Moreover, we used that $(1-p_{s'})$ and $(1-p_{s''})$ commute, if im s' and im s'' are disjoint. Note that C does not depend on ε, but only on the fixed spin web.
Hence, $\lim_{l\to\infty}(1-p_{s_0})\cdots(1-p_{s_l})(T_{w,\vec\varphi})^{\boldsymbol{i}}_{\boldsymbol{j}} = 0$ for all $\boldsymbol{i},\boldsymbol{j}$. By linearity we get

$$
\lim_{l\to\infty}(1-p_{s_0})\cdots(1-p_{s_l})\Psi \;=\; 0
$$

for all $\Psi \in \mathcal{H}_{w,\vec\varphi}$. **qed**

We remark finally that the Lewandowski-Thiemann conjecture can be extended even to arbitrary connected compact Lie groups $\mathbf{G}$ – with one restriction, of course: In general, it is only true for webs w where V_w generates full $\mathbb{R}^{\#w}$. In fact, then we have $[\mathbf{G}_{V_w}]^{\bullet q} = \mathbf{G}^n$ for some $q \in \mathbb{N}$. [9] This has been the crucial ingredient for the proof of Lemma 7.2. In the proof of the Lewandowski-Thiemann conjecture itself, the assumption of semisimplicity has been used only indirectly to guarantee the applicability of the lemma just mentioned.

8 "Standard" Example of a Web

The original idea [11] of Lewandowski and Thiemann to prove their conjecture was that it should always be possible to find degenerate segments s_l, such that – in our terminology – the portion of the web between two subsequent intervals corresponds always to P_0, which is given if these portions are measure-theoretically, i.e. in a certain sense "strongly" independent. They argued that, for that purpose, it ought to be sufficient to prove just the holonomical independence of these portions. Unfortunately, this is not the case as we will see in this section. Therefore, the article [9], where the holonomical independence has been established, cannot prove the Lewandowski-Thiemann conjecture yet. However, all this is not a real problem, since we have now been able to prove in the present article that these portions can be chosen, such that the corresponding operators are sufficiently close to P_0 which still gives the proof.

In this final section we consider $\mathbf{G} = SU(2)$. Let now $V_1 := \{(1,1,0,0),(0,0,1,1)\}$ and $V_2 := \{(1,0,1,0),(0,1,0,1)\}$ be two 4-splittings. Moreover, let the quadrupel $\vec\varphi = (\frac12,\frac12,\frac12,\frac12)$ consist of spin-$\frac12$ representations of $SU(2)$.

Lemma 8.1 We have $P_{V_1} Y \cap P_{V_2} Y = P_0 Y$, but $P_{V_1} P_{V_2} \neq P_0$.

Proof Of course, $V := V_1 \cup V_2$ is rich. Hence, by Theorem 3.1, we have $[\mathbf{G}_V]^{\bullet q(4)} = \mathbf{G}^4$. Lemma 6.4 now gives $P_{V_1} Y \cap P_{V_2} Y = P_0 Y$.
Let us now prove $P_{V_1} P_{V_2} \neq P_0$. We have for every integrable function f on $\mathbf{G}^4$

$$\begin{aligned} & \int_{\mathbf{G}^4} f\bigl(g_{1+}g_{2+},\, g_{1+}g_{2-},\, g_{1-}g_{2+},\, g_{1-}g_{2-}\bigr)\, \mathrm{d}\mu^4_{\mathrm{Haar}} \\ = & \int_{\mathbf{G}^4} f\bigl(g_{1+}g_{1-}g_{2+},\, g_{1+}g_{1-}g_{2-},\, g_{1-}g_{2+},\, g_{1-}g_{2-}\bigr)\, \mathrm{d}\mu^4_{\mathrm{Haar}} \\ & \qquad\qquad \text{(Translation invariance w.r.t. } g_{1+} \longmapsto g_{1+}g_{1-}\text{)} \\ = & \int_{\mathbf{G}^3} f\bigl(g_{1+}g_{2+},\, g_{1+}g_{2-},\, g_{2+},\, g_{2-}\bigr)\, \mathrm{d}\mu^3_{\mathrm{Haar}} \\ & \text{(Translation invariance w.r.t. } g_{2\pm} \longmapsto (g_{1-})^{-1}g_{2\pm}\text{; Normalization)} \\ = & \int_{\mathbf{G}^3} f\bigl(g_1g_2,\, g_1g_3,\, g_2,\, g_3\bigr)\, \mathrm{d}\mu^3_{\mathrm{Haar}}. \qquad \text{(Renumeration)} \end{aligned}$$

