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Global existence of classical solutions for a hyperbolic chemotaxis model and its parabolic limit

by

Hyung Ju Hwang, Kyungkeun Kang, and Angela Stevens

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GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS FOR A HYPERBOLIC CHEMOTAXIS MODEL AND ITS PARABOLIC LIMIT

H.J. HWANG, K. KANG, AND A. STEVENS

Abstract. We consider a one dimensional hyperbolic system for chemosensitive movement, especially for chemotactic behavior. The model consists of two hyperbolic differential equations for the chemotactic species and is coupled with either a parabolic or an elliptic equation for the dynamics of the external chemical signal. The speed of the chemotactic species is allowed to depend on the external signal and the turning rates may depend on the signal and its gradients in space and time, as observed in experiments. Global classical solutions are established for regular initial data and a parabolic limit is proved.

1. Introduction

Changes in the pattern of movement in dependence of external chemical signals is a common mechanism for biological organisms to respond to their environment. The directed motion to higher concentrations of chemical signals is described by positive chemotaxis. Chemosensitivity describes the more general changes of speed of motion and orientation of the individuals in dependence of the chemical environment. This behavior can lead to different states of pattern formation and self-organization. Well known examples are the bacteria *Escherichia coli* and the slime mold amoebae *Dictyostelium discoideum*

The classical chemotaxis model discussed by Keller and Segel, [14] is a parabolic system. A related one dimensional hyperbolic model for chemotaxis was introduced in [17]. It is based on the Goldstein-Kac model [8, 13] for one-dimensional correlated random walks. In [9] the following hyperbolic model for chemotaxis with suitable boundary conditions was analyzed

$$
\begin{aligned}
u_t^+ + \gamma u_x^+ &= -\mu^+(s_x)u^+ + \mu^-(s_x)u^-,\\
u_t^- - \gamma u_x^- &= \mu^+(s_x)u^+ - \mu^-(s_x)u^-,\\
\tau s_t &= D s_{xx} + u^+ + u^- \quad \tau \geq 0, \quad t > 0, \;\; x \in (-1,1)
\end{aligned}
$$

where γ is the constant speed of the right and left moving cells u^+ and u^-, and μ^+, μ^- are the turning rates, which in this case depend linearly on the spatial gradient of the given chemical signal s. In [9] the gradient of s was expressed by a quasistationary approximation in the asymptotic limit $\tau \to 0$ and thus a quasilinear hyperbolic conservation law for $U(x,t) = \int_{-1}^{x} u(\xi,t)d\xi$ resulted.

Here we are concerned with the original and more general hyperbolic models for chemosensitive movement. Again, the density for the right moving particles is denoted by u^+, for the left moving particles by u^- and the external signal is s:

$$u_t^+ + (\gamma(s)u^+)_x = -\mu^+(s, s_t, s_x, s_{xx})u^+ + \mu^-(s, s_t, s_x, s_{xx})u^-, \tag{1.1}$$

$$u_t^- - (\gamma(s)u^-)_x = \mu^+(s, s_t, s_x, s_{xx})u^+ - \mu^-(s, s_t, s_x, s_{xx})u^-, \tag{1.2}$$

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$$\tau s_t = D s_{xx} + f(s, u^+ + u^-), \quad \tau \geq 0, \quad t > 0, \ x \in \mathbb{R} \tag{1.3}$$

$$u^\pm(0,x) = u_0^\pm(x), \quad s(0,x) = s_0(x) \tag{1.4}$$

where $u_0^\pm$ are assumed to have compact support, and s_0 and u_0 satisfy a compatibility condition. Typically f is given as follows:

$$f(s, u^+ + u^-) = \alpha(u^+ + u^-) - \beta s.$$

The diffusion rate of the external signal s and its production, and degradation rate are denoted by $D > 0$, $\alpha > 0$, and $\beta \geq 0$, respectively. Here we study the fully parabolic equation (1.3) for the external signal s, and the turning rates $\mu^\pm$ in (1.1 ,1.2) depend not only on the spatial derivatives of s but also on its time derivative and s itself. This is reasonable to assume since in Soll's studies [18] it turned out that the turning behavior and the speed of the slime mold amoebae Dictyostelium discoideum are dependent on both, the temporal and the spatial gradient of the cAMP concentration. Chen et al. [4, 5] analyzed data of E.coli and found out that the bacterial speed is close to constant, whereas the turning frequencies depend on the temporal gradient of the external signal. Their model was set into context with a one-dimensional projection of a 3D model for chemosensitive movement given by Alt, [1].

A general model of the kind described above, also with $\gamma = \gamma(s, s_x, s_t)$ was already introduced in [12] and a formal parabolic limit was derived. Local and global existence of solutions was proved for a simplified version of this system, namely for constant speed γ and turning rates $\mu^\pm = \mu^\pm(s, s_x)$. The dynamics for the chemical s were discussed for both cases, $\tau = 0$ and $\tau \neq 0$.

In [11] the case $\gamma = \gamma(s)$ and, as before $\mu^\pm = \mu^\pm(s, s_x)$ was discussed. For $\tau = 0$, which means elliptic dynamics for the chemical signal, existence of weak solutions could be proved.

In this paper we extend this result further in several ways. We consider $\mu^\pm = \mu^\pm(s, s_x, s_t, s_{xx})$. So also the dependency of the turning rates on chemical gradients in time are taken into account. The dynamics of the chemical can be considered to be parabolic ($\tau \neq 0$) as well as elliptic ($\tau = 0$), and global existence of classical solutions is proved. The results in [11] are a special case of our discussion here.

Our main result reads

Main Theorem Let $u_0^\pm \geq 0$, $s_0 \geq 0$ be smooth and bounded, and $u_0^\pm$ be compactly supported and not identically zero. Let $u_0^\pm$ and s_0 satisfy some compatibility condition. Then there exists a unique smooth solution $u^\pm$ and s of (1.1,1.2,1.3) with (1.4).

This paper is arranged as follows. We start with assumptions and notations in Section 2. A priori estimates on L^p are derived in Section 3, followed by the estimates of higher derivatives $W^{k,p}$ in Section 4. Finally, we attain global classical solutions for the hyperbolic chemotaxis model and rigorously derive a parabolic limit for the system.

2. Assumptions and Notations

Here we introduce notations which will be used throughout this article and give assumptions on the initial data, turning rates, and speed.

NOTATIONS:

(1) By Γ we denote the fundamental solution of the differential operator $\partial_t - \partial_{xx} + \beta$ in $\mathbb{R} \times \mathbb{R}$

$$\Gamma(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t} - \beta t\right).$$

(2) For $1 \leq p \leq \infty$, $L^p(\mathbb{R})$ denotes the Banach space of measurable functions with the finite norms

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}, \ p < \infty \text{ and } \|f\|_{L^\infty(\mathbb{R})} = \text{ess } \sup_{\mathbb{R}} |f|.$$

Let $W^{k,p}(\mathbb{R})$ denote the usual Sobolev space with its norm $\|f\|_{W^{k,p}(\mathbb{R})}$, i.e.,

$$W^{k,p}(\mathbb{R}) = \{ f | \partial^\alpha f \in L^p(\mathbb{R}), \ 0 \leq |\alpha| \leq k \}.$$

(3) For $0 < \delta < 1$, $C^\delta(\mathbb{R})$ denotes the Banach space of functions that are Hölder continuous with exponent δ and by $C^{k,\delta}(\mathbb{R})$ we denote the space of all functions whose derivatives up to k-th order are Hölder continuous with exponent $0 < \delta < 1$.

(4) Let $\Omega_t = \mathbb{R} \times (0,t)$ for $t \in [0,T]$. For $1 \leq p \leq \infty$, $L^p(\Omega_t)$ denotes the Banach space of all measurable functions with the finite norms

$$\|f\|_{L^p(\Omega_t)} = \left(\int_0^t \int_{\mathbb{R}} |f(x,t)|^p \, dxdt \right)^{1/p}, p < \infty \text{ and } \|f\|_{L^\infty(\Omega_t)} = \text{ess } \sup_{\Omega_t} |f|.$$

$W^{k,p}(\Omega_t)$, $C^\delta(\Omega_t)$, and $C^{k,\delta}(\Omega_t)$ denote the usual Sobolev and Hölder spaces in Ω_t.

(5) By $\|f\|_{L^p(\mathbb{R})}$ for $1 \leq p \leq \infty$, we denote the L^p norm of f with respect to x for fixed time t and $\sup_{0 \leq \tau \leq t} \|f\|_{L^p(\mathbb{R})}$ denotes the L^∞ norm of $\|f\|_{L^p(\mathbb{R})}$ with respect to time in $[0,t]$.

(6) By $C = C(\alpha, \beta, ...)$ we denote a constant depending on the prescribed quantities $\alpha, \beta, ...$.

ASSUMPTIONS:

(A1): The initial values $u_0^\pm \in C^\infty(\mathbb{R})$ have compact support and $u_0^\pm \geq 0$. We use the following compatibility condition: $s_0 \in C^\infty(\mathbb{R})$ is the unique solution of

$$0 = Ds_{0,xx} - \beta s_0 + \alpha \left(u_0^+ + u_0^- \right), \quad s_0(\pm\infty) = 0. \tag{2.1}$$

(A2): The turning rates are nonnegative and symmetric with respect to s_x

$$\mu^+, \mu^- \geq 0, \qquad \mu^+(s, s_t, s_x, s_{xx}) = \mu^-(s, s_t, -s_x, s_{xx}).$$

(A3): The turning rates satisfy $\mu^\pm \in C^\infty(\mathbb{R}^4)$ and are bounded

$$||D^{(\kappa)} \mu^\pm||_{L^\infty} \leq C_{|\kappa|}, \quad \text{,where } \kappa \text{ is a multi-index}$$

$$0 \leq \mu^\pm(s, s_t, s_x, s_{xx}) \leq C(1 + ||s||_{W^{1,\infty}(\mathbb{R})}).$$

(A4): The speed function satisfies $\gamma = \gamma(s) \in C^\infty(\mathbb{R})$ with

$$||\gamma^{(k)}||_{L^\infty} \leq C_k, \quad \text{,where } k \in \mathbb{N}_+.$$

From **(A4)**, it follows that $||\gamma(s)||_{L^\infty(\mathbb{R})} \leq C(1 + \|s\|_{L^\infty(\mathbb{R})})$.

