Global existence of classical solutions for a hyperbolic chemotaxis model and its parabolic limit

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GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS FOR A HYPERBOLIC CHEMOTAXIS MODEL AND ITS PARABOLIC LIMIT

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Abstract. We consider a one dimensional hyperbolic system for chemosensitive movement, especially for chemotactic behavior. The model consists of two hyperbolic differential equations for the chemotactic species and is coupled with either a parabolic or an elliptic equation for the dynamics of the external chemical signal. The speed of the chemotactic species is allowed to depend on the external signal and the turning rates may depend on the signal and its gradients in space and time, as observed in experiments. Global classical solutions are established for regular initial data and a parabolic limit is proved.

1. Introduction

Changes in the pattern of movement in dependence of external chemical signals is a common mechanism for biological organisms to respond to their environment. The directed motion to higher concentrations of chemical signals is described by positive chemotaxis. Chemosensitivity describes the more general changes of speed of motion and orientation of the individuals in dependence of the chemical environment. This behavior can lead to different states of pattern formation and self-organization. Well known examples are the bacteria *Escherichia coli* and the slime mold amoebae *Dictyostelium discoideum*

The classical chemotaxis model discussed by Keller and Segel, [14] is a parabolic system. A related one dimensional hyperbolic model for chemotaxis was introduced in [17]. It is based on the Goldstein-Kac model [8, 13] for one-dimensional correlated random walks. In [9] the following hyperbolic model for chemotaxis with suitable boundary conditions was analyzed

\[
\begin{align*}
    u^+_t + \gamma u^+_x &= -\mu^+(s_x)u^+_x + \mu^-(s_x)u^-_x, \\
    u^-_t - \gamma u^-_x &= \mu^+(s_x)u^+_x - \mu^-(s_x)u^-_x, \\
    \tau s_t &= D s_{xx} + u^+_x + u^-_x \quad \tau \geq 0, \quad t > 0, \quad x \in (-1, 1)
\end{align*}
\]

where \( \gamma \) is the constant speed of the right and left moving cells \( u^+ \) and \( u^- \), and \( \mu^+, \mu^- \) are the turning rates, which in this case depend linearly on the spatial gradient of the given chemical signal \( s \). In [9] the gradient of \( s \) was expressed by a quasistationary approximation in the asymptotic limit \( \tau \to 0 \) and thus a quasilinear hyperbolic conservation law for \( U(x, t) = \int_{-1}^x u(\xi, t)d\xi \) resulted.

Here we are concerned with the original and more general hyperbolic models for chemosensitive movement. Again, the density for the right moving particles is denoted by \( u^+ \), for the left moving particles by \( u^- \) and the external signal is \( s \):

\[
\begin{align*}
    u^+_t + (\gamma(s)u^+_x)_x &= -\mu^+(s, s_t, s_x, s_{xx})u^+_x + \mu^-(s, s_t, s_x, s_{xx})u^-_x, \\
    u^-_t - (\gamma(s)u^-_x)_x &= \mu^+(s, s_t, s_x, s_{xx})u^+_x - \mu^-(s, s_t, s_x, s_{xx})u^-_x,
\end{align*}
\]

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\[ \tau_s = D s_{xx} + f(s,u^+ + u^-), \quad \tau \geq 0, \quad t > 0, \quad x \in \mathbb{R} \]
\[ u^\pm(0,x) = u^\pm_0(x), \quad s(0,x) = s_0(x) \]
where \( u^\pm_0 \) are assumed to have compact support, and \( s_0 \) and \( u_0 \) satisfy a compatibility condition. Typically \( f \) is given as follows:
\[ f(s,u^+ + u^-) = \alpha(u^+ + u^-) - \beta s. \]

The diffusion rate of the external signal \( s \) and its production, and degradation rate are denoted by \( D > 0, \alpha > 0, \) and \( \beta \geq 0, \) respectively. Here we study the fully parabolic equation (1.3) for the external signal \( s \), and the turning rates \( \mu^\pm \) in (1.1, 1.2) depend not only on the spatial derivatives of \( s \) but also on its time derivative and \( s \) itself. This is reasonable to assume since in Soll’s studies [18] it turned out that the turning behavior and the speed of the slime mold amoebae Dictyostelium discoideum are dependent on both, the temporal and the spatial gradient of the cAMP concentration. Chen et al. [4, 5] analyzed data of E.coli and found out that the bacterial speed is close to constant, whereas the turning frequencies depend on the temporal gradient of the external signal. Their model was set into context with a one-dimensional projection of a 3D model for chemosensitive movement given by Alt, [1].

A general model of the kind described above, also with \( \gamma = \gamma(s,s_x,s_t) \) was already introduced in [12] and a formal parabolic limit was derived. Local and global existence of solutions was proved for a simplified version of this system, namely for constant speed \( \gamma \) and turning rates \( \mu^\pm = \mu^\pm(s,s_x) \). The dynamics for the chemical \( s \) were discussed for both cases, \( \tau = 0 \) and \( \tau \neq 0 \).

In [11] the case \( \gamma = \gamma(s) \) and, as before \( \mu^\pm = \mu^\pm(s,s_x) \) was discussed. For \( \tau = 0 \), which means elliptic dynamics for the chemical signal, existence of weak solutions could be proved.

In this paper we extend this result further in several ways. We consider \( \mu^\pm = \mu^\pm(s,s_x,s_t,s_{xx}) \). So also the dependency of the turning rates on chemical gradients in time are taken into account. The dynamics of the chemical can be considered to be parabolic \((\tau \neq 0)\) as well as elliptic \((\tau = 0)\), and global existence of classical solutions is proved. The results in [11] are a special case of our discussion here.

Our main result reads

**Main Theorem** Let \( u^\pm_0 \geq 0, s_0 \geq 0 \) be smooth and bounded, and \( u^\pm_0 \) be compactly supported and not identically zero. Let \( s_0 \) satisfy some compatibility condition. Then there exists a unique smooth solution \( u^\pm \) and \( s \) of (1.1, 1.2, 1.3) with (1.4).

This paper is arranged as follows. We start with assumptions and notations in Section 2. A priori estimates on \( L^p \) are derived in Section 3, followed by the estimates of higher derivatives \( W^{k,p} \) in Section 4. Finally, we attain global classical solutions for the hyperbolic chemotaxis model and rigorously derive a parabolic limit for the system.

2. **Assumptions and Notations**

Here we introduce notations which will be used throughout this article and give assumptions on the initial data, turning rates, and speed.