Consequently,

$$\begin{aligned} & \bigl(P_{V_1}P_{V_2}\bigr)^{\boldsymbol{ik}}_{\boldsymbol{jl}} \\ = & \int_{\mathbf{G}^4} \overline{(g_{1+}g_{2+})^{i_1}_{j_1}\,(g_{1+}g_{2-})^{i_2}_{j_2}\,(g_{1-}g_{2+})^{i_3}_{j_3}\,(g_{1-}g_{2-})^{i_4}_{j_4}}\;\cdot \\ & \qquad \cdot\; (g_{1+}g_{2+})^{k_1}_{l_1}\,(g_{1+}g_{2-})^{k_2}_{l_2}\,(g_{1-}g_{2+})^{k_3}_{l_3}\,(g_{1-}g_{2-})^{k_4}_{l_4}\;\mathrm{d}\mu^4_{\mathrm{Haar}} \\ = & \int_{\mathbf{G}^3} \overline{(g_1g_2)^{i_1}_{j_1}\,(g_1g_3)^{i_2}_{j_2}\,(g_2)^{i_3}_{j_3}\,(g_3)^{i_4}_{j_4}}\,(g_1g_2)^{k_1}_{l_1}\,(g_1g_3)^{k_2}_{l_2}\,(g_2)^{k_3}_{l_3}\,(g_3)^{l_4}_{l_4}\;\mathrm{d}\mu^3_{\mathrm{Haar}} \\ = & \int_{\mathbf{G}^3} \overline{(g_1)^{i_1}_{m_1}}\,\overline{(g_2)^{m_1}_{j_1}}\;\overline{(g_1)^{i_2}_{m_2}}\,\overline{(g_3)^{m_2}_{j_2}}\;\overline{(g_2)^{i_3}_{j_3}}\;\overline{(g_3)^{i_4}_{j_4}}\;\cdot \\ & \qquad \cdot\; (g_1)^{k_1}_{n_1}\,(g_2)^{n_1}_{l_1}\;(g_1)^{k_2}_{n_2}\,(g_3)^{n_2}_{l_2}\;(g_2)^{k_3}_{l_3}\;(g_3)^{k_4}_{l_4}\;\mathrm{d}\mu^3_{\mathrm{Haar}} \\ = & \;\langle g^{i_1}_{m_1} g^{i_2}_{m_2}, g^{k_1}_{n_1} g^{k_2}_{n_2}\rangle_{\mathrm{Haar},1}\;\langle g^{m_1}_{j_1} g^{i_3}_{j_3}, g^{n_1}_{l_1} g^{k_3}_{l_3}\rangle_{\mathrm{Haar},2}\;\langle g^{i_4}_{j_4} g^{m_2}_{j_2}, g^{k_4}_{l_4} g^{n_2}_{l_2}\rangle_{\mathrm{Haar},3}. \end{aligned}$$

Now, we set $j_4 := l_2 := 2$ and the remaining indices of $P_{V_1}P_{V_2}$ equal to 1:

$$\begin{aligned} & \bigl(P_{V_1}P_{V_2}\bigr)^{1111\,1111}_{1112\,1211} \\ = & \;\langle g^1_{m_1} g^1_{m_2}, g^1_{n_1} g^1_{n_2}\rangle_{\mathrm{Haar},1}\;\langle g^{m_1}_1 g^1_1, g^{n_1}_1 g^1_1\rangle_{\mathrm{Haar},2}\;\langle g^1_2 g^{m_2}_1, g^1_1 g^{n_2}_2\rangle_{\mathrm{Haar},3} \\ = & \;\textstyle\sum_{m_1,m_2} \langle g^1_{m_1} g^1_{m_2}, g^1_{m_1} g^1_{m_2}\rangle_{\mathrm{Haar},1}\;\langle g^{m_1}_1 g^1_1, g^{m_1}_1 g^1_1\rangle_{\mathrm{Haar},2}\;\langle g^1_2 g^{m_2}_1, g^1_1 g^{m_2}_2\rangle_{\mathrm{Haar},3} \\ = & \;\textstyle\sum_{m_1,m_2} \frac{1}{6}(1+\delta_{m_1m_2})\,\frac{1}{6}(3-m_1)\,\frac{1}{6}(-1)^{m_2+1} \\ = & \;\frac{1}{6^3}. \end{aligned}$$

Here, in the second step we used that, by Lemma C.1, only those scalar products are non-zero, where the sum of the first two upper (lower) indices equals that of the last two upper (lower) indices. Thus, by the third scalar product, only $m_2 = n_2$ contributes. Analogously, $m_1 = n_1$ by the second scalar product. Finally, we used Lemma C.2 and Lemma C.3. Since, as seen in the proof of Lemma 6.5, we have $(P_0)^{1111\,1111}_{1112\,1211} = 0$, we get $P_{V_1}P_{V_2} \neq P_0$. **qed**

Before stating the final result of this paper, let us recall

Definition 8.1 Let $\boldsymbol{\gamma}$ be some tuple of paths.