The existence of a unique solution of (2.1) is clear from standard arguments for elliptic equations. The maximum principle for elliptic equations together with the positivity of $u_0^\pm$ leads to $s_0(x) \geq 0$ for all $x \in \mathbb{R}$. Using the method of vanishing viscosity, we consider the following model

$$u_t^{\epsilon+} - \epsilon u_{xx}^{\epsilon+} = -(\gamma(s^\epsilon) u^{\epsilon+})_x - \mu^+(s^\epsilon, s_t^\epsilon, s_x^\epsilon, s_{xx}^\varepsilon) u^{\epsilon+} + \mu^-(s^\epsilon, s_t^\epsilon, s_x^\epsilon, s_{xx}^\varepsilon) u^{\epsilon-}, \tag{2.2}$$

$$u_t^{\epsilon-} - \epsilon u_{xx}^{\epsilon-} = (\gamma(s^\epsilon) u^{\epsilon-})_x + \mu^+(s^\epsilon, s_t^\epsilon, s_x^\epsilon, s_{xx}^\varepsilon) u^{\epsilon+} - \mu^-(s^\epsilon, s_t^\epsilon, s_x^\epsilon, s_{xx}^\varepsilon) u^{\epsilon-}, \tag{2.3}$$

$$\tau s_t^\epsilon = Ds_{xx}^\epsilon - \beta s^\epsilon + \alpha(u^{\epsilon+} + u^{\epsilon-}), \tag{2.4}$$

$$u^{\epsilon+}(0,\cdot)=u_0^+,\quad u^{\epsilon-}(0,\cdot)=u_0^-,\quad s^\epsilon(0,\cdot)=s_0, \tag{2.5}$$

where $u_0^\pm$ and s_0 satisfy the compatibility condition (**A**1). Introducing the total population density $u^\varepsilon=u^{\varepsilon+}+u^{\varepsilon-}$ and the density flow $v^\varepsilon=u^{\varepsilon+}-u^{\varepsilon-}$, the system reads:

$$u_t^\epsilon-\epsilon u_{xx}^\epsilon=-(\gamma(s^\epsilon)v^\epsilon)_x, \tag{2.6}$$

$$v_t^\epsilon-\epsilon v_{xx}^\epsilon=-(\gamma(s^\epsilon)u^\epsilon)_x-\xi(s^\epsilon,s_t^\epsilon,s_x^\epsilon,s_{xx}^\varepsilon)u^\epsilon-\eta(s^\epsilon,s_t^\epsilon,s_x^\epsilon,s_{xx}^\varepsilon)v^\epsilon, \tag{2.7}$$

$$\tau s_t^\epsilon=Ds_{xx}^\epsilon-\beta s^\epsilon+\alpha u^\epsilon, \tag{2.8}$$

where

$$u^\epsilon(0,\cdot)=u_0=u_0^++u_0^-,\quad v^\epsilon(0,\cdot)=v_0=u_0^+-u_0^-,\quad s^\epsilon(0,\cdot)=s_0$$

and $\xi=\mu^+-\mu^-,\eta=\mu^++\mu^-$.

3. A priori estimates on L^p

Lemma 1. *Let $a(t)$ and $b(t)$ be positive functions. Let $y(t)>0$ be differentiable in t and satisfy*

$$y'\le a(t)\,y\ln y+b(t)\,y.$$

Then

$$y(t)\le\left[y(0)\exp\left(\int_0^t b(s)\,e^{-\int_0^s a(\tau)d\tau}ds\right)\right]^{\exp\left(\int_0^t a(s)ds\right)}.$$

Proof. Dividing both sides of the inequality by y, we get a typical Gronwall inequality for $z=\ln y$

$$z'\le a(t)\,z+b(t).$$

Therefore, we deduce the lemma. □

Lemma 2. [Gronwall's inequality] *Let g and h be positive functions. Suppose that f is an integrable function in t and satisfies*

$$f(t)\le g(t)+h(t)\int_0^t f(s)\,ds.$$

Then we have

$$f(t)\le g(t)+h(t)\int_0^t g(s)\exp\left(\int_s^t h(\tau)\,d\tau\right)ds.$$

Proof. Computations are straightforward and hence we omit details (see e.g. [7]). □

Throughout this paper we consider only the case $\beta>0$, for simplicity. We remark, however, our main result can be easily extended to the case $\beta=0$ (see Remark 3 for more details). For convenience, we will use $u^\pm$ without ε for $u^{\varepsilon\pm}$ from now on. Without loss of generality, we assume that $\tau=1$ and $D=1$. Let $1<p<\infty$. For given $u\in L^p(\Omega_t)$, we study the parabolic equation:

$$s_t-s_{xx}=-\beta s+\alpha u. \tag{3.1}$$

Using potential estimates similar to the heat kernel, we have:

Lemma 3. *Let $u \in L^p(\Omega_t)$ and $1 < p < \infty$. Then there exists a constant $C_p = C(\alpha, \beta, p)$ such that the following estimate holds*

$$\|s_t\|_{L^p(\Omega_t)} + \|s_{xx}\|_{L^p(\Omega_t)} + \|s\|_{L^p(\Omega_t)} \leq C_p \|u\|_{L^p(\Omega_t)}.$$

Proof. Since the above estimate is standard (e.g. see [15] and [16]), we omit the details. □

Next we estimate $\|s\|_{W^{1,\infty}(\mathbb{R})}$.

Lemma 4. *If $u \in L^\infty\left([0,\infty) : L^1(\mathbb{R}) \cap L^2(\mathbb{R})\right)$, then the solution s in (3.1) satisfies*

$$\|s\|_{L^\infty(\mathbb{R})} \leq C(\alpha, \beta) \sup_{0 \leq \tau \leq t} \|u\|_{L^1(\mathbb{R})} = C(\alpha, \beta) \|u_0\|_{L^1(\mathbb{R})},$$

$$\|s_x\|_{L^\infty(\mathbb{R})} \leq C(\alpha, \beta) \left[1 + \|u_0\|_{L^1(\mathbb{R})} \left(1 + (\ln t)_+ + \left|\ln\left(\sup_{0 \leq \tau \leq t} \|u\|_{L^2(\mathbb{R})}\right)\right|\right)\right],$$

where $(\cdot)_+$ means the positive part and

$$\lim_{|x| \to \infty} s(x,t) = 0 \quad \textit{for all } t.$$

Proof. The fundamental solution of the operator $\partial_t - \partial_{xx} + \beta$ is

$$\Gamma(x,t) = \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t} - \beta t\right)$$

and its Fourier transform is

$$\hat{\Gamma}(\xi, t) = \exp\left(-t\left(4\xi^2 + \beta\right)\right).$$

By Duhamel's principle, we obtain

$$\begin{aligned} s(x,t) &= \int_0^t (\Gamma * \alpha u)(x, t-\tau)\, d\tau \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{t-\tau}} \exp\left(-\frac{(x-y)^2}{4(t-\tau)} - \beta(t-\tau)\right) \alpha u(y,\tau)\, dy d\tau. \end{aligned}$$

Next we estimate

$$\begin{aligned} &\int_{-\infty}^{\infty} |\hat{s}(\xi,t)|\, d\xi = \int_{-\infty}^{\infty} \left| \int_0^t \hat{\Gamma} \alpha \hat{u}\, d\tau \right| d\xi \\ &\leq \alpha \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} \int_0^t e^{-(t-\tau)\left(4\xi^2 + \beta\right)} d\tau d\xi \\ &\leq \alpha \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} \frac{1}{4\xi^2 + \beta} d\xi \\ &\leq C(\alpha, \beta) \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

and we have

$$|\hat{s}(\xi,t)| \leq \alpha \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^\infty(\mathbb{R})} \frac{1}{4\xi^2 + \beta}. \tag{3.2}$$

Therefore, using the inverse Fourier-transform for $\hat{s}$, we have $\lim_{|x| \to \infty} s(x) = 0$ and

$$\|s\|_{L^\infty(\mathbb{R})} \leq \|\hat{s}\|_{L^1(\mathbb{R})} \leq C(\alpha, \beta) \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^\infty(\mathbb{R})} \leq C(\alpha, \beta) \sup_{0 \leq \tau \leq t} \|u\|_{L^1(\mathbb{R})}.$$

Next, we estimate $\|s_x\|_{L^\infty(\mathbb{R})}$:

$$\|s_x\|_{L^\infty(\mathbb{R})} \le \|\xi\hat{s}\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty}\int_0^t |\xi| \left|\hat{\Gamma}\alpha\hat{u}\right| (\xi, t-\tau)\, d\tau d\xi$$
$$= \alpha \int_0^t \int_{-\infty}^{\infty} |\xi| \exp\left(-\tau\left(4\xi^2+\beta\right)\right) |\hat{u}(\xi,\tau)|\, d\xi d\tau.$$

The integration is done by splitting the time integration into two:

$$\int_0^t \int_{-\infty}^{\infty} |\xi| \exp\left(-\tau\left(4\xi^2+\beta\right)\right) |\hat{u}(\xi,\tau)|\, d\xi d\tau = \int_0^r \cdots + \int_r^t \cdots = I_1 + I_2,$$

where $r > 0$ will be chosen later.

(1) For $0 < \tau < r$, we use Hölder's inequality with $p = q = 2$:

$$\int_{-\infty}^{\infty} |\xi| \exp\left(-\tau\left(4\xi^2+\beta\right)\right) |\hat{u}(\xi,\tau)|\, d\xi \le \left(\int_{-\infty}^{\infty} \xi^2 \exp\left(-2\tau\left(4\xi^2+\beta\right)\right) d\xi\right)^{1/2} \|\hat{u}\|_{L^2(\mathbb{R})}$$
$$= \left(2\int_0^{\infty} \xi^2 \exp\left(-2\tau\left(4\xi^2+\beta\right)\right) d\xi\right)^{1/2} \|u\|_{L^2(\mathbb{R})},$$

where we used the Plancherel's equality for L^2. By integration by parts, we have

$$\int_0^{\infty} \xi^2 \exp\left(-2\tau\left(4\xi^2+\beta\right)\right) d\xi = \frac{1}{16\tau}\int_0^{\infty} \exp\left(-2\tau\left(4\xi^2+\beta\right)\right) d\xi$$
$$= \frac{\sqrt{\pi}}{64\sqrt{2}}\tau^{-3/2}e^{-2\beta\tau}.$$

Hence, we obtain

$$I_1 \le \frac{\pi^{1/4}}{4\sqrt[4]{2^3}} \sup_{0\le\tau\le t} \|u\|_{L^2(\mathbb{R})} \int_0^r \tau^{-3/4} e^{-\beta\tau} d\tau \le \frac{\pi^{1/4}}{\sqrt[4]{2^3}} r^{1/4} \sup_{0\le\tau\le t} \|u\|_{L^2(\mathbb{R})}.$$

(2) For $r \le \tau \le t$, we use Hölder's inequality with $p = 1,\ q = \infty$:

$$\int_{-\infty}^{\infty} |\xi| \exp\left(-\tau\left(4\xi^2+\beta\right)\right) |\hat{u}(\xi,\tau)|\, d\xi$$
$$\le \|\hat{u}\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} |\xi| \exp\left(-\tau\left(4\xi^2+\beta\right)\right) d\xi$$
$$= \frac{1}{4\tau} e^{-\beta\tau} \|\hat{u}\|_{L^\infty(\mathbb{R})}.$$

So, we have

$$I_2 \le \frac{1}{4} \sup_{0\le\tau\le t} \|\hat{u}\|_{L^\infty(\mathbb{R})} \int_r^t \frac{1}{\tau} e^{-\beta\tau} d\tau \le \frac{1}{4} \sup_{0\le\tau\le t} \|u\|_{L^1(\mathbb{R})} |\ln t - \ln r|.$$

Therefore, we get

$$\|s_x\|_{L^\infty(\mathbb{R})} \le C\alpha\left(r^{1/4} \sup_{0\le\tau\le t} \|u\|_{L^2(\mathbb{R})} + \sup_{0\le\tau\le t} \|u\|_{L^1(\mathbb{R})} |\ln t - \ln r|\right).$$

We optimize the upper bound for the above inequality by choosing

$$r=\min\left\{\left(\sup_{0\le\tau\le t}\|u\|_{L^2(\mathbb{R})}\right)^{-4},t\right\}.$$

If $r=t$ and for $t\le 1$ we have $t\le\left(\sup_{0\le\tau\le t}\|u\|_{L^2(\mathbb{R})}\right)^{-4}$ and $\|s_x\|_{L^\infty(\mathbb{R})}\le C\alpha$.
If $r=\left(\sup_{0\le\tau\le t}\|u\|_{L^2(\mathbb{R})}\right)^{-4}$, then $4\left|\ln\left(\sup_{0\le\tau\le t}\|u\|_{L^2(\mathbb{R})}\right)\right|\ge|\ln t|$ and

$$\|s_x\|_{L^\infty(\mathbb{R})}\le C\alpha\left[1+\sup_{0\le\tau\le t}\|u\|_{L^1(\mathbb{R})}\left|\ln\left(\sup_{0\le\tau\le t}\|u\|_{L^2(\mathbb{R})}\right)\right|\right].$$

For $t\ge 1$, we have $|\ln t-\ln r|\le C\left(|\ln t|+\left|\ln\left(\sup_{0\le\tau\le t}\|u\|_{L^2(\mathbb{R})}\right)\right|\right)$.