**NOTATIONS:**

1. By \( \Gamma \) we denote the fundamental solution of the differential operator \( \partial_t - \partial_{xx} + \beta \) in \( \mathbb{R} \times \mathbb{R} \)
\[ \Gamma(x,t) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} - \beta t \right). \]
ASSUMPTIONS:

(A1): The initial values \( u_0^\pm \in C^\infty (\mathbb{R}) \) have compact support and \( u_0^\pm \geq 0 \). We use the following compatibility condition: \( s_0 \in C^\infty (\mathbb{R}) \) is the unique solution of

\[
0 = Ds_{0,xx} - \beta s_0 + \alpha (u_0^+ + u_0^-), \quad s_0 (\pm \infty) = 0. \tag{2.1}
\]

(A2): The turning rates are nonnegative and symmetric with respect to \( s_x \)

\[
\mu^+, \mu^- \geq 0, \quad \mu^+(s, s_t, s_x, s_{xx}) = \mu^-(s, s_t, -s_x, s_{xx}).
\]

(A3): The turning rates satisfy \( \mu^\pm \in C^\infty (\mathbb{R}^4) \) and are bounded

\[
||D^{(\kappa)} \mu^\pm||_{L^\infty} \leq C_{|\kappa|}, \quad \text{where } \kappa \text{ is a multi-index}
\]

\[
0 \leq \mu^\pm (s, s_t, s_x, s_{xx}) \leq C(1 + ||s||_{W^{1,\infty}(\mathbb{R})}).
\]

(A4): The speed function satisfies \( \gamma = \gamma (s) \in C^\infty (\mathbb{R}) \) with

\[
||\gamma (k)||_{L^\infty} \leq C_k, \quad \text{where } k \in \mathbb{N}_+.
\]

From (A4), it follows that \( ||\gamma (s)||_{L^\infty(\mathbb{R})} \leq C(1 + ||s||_{L^\infty(\mathbb{R})}). \)

The existence of a unique solution of (2.1) is clear from standard arguments for elliptic equations. The maximum principle for elliptic equations together with the positivity of \( u_0^\pm \) leads to \( s_0(x) \geq 0 \) for all \( x \in \mathbb{R} \). Using the method of vanishing viscosity, we consider the following model

\[
\begin{align*}
u_0^+ - cu_0^+ &= -(\gamma (s^e)u^e^+)_{xx} - \mu^+(s^e, s_t^e, s_x^e, s_{xx}^e)u^e^+ + \mu^-(s^e, s_t^e, s_x^e, s_{xx}^e)u^e^-,

\tau s_t^e &= Ds_{xx}^e - \beta s^e + \alpha (u^e^+ + u^e^-),
\end{align*}
\]
(2.5) \[ u^+(0, \cdot) = u^+_0, \quad u^-(0, \cdot) = u^-_0, \quad s^+(0, \cdot) = s_0, \]
where \( u^\pm_0 \) and \( s_0 \) satisfy the compatibility condition (A1). Introducing the total population density \( u^\varepsilon = u^\varepsilon^+ + u^\varepsilon^- \) and the density flow \( v^\varepsilon = u^\varepsilon^+ - u^\varepsilon^- \), the system reads:

(2.6) \[ u^\varepsilon_t - \varepsilon u^\varepsilon_{xx} = -\gamma(s^\varepsilon) v^\varepsilon_x, \]

(2.7) \[ v^\varepsilon_t - \varepsilon v^\varepsilon_{xx} = -\gamma(s^\varepsilon) u^\varepsilon_x - \eta(s^\varepsilon, s^\varepsilon_t, s^\varepsilon_x, s^\varepsilon_{xx}) v^\varepsilon, \]

(2.8) \[ \tau s^\varepsilon_t = Ds^\varepsilon_{xx} - \beta s^\varepsilon + \alpha u^\varepsilon, \]
where \( u^\varepsilon(0, \cdot) = u_0 = u^+_0 + u^-_0, \quad v^\varepsilon(0, \cdot) = v_0 = u^+_0 - u^-_0, \quad s^\varepsilon(0, \cdot) = s_0 \) and \( \xi = \mu^+ - \mu^-, \eta = \mu^+ + \mu^- \).

3. A priori estimates on \( L^p \)

**Lemma 1.** Let \( a(t) \) and \( b(t) \) be positive functions. Let \( y(t) > 0 \) be differentiable in \( t \) and satisfy

\[ y' \leq a(t) y \ln y + b(t) y. \]

Then

\[ y(t) \leq y(0) \exp \left( \int_0^t b(s) e^{-\int_0^s a(\tau) d\tau} ds \right) \exp \left( \int_0^t a(s) ds \right). \]

**Proof.** Dividing both sides of the inequality by \( y \), we get a typical Gronwall inequality for \( z = \ln y \)

\[ z' \leq a(t) z + b(t). \]

Therefore, we deduce the lemma. \( \square \)

**Lemma 2.** [Gronwall’s inequality] Let \( g \) and \( h \) be positive functions. Suppose that \( f \) is an integrable function in \( t \) and satisfies

\[ f(t) \leq g(t) + h(t) \int_0^t f(s) ds. \]

Then we have

\[ f(t) \leq g(t) + h(t) \int_0^t g(s) \exp \left( \int_s^t h(\tau) d\tau \right) ds. \]

**Proof.** Computations are straightforward and hence we omit details (see e.g. [7]). \( \square \)

Throughout this paper we consider only the case \( \beta > 0 \), for simplicity. We remark, however, our main result can be easily extended to the case \( \beta = 0 \) (see Remark 3 for more details). For convenience, we will use \( u^\pm \) without \( \varepsilon \) for \( u^\varepsilon^\pm \) from now on. Without loss of generality, we assume that \( \tau = 1 \) and \( D = 1 \). Let \( 1 < p < \infty \). For given \( u \in L^p(\Omega_t) \), we study the parabolic equation:

(3.1) \[ s_t - s_{xx} = -\beta s + \alpha u. \]

Using potential estimates similar to the heat kernel, we have:
Lemma 3. Let \( u \in L^p(\Omega_t) \) and \( 1 < p < \infty \). Then there exists a constant \( C_p = C(\alpha, \beta, p) \) such that the following estimate holds
\[
\|s_t\|_{L^p(\Omega_t)} + \|s_{xx}\|_{L^p(\Omega_t)} + \|s\|_{L^p(\Omega_t)} \leq C_p \|u\|_{L^p(\Omega_t)}.
\]

Proof. Since the above estimate is standard (e.g. see [15] and [16]), we omit the details. \(\square\)

Next we estimate \( \|s\|_{W^{1,\infty}(\mathbb{R})} \).

Lemma 4. If \( u \in L^\infty([0, \infty) : L^1(\mathbb{R}) \cap L^2(\mathbb{R})) \), then the solution \( s \) in (3.1) satisfies
\[
\|s\|_{L^\infty(\mathbb{R})} \leq C(\alpha, \beta) \sup_{0 \leq \tau \leq t} \|u\|_{L^1(\mathbb{R})} = C(\alpha, \beta) \|u_0\|_{L^1(\mathbb{R})},
\]

\[
\|s_x\|_{L^\infty(\mathbb{R})} \leq C(\alpha, \beta) \left[ 1 + \|u_0\|_{L^1(\mathbb{R})} \left( 1 + (\ln t)_+ + \left| \ln \left( \sup_{0 \leq \tau \leq t} \|u\|_{L^2(\mathbb{R})} \right) \right) \right],
\]

where \((\cdot)_+\) means the positive part and
\[
\lim_{|x| \to \infty} s(x, t) = 0 \text{ for all } t.
\]

Proof. The fundamental solution of the operator \( \partial_t - \partial_{xx} + \beta \) is
\[
\Gamma(x, t) = \frac{1}{\sqrt{t}} \exp \left( -\frac{x^2}{4t} - \beta t \right)
\]
and its Fourier transform is
\[
\hat{\Gamma}(\xi, t) = \exp(-t(4\xi^2 + \beta)).
\]