- $\boldsymbol{\gamma}$ is called **measure-theoretically independent** iff $\pi_{\boldsymbol{\gamma}*}\mu_0 = \mu^{\#\boldsymbol{\gamma}}_{\mathrm{Haar}}$.
- $\boldsymbol{\gamma}$ is called **holonomically independent** iff for every $\vec{g} \in \mathbf{G}^{\#\boldsymbol{\gamma}}$ there is some smooth connection $A \in \mathcal{A}$ such that $h_A(\boldsymbol{\gamma}) = \vec{g}$.

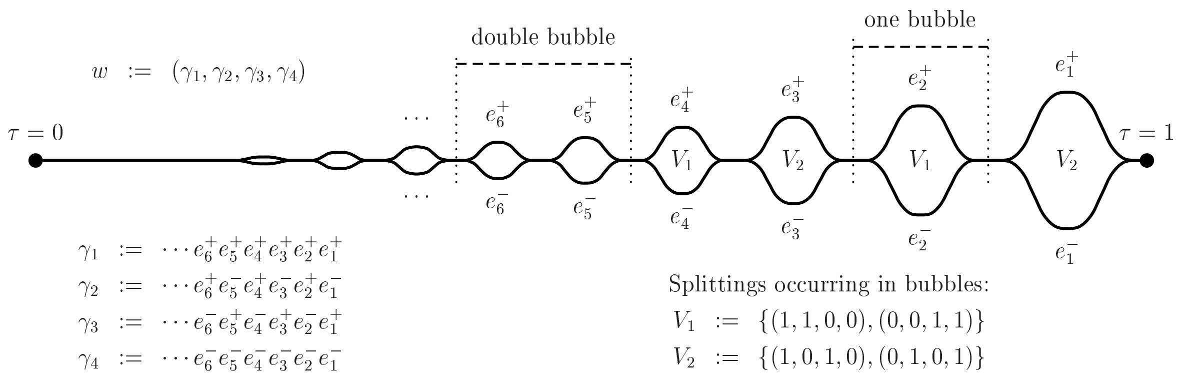

Figure 1: Standard Example of a Web [4, 11]

Note that the holonomical independence of $\boldsymbol{\gamma}$ is independent of the ultralocal trivialization chosen to define the group values $h_A(\boldsymbol{\gamma})$ of parallel transports for A.

Proposition 8.2 Let $\mathbf{G} = SU(2)$ and let w be the web of Figure 1, where each of the four paths in w is labelled by the $\frac{1}{2}$-representation of $SU(2)$. Then we have:

1. This spin web $(w, \vec{\varphi})$ is weakly degenerate.
2. $w|_I$ is holonomically independent, but *not* measure-theoretically independent for every interval $I \subseteq (0,1]$ whose image under w contains at least four subsequent bubbles.

We remark that $w|_I$ is measure-theoretically independent if and only if 0 is contained in I (and I is nontrivial, of course).

Proof

- The weak degeneracy of $(w, \vec{\varphi})$ is clear.
- $w|_I$ is not measure-theoretically independent.
  Applying the terminology of Section 6 to the case of the given spin web, we see that
  $$\langle \boldsymbol{\varphi}^{\boldsymbol{i}}_{\boldsymbol{j}} \circ \pi_{w|_I}, \boldsymbol{\varphi}^{\boldsymbol{k}}_{\boldsymbol{l}} \circ \pi_{w|_I} \rangle = \bigl(P'(P_{V_1} P_{V_2})^B P''\bigr)^{\boldsymbol{ik}}_{\boldsymbol{jl}}.$$
  Here, V_1 and V_2 are again given as above. These are precisely the two splittings that occur in w for w-regular parameter values. P' is the identity, if the bubble, that is (at least partially, but nontrivially) passed first by $w|_I$ (when running through I with increasing parameter values), corresponds to splitting V_1. It equals P_{V_2} otherwise. Analogously, P'' is the identity, if the last (partially) passed bubble is of splitting V_2, and equals P_{V_1} otherwise. Finally, B is the number of double bubbles of "type" (V_1, V_2) passed by $w|_I$ (one bubble may be passed only partially). Note that I does not contain 0, hence B is indeed finite. If $w|_I$ were measure-theoretically independent, we would get
  $$\langle \boldsymbol{\varphi}^{\boldsymbol{i}}_{\boldsymbol{j}} \circ \pi_{w|_I}, \boldsymbol{\varphi}^{\boldsymbol{k}}_{\boldsymbol{l}} \circ \pi_{w|_I} \rangle = \langle \boldsymbol{\varphi}^{\boldsymbol{i}}_{\boldsymbol{j}}, \boldsymbol{\varphi}^{\boldsymbol{k}}_{\boldsymbol{l}} \rangle_{\mathbf{G}^4} = (P_0)^{\boldsymbol{ik}}_{\boldsymbol{jl}}.$$
  This, however, is a contradiction since, by Lemma 8.1, we know that $P_{V_1} P_{V_2} \neq P_0$, hence $P'(P_{V_1} P_{V_2})^B P'' \neq P_0$ by Lemma A.3.
- $w|_I$ is holonomically independent.
  As one checks quite easily, we have $\mathbf{G}_{V_1}\mathbf{G}_{V_2}\mathbf{G}_{V_1}\mathbf{G}_{V_2} = \mathbf{G}^4 = \mathbf{G}_{V_2}\mathbf{G}_{V_1}\mathbf{G}_{V_2}\mathbf{G}_{V_1}$ for every connected semisimple $\mathbf{G}$. Consequently, the results shown in [9] imply that if two double bubbles (i.e. twice the sequence (V_1, V_2) or (V_2, V_1) of splittings) are passed, then the web, restricted to these two double bubbles, is strongly holonomically independent. Since $w|_I$ passes at least two double bubbles, we get the assertion. **qed**