Finally $\sup_{0\le\tau\le t}\|u\|_{L^1}(\mathbb{R})=\|u_0\|_{L^1}(\mathbb{R})$ since the total population size is preserved, namely

$$\int_{\mathbb{R}}u(x,t)dx=\int_{\mathbb{R}}u_0(x)dx=\int_{\mathbb{R}}(u_0^++u_0^-)(x)dx\qquad\text{for all } t.$$

because

$$\frac{d}{dt}\int_{\mathbb{R}}u(x,t)dx=\int_{\mathbb{R}}[-(\gamma(s)u^+(x,t))_x+(\gamma(s)u^-(x,t))_x]dx=0.$$

This completes the proof. □

Lemma 5. *Let $U_0=\int_{\mathbb{R}}u(x,t)\,dx<\infty$ and $S_0=\int_{\mathbb{R}}s_0(x)\,dx$, then $s\in L^p(\Omega_t)$ for $2\le p\le\infty$ and $s_x\in L^2(\Omega_t)$ in (3.1) with*

$$\|s\|_{L^p(\Omega_t)}\le C(\alpha,\beta)\,t^{\frac{1+2p}{2p}}U_0,\quad \|s_x\|_{L^2(\Omega_t)}\le C(\alpha,\beta)\,t^{\frac12}U_0$$

$$\int_{\mathbb{R}}s(x,t)\,dx=\frac{\alpha U_0}{\beta}+\left(S_0-\frac{\alpha U_0}{\beta}\right)e^{-\beta t},\tag{3.3}$$

where

$$\lim_{|x|\to\infty}s_x(x,t)=0\qquad\textit{for all } t.$$

Proof. From (3.2), we easily see that $\|s\|_{L^2(\Omega_t)}=\|\hat{s}\|_{L^2(\Omega_t)}\le C(\alpha,\beta)\,t^{\frac12}U_0$ and

$$\|s_x\|_{L^2(\Omega_t)}=\|\hat{s}_x\|_{L^2(\Omega_t)}=\|\xi\hat{s}\|_{L^2(\Omega_t)}\le C(\alpha,\beta)\,t^{\frac12}U_0$$

with $\lim_{|x|\to\infty}s_x(x,t)=0$.
Since $\|s\|_{L^\infty(\Omega_t)}\le C(\alpha,\beta)\,U_0$ by Corollary 4, we have $\|s\|_{L^p(\Omega_t)}\le C(\alpha,\beta)\,t^{\frac1p}U_0$ for $2\le p\le\infty$ by interpolation. From (1.3), we have,

$$\int_{\mathbb{R}}s(x,\tau)\,dx+\beta\int_0^t\int_{\mathbb{R}}s(x,\tau)\,dxd\tau=\int_{\mathbb{R}}s_0(x)\,dx+t\alpha\int_{\mathbb{R}}u_0(x)\,dx=S_0+t\alpha U_0.$$

For convenience, we set $S(t)=\int_{\mathbb{R}}s(x,t)\,dx$. Then we have

$$S'(t)=-\beta S(t)+\alpha U_0.\tag{3.4}$$

Solving the ordinary differential equation (3.4), we obtain

$$\int_{\mathbb{R}}s(x,t)\,dx=S_0e^{-\beta t}+\alpha U_0\int_0^t e^{\beta(\tau-t)}d\tau.$$

By integrating the last term, we have (3.3). The proof is complete. □

We have the invariance of positivity of $u^{\pm}$ and s.

Lemma 6. *Assume $u^{\pm} \geq 0$ in Ω_t. Then $s \geq 0$ in Ω_t.*

Proof. This is an easy consequence of the parabolic maximum principle (e.g. see [7] or [16]). □

Lemma 7. *If $u_0^{\pm} \geq 0$, then the solution (u^+, u^-, s) of (2.3)-(2.5) satisfies $u^{\pm} \geq 0$ as long as (u^+, u^-, s) exists.*

Proof. Assumption **(A2)** on the non-negativity of the turning rates ensures our lemma from the concept of invariant regions for parabolic systems (e.g. see [6]). □

Remark 1. *For the conserved total population density U_0, we have $s \in L^1(\Omega_t)$. Indeed, since $s \geq 0$, we have by (3.3)*

$$
\begin{aligned}
\|s\|_{L^1(\Omega_t)} &= \int_0^t \int s(x,\tau)\, dx d\tau = \int_0^t \frac{\alpha U_0}{\beta} + \left(S_0 - \frac{\alpha U_0}{\beta}\right) e^{-\beta\tau} d\tau \\
&= \frac{\alpha U_0 t}{\beta} + \frac{1}{\beta}\left(S_0 - \frac{\alpha U_0}{\beta}\right)\left(1 - e^{-\beta t}\right) < \infty.
\end{aligned}
$$

So, combining the results of Lemma 5, we have $s \in L^p(\Omega_t)$ for all $1 \leq p \leq \infty$. □

Now let $\mathcal{K} = L^2(\mathbb{R}) \cap \mathcal{C}_0^2(\mathbb{R})$, where

$$
\mathcal{C}_0 = \{u \in L^\infty(\mathbb{R}) : \lim_{|x|\to\infty} u(x) = 0\}, \qquad \mathcal{C}_0^k = \{u : D_j u \in \mathcal{C}_0,\ j = 0, ..., k\}
$$

We state the local existence result for $u^{\pm}$:

Lemma 8. *For initial values $u_0^{\pm} \in \mathcal{K}$ there exists a unique solution of (2.2), (2.3) with*

$$
\left(u^+, u^-\right) \in C\left([0, T_0], \mathcal{K}\right)
$$

for some time $T_0 > 0$.

Proof. Theorem can be proved by following a similar procedure as in Corollary 3.1 and Theorem 3.1 in [11, see page 180-182]. Therefore, details are omitted. □

Next we give growth rates for the L^2-norms of $u^{\pm}$ which ensure global existence. For simplicity, we denote $u^{\epsilon\pm}$ and s^ϵ in (2.2-2.4) by $u^{\pm}$ and s, respectively, in case no confusion is to be expected.

Lemma 9. *Let $u_0^{\pm} \in L^1 \cap L^2$ with $\int_{\mathbb{R}} u_0^+ + u_0^- = U_0$. Assume (A1)-(A4). Then the solution (u^+, u^-) of (2.2,2.3) exists globally in $C\left([0,\infty), L^1 \cap L^2\right)$ and there exist constants $K = K(\alpha, \beta, U_0)$ and $C = C(\alpha, \beta, U_0)$ which are independent of $\varepsilon > 0$ such that for all $t \geq 0$,*

$$
\left\|\left(u^+, u^-\right)\right\|_{L^2(\mathbb{R})} \leq \left[C\left\|\left(u_0^+, u_0^-\right)\right\|_{L^2(\mathbb{R})}\right]^{e^{Kt}}. \tag{3.5}
$$

Proof. Using (2.2), (2.3) and applying Hölder's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|u^{+}\right\|_{L^{2}(\mathbb{R})}^{2}+\left\|u^{-}\right\|_{L^{2}(\mathbb{R})}^{2}\right)=2\int_{\mathbb{R}}\left(u^{+}u_{t}^{+}+u^{-}u_{t}^{-}\right)dx\\
&=-2\varepsilon\int_{\mathbb{R}}\left[\left|u^{+}\right|^{2}+\left|u^{-}\right|^{2}\right]dx+2\int_{\mathbb{R}}\left[-\left(\gamma u^{+}\right)_{x}u^{+}-\mu^{+}\left|u^{+}\right|^{2}+\mu^{-}u^{-}u^{+}\right]dx\\
&\quad+\int_{\mathbb{R}}\left[\left(\gamma u^{-}\right)_{x}u^{-}+\mu^{+}u^{+}u^{-}-\mu^{-}\left|u^{-}\right|^{2}\right]dx\\
&\leq 2\int_{\mathbb{R}}\left[-\left(\gamma u^{+}\right)_{x}u^{+}+\mu^{-}u^{-}u^{+}\right]dx+2\int_{\mathbb{R}}\left[\left(\gamma u^{-}\right)_{x}u^{-}+\mu^{+}u^{+}u^{-}\right]dx\\
&\leq\left\|\gamma_{x}\right\|_{L^{\infty}}\int_{\mathbb{R}}\left[\left|u^{+}\right|^{2}+\left|u^{-}\right|^{2}\right]dx+\left(\left\|\mu^{+}\right\|_{L^{\infty}}+\left\|\mu^{-}\right\|_{L^{\infty}}\right)\int_{\mathbb{R}}\left[\left|u^{+}\right|^{2}+\left|u^{-}\right|^{2}\right]dx\\
&\leq C\left(1+\|s\|_{W^{1,\infty}(\mathbb{R})}\right)\left(\left\|u^{+}\right\|_{L^{2}(\mathbb{R})}^{2}+\left\|u^{-}\right\|_{L^{2}(\mathbb{R})}^{2}\right).
\end{aligned}$$

Therefore, by Lemma 4, we have

$$\frac{d}{dt}\left(\left\|u^{+}\right\|_{L^{2}(\mathbb{R})}^{2}+\left\|u^{-}\right\|_{L^{2}(\mathbb{R})}^{2}\right)\leq$$

$$C\left(\alpha,\beta,U_{0}\right)\left(1+(\ln t)_{+}+\left|\ln\left(\sup_{0\leq\tau\leq t}\left[\left\|u^{+}\right\|_{L^{2}(\mathbb{R})}^{2}+\left\|u^{-}\right\|_{L^{2}(\mathbb{R})}^{2}\right]\right)\right|\right)\left(\left\|u^{+}\right\|_{L^{2}(\mathbb{R})}^{2}+\left\|u^{-}\right\|_{L^{2}(\mathbb{R})}^{2}\right).$$

Setting $y\left(t\right)=\sup_{0\leq\tau\leq t}\left(\|u^{+}\|_{L^{2}(\mathbb{R})}^{2}+\|u^{-}\|_{L^{2}(\mathbb{R})}^{2}\right)$, we obtain

$$y'\leq K\left(\alpha,\beta,U_{0}\right)y\left|\ln y\right|+K\left(\alpha,\beta,U_{0}\right)y\left(1+(\ln t)_{+}\right).$$

Hence, applying Lemma 1 with $a\left(t\right)=K,\ b\left(t\right)=K\left(1+(\ln t)_{+}\right)$, we complete the proof. □

An easy consequence of the above result is the following.