By Duhamel’s principle, we obtain
\[
s(x, t) = \int_0^t (\Gamma \ast u)(x, t - \tau) \, d\tau \]
\[
= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{t - \tau}} \exp \left( -\frac{(x - y)^2}{4(t - \tau)} - \beta (t - \tau) \right) \alpha u(y, \tau) \, dy \, d\tau.
\]

Next we estimate
\[
\left| \int_{-\infty}^{\infty} \hat{s}(\xi, t) \, d\xi \right| = \left| \int_{-\infty}^{\infty} \int_0^t \hat{\Gamma}(\xi, \tau) u(\xi, \tau) \, d\tau \, d\xi \right|
\]
\[
\leq \alpha \sup_{0 \leq \tau \leq t} \|u\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} \int_0^t e^{-(t-\tau)(4\xi^2 + \beta)} \, d\tau \, d\xi
\]
\[
\leq \alpha \sup_{0 \leq \tau \leq t} \|u\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} \frac{1}{4\xi^2 + \beta} \, d\xi
\]
\[
\leq C(\alpha, \beta) \sup_{0 \leq \tau \leq t} \|u\|_{L^\infty(\mathbb{R})},
\]
and we have
\[
(3.2) \quad \left| \hat{s}(\xi, t) \right| \leq \alpha \sup_{0 \leq \tau \leq t} \|u\|_{L^\infty(\mathbb{R})} \frac{1}{4\xi^2 + \beta}.
\]

Therefore, using the inverse Fourier-transform for \( \hat{s} \), we have \( \lim_{|x| \to \infty} s(x) = 0 \) and
\[
\|s\|_{L^\infty(\mathbb{R})} \leq \|\hat{s}\|_{L^1(\mathbb{R})} \leq C(\alpha, \beta) \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^\infty(\mathbb{R})} \leq C(\alpha, \beta) \sup_{0 \leq \tau \leq t} \|u\|_{L^1(\mathbb{R})}.
\]
Next, we estimate $\|s_x\|_{L^\infty(\mathbb{R})}$:

$$
\|s_x\|_{L^\infty(\mathbb{R})} \leq \|\xi \hat{s}\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} \left| \int_0^t \xi|\hat{\alpha}u| (\xi, t - \tau) \, d\tau \right| d\xi
$$

$$
= \alpha \int_0^t \int_{-\infty}^{\infty} |\xi| \exp \left( -\tau (4\xi^2 + \beta) \right) |\hat{u}(\xi, \tau)| \, d\xi \, d\tau.
$$

The integration is done by splitting the time integration into two:

$$
\int_0^t \int_{-\infty}^{\infty} |\xi| \exp \left( -\tau (4\xi^2 + \beta) \right) |\hat{u}(\xi, \tau)| \, d\xi \, d\tau = \int_0^r \cdots + \int_r \cdots = I_1 + I_2,
$$

where $r > 0$ will be chosen later.

1. For $0 < \tau < r$, we use Hölder’s inequality with $p = q = 2$:

$$
\int_{-\infty}^{\infty} |\xi| \exp \left( -\tau (4\xi^2 + \beta) \right) |\hat{u}(\xi, \tau)| \, d\xi \leq \left( \int_{-\infty}^{\infty} \xi^2 \exp \left( -2\tau (4\xi^2 + \beta) \right) \, d\xi \right)^{1/2} \|\hat{u}\|_{L^2(\mathbb{R})}
$$

$$
= \left( 2 \int_{-\infty}^{\infty} \xi^2 \exp \left( -2\tau (4\xi^2 + \beta) \right) \, d\xi \right)^{1/2} \|u\|_{L^2(\mathbb{R})},
$$

where we used the Plancherel’s equality for $L^2$. By integration by parts, we have

$$
\int_{-\infty}^{\infty} \xi^2 \exp \left( -2\tau (4\xi^2 + \beta) \right) \, d\xi = \frac{1}{16\tau} \int_0^\infty \exp \left( -2\tau (4\xi^2 + \beta) \right) \, d\xi
$$

$$
= \frac{\sqrt{\pi}}{64\sqrt{2}} e^{-2\beta \tau}.
$$

Hence, we obtain

$$
I_1 \leq \frac{\pi^{1/4}}{4\sqrt{2^3}} \sup_{0 \leq \tau \leq t} \|u\|_{L^2(\mathbb{R})} \int_0^r \tau^{-3/4} e^{-\beta \tau} \, d\tau \leq \frac{\pi^{1/4}}{\sqrt{2^3}} \left( \pi^{1/4} \sup_{0 \leq \tau \leq t} \|u\|_{L^2(\mathbb{R})} \right).
$$

2. For $r \leq \tau \leq t$, we use Hölder’s inequality with $p = 1$, $q = \infty$:

$$
\int_{-\infty}^{\infty} |\xi| \exp \left( -\tau (4\xi^2 + \beta) \right) |\hat{u}(\xi, \tau)| \, d\xi
$$

$$
\leq \|\hat{u}\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} |\xi| \exp \left( -\tau (4\xi^2 + \beta) \right) \, d\xi
$$

$$
= \frac{1}{4\tau} e^{-\beta \tau} \|\hat{u}\|_{L^\infty(\mathbb{R})}.
$$

So, we have

$$
I_2 \leq \frac{1}{4} \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^\infty(\mathbb{R})} \int_r^t \frac{1}{\tau} e^{-\beta \tau} \, d\tau \leq \frac{1}{4} \sup_{0 \leq \tau \leq t} \|u\|_{L^1(\mathbb{R})} \ln t - \ln r.
$$

Therefore, we get

$$
\|s_x\|_{L^\infty(\mathbb{R})} \leq C \alpha \left( r^{1/4} \sup_{0 \leq \tau \leq t} \|u\|_{L^2(\mathbb{R})} + \sup_{0 \leq \tau \leq t} \|u\|_{L^1(\mathbb{R})} \ln t - \ln r \right).
$$
We optimize the upper bound for the above inequality by choosing

$$r = \min \left\{ \left( \sup_{0 \leq \tau \leq t} \| u \|_{L^2(\mathbb{R})} \right)^{-4}, t \right\}.$$  

If $r = t$ and for $t \leq 1$ we have $t \leq \left( \sup_{0 \leq \tau \leq t} \| u \|_{L^2(\mathbb{R})} \right)^{-4}$ and $\| s_x \|_{L^\infty(\mathbb{R})} \leq C \alpha$.