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Appendix

A Convergence of Projector Products

Proposition A.1 Let H be a finite-dimensional Hilbert space and let $P_1, \ldots, P_n$ be (self-adjoint) projections on H. Moreover, let $H_i := P_i H$, $i = 1, \ldots, n$, be the corresponding projection spaces. Now, define $H_0 := \bigcap_{i=1}^n H_i$ and denote the projector from H to H_0 by P_0. Next, let $I \subseteq \{1, \ldots, n\}$ be some subset, such that $\bigcap_{i \in I} H_i = H_0$. Finally, let $(j_k)_{k \in \mathbb{N}_+}$ be a sequence of integers, such that

- $1 \leq j_k \leq n$ for all $k \in \mathbb{N}_+$;
- every $i \in I$ occurs infinitely many times in $(j_k)_{k \in \mathbb{N}}$.

Then both $\prod_{k=1}^N P_{j_k}$ and $\prod_{k=N}^1 P_{j_k}$ converge for $N \to \infty$ in the operator norm to P_0.

Proof First let us assume $H_0 = 0$.

- Let a nonempty subset $L \subseteq \{1, \ldots, n\}$ be called *full* iff $\bigcap_{i \in L} H_i = 0$. Then by [12] for all full L there is some constant $\vartheta_L \in [0, 1)$, such that
$$\|P_{l_1} P_{l_2} \cdots P_{l_N}\| \leq \vartheta_L$$
for all N and for all finite sequences $l_1, \ldots, l_N$ of elements in L where every element of L occurs at least once.[7]
- The number of full subsets $L \subseteq \{1, \ldots, n\}$ is again finite. Let ϑ be the maximum of all these corresponding ϑ_L. Consequently,
$$\|P_{l_1} P_{l_2} \cdots P_{l_N}\| \leq \vartheta$$
for all N and for all sequences $l_1, \ldots, l_N$ with $\bigcap_{k=1}^N H_{l_k} = 0$. Of course, $\vartheta < 1$.
- Let now (j_k) be a sequence as given in the assumptions. Since I is full, there exists a strictly increasing sequence $(N_q)_{q \in \mathbb{N}}$ of natural numbers with $N_0 = 0$, such that
$$H_{j_{N_q+1}} \cap \ldots \cap H_{j_{N_{q+1}}} = 0$$
for all $q \in \mathbb{N}$. By the preceding step we have $\|P_{j_{N_q+1}} \cdots P_{j_{N_{q+1}}}\| \leq \vartheta$ for all $q \in \mathbb{N}$.
- Setting $A_N := \prod_{k=1}^N P_{j_k}$, we get for $Q \in \mathbb{N}_+$
$$\|A_{N_Q}\| = \Bigl\| \prod_{q=0}^{Q-1} \prod_{s=N_q+1}^{N_{q+1}} P_s \Bigr\| \leq \prod_{q=0}^{Q-1} \Bigl\| \prod_{s=N_q+1}^{N_{q+1}} P_s \Bigr\| \leq \prod_{q=0}^{Q-1} \vartheta = \vartheta^Q.$$
Consequently, $\|A_{N_Q}\| \to 0$ for $Q \to \infty$. Since $\|A_{N+1}\| = \|A_N P_{j_{N+1}}\| \leq \|A_N\|$, i.e., since the sequence $\|A_N\|$ is non-decreasing, we have $\|A_N\| \to 0$ for $N \to \infty$.

Let now $H_0 \neq 0$. Denote by H_i' the orthogonal complement of H_0 in H_i and by P_i' the corresponding projector. Using $P_i = P_0 + P_i'$ and $P_0 P_i' = P_i' P_0 = 0$ for all i, we get $\prod_{k=1}^N P_{j_k} = P_0 + \prod_{k=1}^N P_{j_k}'$ for all N. By $\bigcap_{i \in I} H_i' = 0$ we have $\prod_{k=1}^N P_{j_k}' \to 0$ and thus finally $\prod_{k=1}^N P_{j_k} \to P_0$ for $N \to \infty$.