Corollary 1. *Let s be the solution of (2.8). Then s satisfies*

$$\left\|s_{x}\right\|_{L^{\infty}(\mathbb{R})}\leq C\left(\alpha,\beta,U_{0}\right)e^{Kt},$$

where $K=K\left(\alpha,\beta,U_{0}\right)$.

Proof. This is a combination of the a priori estimate for $||s_x||_{L^\infty(\mathbb{R})}$ in Lemma 4 and the estimate (3.5) in Lemma 9. □

Next we prove L^p estimates. For convenience, we denote $(u^{\pm})^p$ by $u^{\pm,p}$.

Lemma 10. *Let $u_0^{\pm}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ with $\int_{\mathbb{R}}u_0^{+}+u_0^{-}=U_0$. Assume (A1)-(A4). Then there exist constants $C_1=C_1\left(u_0^{\pm}\right)$, $C_2=C_2\left(u_0^{\pm},\alpha,\beta,U_0\right)$ and $K=K\left(\alpha,\beta,U_0\right)$ such that*

$$\left\|u^{+}\right\|_{L^{p}(\mathbb{R})}+\left\|u^{-}\right\|_{L^{p}(\mathbb{R})}\leq C_{1}\exp\left(C_{2}\int_{0}^{t}e^{Ks}ds\right)\leq C_{1}\exp\left(C_{2}\exp Kt\right).$$

for all $2\leq p<\infty$ and $0\leq t<\infty$.

Proof. Multiplying $p\,(u^+)^{p-1}$ and $p\,(u^-)^{p-1}$ to (2.2) and (2.3), we have

$$\begin{aligned}
&\frac{d}{dt}\left\|u^+\right\|^p_{L^p(\mathbb{R})}+\int_{\mathbb{R}}\left(\gamma u^{+,p}\right)_x dx+\int_{\mathbb{R}}(p-1)\,\gamma_x u^{+,p}dx\\
&=p\int_{\mathbb{R}}\left(-\mu^+u^{+,p}+\mu^-u^-u^{+,(p-1)}\right)dx-\varepsilon p\,(p-1)\int_{\mathbb{R}}u^{+,p-2}\left(u^+_x\right)^2dx,\\
&\frac{d}{dt}\left\|u^-\right\|^p_{L^p(\mathbb{R})}-\int_{\mathbb{R}}\left(\gamma u^{-,p}\right)_x dx-\int_{\mathbb{R}}(p-1)\,\gamma_x u^{-,p}dx\\
&=p\int_{\mathbb{R}}\left(\mu^+u^+u^{-,p-1}-\mu^-u^{-,p}\right)dx-\varepsilon p\,(p-1)\int_{\mathbb{R}}u^{-,p-2}\left(u^-_x\right)^2dx,
\end{aligned}$$

where $u^{\pm,p}$ denotes $(u^{\pm})^p$ and we have used

$$p\left(\gamma u^{\pm}\right)_x u^{\pm,p-1}=\left(\gamma u^{\pm,p}\right)_x+(p-1)\,\gamma_x u^{\pm,p}.$$

Therefore, applying the Hölder's inequality we deduce

$$\begin{aligned}
&\frac{d}{dt}\left\|u^+\right\|^p_{L^p(\mathbb{R})}\le Cp\left(1+\|s\|_{W^{1,\infty}(\mathbb{R})}\right)\left(\left\|u^+\right\|_{L^p(\mathbb{R})}+\left\|u^-\right\|_{L^p(\mathbb{R})}\right)\left\|u^+\right\|^{p-1}_{L^p(\mathbb{R})}\\
&\le Cpe^{Kt}\left(1+\left|\ln\left(\left\|\left(u^+_0,u^-_0\right)\right\|_{L^2(\mathbb{R})}\right)\right|\right)\left(\left\|u^+\right\|_{L^p(\mathbb{R})}+\left\|u^-\right\|_{L^p(\mathbb{R})}\right)\left\|u^+\right\|^{p-1}_{L^p(\mathbb{R})}.
\end{aligned}$$

A similar estimate holds for $||u^-||_{L^p(\mathbb{R})}$. Therefore, we have

$$\frac{d}{dt}\left(\left\|u^+\right\|_{L^p(\mathbb{R})}+\left\|u^-\right\|_{L^p(\mathbb{R})}\right)\le C\left(1+\left|\ln\left(\left\|\left(u^+_0,u^-_0\right)\right\|_{L^2(\mathbb{R})}\right)\right|\right)e^{Kt}\left(\left\|u^+\right\|_{L^p(\mathbb{R})}+\left\|u^-\right\|_{L^p(\mathbb{R})}\right).$$

Then the standard Gronwall inequality, Lemma 2, implies

$$\left\|u^+\right\|_{L^p(\mathbb{R})}+\left\|u^-\right\|_{L^p(\mathbb{R})}\le\left(\left\|u^+_0\right\|_{L^p(\mathbb{R})}+\left\|u^-_0\right\|_{L^p(\mathbb{R})}\right)\exp\left(C\left|\ln\left(\left\|\left(u^+_0,u^-_0\right)\right\|_{L^2(\mathbb{R})}\right)\right|\int_0^t e^{Ks}ds\right).$$

Since the initial data u_0 are estimated as follows

$$||u^+_0||_{L^p(\mathbb{R})}+||u^-_0||_{L^p(\mathbb{R})}\le||u_0||_{L^p(\mathbb{R})}\le||u_0||^{\frac{p-1}{p}}_{L^\infty(\mathbb{R})}||u_0||^{\frac{1}{p}}_{L^1(\mathbb{R})}\le||u_0||^{\frac{p-1}{p}}_{L^\infty(\mathbb{R})}U_0^{\frac{1}{p}},$$

we have

$$\left\|u^+\right\|_{L^p(\mathbb{R})}+\left\|u^-\right\|_{L^p(\mathbb{R})}\le C_1\exp\left(C_2\int_0^t e^{Ks}ds\right)\le C_1\exp\left(C_2\exp Kt\right),$$

where $C_1\equiv(||u_0||_{L^\infty(\mathbb{R})}+1)(U_0+1)$ and $C_2\equiv C(1+|\ln C_1|)$. This completes the proof. □

We also have L^∞ estimates.

Lemma 11. *Let $u^{\pm}_0\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ with $\int_{\mathbb{R}}u^+_0+u^-_0=U_0$. Assume (A1)-(A4). There exist constants $C_1=C_1\left(u^{\pm}_0\right)$, $C_2=C_2\left(u^{\pm}_0,\alpha,\beta,U_0\right)$ and $K=K\left(\alpha,\beta,U_0\right)$ such that for all $t\ge 0$,*

$$\left\|u^+\right\|_\infty+\left\|u^-\right\|_\infty\le C_1\exp\left(C_2\exp Kt\right). \tag{3.6}$$

Proof. In the L^p-estimate of the previous lemma, the constants C_1, C_2 and K are uniformly bounded and independent of p. Thus, (3.6) is obvious. □

4. $W^{k,p}$-ESTIMATES

In this section, we study L^p-estimates for all higher derivatives of $u^\pm$. We first present standard regularity estimates for parabolic equations without proof. For conveninence, we denote $u_x^+ = v^+$ and $u_x^- = v^-$.

Lemma 12. *There exists a constant $C = C(p)$ such that for all $1 < p < \infty$,*

$$\|s_{tx}\|_{L^p(\Omega_t)} + \|s_{xxx}\|_{L^p(\Omega_t)} + \|s_x\|_{L^p(\Omega_t)} \le C(||v^+||_{L^p(\Omega_t)} + ||v^-||_{L^p(\Omega_t)}).$$

Lemma 13. *Let $u^\pm, s$ be solutions of (2.2)-(2.5) and let the initial data fulfill $u_{0x}^\pm \in L^2$ Then (u_x^+, u_x^-) exists globally in $C\left([0,\infty), L^2(\mathbb{R})\right)$ and (u_x^+, u_x^-) satisfies*

$$\left\|\left(u_x^+, u_x^-\right)\right\|_{L^2(\mathbb{R})} \le \left\|\left(u_{0x}^+, u_{0x}^-\right)\right\|_{L^2(\mathbb{R})} + C\exp\left(C\exp\left(C\exp Kt\right)\right),$$

where $C = C(\alpha, \beta, U_0)$, $K = K(\alpha, \beta, U_0)$.

Proof. From (2.2) and (2.3), we obtain

$$v_t^+ - \varepsilon v_{xx}^+ + (\gamma v^+)_x = -(\gamma_x u^+)_x + (-\mu^+ u^+ + \mu^- u^-)_x \tag{4.1}$$

$$v_t^- - \varepsilon v_{xx}^- - (\gamma v^-)_x = (\gamma_x u^-)_x + (\mu^+ u^+ - \mu^- u^-)_x \tag{4.2}$$

Multiplying v^+ and v^- to (4.1) and (4.2) respectively, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} |v^+(\cdot,t)|^2 dx = \frac{1}{2}\int_{\mathbb{R}} |v^+(\cdot,0)|^2 dx + \int_0^t\int_{\mathbb{R}} v_t^+ v^+ dx \\
&= \frac{1}{2}\int_{\mathbb{R}} |v^+(\cdot,0)|^2 dx + \int_0^t\int_{\mathbb{R}} \gamma v^+ v_x^+ dx - \int_0^t\int_{\mathbb{R}} (\gamma_x u^+)_x v^+ dx \\
&\quad + \int_0^t\int_{\mathbb{R}} (-\mu^+ u^+ + \mu^- u^-)_x v^+ dxds - \varepsilon\int_0^t\int_{\mathbb{R}} \left|v_x^+(\cdot,s)\right|^2 dxds \\
&= \frac{1}{2}\int_{\mathbb{R}} |v^+(\cdot,0)|^2 dx + \int_0^t\int_{\mathbb{R}} -\gamma_x \frac{|v^+|^2}{2} dxds - \int_0^t\int_{\mathbb{R}} \left[\gamma_{xx} u^+ v^+ + \gamma_x |v^+|^2\right] dxds \\
&\quad + \int_0^t\int_{\mathbb{R}} \left[-\mu^+ |v^+|^2 + \mu^- v^- v^+ - \mu_x^+ u^+ v^+ + \mu_x^- u^- v^+\right] dxds - \varepsilon\int_0^t\int_{\mathbb{R}} \left|v_x^+(\cdot,s)\right|^2 dxds \\
&\le \frac{1}{2}\int_{\mathbb{R}} |v^+(\cdot,0)|^2 dx + C\left\|\gamma'\right\|_{L^\infty} \|s_x\|_{L^\infty(\mathbb{R})} \int_0^t\int_{\mathbb{R}} |v^+|^2 dxds \\
&\quad + C\left(\left\|\mu^+\right\|_{L^\infty} + \left\|\mu^-\right\|_{L^\infty}\right)\int_0^t\int_{\mathbb{R}} \left[|v^+|^2 + |v^-|^2\right] dxds \\
&\quad + C\left(\left\|u^+\right\|_{L^\infty(\mathbb{R})} + \left\|u^-\right\|_{L^\infty(\mathbb{R})}\right)\int_0^t\int_{\mathbb{R}} \left[|v^+|^2 + |\gamma_{xx}|^2 + |\mu_x^+|^2 + |\mu_x^-|^2\right] dxds.
\end{aligned}$$