If $r = \left( \sup_{0 \leq \tau \leq t} \| u \|_{L^2(\mathbb{R})} \right)^{-4}$, then

$$\| s_x \|_{L^\infty(\mathbb{R})} \leq C \alpha \left[ 1 + \sup_{0 \leq \tau \leq t} \| u \|_{L^1(\mathbb{R})} \right] \ln \left( \sup_{0 \leq \tau \leq t} \| u \|_{L^2(\mathbb{R})} \right).$$

For $t \geq 1$, we have $\| u \|_{L^1(\mathbb{R})} \leq \left( \sup_{0 \leq \tau \leq t} \| u \|_{L^2(\mathbb{R})} \right)$ since the total population size is preserved, namely

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx = \int_{\mathbb{R}} (u^+_0 + u^-_0)(x)dx \quad \text{for all } t.$$  

because

$$\frac{d}{dt} \int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} \left[ -\left( \gamma(s)u^+(x,t) \right)_x + \left( \gamma(s)u^-(x,t) \right)_x \right]dx = 0.$$  

This completes the proof. $\Box$

**Lemma 5.** Let $U_0 = \int_{\mathbb{R}} u(x,t)dx < \infty$ and $S_0 = \int_{\mathbb{R}} s_0(x)dx$, then $s \in L^p(\Omega)$ for $2 \leq p \leq \infty$ and $s_x \in L^2(\Omega)$ in (3.1) with

$$\| s \|_{L^p(\Omega)} \leq C(\alpha, \beta) t^{\frac{1 + 2\alpha}{\beta}} U_0, \quad \| s_x \|_{L^2(\Omega)} \leq C(\alpha, \beta) t^{\frac{1}{2}} U_0$$

(3.3)

$$\int_{\mathbb{R}} s(x,t)dx = \frac{\alpha U_0}{\beta} + \left( S_0 - \frac{\alpha U_0}{\beta} \right) e^{-\beta t},$$

where

$$\lim_{|x| \to \infty} s_x(x,t) = 0 \quad \text{for all } t.$$  

**Proof.** From (3.2), we easily see that $\| s \|_{L^2(\Omega)} = \| \hat{s} \|_{L^2(\Omega)} \leq C(\alpha, \beta) t^{\frac{1}{2}} U_0$ and

$$\| s_x \|_{L^2(\Omega)} = \| \hat{s}_x \|_{L^2(\Omega)} = \| \xi \hat{s} \|_{L^2(\Omega)} \leq C(\alpha, \beta) t^{\frac{1}{2}} U_0$$

with $\lim_{|x| \to \infty} s_x(x,t) = 0$.

Since $\| s \|_{L^\infty(\Omega)} \leq C(\alpha, \beta) U_0$ by Corollary 4, we have $\| s \|_{L^p(\Omega)} \leq C(\alpha, \beta) t^{\frac{1}{p}} U_0$ for $2 \leq p \leq \infty$ by interpolation. From (1.3), we have,

$$\int_{\mathbb{R}} s(x,\tau)dx + \beta \int_{0}^{t} \int_{\mathbb{R}} s(x,\tau)dx d\tau = \int_{\mathbb{R}} s_0(x)dx + t \alpha \int_{\mathbb{R}} u_0(x)dx = S_0 + t \alpha U_0.$$  

For convenience, we set $S(t) = \int_{\mathbb{R}} s(x,t)dx$. Then we have

$$S'(t) = -\beta S(t) + \alpha U_0.$$  

(3.4)

Solving the ordinary differential equation (3.4), we obtain

$$\int_{\mathbb{R}} s(x,t)dx = S_0 e^{-\beta t} + \alpha U_0 \int_{0}^{t} e^{\beta(\tau-t)}d\tau.$$
By integrating the last term, we have (3.3). The proof is complete. □

We have the invariance of positivity of \( u^\pm \) and \( s \).

Lemma 6. Assume \( u^\pm \geq 0 \) in \( \Omega_t \). Then \( s \geq 0 \) in \( \Omega_t \).

Proof. This is an easy consequence of the parabolic maximum principle (e.g. see [7] or [16]). □

Lemma 7. If \( u_0^\pm \geq 0 \), then the solution \( (u^+, u^-, s) \) of (2.3)-(2.5) satisfies \( u^\pm \geq 0 \) as long as \( (u^+, u^-, s) \) exists.

Proof. Assumption (A2) on the non-negativity of the turning rates ensures our lemma from the concept of invariant regions for parabolic systems (e.g. see [6]). □

Remark 1. For the conserved total population density \( U_0 \), we have \( s \in L^1(\Omega_t) \). Indeed, since \( s \geq 0 \), we have by (3.3)

\[
\|s\|_{L^1(\Omega_t)} = \int_0^t \int \int_s(x, \tau) dx d\tau = \int_0^t \frac{\alpha U_0}{\beta} + \left( S_0 - \frac{\alpha U_0}{\beta} \right) e^{-\beta \tau} d\tau = \frac{\alpha U_0 t}{\beta} + \frac{1}{\beta} \left( S_0 - \frac{\alpha U_0}{\beta} \right) \left( 1 - e^{-\beta t} \right) < \infty.
\]

So, combining the results of Lemma 5, we have \( s \in L^p(\Omega_t) \) for all \( 1 \leq p \leq \infty \).

Now let \( \mathcal{K} = L^2(\mathbb{R}) \cap C^2_0(\mathbb{R}) \), where

\[
\mathcal{C}_0 = \{ u \in L^\infty(\mathbb{R}) : \lim_{|x| \to \infty} u(x) = 0 \}, \quad \mathcal{C}_0^k = \{ u : D_j u \in \mathcal{C}_0, \ j = 0, \ldots, k \}
\]

We state the local existence result for \( u^\pm \):

Lemma 8. For initial values \( u_0^\pm \in \mathcal{K} \) there exists a unique solution of (2.2), (2.3) with

\[
(u^+, u^-) \in C([0, T_0), \mathcal{K})
\]

for some time \( T_0 > 0 \).

Proof. Theorem can be proved by following a similar procedure as in Corollary 3.1 and Theorem 3.1 in [11, see page 180-182]. Therefore, details are omitted. □

Next we give growth rates for the \( L^2 \)-norms of \( u^\pm \) which ensure global existence. For simplicity, we denote \( u^\pm \) and \( s^\prime \) in (2.2-2.4) by \( \overline{u^\pm} \) and \( s \), respectively, in case no confusion is to be expected.

Lemma 9. Let \( u_0^\pm \in L^1 \cap L^2 \) with \( \int_\mathbb{R} u_0^+ + u_0^- = U_0 \). Assume (A1)-(A4). Then the solution \( (u^+, u^-) \) of (2.2,2.3) exists globally in \( C([0, \infty), L^1 \cap L^2) \) and there exist constants \( K = K(\alpha, \beta, U_0) \) and \( C = C(\alpha, \beta, U_0) \) which are independent of \( \varepsilon > 0 \) such that for all \( t \geq 0 \),