The proof of $\prod_{k=N}^1 P_{j_k} \to P_0$ is now clear. **qed**

[7] If $\#L = 1$, i.e. $L = \{i\}$, then $H_i = 0$ and $P_i = 0$. Consequently, $\|P_{l_1} P_{l_2} \cdots P_{l_N}\| = \|P_i^N\| = 0 =: \vartheta_L < 1$ for all sequences $l_1, \ldots, l_N$.

Proposition A.2 Let H be some Hilbert space, $N \in \mathbb{N}$ and $\varepsilon > 0$. Moreover, let A, A_i and B_i be linear continuous operators on H, such that for all $i = 1, \ldots, N$

- $\|A_i - A\| \leq (1+\varepsilon)^{2^{-i}} - 1$ and
- $\|B_i\| \leq 1$.

If additionally $\|A\| = 1$, then we have
$$\Bigl\|\prod_{i=1}^N A_i B_i - \prod_{i=1}^N A B_i\Bigr\| < \varepsilon \quad \text{and} \quad \Bigl\|\prod_{i=1}^N B_i A_i - \prod_{i=1}^N B_i A\Bigr\| < \varepsilon.$$

Proof We have
$$\begin{aligned}
\|\textstyle\prod_{i=1}^N A_i B_i - \prod_{i=1}^N A B_i\| &= \|\textstyle\prod_{i=1}^N (A + [A_i - A])B_i - \prod_{i=1}^N A B_i\| \\
&\leq \textstyle\prod_{i=1}^N \bigl(\|AB_i\| + \|(A_i - A)B_i\|\bigr) - \prod_{i=1}^N \|AB_i\| \\
&\leq \textstyle\prod_{i=1}^N \bigl(\|A\| + \|A_i - A\|\bigr) - \prod_{i=1}^N \|A\| \\
&\leq \textstyle\prod_{i=1}^N \bigl(1 + (1+\varepsilon)^{2^{-i}} - 1\bigr) - \prod_{i=1}^N 1 \\
&= (1+\varepsilon)^{\sum_{i=1}^N 2^{-i}} - 1 \\
&< \varepsilon.
\end{aligned}$$

The proof for the opposite factor ordering is completely analogous. **qed**

Finally, we consider the special case of two projectors.

Lemma A.3 Let P_1 and P_2 be orthogonal projections on some Hilbert space H and let P_0 be the orthogonal projection from H onto $P_1 H \cap P_2 H$.
Then we have for every $n \in \mathbb{N}_+$
$$(P_1 P_2)^n = P_0 \implies P_1 P_2 = P_0.$$

Proof

- Assume first $P_0 = 0$.
  Since P_1 and P_2 are hermitian (i.e., in the real case, they equal their respective transposes), $(P_1 P_2)^m P_1$ is hermitian for $m \in \mathbb{N}$. Since $\|A^2\| = \|A\|^2$ for all hermitian operators A, we have
  $$\|(P_1 P_2)^{2m} P_1\| = \|(P_1 P_2)^m P_1 (P_1 P_2)^m P_1\| = \|(P_1 P_2)^m P_1\|^2,$$
  hence for all $s \in \mathbb{N}$
  $$\|(P_1 P_2)^{2^s} P_1\| = \|P_1 P_2 P_1\|^{2^s}.$$
  Choosing some s with $n \leq 2^s$, we get $P_1 P_2 P_1 = 0$ from $(P_1 P_2)^n = 0$.
  Therefore, $\langle P_2 P_1 x, P_2 P_1 x\rangle = \langle x, P_1 P_2 P_1 x\rangle = 0$ for all $x \in H$, hence $P_2 P_1 = 0$ which implies $P_1 P_2 = 0$.
- Let now P_0 be arbitrary.
  Let P_i' for $i = 1, 2$ be the orthogonal projector from H onto the orthogonal complement of $P_0 H$ in $P_i H$. By $P_i = P_0 + P_i'$ and $P_0 P_i' = P_i' P_0 = 0$, we get $P_0 = (P_1 P_2)^n = P_0 + (P_1' P_2')^n$, hence $(P_1' P_2')^n = 0$. As shown above, $P_1' P_2' = 0$, thus $P_1 P_2 = P_0 + P_1' P_2' = P_0$. **qed**

B Integrals of Operator Products

Lemma B.1 Let $\boldsymbol{\gamma}^{(i)}$, $i = 1, \ldots, k$, be finite tuples of edges and let $\upsilon^{(i)}$ for every $i = 1, \ldots, k$ be some hyph with $\boldsymbol{\gamma}^{(i)} \leq \upsilon^{(i)}$, such that

- $\upsilon^{(i)} \cap \upsilon^{(j)} = \varnothing$ for all $i \neq j$ and
- $\bigcup_i \upsilon^{(i)}$ is a hyph.