Note that

$$\begin{aligned}
|\gamma_{xx}| &\le \left\|\gamma'\right\|_{L^\infty} |s_{xx}| + \left\|\gamma''\right\|_{L^\infty} |s_x|^2, \\
|\mu_x^+| &\le \left\|D_j\mu\right\|_{L^\infty} \left(|s_x| + |s_{xx}| + |s_{xxx}| + |s_{tx}|\right).
\end{aligned}$$

Here, as in Lemma 12, we use the following L^2-estimate for s.

$$||s_{tx}||_{L^2(\Omega_t)} + ||s_{xxx}||_{L^2(\Omega_t)} + \|s_x\|_{L^2(\Omega_t)} \le C\left(||v^+||_{L^2(\Omega_t)} + ||v^-||_{L^2(\Omega_t)}\right).$$

For s_{xx}, we have from Lemma 3 and Lemma 10,

$$\|s_{xx}\|_{L^2(\Omega_t)} \le C\,\|u\|_{L^2(\Omega_t)} \le C\exp\left(C\exp Kt\right).$$

By Corollary 1 and Lemma 11, we have

$$\|s_x\|_{L^\infty(\mathbb{R})} \le Ce^{Kt}, \qquad \left\|u^\pm\right\|_{L^\infty(\mathbb{R})} \le C\exp\left(C\exp Kt\right).$$

Therefore

$$\int_{\mathbb{R}} |v^+(\cdot,t)|^2 dx \le \int_{\mathbb{R}} |v^+(\cdot,0)|^2 dx + C\exp\left(C\exp Kt\right)$$
$$+ C\exp\left(C\exp Kt\right)\int_0^t \int_{\mathbb{R}} \left[|v^+|^2 + |v^-|^2\right] dxds.$$

In a similar manner, we deduce

$$\int_{\mathbb{R}} |v^-(\cdot,t)|^2 \le \int_{\mathbb{R}} |v^+(\cdot,0)|^2 + C\exp\left(C\exp Kt\right)$$
$$+ C\exp\left(C\exp Kt\right)\int_0^t \int_{\mathbb{R}} \left[|v^+|^2 + |v^-|^2\right] dxds.$$

Gronwall's inequality, Lemma 2, implies that

$$\left\|\left(u_x^+, u_x^-\right)\right\|_{L^2(\mathbb{R})} \le \left\|\left(u_{0x}^+, u_{0x}^-\right)\right\|_{L^2(\mathbb{R})} + C\exp\left(C\exp\left(C\exp Kt\right)\right).$$

This completes the proof. □

Next we show $W^{1,p}$ estimates.

Lemma 14. *Let $2 \le p < \infty$. Let $u^\pm, s$ be solutions of (2.2)-(2.5) and $u_{0x}^\pm \in L^p$. Then (u_x^+, u_x^-) from (2.2,2.3 ,2.4) exist globally in $C\left([0,\infty), L^p\left(\mathbb{R}\right)\right)$ and satisfy*

$$\left\|\left(u_x^+, u_x^-\right)\right\|_{L^p(\mathbb{R})} \le \left\|\left(u_{0x}^+, u_{0x}^-\right)\right\|_{L^p(\mathbb{R})} + C_p\exp\left(C_p\exp\left(C\exp Kt\right)\right),$$

where $K = K\left(\alpha,\beta,U_0\right)$, $C = C\left(\alpha,\beta,U_0\right)$ and $C_p = C\left(\alpha,\beta,U_0,p\right)$.

Proof. We multiply $pv^{+,p-1}$ and $pv^{-,p-1}$ to (4.1) and (4.2) respectively. Then we have

$$||v^+||^p_{L^p(\mathbb{R})} = ||v^+\left(0\right)||^p_{L^p(\mathbb{R})} - \int_0^t \int_{\mathbb{R}} \left[(\gamma v^{+,p})_x - (p-1)\gamma_x v^{+,p}\right] dxds$$
$$- \int_0^t \int_{\mathbb{R}} (\gamma_{xx} u^+ + \gamma_x v^+) pv^{+,p-1} dxds + \int_0^t \int_{\mathbb{R}} (\mu_x^+ u^+ + \mu_x^- u^-) pv^{+,p-1} dxds$$
$$+ \int_0^t \int_{\mathbb{R}} (\mu^+ v^+ + \mu^- v^-) pv^{+,p-1} dxds.$$

Using the following estimates

$$||\gamma_{xx}||_{L^p(\Omega_t)} + ||\mu_x^+||_{L^p(\Omega_t)} + ||\mu_x^-||_{L^p(\Omega_t)} \le C_p e^{Kt}(||v^+||_{L^p(\Omega_t)} + ||v^-||_{L^p(\Omega_t)} + \|u\|_{L^p(\Omega_t)}),$$
$$\|\gamma_x\|_{L^\infty(\mathbb{R})} \le Ce^{Kt}, \left\|u^\pm\right\|_{L^\infty(\mathbb{R})} \le C\exp\left(C\exp Kt\right),$$

we have

$$||v^+||^p_{L^p(\mathbb{R})} \le ||v^+\left(0\right)||^p_{L^p(\mathbb{R})} + C_p\exp\left(C\exp Kt\right)$$
$$+ C_p\exp\left(C\exp Kt\right)\int_0^t ||v^+||^p_{L^p(\mathbb{R})} + ||v^-||^p_{L^p(\mathbb{R})} ds.$$

Similarly, we get the same estimate for v^-, and thus

$$||v^+||^p_{L^p(\mathbb{R})} + ||v^+||^p_{L^p(\mathbb{R})} \leq ||v^+(0)||^p_{L^p(\mathbb{R})} + ||v^-(0)||^p_{L^p(\mathbb{R})} + C_p \exp(C \exp Kt)$$
$$+ C_p \exp(C \exp Kt) \int_0^t \left[||v^+||^p_{L^p(\mathbb{R})} + ||v^-||^p_{L^p(\mathbb{R})} \right] ds.$$

Gronwall's inequality, Lemma 2, implies $v^\pm = u_x^\pm$ is L^p. Thus the proof is complete. $\square$

For simplicity, we denote by w_k the k-th spatial derivative of w. We present $W^{k,p}-$estimates of $u^\pm$ where $k \geq 2$ is an integer and $2 \leq p < \infty$.

Lemma 15. *Let $2 \leq p < \infty$. Let $u^\pm, s$ be solutions of (2.2)-(2.5) and initial data $u^\pm_{0k} \in L^p(\mathbb{R})$. Then $\left(u_k^+, u_k^-\right)$ exists globally in $C([0,\infty), L^p(\mathbb{R}))$ and (u_k^+, u_k^-) satisfies*

$$\left\| \left(u_k^+, u_k^-\right) \right\|_{L^p(\mathbb{R})} \leq \left\| \left(u_{0k}^+, u_{0k}^-\right) \right\|_{L^p(\mathbb{R})} + C_p \exp\left(\underbrace{\cdots}_{k} C \exp Kt \right),$$

where $K = K(\alpha, \beta, U_0)$ and $C = C(\alpha, \beta, U_0, k)$ and $C_p = C(\alpha, \beta, U_0, p, k)$.

Proof. We take the k-th derivative repeatedly of equations (1.1) -(1.2) and obtain

$$\begin{aligned}
u^+_{k,t} &= -\left(\gamma u^+\right)_{k+1} + \left(-\mu^+ u^+ + \mu^- u^-\right)_k \\
&= -\sum_{l=0}^{k+1} \binom{k+1}{l} \gamma_l u^+_{k+1-l} \mp \sum_{l=0}^{k} \binom{k}{l} \mu_l^\pm u^\pm_{k-l}, \\
u^-_{k,t} &= (\gamma u^-)_{k+1} + \left(\mu^+ u^+ - \mu^- u^-\right)_k \\
&= -\sum_{l=0}^{k+1} \binom{k+1}{l} \gamma_l u^-_{k+1-l} \pm \sum_{l=0}^{k} \binom{k}{l} \mu_l^\pm u^\pm_{k-l}.
\end{aligned} \tag{4.3}$$

We note that

$$\|\gamma_x\|_{L^\infty(\mathbb{R})} \leq C\left(1 + \|s\|_{W^{1,\infty}(\mathbb{R})}\right) \leq C \exp Kt,$$
$$\|\mu\|_{L^\infty(\mathbb{R})} \leq C\left(1 + \|s\|_{W^{1,\infty}(\mathbb{R})}\right) \leq C \exp Kt.$$

By induction we deduce for $l \geq 2$,

$$\|\gamma_l\|_{L^q(\Omega_t)} \leq C \sum_{j=1}^{l} \|s\|^j_{l+1-j,jq} \leq C \exp\left(\underbrace{\cdots}_{l-2} C \exp Kt \right).$$

Similarly, we get for $l \geq 1$,

$$\left\|\mu_l^\pm\right\|_{L^q(\Omega_t)} \leq C\left(\|u_l\|_{L^q(\mathbb{R})} + \exp\left(\underbrace{\cdots}_{l-1} C \exp Kt \right)\right). \tag{4.4}$$

The estimate in (4.4) is obtained from the dependence of μ on derivatives of s up to s_{xx} and from (3.1). Multiplying $pu_k^{\pm,p-1}$ to (4.3), integrating with respect to x and t and using induction on k, the left-hand-side reads $\int_\mathbb{R} |u_k^+(\cdot,t)|^2 dx - \int_\mathbb{R} |u_k^+(\cdot,0)|^2 dx$. We now estimate the right-hand-side term by term:

- $l=0:$

$$p\int_0^t\int \gamma u_{k+1}^{+}u_k^{+,p-1}dxds=-\int_0^t\int \gamma_x\left|u_k^{+}\right|^p dxds\le C\exp\left(Kt\right)\left\|u_k^{+}\right\|_{L^p(\mathbb{R})}^p,$$

$$p\int_0^t\int \mu^{\pm}u_k^{\pm}u_k^{+,p-1}dxds\le C\exp\left(Kt\right)\left\|\left(u_k^{+},u_k^{-}\right)\right\|_{L^p(\mathbb{R})}^p.$$

- $l=1:$

$$p\int_0^t\int \gamma_1 u_k^{+}u_k^{+,p-1}dxds\le C\exp\left(Kt\right)\left\|u_k^{+}\right\|_{L^p(\mathbb{R})}^p,$$

$$\begin{aligned}p\int_0^t\int \mu_1^{\pm}u_{k-1}^{\pm}u_k^{+,p-1}dxds&\le C\left\|\mu_1^{\pm}\right\|_{L^{2p}(\mathbb{R})}\left\|u_{k-1}^{\pm}\right\|_{L^{2p}}\left\|u_k^{+,p-1}\right\|_{L^{p/(p-1)}}\\&\le C\exp\left(\underbrace{\cdots}_{k-1}C\exp Kt\right)\left\|u_k^{+}\right\|_{L^p(\mathbb{R})}^{p-1},\end{aligned}\tag{4.5}$$

where we used the induction on $k-1$ with $2p$.