\[
\|(u^+, u^-)\|_{L^2(\mathbb{R})} \leq \left[ C \| (u_0^+, u_0^-) \|_{L^2(\mathbb{R})} \right] e^{Kt}.
\]
Proof. Using (2.2), (2.3) and applying Hölder’s inequality, we have
\[
\frac{d}{dt} \left( \|u^+\|^2_{L^2(\mathbb{R})} + \|u^-\|^2_{L^2(\mathbb{R})} \right) = 2 \int_{\mathbb{R}} \left( u^+_t u^+ + u^-_t u^- \right) dx
\]
\[
= -2 \varepsilon \int_{\mathbb{R}} \left| u^+_t \right|^2 + \left| u^-_t \right|^2 dx + 2 \int_{\mathbb{R}} \left[ -\left( \gamma u^+_x \right) u^+ - \mu^+ \left| u^+ \right|^2 + \mu^- u^+ u^- \right] dx
\]
\[
+ \int_{\mathbb{R}} \left[ (\gamma u^-)_x u^- + \mu^+ u^+ u^- - \mu^- \left| u^- \right|^2 \right] dx
\]
\[
\leq 2 \int_{\mathbb{R}} \left[ -\left( \gamma u^+_x \right) u^+ + \mu^+ u^- u^+ \right] dx + 2 \int_{\mathbb{R}} \left[ (\gamma u^-)_x u^- + \mu^+ u^+ u^- \right] dx
\]
\[
\leq \|\gamma\|_{L^\infty} \int_{\mathbb{R}} \left( |u^+_t|^2 + |u^-_t|^2 \right) dx + (\|\mu^+\|_{L^\infty} + \|\mu^-\|_{L^\infty}) \int_{\mathbb{R}} \left( |u^+_t|^2 + |u^-_t|^2 \right) dx
\]
\[
\leq C \left( 1 + \|s\|_{W^{1,\infty}(\mathbb{R})} \right) \left( \|u^+\|^2_{L^2(\mathbb{R})} + \|u^-\|^2_{L^2(\mathbb{R})} \right).
\]
Therefore, by Lemma 4, we have
\[
\frac{d}{dt} \left( \|u^+\|^2_{L^2(\mathbb{R})} + \|u^-\|^2_{L^2(\mathbb{R})} \right) \leq C (\alpha, \beta, U_0) \left( 1 + (\ln t) \right) + \left| \ln \left( \sup_{0 \leq \tau \leq t} \left[ \|u^+\|^2_{L^2(\mathbb{R})} + \|u^-\|^2_{L^2(\mathbb{R})} \right] \right) \right) \left( \|u^+\|^2_{L^2(\mathbb{R})} + \|u^-\|^2_{L^2(\mathbb{R})} \right),
\]
Setting \( y(t) = \sup_{0 \leq \tau \leq t} \left( \|u^+\|^2_{L^2(\mathbb{R})} + \|u^-\|^2_{L^2(\mathbb{R})} \right) \), we obtain
\[
y' \leq K (\alpha, \beta, U_0) y \ln y + K (\alpha, \beta, U_0) \left( 1 + (\ln t) \right).
\]
Hence, applying Lemma 1 with \( a(t) = K, b(t) = K (1 + (\ln t)) \), we complete the proof. \( \square \)

An easy consequence of the above result is the following.

**Corollary 1.** Let \( s \) be the solution of (2.8). Then \( s \) satisfies
\[
\|s_x\|_{L^\infty(\mathbb{R})} \leq C (\alpha, \beta, U_0) e^{Kt},
\]
where \( K = K (\alpha, \beta, U_0) \).

**Proof.** This is a combination of the a priori estimate for \( \|s_x\|_{L^\infty(\mathbb{R})} \) in Lemma 4 and the estimate (3.5) in Lemma 9. \( \square \)

Next we prove \( L^p \) estimates. For convenience, we denote \((u^\pm)^p \) by \( u^\pm_p \).

**Lemma 10.** Let \( u_0^+ \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) with \( \int_{\mathbb{R}} u_0^+ + u_0^- = U_0 \). Assume (A1)-(A4). Then there exist constants \( C_1 = C_1 (u_0^+) \), \( C_2 = C_2 (u_0^+, \alpha, \beta, U_0) \) and \( K = K (\alpha, \beta, U_0) \) such that
\[
\|u^+\|^p_{L^p(\mathbb{R})} + \|u^-\|^p_{L^p(\mathbb{R})} \leq C_1 \exp \left( C_2 \int_0^t e^{Ks} ds \right) \leq C_1 \exp (C_2 \exp Kt).
\]
for all \( 2 \leq p < \infty \) and \( 0 \leq t < \infty \).
Proof. Multiplying $p(u^+)^{p-1}$ and $p(u^-)^{p-1}$ to (2.2) and (2.3), we have
\[ \frac{d}{dt} \|u^+\|_{L^p(R)}^p + \int_R (\gamma u^+) \, dx + \int_R (p-1) \gamma x u^+ \, dx = p \int_R (-\mu^+ u^+ + \mu^- u^- u^+(p-1)) \, dx - \varepsilon p (p-1) \int_R u^{p-2} (u_x^+)^2 \, dx , \]
\[ \frac{d}{dt} \|u^-\|_{L^p(R)}^p - \int_R (\gamma u^-) \, dx - \int_R (p-1) \gamma x u^- \, dx = p \int_R (\mu^+ u^+ u^- u^- - \mu^- u^- u^-) \, dx - \varepsilon p (p-1) \int_R u^{p-2} (u_x^-)^2 \, dx , \]
where $u^\pm$ denotes $(u^\pm)^p$ and we have used
\[ p(\gamma u^+) \cdot u^\pm, (p-1) \gamma x u^+ . \]
Therefore, applying the Hölder’s inequality we deduce
\[ \frac{d}{dt} \|u^+\|_{L^p(R)}^p \leq C p \left( 1 + \|u\|_{W^{1,\infty}(R)} \right) \left( \|u^+\|_{L^p(R)} + \|u^-\|_{L^p(R)} \right) \|u^+\|_{L^p(R)}^{p-1} \]
\[ \leq C p e^{Kt} \left( 1 + \ln \left( \|u_0^+ \|_{L^2(R)} \right) \right) \left( \|u^+\|_{L^p(R)} + \|u^-\|_{L^p(R)} \right) \|u^+\|_{L^p(R)}^{p-1} . \]
A similar estimate holds for $\|u^-\|_{L^p(R)}$. Therefore, we have
\[ \frac{d}{dt} \left( \|u^+\|_{L^p(R)} + \|u^-\|_{L^p(R)} \right) \leq C \left( 1 + \ln \left( \|u_0^+ \|_{L^2(R)} \right) \right) e^{Kt} \left( \|u^+\|_{L^p(R)} + \|u^-\|_{L^p(R)} \right) . \]
Then the standard Gronwall inequality, Lemma 2, implies
\[ \|u^+\|_{L^p(R)} + \|u^-\|_{L^p(R)} \leq \left( \|u_0^+\|_{L^p(R)} + \|u_0^-\|_{L^p(R)} \right) \exp \left( C \ln \left( \|u_0^+\|_{L^2(R)} \right) \right) \int_0^t e^{Ks} \, ds . \]
Since the initial data $u_0$ are estimated as follows
\[ \|u_0^+\|_{L^p(R)} + \|u_0^-\|_{L^p(R)} \leq \|u_0\|_{L^p(R)} \leq \|u_0\|_{L^\infty(R)} \leq \|u_0\|_{L^p(R)} \leq \|u_0\|_{L^p(R)} \leq \|u_0\|_{L^p(R)} \leq \|u_0\|_{L^p(R)} , \]
we have
\[ \|u^+\|_{L^p(R)} + \|u^-\|_{L^p(R)} \leq C_1 \exp \left( C_2 \int_0^t e^{Ks} \, ds \right) \leq C_1 \exp \left( C_2 \exp Kt \right) , \]
where $C_1 \equiv (\|u_0\|_{L^\infty(R)} + 1)(U_0 + 1)$ and $C_2 \equiv C(1 + \ln C_1)$. This completes the proof. \hfill \Box

We also have L∞ estimates.