Then we have for all continuous $f_i : \mathbf{G}^{\#\boldsymbol{\gamma}^{(i)}} \longrightarrow \mathbb{C}$

$$\int_{\overline{\mathcal{A}}} \prod_i \bigl(f_i \circ \pi_{\boldsymbol{\gamma}^{(i)}}\bigr) \,\mathrm{d}\mu_0 \quad = \quad \prod_i \int_{\overline{\mathcal{A}}} f_i \circ \pi_{\boldsymbol{\gamma}^{(i)}} \,\mathrm{d}\mu_0.$$

Proof Define $\upsilon := \bigcup_i \upsilon^{(i)}$. Due to $\boldsymbol{\gamma}^{(i)} \leq \upsilon^{(i)} \leq \upsilon$ we have

$$\begin{aligned}
\int_{\overline{\mathcal{A}}} \prod_i \bigl(f_i \circ \pi_{\boldsymbol{\gamma}^{(i)}}\bigr) \,\mathrm{d}\mu_0 &= \int_{\overline{\mathcal{A}}} \prod_i \Bigl(\bigl[\bigl(f_i \circ \pi^{\upsilon^{(i)}}_{\boldsymbol{\gamma}^{(i)}}\bigr) \circ \pi^{\upsilon}_{\upsilon^{(i)}}\bigr] \circ \pi_{\upsilon}\Bigr) \,\mathrm{d}\mu_0 \\
&= \int_{\mathbf{G}^{\#\upsilon}} \prod_i \bigl[\bigl(f_i \circ \pi^{\upsilon^{(i)}}_{\boldsymbol{\gamma}^{(i)}}\bigr) \circ \pi^{\upsilon}_{\upsilon^{(i)}}\bigr] \,\mathrm{d}\mu_{\mathrm{Haar}} \\
&= \prod_i \int_{\mathbf{G}^{\#\upsilon^{(i)}}} f_i \circ \pi^{\upsilon^{(i)}}_{\boldsymbol{\gamma}^{(i)}} \,\mathrm{d}\mu_{\mathrm{Haar}} \\
&\qquad (\upsilon \text{ is the disjoint union of the } \upsilon^{(i)}.) \\
&= \prod_i \int_{\overline{\mathcal{A}}} f_i \circ \pi^{\upsilon^{(i)}}_{\boldsymbol{\gamma}^{(i)}} \circ \pi_{\upsilon^{(i)}} \,\mathrm{d}\mu_0 \\
&= \prod_i \int_{\overline{\mathcal{A}}} f_i \circ \pi_{\boldsymbol{\gamma}^{(i)}} \,\mathrm{d}\mu_0.
\end{aligned}$$

qed

Corollary B.2 Let finitely many $\tau_i \in [0,1]$ with $0 = \tau_0 < \tau_1 < \ldots < \tau_N = 1$ be given. Let $\boldsymbol{\gamma}$ be an n-tuple of edges and define $\boldsymbol{\gamma}^{(i)} := \boldsymbol{\gamma}|_{[\tau_{i-1},\tau_i]}$. Assume, moreover, that the reductions $\upsilon^{(i)} := \mathcal{R}(\boldsymbol{\gamma}^{(i)})$ have the following two properties:

- $\upsilon^{(i)} \cap \upsilon^{(j)} = \varnothing$ for all $i \neq j$ and
- $\bigcup_i \upsilon^{(i)}$ is a hyph.

Let now X be a finite-dimensional Hilbert space and let $F : \overline{\mathcal{A}} \longrightarrow \mathrm{End}\, X$ be some function. Equip $\mathrm{End}\, X$ with the standard operator norm induced by the norm on X. Assume finally, that there are continuous functions $F_i : \mathbf{G}^n \longrightarrow \mathrm{End}\, X$, such that $F = \prod_i \bigl(F_i \circ \pi_{\boldsymbol{\gamma}^{(i)}}\bigr)$.
Then

$$\int_{\overline{\mathcal{A}}} F \,\mathrm{d}\mu_0 \quad = \quad \prod_i \int_{\overline{\mathcal{A}}} F_i \circ \pi_{\boldsymbol{\gamma}^{(i)}} \,\mathrm{d}\mu_0.$$