- $2\le l\le k-1:$ the same method as in (4.5) applies
- $l=k:$ For the term with γ_k, the same method applies. For the term with $\mu_k^{\pm}$, we have

$$\begin{aligned}p\int_0^t\int \mu_k^{\pm}u^{\pm}u_k^{+,p-1}dxds&\le C\left\|u^{\pm}\right\|_{L^\infty(\mathbb{R})}\left\|\mu_k^{\pm}\right\|_{L^p(\mathbb{R})}\left\|u_k^{+,p-1}\right\|_{L^{p/(p-1)}}\\&\le C\exp\left(\underbrace{\cdots}_{k-1}C\exp Kt\right)\left\|\left(u_k^{+},u_k^{-}\right)\right\|_{L^p(\mathbb{R})}^{p-1}\\&\quad+C\left\|\left(u_k^{+},u_k^{-}\right)\right\|_{L^p(\mathbb{R})}^p.\end{aligned}$$

- $l=k+1:$ We can deal with this case in a similar way.

The terms with u^- can be estimated similarly. Therefore, by applying Gronwall's inequality, we complete the proof. □

Taking a similar procedure as given in [11, see page 188-190] we can also obtain $W^{1,1}$ estimates. The difference is that we use boundedness of $\|u^{\pm}\|_{L^2}$ and the L^2 norms of the derivatives of the chemical signal s, which are bounded by $\|u^{\pm}\|_{W^{1,2}}$. Similarly, we can have $W^{k,1}$ estimates. To sum up, we have

Lemma 16. *Let $u^{\pm},s$ be solutions of* (2.2)-(2.5) *and the initial data $u_{0k}^{\pm}\in L^p(\mathbb{R})$. Let $1\le p<\infty$. There exists a constant $C=C\left(\alpha,\beta,U_0,k,p,T\right)$ such that for all $t\in[0,T]$*

$$\left\|u_k^{+}\right\|_{L^p(\mathbb{R})}+\left\|u_k^{-}\right\|_{L^p(\mathbb{R})}\le C.$$

By standard embedding arguments, we finally have

Theorem 1. *Let $0<T\le\infty$ and $\Omega_T=\mathbb{R}\times(0,T)$. Suppose that $u_0^{\pm}\in W^{k,p}(\mathbb{R})$ for all $k\ge0$ and all $1\le p\le\infty$. Then we have for (2.2)-(2.5) a solution $(u^{\varepsilon+},u^{\varepsilon-},s^{\varepsilon})\in\left[C\left([0,T),W^{k,p}(\mathbb{R})\right)\right]^3$ for all $k\ge0$ and $1\le p\le\infty$, where the bound is independent of ε in $\left[C\left([0,T),W^{k,p}(\mathbb{R})\right)\right]^3$ for each k and p.*

5. The Vanishing Viscosity Limit, $\varepsilon \to 0$

Now we are back to the notation $u^{\varepsilon+}, u^{\varepsilon-}, s^{\varepsilon}$.

Lemma 17. *Let $0 < T < \infty$ and $\Omega_T = \mathbb{R} \times (0,T)$ then there exists a $s(x,t) \in C^{k,\delta}\left(\bar{\Omega}_T\right)$ with $s_t(x,t) \in C^{k-2,\delta}\left(\bar{\Omega}_T\right)$ for any small $\delta > 0$ and all $k \geq 2$ such that*

$$s^{\varepsilon_m}(x,t) \to s(x,t) \quad \text{in } C^{k,\delta}\left(\bar{\Omega}_T\right),$$
$$s_t^{\varepsilon_m}(\cdot,t) \to s_t(\cdot,t) \ \text{ in } C^{k-2,\delta}\left(\bar{\Omega}_T\right)$$

for some sequence $\varepsilon_m \to 0$ with $s(x,t) \in W^{k+1,p}\left(\bar{\Omega}_T\right) \cap W^{k,\infty}\left(\bar{\Omega}_T\right)$ and $s_t(x,t) \in W^{k-1,p}\left(\bar{\Omega}_T\right) \cap W^{k-2,\infty}\left(\bar{\Omega}_T\right)$for all $1 \leq p \leq \infty$ and all $k \geq 2$.

Proof. By Theorem 1, $\{s^{\varepsilon}(x,t)\}$ is bounded in $W^{k+1,p}\left(\bar{\Omega}_T\right)$ and $\{s_t^{\varepsilon}(x,t)\}$ in $W^{k-1,p}\left(\bar{\Omega}_T\right)$, $1 \leq p < \infty$, and $0 < \varepsilon \leq 1$. Since the embeddings $W^{k+1,p} \to C^{k,\delta}$ and $W^{k-1,p} \to C^{k-2,\delta}$ are compact, there is a convergent subsequence. Moreover, all the previous estimates, which are independent of ε imply the spaces which s and s_t belong to. □

Now we are ready to prove our main result:

Proof of Main Theorem By Theorem 1, we have for all $0 < \varepsilon \leq 1$ and all $T > 0$ a classical solution $(u^{\varepsilon+}, u^{\varepsilon-}, s^{\varepsilon})$ of the parabolic-parabolic Cauchy problem (2.2)-(2.4). We consider for $m \in \mathbb{N}$ a sequence ε_m as in the previous Lemma . Similar arguments apply to $\{u^{\varepsilon_m \pm}\}$ which are bounded in $W^{k,p}$ for all $k \geq 0$ and $1 \leq p \leq \infty$. Since for any small $\delta > 0$ the embedding $W^{k,p} \subset C^{k-1,\delta}$ is compact, we further extract a subsequence ε_{m_k} with $\varepsilon_{m_k} \to \infty$ from $\{\varepsilon_m\}$ such that $u^{\varepsilon_{m_k} \pm} \to u^{\pm}$ in $C^{k-1,\delta}\left(\bar{\Omega}_T\right)$. For convenience, we denote $\{u^{\varepsilon_m+}, u^{\varepsilon_m-}, s^{\varepsilon_m}\}$ as the convergent subsequence. We now show that the limit $\{u^+, u^-, s\}$ of $\{u^{\varepsilon_m+}, u^{\varepsilon_m-}, s^{\varepsilon_m}\}$ is the desired classical solution of the original hyperbolic-parabolic Cauchy problem (1.1)- (1.4). By the smoothness of γ, $\mu^{\pm}$ and the convergence of $u^{\varepsilon_m \pm}$ and s^{ε_m} to $u^{\pm}$ and s respectively in the Hölder spaces $C^{k,\delta}\left(\bar{\Omega}_T\right)$, (u^+, u^-, s) clearly satisfies (1.1)-(1.3). Since $u^{\varepsilon_m \pm}$ and s^{ε_m} converge in C^{δ} to $u^{\pm}$ and s, $u^{\pm}$ and s satisfy the initial conditions (1.4) and (2.1). A priori estimates imply that $(u^+, u^-, s) \in \left[W^{k,p}\left(\bar{\Omega}_T\right)\right]^3$ for all $k \geq 0$ and all $1 \leq p \leq \infty$. □

Remark 2. *Every estimate works in a similar way for the stationary case with dependence of the turning rates $\mu^{\pm} = \mu^{\pm}(s, s_t, s_x, s_{xx})$ also on the highest derivative s_{xx} and the temporal gradient s_t of s. Thus we also have global classical solutions in the case, which covers and is even a stronger result than in* [11]. □

Remark 3. *So far we have considered the system* (1.1)-(1.4) *for $\beta > 0$. As mentioned at the beginning, the case $\beta = 0$ can be proved by slight modification of our arguments. One minor change, in case $\beta = 0$, will be the L^{∞}-estimate of s in Lemma 4, which should be replaced by*

$$\|s\|_{L^{\infty}(\mathbb{R})} \leq C(\alpha)\, t^{\frac{1}{2}} \|u_0\|_{L^1(\mathbb{R})}. \tag{5.1}$$

Indeed, using the Fourier transform of the heat kernel, we have

$$\int_{-\infty}^{\infty} |\hat{s}(\xi,t)|\, d\xi \leq \alpha \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^{\infty}(\mathbb{R})} \int_0^t \int_{-\infty}^{\infty} e^{-(t-\tau)4\xi^2} d\xi d\tau \leq C(\alpha)\, t^{\frac{1}{2}} \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^{\infty}(\mathbb{R})},$$

where we used the change of variables in the last inequality. The above calculation automatically implies (5.1). *To prove our main result for the case $\beta = 0$ needs simple modifications, like the one above. But since these are obvious, we omit the details.* □

6. The Parabolic limit

In [12] a formal parabolic limit was derived from the general hyperbolic model for chemotaxis. Similar to considerations done for higher dimensional analoga of this model, compare [2, 3, 10], in this section we rigorously derive a parabolic Keller-Segel type system in one dimension from the kinetic model where γ is assumed to be constant.