**Lemma 11.** Let $u_0^\pm \in L^1(R) \cap L^\infty(R)$ with $\int_R u_0^+ + u_0^- = U_0$. Assume (A1)-(A4). There exist constants $C_1 = C_1(u_0^\pm)$, $C_2 = C_2(u_0^\pm, \alpha, \beta, U_0)$ and $K = K(\alpha, \beta, U_0)$ such that for all $t \geq 0$,
\[ \|u^+\|_\infty + \|u^-\|_\infty \leq C_1 \exp \left( C_2 \exp Kt \right) . \]

**Proof.** In the Lp-estimate of the previous lemma, the constants $C_1$, $C_2$ and $K$ are uniformly bounded and independent of $p$. Thus, (3.6) is obvious. \hfill \Box
4. $W^{k,p}$-estimates

In this section, we study $L^p$-estimates for all higher derivatives of $u^\pm$. We first present standard regularity estimates for parabolic equations without proof. For convenience, we denote $u^+_x = v^+$ and $u^-_x = v^-.$

**Lemma 12.** There exists a constant $C = C (p)$ such that for all $1 < p < \infty$,
\[
\|s_{tx}\|_{L^p(\Omega_t)} + \|s_{txx}\|_{L^p(\Omega_t)} + \|s_x\|_{L^p(\Omega_t)} \leq C (\|v^+\|_{L^p(\Omega_t)} + \|v^-\|_{L^p(\Omega_t)}).
\]

**Lemma 13.** Let $v^\pm, s$ be solutions of (2.2)-(2.5) and let the initial data fulfill $u^\pm_{0x} \in L^2$. Then $(u^+_x, u^-_x)$ exists globally in $C \left( [0, \infty), L^2(\mathbb{R}) \right)$ and $(u^+_x, u^-_x)$ satisfies
\[
\| (u^+_x, u^-_x) \|_{L^2(\mathbb{R})} \leq \| (u^+_{0x}, u^-_{0x}) \|_{L^2(\mathbb{R})} + C \exp \left( C \exp \left( C \exp Kt \right) \right),
\]
where $C = C (\alpha, \beta, U_0), K = K (\alpha, \beta, U_0)$.

**Proof.** From (2.2) and (2.3), we obtain
\[
\begin{align*}
  v^+_t &- \varepsilon v^+_{xx} + (\gamma v^+_x)_x = -(\gamma_x u^+_x)_x + (-\mu^+ u^+_x + \mu^- u^-_x)_x, \\
  v^-_t &- \varepsilon v^-_{xx} - (\gamma v^-)_x = (\gamma_x u^-_x)_x + (\mu^+ u^+_x - \mu^- u^-_x)_x.
\end{align*}
\]

Multiplying $v^+$ and $v^-$ to (4.1) and (4.2) respectively, we have
\[
\begin{align*}
  \frac 12 \int_{\mathbb{R}} |v^+| \partial_t \, dx &= \frac 12 \int_{\mathbb{R}} |v^+| \partial_t \, dx + \int_0^t \int_{\mathbb{R}} v^+_x v^+ \, dx \\
  &\leq \frac 12 \int_{\mathbb{R}} |v^+| \partial_t \, dx + \int_0^t \int_{\mathbb{R}} \gamma v^+_x v^+_x \, dx - \int_0^t \int_{\mathbb{R}} (\gamma_x u^+_x)_x v^+ \, dx \\
  &\quad + \int_0^t \int_{\mathbb{R}} (-\mu^+ u^+_x + \mu^- u^-_x)_x v^+ \, dx ds - \varepsilon \int_0^t \int_{\mathbb{R}} |v^+_x|^2 \, ds \\
  &= \frac 12 \int_{\mathbb{R}} |v^+| \partial_t \, dx + \int_0^t \int_{\mathbb{R}} \gamma_x u^+_x v^+ + \gamma_x (v^+)^2 \, dx ds \\
  &\quad + \int_0^t \int_{\mathbb{R}} [-\mu^+ v^+_x]^2 + \mu^+ v^+_x v^- - \mu^+_x u^+_x v^+ + \mu^- u^-_x v^-] \, dx ds - \varepsilon \int_0^t \int_{\mathbb{R}} |v^+_x|^2 \, ds \\
  \leq \frac 12 \int_{\mathbb{R}} |v^+| \partial_t \, dx + C \|\gamma\|_{L^\infty(\mathbb{R})} \|s_{xx}\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}} (v^+)^2 \, dx ds \\
  &\quad + C \left( \|\mu^+\|^2_{L^\infty(\mathbb{R})} + \|\mu^-\|^2_{L^\infty(\mathbb{R})} \right) \int_0^t \int_{\mathbb{R}} (v^+)^2 + |v^-|^2 \, dx ds \\
  &\quad + C \left( \|u^+\|_{L^p(\mathbb{R})} + \|u^-\|_{L^p(\mathbb{R})} \right) \int_0^t \int_{\mathbb{R}} (v^+)^2 + |\gamma_{xx}|^2 + |\mu^+_x|^2 + |\mu^-_x|^2 \, dx ds.
\end{align*}
\]

Note that
\[
|\gamma_{xx}| \leq \|\gamma\|_{L^\infty(\mathbb{R})} |s_{xx}| + \|\gamma''\|_{L^\infty(\mathbb{R})} |s_x|^2,
\]
\[
|\mu^+_x| \leq \|D_j \mu\|_{L^\infty(\mathbb{R})} (|s_x| + |s_{xx}| + |s_{xxx}| + |s_{xxxx}|).
\]

Here, as in Lemma 12, we use the following $L^2$-estimate for $s$.
\[
\|s_{tx}\|_{L^2(\Omega_t)} + \|s_{txx}\|_{L^2(\Omega_t)} + \|s_x\|_{L^2(\Omega_t)} \leq C (\|v^+\|_{L^2(\Omega_t)} + \|v^-\|_{L^2(\Omega_t)}).
\]
For $s_{xx}$, we have from Lemma 3 and Lemma 10,

$$
\|s_{xx}\|_{L^2(\Omega_t)} \leq C \|u\|_{L^2(\Omega)} \leq C \exp(C \exp K t).
$$

By Corollary 1 and Lemma 11, we have

$$
\|s_x\|_{L^\infty(\mathbb{R})} \leq Ce^{Kt}, \quad \|u^\pm\|_{L^\infty(\mathbb{R})} \leq C \exp(C \exp K t).
$$

Therefore

$$
\int_\mathbb{R} |v^+(\cdot,t)|^2 \, dx \leq \int_\mathbb{R} |v^+(\cdot,0)|^2 \, dx + C \exp(C \exp K t)
$$

$$
+ C \exp(C \exp K t) \int_0^t \int_\mathbb{R} [\|v^+\|^2 + |v^-|^2] \, dxds.
$$

In a similar manner, we deduce

$$
\int_\mathbb{R} |v^-(\cdot,t)|^2 \, dx \leq \int_\mathbb{R} |v^+(\cdot,0)|^2 + C \exp(C \exp K t)
$$

$$
+ C \exp(C \exp K t) \int_0^t \int_\mathbb{R} [\|v^+\|^2 + |v^-|^2] \, dxds.
$$

Gronwall’s inequality, Lemma 2, implies that

$$
\left\|\left(\frac{u^+_0}{u^-_0}\right)\right\|_{L^2(\mathbb{R})} \leq \left\|\left(\frac{u^+_0}{u^-_0}\right)\right\|_{L^2(\mathbb{R})} + C \exp(C \exp K t).
$$

This completes the proof.