Proof Using Lemma B.1 we have for all indices k, l

$$\begin{aligned}
\Bigl(\int_{\overline{\mathcal{A}}} F \,\mathrm{d}\mu_0\Bigr)^k_l &= \int_{\overline{\mathcal{A}}} \Bigl(\prod_i F_i \circ \pi_{\boldsymbol{\gamma}^{(i)}}\Bigr)^k_l \,\mathrm{d}\mu_0 \\
&= \delta^k_{j_0} \delta^{j_N}_l \int_{\overline{\mathcal{A}}} \prod_i (F_i)^{j_{i-1}}_{j_i} \circ \pi_{\boldsymbol{\gamma}^{(i)}} \,\mathrm{d}\mu_0 \\
&= \delta^k_{j_0} \delta^{j_N}_l \prod_i \int_{\overline{\mathcal{A}}} (F_i)^{j_{i-1}}_{j_i} \circ \pi_{\boldsymbol{\gamma}^{(i)}} \,\mathrm{d}\mu_0 \\
&= \Bigl(\prod_i \int_{\overline{\mathcal{A}}} F_i \circ \pi_{\boldsymbol{\gamma}^{(i)}} \,\mathrm{d}\mu_0\Bigr)^k_l.
\end{aligned}$$

Note that the independence of $\bigcup_i \upsilon^{(i)}$ implies that of every $\upsilon^{(i)}$. **qed**

C $SU(2)$ Integral Formulae

The basic formula [6] we will exploit below is

$$
\begin{aligned}
6\langle g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}, g^{\rho_1}_{\sigma_1} g^{\rho_2}_{\sigma_2}\rangle_{\mathrm{Haar}} &= 2\bigl(\delta^{\mu_1\rho_1}\delta_{\nu_1\sigma_1}\delta^{\mu_2\rho_2}\delta_{\nu_2\sigma_2} + \delta^{\mu_1\rho_2}\delta_{\nu_1\sigma_2}\delta^{\mu_2\rho_1}\delta_{\nu_2\sigma_1}\bigr) \\
&\qquad - \bigl(\delta^{\mu_1\rho_1}\delta_{\nu_1\sigma_2}\delta^{\mu_2\rho_2}\delta_{\nu_2\sigma_1} + \delta^{\mu_1\rho_2}\delta_{\nu_1\sigma_1}\delta^{\mu_2\rho_1}\delta_{\nu_2\sigma_2}\bigr).
\end{aligned}
$$

Here, g^μ_ν, as usual, denotes some matrix function on $SU(2)$. Set $S := 6\langle g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}, g^{\rho_1}_{\sigma_1} g^{\rho_2}_{\sigma_2}\rangle_{\mathrm{Haar}}$.

Lemma C.1 $6\langle g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}, g^{\rho_1}_{\sigma_1} g^{\rho_2}_{\sigma_2}\rangle_{\mathrm{Haar}} \neq 0$ iff $\mu_1 + \mu_2 = \rho_1 + \rho_2$ and $\nu_1 + \nu_2 = \sigma_1 + \sigma_2$.

Proof Observe first that $S = 0$ iff either both brackets are zero or the first equals 1 and the second equals 2. However, if the second were 2, then $\mu_1 = \mu_2 = \rho_1 = \rho_2$ and $\nu_1 = \nu_2 = \sigma_1 = \sigma_2$, hence the first bracket were 2 implying $S = 2$. Consequently, $S = 0$ iff both brackets are zero. By positivity, $S = 0$ iff each of the four Kronecker products vanishes. Hence, $S = 0$ iff

$$
\begin{aligned}
0 &= \delta^{\mu_1\rho_1}\delta_{\nu_1\sigma_1}\delta^{\mu_2\rho_2}\delta_{\nu_2\sigma_2} + \delta^{\mu_1\rho_2}\delta_{\nu_1\sigma_2}\delta^{\mu_2\rho_1}\delta_{\nu_2\sigma_1} \\
&\quad + \delta^{\mu_1\rho_1}\delta_{\nu_1\sigma_2}\delta^{\mu_2\rho_2}\delta_{\nu_2\sigma_1} + \delta^{\mu_1\rho_2}\delta_{\nu_1\sigma_1}\delta^{\mu_2\rho_1}\delta_{\nu_2\sigma_2} \\
&= \bigl(\delta^{\mu_1\rho_1}\delta^{\mu_2\rho_2} + \delta^{\mu_1\rho_2}\delta^{\mu_2\rho_1}\bigr)\bigl(\delta_{\nu_1\sigma_1}\delta_{\nu_2\sigma_2} + \delta_{\nu_1\sigma_2}\delta_{\nu_2\sigma_1}\bigr).
\end{aligned}
$$

The assertion can now be verified immediately. **qed**

Lemma C.2 We have

$$
6\langle g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}, g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}\rangle_{\mathrm{Haar}} = \begin{cases} 2 & \text{iff } \mu_1 + \mu_2 + \nu_1 + \nu_2 \equiv_2 0 \\ 1 & \text{iff } \mu_1 + \mu_2 + \nu_1 + \nu_2 \equiv_2 1 \end{cases}.
$$