$$u_t^+ + \gamma u_x^+ = -\mu^+(s, s_t, s_x, s_{xx})u^+ + \mu^-(s, s_t, s_x, s_{xx})u^-$$
$$u_t^- - \gamma u_x^- = \mu^+(s, s_t, s_x, s_{xx})u^+ - \mu^-(s, s_t, s_x, s_{xx})u^-$$

coupled with the equation for the chemotactic signal (1.3). Using a diffusive scaling of time and space, the kinetic equations in non-dimensional form become

$$u_t^{\epsilon+} + \epsilon^{-1}\gamma u_x^{\epsilon+} = -\epsilon^{-2}\mu^+(s^\epsilon, s_t^\epsilon, s_x^\epsilon, s_{xx}^\epsilon)u^{\epsilon+} + \epsilon^{-2}\mu^-(s^\epsilon, s_t^\epsilon, s_x^\epsilon, s_{xx}^\epsilon)u^{\epsilon-} \tag{6.1}$$

$$u_t^{\epsilon-} - \epsilon^{-1}\gamma u_x^{\epsilon-} = \epsilon^{-2}\mu^+(s^\epsilon, s_t^\epsilon, s_x^\epsilon, s_{xx}^\epsilon)u^{\epsilon+} - \epsilon^{-2}\mu^-(s^\epsilon, s_t^\epsilon, s_x^\epsilon, s_{xx}^\epsilon)u^{\epsilon-} \tag{6.2}$$

$$s_t^\epsilon - \Delta s^\epsilon = -\beta s^\epsilon + \alpha u^\epsilon. \tag{6.3}$$

$$u^\epsilon(\cdot, 0) = u_0, \qquad s^\epsilon(\cdot, 0) = s_0, \tag{6.4}$$

where ϵ is a non-dimensional small parameter. Here we note that u^ϵ is regular for each fixed $\epsilon > 0$ under the assumptions (**A2**) and (**A3**). Let $\xi = \mu^+ - \mu^-$ and $\eta = \mu^+ + \mu^-$. Then by adding and subtracting (6.1) and (6.2), we obtain

$$u_t^\epsilon + \epsilon^{-1}\gamma v_x^\epsilon = 0, \qquad v_t^\epsilon + \epsilon^{-1}\gamma u_x^\epsilon = -\epsilon^{-2}\xi u^\epsilon - \epsilon^{-2}\eta v^\epsilon, \tag{6.5}$$

where $u^\epsilon = u^{\epsilon+} + u^{\epsilon-}$ and $v^\epsilon = u^{\epsilon+} - u^{\epsilon-}$. The following analysis is based on an asymptotic expansion of the turning rates $\mu^\pm = \mu^{(0)\pm} + \epsilon\mu^{(1)\pm} + \epsilon^2\mu^{(2)\pm} + O(\epsilon^3)$. Our goal is, to derive equations for the leading order terms of $u^\epsilon = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + O(\epsilon^3)$, $v^\epsilon = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + O(\epsilon^3)$, and $s^\epsilon = s^{(0)} + \epsilon s^{(1)} + \epsilon^2 s^{(2)} + O(\epsilon^3)$. Due to our definition, we have $\xi = \xi^{(0)} + \epsilon\xi^{(1)} + \epsilon^2\xi^{(2)} + O(\epsilon^3)$, where $\xi^{(0)} = 0$, $\xi^{(1)} = 2\mu^{(1)+}$, $\xi^{(2)} = \mu^{(2)+} - \mu^{(2)-}$, and $\eta = \eta^{(0)} + \epsilon\eta^{(1)} + \epsilon^2\eta^{(2)} + O(\epsilon^3)$, where $\eta^{(0)} = 2\mu^{(0)+}$, $\eta^{(1)} = 0$, $\eta^{(2)} = \mu^{(2)+} + \mu^{(2)-}$. Before proceeding further, we give some structure conditions on the turning rates $\mu^\pm$.

Assumption 1. *The leading order terms $\mu^{(0)\pm}$ are balanced and strictly positive, and the first order term $\mu^{(1)\pm}$ have opposite sign*

$$\mu^{(0)+} = \mu^{(0)-} > C > 0, \qquad \mu^{(1)+} = -\mu^{(1)-},$$

where C is a positive constant. Moreover, there exist constants $C_1, C_2 > 0$ such that

$$0 < C_1 \le \mu^\pm(s, s_x, s_{xx}, s_t) \le C_2(1 + s(x,t) + s(x + \epsilon\gamma, t) + s(x - \epsilon\gamma, t) + |s_x(x,t)|) \tag{6.6}$$

for any $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ and $s \in L^\infty_{\mathrm{loc}}(W^{1,\infty}(\mathbb{R}))$. □

We first derive the Keller-Segel type system formally from (6.1) and (6.2) for $\epsilon \to 0$. For convenience we define $\mu_0 \equiv \mu^{(0)+}$ and $\mu_1 \equiv \mu^{(1)+}$. Comparing the coefficients of ϵ^{-2} in the second equation of (6.5), we have

$$-\xi^{(0)}u^{(0)} - \eta^{(0)}v^{(0)} = 0.$$

Therefore, since $\xi^{(0)} = 0$ and $\eta^{(0)} = 2\mu_0 > 0$, we have $v^{(0)} = 0$. Comparing the coefficients of ϵ^{-1} in the second equation of (6.5), we get

$$v^{(1)} = -\frac{\xi^{(1)}}{\eta^{(0)}}u^{(0)} - \frac{\gamma}{\eta^{(0)}}u_x^{(0)}. \tag{6.7}$$

Now we consider the zero order terms in both equations of (6.5). After simple computations, we have

$$0 = u_t^{(0)} + \gamma v_x^{(1)}, \qquad \eta^{(0)} v^{(2)} = -\xi^{(2)} u^{(0)} - \xi^{(1)} u^{(1)} - \gamma u_x^{(1)}. \tag{6.8}$$

Due to (6.7) and the first equation in (6.8), the diffusion limit reads

$$0 = u_t^{(0)} + \gamma v_x^{(1)} = u_t^{(0)} - \gamma(\frac{\xi^{(1)}}{\eta^{(0)}} u^{(0)} + \frac{\gamma}{\eta^{(0)}} u_x^{(0)})_x = u_t^{(0)} - \gamma(\frac{\mu_1}{\mu_0} u^{(0)} + \frac{\gamma}{2\mu_0} u_x^{(0)})_x.$$

For the second equation in (6.8) we use $u_t^{(1)} + \gamma v_x^{(2)} = 0$ from (6.5). By taking the derivative with respect to x of the second equation of (6.8), we have

$$v_x^{(2)} = -\left(\frac{\xi^{(2)}}{\eta^{(0)}} u^{(0)}\right)_x - \left(\frac{\xi^{(1)}}{\eta^{(0)}} u^{(1)}\right)_x - \left(\frac{\gamma}{\eta^{(0)}} u_x^{(1)}\right)_x.$$

Using $\gamma v_x^{(2)} = -u_t^{(1)}$, we obtain

$$u_t^{(1)} = \gamma \left(\frac{\xi^{(2)}}{\eta^{(0)}} u^{(0)}\right)_x + \gamma \left(\frac{\xi^{(1)}}{\eta^{(0)}} u^{(1)}\right)_x + \gamma^2 \left(\frac{u_x^{(1)}}{\eta^{(0)}}\right)_x.$$

Note that this equation is non-degenerate second order parabolic equation with smooth coefficients, because we proved that u^ϵ and s^ϵ are regular for each $\epsilon > 0$ in the previous sections and $\eta^{(0)} > 0$, see Assumption 1. Therefore, $u^{(1)}$ can be solved, which implies that $v^{(2)}$ can be automatically recovered from (6.8). Equation (6) compares exactly to (6) in case $\xi^{(2)} = 0$.
To sum up, the formal parabolic limit leads to

$$u_t^{(0)} = (D u_x^{(0)})_x + (\mathcal{H} u^{(0)})_x, \tag{6.9}$$

where the diffusion coefficient D and the drift coefficient $\mathcal{H}$ are

$$D = \frac{\gamma^2}{\eta^{(0)}(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})} = \frac{\gamma^2}{2\mu_0(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})}, \tag{6.10}$$

$$\mathcal{H} = \gamma \frac{\xi^{(1)}(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})}{\eta^{(0)}(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})} = \gamma \frac{\mu_1(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})}{\mu_0(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})} \tag{6.11}$$

which compares to the formulations given in [12], for $\gamma = const.$. How $\mathcal{H}$ relates to the chemotactic sensitivity χ times s_x, which is the classical parabolic formulation used for chemotaxis, we will see later in an example. The formal limit of (6.1) and (6.2) is (6.9), coupled to chemo-attractant equation for $s^{(0)}$

$$s_t^{(0)} - \Delta s^{(0)} = \alpha u^{(0)} - \beta s^{(0)}.$$

Now we rigorously prove the convergence. First, under Assumption 1, we show uniform estimates, independently of ϵ.

Lemma 18. *Let $\Psi \in L^\infty_{\rm loc}([0,\infty))$ be a measurable function satisfying the linear growth condition at infinity, e.g. $|\Psi(x)| \le C(1+|x|)$. Suppose Assumption 1 holds and $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Assume further that there exists C, independent of ϵ, such that*

$$\mu^+ + \mu^- = \eta \ge C(1 - \epsilon\Psi(||s||_{W^{1,\infty}})), \qquad |\mu^+ - \mu^-|^2 = |\xi|^2 \le C\epsilon^2 \Psi(||s||_{W^{1,\infty}}). \tag{6.12}$$

Then the solution (u^ϵ, s^ϵ) in (6.1)-(6.4) satisfies, uniformly in ϵ,

$$u^\epsilon \in L^\infty_{\rm loc}((0,\infty); L^2(\mathbb{R})), \qquad s^\epsilon \in L^\infty_{\rm loc}((0,\infty); L^p(\mathbb{R}) \cap \mathcal{C}^{1,\alpha}(\mathbb{R})),$$

where $1 \le p < \infty$ and $0 < \alpha \le 1/2$.

Proof. First we note that mass is conserved

$$||u^\epsilon(\cdot,t)||_{L^1(\mathbb{R})} = ||u_0||_{L^1(\mathbb{R})}.$$

Multiply u^ϵ and v^ϵ to the first and second equation in (6.5), respectively, then we have

$$(\frac{|u^\epsilon|^2}{2})_t + \epsilon^{-1}\gamma v^\epsilon_x u^\epsilon = 0, \quad (\frac{|v^\epsilon|^2}{2})_t + \epsilon^{-1}\gamma u^\epsilon_x v^\epsilon = -\epsilon^{-2}\xi u^\epsilon v^\epsilon - \epsilon^{-2}\eta|v^\epsilon|^2.$$

Adding together and integrating in space and time, we obtain

$$\frac{1}{2}\int_{\mathbb{R}}(|u^\epsilon|^2+|v^\epsilon|^2)dx + \int_0^t\int_{\mathbb{R}}\epsilon^{-2}\eta|v^\epsilon|^2dxdt = \frac{1}{2}\int_{\mathbb{R}}(|u_0|^2+|v_0|^2)dx - \int_0^t\int_{\mathbb{R}}\epsilon^{-2}\xi u^\epsilon v^\epsilon dxdt.$$

Since $|\xi| \le 2^{-1}\eta + 2^{-1}|\xi|^2\eta^{-1}$, we have

$$\int_{\mathbb{R}}(|u^\epsilon|^2+|v^\epsilon|^2)dx \le \int_{\mathbb{R}}(|u_0|^2+|v_0|^2)dx + \int_0^t\int_{\mathbb{R}}\epsilon^{-2}\frac{|\xi|^2}{\eta}|u^\epsilon|^2dxdt.$$

Due to (6.12), we have

$$\int_{\mathbb{R}}|u^\epsilon(\cdot,t)|^2dx \le 4\int_{\mathbb{R}}|u_0|^2dx + C\Psi(||s^\epsilon||_{W^{1,\infty}})\int_0^t\int_{\mathbb{R}}|u^\epsilon|^2dxdt.$$