Next we show $W^{1,p}$ estimates.

**Lemma 14.** Let $2 \leq p < \infty$. Let $u^\pm, s$ be solutions of (2.2)-(2.5) and $u^\pm_{0x} \in L^p$. Then $(u^+_x, u^-_x)$ from (2.2,2.3 ,2.4) exist globally in $C([0, \infty), L^p(\mathbb{R}))$ and satisfy

$$
\|u^+_x, u^-_x\|_{L^p(\mathbb{R})} \leq \|u^+_0, u^-_0\|_{L^p(\mathbb{R})} + C_p \exp(C \exp K t),
$$

where $K = K(\alpha, \beta, U_0)$, $C = C(\alpha, \beta, U_0)$ and $C_p = C(\alpha, \beta, U_0, p)$. 

**Proof.** We multiply $pv^+, p-1$ and $pv^-, p-1$ to (4.1) and (4.2) respectively. Then we have

$$
\|v^+\|_{L^p(\mathbb{R})} = \|v^+(0)\|_{L^p(\mathbb{R})} - \int_0^t \int_\mathbb{R} \left( (\gamma v^+, \gamma v^+) - (p - 1) \gamma x v^+, \gamma x v^+ \right) \, dxds
$$

$$
- \int_0^t \int_\mathbb{R} (\mu v^+, \mu v^+ + \gamma v^+) \, dxds + \int_0^t \int_\mathbb{R} \left( \mu v^+, \mu v^+ + \gamma v^+ \right) \, dxds
$$

$$
+ \int_0^t \int_\mathbb{R} \left( \mu v^+, \mu v^+ + \gamma v^+ \right) \, dxds.
$$

Using the following estimates

$$
\|\gamma x\|_{L^p(\mathbb{R})} + ||\mu v^+_x||_{L^p(\mathbb{R})} + ||\mu v^-||_{L^p(\mathbb{R})} \leq C_p e^{Kt} \left( \|v^+\|_{L^p(\mathbb{R})} + \|v^-\|_{L^p(\mathbb{R})} \right),
$$

$$
\|\gamma x\|_{L^\infty(\mathbb{R})} \leq C e^{Kt}, \quad \|u^\pm\|_{L^\infty(\mathbb{R})} \leq C \exp(C \exp K t),
$$

we have

$$
\|v^+\|_{L^p(\mathbb{R})} \leq \|v^+(0)\|_{L^p(\mathbb{R})} + C_p \exp(C \exp K t)
$$

$$
+ C_p \exp(C \exp K t) \int_0^t \|v^+\|_{L^p(\mathbb{R})} + \|v^-\|_{L^p(\mathbb{R})} \, ds.
$$
Similarly, we get the same estimate for $v^-$, and thus

\[
\|v^+\|_{L^p(\mathbb{R})}^p + \|v^-\|_{L^p(\mathbb{R})}^p \leq \|v^+(0)\|_{L^p(\mathbb{R})}^p + \|v^-(0)\|_{L^p(\mathbb{R})}^p + C_p \exp(C \exp Kt)
\]

\[+ C_p \exp(C \exp Kt) \int_0^t \left[ \|v^+\|_{L^p(\mathbb{R})}^p + \|v^-\|_{L^p(\mathbb{R})}^p \right] ds.
\]

Gronwall’s inequality, Lemma 2, implies $v^\pm = u_{\pm}^k$ is $L^p$. Thus the proof is complete. \(\square\)

For simplicity, we denote by $u_k$ the $k$-th spatial derivative of $w$. We present $W^{k,p}$-estimates of $u^\pm$ where $k \geq 2$ is an integer and $2 \leq p < \infty$.

**Lemma 15.** Let $2 \leq p < \infty$. Let $w^\pm, s$ be solutions of (2.2)-(2.5) and initial data $u_{0k}^\pm \in L^p(\mathbb{R})$. Then $(u_k^+, u_k^-)$ exists globally in $C([0, \infty), L^p(\mathbb{R}))$ and $(u_k^+, u_k^-)$ satisfies

\[
\left\| \left( u_k^+, u_k^- \right) \right\|_{L^p(\mathbb{R})} \leq \left\| \left( u_{0k}^+, u_{0k}^- \right) \right\|_{L^p(\mathbb{R})} + C_p \exp \left( \sum_{k \geq 2} C \exp Kt \right)
\]

where $K = K(\alpha, \beta, U_0)$ and $C = C(\alpha, \beta, U_0, k)$ and $C_p = C(\alpha, \beta, U_0, p, k)$.

**Proof.** We take the $k$-th derivative repeatedly of equations (1.1) -(1.2) and obtain

\[
u_{k,t}^+ = - \left( \gamma u^+ \right)_{k+1} + (-\mu^+ u^+ + \mu^- u^-)_k
\]

\[= - \sum_{l=0}^{k+1} \binom{k+1}{l} \gamma^l u_{k+1-l}^+ \pm \sum_{l=0}^{k} \binom{k}{l} \mu^+_l u_{k-l}^+.
\]

\[
u_{k,t}^- = \left( \gamma u^- \right)_{k+1} + (\mu^+ u^+ - \mu^- u^-)_k
\]

\[= - \sum_{l=0}^{k+1} \binom{k+1}{l} \gamma^l u_{k+1-l}^- \pm \sum_{l=0}^{k} \binom{k}{l} \mu^-_l u_{k-l}^-.
\]

We note that

\[
\|\gamma s\|_{L^\infty(\mathbb{R})} \leq C \left( 1 + \|s\|_{W^{1,\infty}(\mathbb{R})} \right) \leq C \exp Kt,
\]

\[
\|\mu s\|_{L^\infty(\mathbb{R})} \leq C \left( 1 + \|s\|_{W^{1,\infty}(\mathbb{R})} \right) \leq C \exp Kt.
\]

By induction we deduce for $l \geq 2$,

\[
\|\gamma i\|_{L^\gamma(\Omega_t)} \leq C \sum_{j=1}^l \|s\|_{L^\gamma(\Omega_{t-j})} \leq C \exp \left( \sum_{l=2} C \exp Kt \right).
\]

Similarly, we get for $l \geq 1$,

\[
\|\mu i\|_{L^\gamma(\Omega_t)} \leq C \left( \|u_i\|_{L^\gamma(\mathbb{R})} + \exp \left( \sum_{l=1} C \exp Kt \right) \right).
\]