Proof We have

$$
\begin{aligned}
6\langle g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}, g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}\rangle_{\mathrm{Haar}} &= 2\bigl(\delta^{\mu_1\mu_1}\delta_{\nu_1\nu_1}\delta^{\mu_2\mu_2}\delta_{\nu_2\nu_2} + \delta^{\mu_1\mu_2}\delta_{\nu_1\nu_2}\delta^{\mu_2\mu_1}\delta_{\nu_2\nu_1}\bigr) \\
&\qquad - \bigl(\delta^{\mu_1\mu_1}\delta_{\nu_1\nu_2}\delta^{\mu_2\mu_2}\delta_{\nu_2\nu_1} + \delta^{\mu_1\mu_2}\delta_{\nu_1\nu_1}\delta^{\mu_2\mu_1}\delta_{\nu_2\nu_2}\bigr) \\
&= 2\bigl(1 + \delta^{\mu_1\mu_2}\delta_{\nu_1\nu_2}\bigr) - \bigl(\delta_{\nu_1\nu_2} + \delta^{\mu_1\mu_2}\bigr).
\end{aligned}
$$

For $\mu_1 = \mu_2$, we get $6\langle g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}, g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}\rangle_{\mathrm{Haar}} = 1 + \delta_{\nu_1\nu_2}$, implying the assertion. Analogously, for $\mu_1 \neq \mu_2$, we have $6\langle g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}, g^{\mu_1}_{\nu_1} g^{\mu_2}_{\nu_2}\rangle_{\mathrm{Haar}} = 2 - \delta_{\nu_1\nu_2}$, again implying the assertion. **qed**

Lemma C.3 $6\langle g^1_2 g^\mu_1, g^1_1 g^\mu_2\rangle_{\mathrm{Haar}} = (-1)^{\mu+1}$ for all μ.

Proof The assertion follows from

$$
\begin{aligned}
6\langle g^1_2 g^\mu_1, g^1_1 g^\mu_2\rangle_{\mathrm{Haar}} &= 2\bigl(\delta^{11}\delta_{21}\delta^{\mu\mu}\delta_{12} + \delta^{1\mu}\delta_{22}\delta^{\mu 1}\delta_{11}\bigr) \\
&\qquad - \bigl(\delta^{11}\delta_{22}\delta^{\mu\mu}\delta_{11} + \delta^{1\mu}\delta_{21}\delta^{\mu 1}\delta_{12}\bigr) \\
&= 2\delta^{1\mu} - 1.
\end{aligned}
$$

qed

References

[1] Abhay Ashtekar and Jerzy Lewandowski: Differential geometry on the space of connections via graphs and projective limits. *J. Geom. Phys.* **17** (1995) 191–230. e-print: hep-th/9412073.

[2] Abhay Ashtekar and Jerzy Lewandowski: Projective techniques and functional integration for gauge theories. *J. Math. Phys.* **36** (1995) 2170–2191. e-print: gr-qc/9411046.

[3] Abhay Ashtekar and Jerzy Lewandowski: Representation theory of analytic holonomy C^* algebras. In: *Knots and Quantum Gravity* (Riverside, CA, 1993), edited by John C. Baez, pp. 21–61, Oxford Lecture Series in Mathematics and its Applications 1 (Oxford University Press, Oxford, 1994). e-print: gr-qc/9311010.

[4] John C. Baez and Stephen Sawin: Diffeomorphism-invariant spin network states. *J. Funct. Anal.* **158** (1998) 253–266. e-print: q-alg/9708005.

[5] John C. Baez and Stephen Sawin: Functional integration on spaces of connections. *J. Funct. Anal.* **150** (1997) 1–26. e-print: q-alg/9507023.

[6] Michael Creutz: *Quarks, Gluons and Lattices*. Cambridge University Press, New York, 1983.

[7] Christian Fleischhack: Hyphs and the Ashtekar-Lewandowski Measure. *J. Geom. Phys.* **45** (2003) 231–251. e-print: math-ph/0001007.

[8] Christian Fleischhack: Mathematische und physikalische Aspekte verallgemeinerter Eichfeldtheorien im Ashtekarprogramm (Dissertation). Universität Leipzig, 2001.

[9] Christian Fleischhack: Parallel Transports in Webs. MIS-Preprint 31/2003, CGPG-03/3-6. e-print: math-ph/0304001.

[10] Christian Fleischhack: Stratification of the Generalized Gauge Orbit Space. *Commun. Math. Phys.* **214** (2000) 607–649. e-print: math-ph/0001006, math-ph/0001008.

[11] Jerzy Lewandowski and Thomas Thiemann: Diffeomorphism invariant quantum field theories of connections in terms of webs. *Class. Quant. Grav.* **16** (1999) 2299–2322. e-print: gr-qc/9901015.

[12] Милан Прагер: Об одном принципе сходимости в пространстве Гильберта. *Чехослов. мат. ж. (Czechoslovak. J. Math.)* **10** (1960) 271–282.