Finally, according to Lemma 4, we have

$$\int_{\mathbb{R}}|u^\epsilon(\cdot,t)|^2dx \le 4\int_{\mathbb{R}}|u_0|^2dx + C(1+\log(||u^\epsilon||_{L^2}))\int_0^t\int_{\mathbb{R}}|u^\epsilon|^2dxdt,$$

where we used the linear growth condition of Ψ. Since the above estimate is independent of ϵ, we have an L^2-estimate of u^ϵ independently of ϵ by Gronwall's inequality. For the chemo-attractant we also have, uniformly in ϵ

$$||s^\epsilon_t||_{L^2(\mathbb{R})} + ||s^\epsilon||_{W^{2,2}(\mathbb{R})} \le C||u^\epsilon||_{L^2(\mathbb{R})}. \tag{6.13}$$

Therefore, combining potential estimate and embedding argument, we obtain

$$||s^\epsilon||_{L^p(\mathbb{R})} + ||s^\epsilon||_{\mathcal{C}^{1,\alpha}(\mathbb{R})} \le C\sup_{0\le\tau\le t}||u^\epsilon||_{L^2(\mathbb{R})}, \qquad 1\le p<\infty,\ \ 0<\alpha\le\frac{1}{2}. \tag{6.14}$$

This completes the proof. □

Theorem 2. *Let the assumption of Lemma 18 hold. Assume that*

$$\mu^{\pm}(s^\epsilon, s^\epsilon_t, s^\epsilon_x, s^\epsilon_{xx}) \longrightarrow \mu^{\pm}(s^{(0)}, s^{(0)}_t, s^{(0)}_x, s^{(0)}_{xx}) \quad \text{as } \epsilon\to 0.$$

Then the solution (u^ϵ, s^ϵ) (6.1)-(6.4) *satisfies, after choosing appropriate subsequences*

$$\begin{aligned} &u^\epsilon \longrightarrow u^{(0)} && \text{in } L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{R})\cap L^2(\mathbb{R})) \text{ weakly,}\\ &s^{\epsilon x} \longrightarrow s^{(0)}_x && \text{in } L^p_{\text{loc}}([0,\infty)\times\mathbb{R})\ \ 1\le p<\infty \text{ weakly,}\\ &s^{\epsilon t}, s^{\epsilon xx} \longrightarrow s^{(0)}_t, s^{(0)}_{xx} && \text{in } L^\infty_{\text{loc}}([0,\infty); L^2(\mathbb{R})) \text{ weakly.}\end{aligned}$$

In addition,

$$\begin{aligned} &s^\epsilon \longrightarrow s^{(0)} && \text{in } L^p_{\text{loc}}([0,\infty)\times\mathbb{R}) \qquad 1\le p<\infty,\\ &s^\epsilon \longrightarrow s^{(0)} && \text{in } L^\infty_{\text{loc}}([0,\infty);\mathcal{C}^{1,\alpha}(\mathbb{R})) \qquad 0<\alpha<\frac{1}{2}.\end{aligned}$$

Proof. Mass conservation and uniform boundedness of the L^2-norm of u^ϵ confirm the weak convergence of u^ϵ to $u^{(0)}$. Moreover, the estimate (6.13) immediately implies the third statement. In addition, due to the potential estimate, one can see that $s_x \in L^1(\mathbb{R})$, again independently of ϵ, and therefore $s_x \in L^p(\mathbb{R})$ for all $1 \le p \le \infty$, uniformly in ϵ. Here we used (6.14) and an interpolation argument. The last two assertions can be achieved by compactness results due to standard embedding arguments. This completes the proof. □

Example:

We consider the following class of turning rates which is similar to those suggested in [2] and [3].

$$\mu_\epsilon^\pm = \phi(s(x,t), s(x \pm \epsilon\gamma, t), s(x \mp \epsilon\gamma, t), s_t(x,t), s_x(x,t)s_{xx}(x,t)). \tag{6.15}$$

Note, that ϕ is an even function with respect to the the variable s_x. Additionally ϕ is strictly positive, decreasing in the second and increasing in the third argument, and we assume the structure condition (6.6) for $\mu_\epsilon^\pm$. Biological experiments for positive chemotaxis reveal that individuals moving up gradients of the chemical signal do turn less often than individuals moving down gradients. This fits exactly to our growth conditions assumptions on the growth of ϕ with respect to its second and third argument. We can easily see that the turning rates $\mu^\pm$ have the asymptotic expansion

$$\mu_\epsilon^\pm = \mu^{(0)\pm} + \epsilon\mu^{(1)\pm} + \epsilon^2\mu^{(2)\pm} + O(\epsilon^3)$$

where

$$\mu^{(0)\pm} = \phi(s,s,s,s_t,s_x,s_{xx}), \qquad \mu^{(1)\pm} = \pm\partial_2\phi(s,s,s,s_t,s_x,s_{xx})\gamma s_x \mp \partial_3\phi(s,s,s,s_t,s_x,s_{xx})\gamma s_x,$$

$$\mu^{(2)\pm} = \frac{1}{2}\left(\partial_2\partial_2\phi(s,s,s,s_t,s_x,s_{xx}) \pm 2\partial_2\partial_3\phi(s,s,s,s_t,s_x,s_{xx}) + \partial_3\partial_3\phi(s,s,s,s_t,s_x,s_{xx})\right)\gamma^2 s_{xx}.$$

Here ∂_2 and ∂_3 indicate differentiation with respect to the second and third argument. As before, we set $\mu_0 = \mu^{(0)+}$ and $\mu_1 = \mu^{(1)+}$. One can easily see that the turning rates (6.15) satisfy Assumption 1. Substituting the expansions $u_\epsilon^\pm = u^{(0)\pm} + \epsilon u^{(1)\pm} + \epsilon^2 u^{(2)\pm} + O(\epsilon^3)$ and $s_\epsilon = s^{(0)} + \epsilon s^{(1)} + \epsilon^2 s^{(2)\pm} + O(\epsilon^3)$ into (6.1) and (6.2) and comparing coefficients of ϵ^{-1} and ϵ^{-2}, we have $u^{(0)+} = u^{(0)-}$ and obtain as before

$$0 = u_t + \gamma(u_1^+ - u_1^-)_x = u_t + \gamma(-\frac{\mu_1}{\mu_0}u - \frac{\gamma}{2\mu_0}u_x)_x = u_t - (Du_x)_x + (\chi s_x u)_x,$$

where the diffusion coefficient D and the chemotactic sensitivity χ are

$$D = \frac{\gamma^2}{2\phi(s,s,s,s_t,s_x,s_{xx})}, \qquad \chi = -2[\partial_2\phi(s,s,s,s_t,s_x,s_{xx}) - \partial_3\phi(s,s,s,s_t,s_x,s_{xx})]D.$$

So $\mathcal{H} = \chi s_x$ in this case. In particular, if we take as a specific ϕ

$$\phi(s(x,t), s(x \pm \epsilon\gamma, t), s(x \mp \epsilon\gamma, t), s_t(x,t), s_x(x,t), s_{xx}(x,t)) = \varphi(s(x \pm \epsilon\gamma, t) - s(x \mp \epsilon\gamma, t))$$

where

$$\varphi(x) = -C_1\frac{x}{\sqrt{1+x^2}} + C_2, \qquad C_2 > C_1 > 0,$$

then we obtain both constant diffusion coefficient $D = \gamma^2/2C_2$ and chemotactic sensitivity $\chi = 4C_1 D$, which is the classical version of the Keller-Segel model in one space dimension.

References

[1] W. Alt, Singular perturbation of differential integral equations describing biased random walks, J. f"ur die reine und angewandte Mathematik, **322** (1981), 15–41.

[2] F. A. C. C. Chalub, P. Markowich, B. Perthame, & C. Schmeiser *Kinetic models for chemotaxis and their drift-diffusion limits*, ANUM preprint 4/02, Vienna Technical University, 2002.

[3] F. A. C. C. Chalub, P. Markowich, B. Perthame, & C. Schmeiser *On the derivation of drift-diffusion model for chemotaxis from kinetic equations*, ANUM preprint 14/02, Vienna Technical University, 2002.

[4] K. C. Chen, R. M. Ford, P.T. Cummings, Mathematical models for motile bacterial transport in clylindrical tubes

[5] K. C. Chen, R. M. Ford, P.T. Cummings, Perturbation expansions of Alt's cell balance equations reduces to Segel's one-dimensional equations for shallow chemoattractant gradients, , SIAM J. Appl. Math. 59 **1** (1999), 35–57.

[6] K. N. Chueh, C. C. Conley, & J. A. Smoller, Positively invariant regions for systems of nonlinear diffusion equations, Indiana Univ. Math. J. **26** (1977), 373-392.

[7] L. Evans, Partial Differential Equations, Amer. Math. Soc., Providence, R.I., 1998.

[8] S. Goldstein, On diffusion by discontinuous movements and the telegragh equation, Quart. J. Mech. Appl. Math. **4** (1951), 129-156.

[9] J. M. Greenberg & W. Alt, Stability Results for a Diffusion Equation with Functional Drift Approximating a Chemotaxis Model , Trans. AMS, **300** (1987), 235–258.

[10] H. Hwang, K. Kang, & A. Stevens, Drift-diffusion limits of kinetic models for chemotaxis: a generalization, MPI MIS, Leipzig, Preprint **19** (2003), submitted for publication.

[11] T. Hillen, C. Rohde & F. Lutscher, Existence of weak solutions for a hyperbolic model of Chemosensitive movement, J. Math. Anal. Appl. **260** (2001), no. 1, 173–199.

[12] T. Hillen & A. Stevens, Hyperbolic models for chemotaxis in 1-D, Nonlinear Anal. Real World Appl's. **1** (2000), 409-433.

[13] M. Kac, A stochastic model related to the telegrapher's equation , Rocky Mountain J. Math., **4** (1956), 497–509.

[14] E. F. Keller & L. A. Segel, Initiation of slime mold aggregation viewed as an instability, , J. Theor. Biol. **26** (1970), 399-415.

[15] O. A. Ladyzhenskaya, V. A. Solonnikov & N. N. Uralceva, Linear and Quasilinear Equations of Parabolic type. Translations of Mathematical Monographs, **23**, Amer. Math. Soc., Providence, R.I., 1968.

[16] G. M. Lieberman, Second order parabolic differential equations, World Scientific Publishing Co., 1996.

[17] L. A. Segel, A theoretical study of receptor mechanisms in bacterial chemotaxis, SIAM Appl. Math. **32** (1977), 653-665.

[18] D. R. Soll, Behavioral studies into the mechanism of eukaryotic chemotaxis , J. Chem. Ecol. **16** (1990), 133–150.

Hyung Ju Hwang
Max-Planck-Institute for Mathematics in the Sciences
Inselstr. 22 - 26, D-04103 Leipzig, Germany
E-mail : hwang@mis.mpg.de

Kyungkeun Kang
Max-Planck-Institute for Mathematics in the Sciences
Inselstr. 22 - 26, D-04103 Leipzig, Germany
E-mail : kkang@mis.mpg.de

Angela Stevens
Max-Planck-Institute for Mathematics in the Sciences
Inselstr. 22 - 26, D-04103 Leipzig, Germany
E-mail : stevens@mis.mpg.de