The estimate in (4.4) is obtained from the dependence of $\mu$ on derivatives of $s$ up to $s_{xx}$ and from (3.1). Multiplying $p d_{k}^{+,-p^{-1}}$ to (4.3), integrating with respect to $x$ and $t$ and using induction on $k$, the left-hand-side reads $\int_{\mathbb{R}} |u_k^+(\cdot, t)|^2 dx - \int_{\mathbb{R}} |u_k^+(\cdot, 0)|^2 dx$. We now estimate the right-hand-side term by term:
There exists a constant $u_0$ such that for all $k \geq 0$ and all $1 \leq p \leq \infty$, we have

$$L^p(\mathbb{R}) \ni u_k^\pm \leq C.$$

By standard embedding arguments, we finally have

$$
\bullet \ l = 0:\ \ \ \ \ \ p \int_0^t \int \gamma u_{k+1}^+ u_k^{p-1} dx ds = - \int_0^t \int \gamma x u_k^+ |^p dx ds \leq C \exp (Kt) \| u_k^+ \|^p_{L^p(\mathbb{R})},
$$

$$p \int_0^t \int \mu^\pm u_k^+ u_k^{p-1} dx ds \leq C \exp (Kt) \| (u_k^+, u_k^-) \|^p_{L^p(\mathbb{R})}.
$$

$$\bullet \ l = 1:\ \ \ \ \ \ p \int_0^t \int \gamma u_k^+ u_k^{p-1} dx ds \leq C \exp (Kt) \| u_k^+ \|^p_{L^p(\mathbb{R})},$$

$$p \int_0^t \int \mu^\pm u_k^- u_k^{p-1} dx ds \leq C \| u_k^+ \|_{L^2(\mathbb{R})} \| u_k^- \|_{L^2} \| u_k^{p-1} \|_{L^{p/(p-1)}}$$

$$\leq C \exp \left( \sum_{k-1} C \exp Kt \right) \| u_k^+ \|^p_{L^p(\mathbb{R})},$$

where we used the induction on $k-1$ with $2p$.

$\bullet \ 2 \leq l \leq k-1:$ the same method as in (4.5) applies

$\bullet \ l = k:$ For the term with $\gamma_k$, the same method applies. For the term with $\mu_k^\pm$, we have

$$p \int_0^t \int \mu^\pm u_k^\pm u_k^{p-1} dx ds \leq C \| u_k^\pm \|_{L^\infty(\mathbb{R})} \| u_k^\pm \|_{L^p(\mathbb{R})} \| u_k^{p-1} \|_{L^{p/(p-1)}}$$

$$\leq C \exp \left( \sum_{k-1} C \exp Kt \right) \| (u_k^+, u_k^-) \|^p_{L^p(\mathbb{R})}$$

$$+ C \| (u_k^+, u_k^-) \|_{L^p(\mathbb{R})}^p.$$

$\bullet \ l = k + 1:$ We can deal with this case in a similar way.

The terms with $u^-$ can be estimated similarly. Therefore, by applying Gronwall’s inequality, we complete the proof.

Taking a similar procedure as given in [11, see page 188-190] we can also obtain $W^{1,1}$ estimates. The difference is that we use boundedness of $\| u^\pm \|_{L^2}$ and the $L^2$ norms of the derivatives of the chemical signal $s$, which are bounded by $\| u^\pm \|_{W^{1,2}}$. Similarly, we can have $W^{k,1}$ estimates. To sum up, we have

**Lemma 16.** Let $u^\pm, s$ be solutions of (2.2)-(2.5) and the initial data $u_{0k}^\pm \in L^p(\mathbb{R})$. Let $1 \leq p < \infty$. There exists a constant $C = C (\alpha, \beta, U_0, k, p, T)$ such that for all $t \in [0, T]$

$$\| u_k^+ \|_{L^p(\mathbb{R})} + \| u_k^- \|_{L^p(\mathbb{R})} \leq C.$$

By standard embedding arguments, we finally have

**Theorem 1.** Let $0 < T \leq \infty$ and $\Omega_T = \mathbb{R} \times (0, T)$. Suppose that $u_0^\pm \in W^{k,p}(\mathbb{R})$ for all $k \geq 0$ and all $1 \leq p \leq \infty$. Then we have for (2.2)-(2.5) a solution $(u^\pm, s^\pm) \in \left[ C (0, T), W^{k,p}(\mathbb{R}) \right]^3$ for all $k \geq 0$ and $1 \leq p \leq \infty$, where the bound is independent of $\varepsilon$ in $\left[ C (0, T), W^{k,p}(\mathbb{R}) \right]^3$ for each $k$ and $p$. 


5. The Vanishing Viscosity Limit, $\varepsilon \to 0$

Now we are back to the notation $u^{+}, u^{-}, s^{\varepsilon}$.

**Lemma 17.** Let $0 < T < \infty$ and $\Omega_T = \mathbb{R} \times (0, T)$ then there exists a $s(x, t) \in C^{k,\delta}(\Omega_T)$ with $s_t(x, t) \in C^{k-2,\delta}(\Omega_T)$ for any small $\delta > 0$ and all $k \geq 2$ such that

\[
s^{\varepsilon m}(x, t) \to s(x, t) \quad \text{in } C^{k,\delta}(\Omega_T),
\]

\[
s^{\varepsilon m}_t(\cdot, t) \to s_t(\cdot, t) \quad \text{in } C^{k-2,\delta}(\Omega_T)
\]

for some sequence $\varepsilon_m \to 0$ with $s(x, t) \in W^{k+1, p}(\Omega_T) \cap W^{k, \infty}(\Omega_T)$ and $s_t(x, t) \in W^{k-1, p}(\Omega_T) \cap W^{k-2, \infty}(\Omega_T)$ for all $1 \leq p \leq \infty$ and all $k \geq 2$.

**Proof.** By Theorem 1, $\{s^{\varepsilon}(x, t)\}$ is bounded in $W^{k+1, p}(\Omega_T)$ and $\{s^{\varepsilon}_t(x, t)\}$ in $W^{k-1, p}(\Omega_T)$, $1 \leq p < \infty$, and $0 < \varepsilon \leq 1$. Since the embeddings $W^{k+1, p} \to C^{k,\delta}$ and $W^{k-1, p} \to C^{k-2,\delta}$ are compact, there is a convergent subsequence. Moreover, all the previous estimates, which are independent of $\varepsilon$ imply the spaces which $s$ and $s_t$ belong to.

Now we are ready to prove our main result:

**Proof of Main Theorem.** By Theorem 1, we have for all $0 < \varepsilon \leq 1$ and all $T > 0$ a classical solution $(u^{+}, u^{-}, s^{\varepsilon})$ of the parabolic-parabolic Cauchy problem (2.2)-(2.4). We consider for $m \in \mathbb{N}$ a sequence $\varepsilon_m$ as in the previous Lemma. Similar arguments apply to $\{u^{\varepsilon m \pm}\}$ which are bounded in $W^{k, p}$ for all $k \geq 0$ and $1 \leq p \leq \infty$. Since for any small $\delta > 0$ the embedding $W^{k, p} \subset C^{k-1,\delta}$ is compact, we further extract a subsequence $\varepsilon_{m_k}$ with $\varepsilon_{m_k} \to \infty$ from $\{\varepsilon_m\}$ such that $u^{\varepsilon_{m_k} \pm} \to u^{\pm}$ in $C^{k-1,\delta}(\Omega_T)$. For convenience, we denote $\{u^{\varepsilon m \pm}, u^{\varepsilon_{m_k} \pm}, s^{\varepsilon_{m_k}}\}$ as the convergent subsequence. We now show that the limit $\{u^{+}, u^{-}, s\}$ of $\{u^{\varepsilon m \pm}, u^{\varepsilon_{m_k} \pm}, s^{\varepsilon_{m_k}}\}$ is the desired classical solution of the original hyperbolic-parabolic Cauchy problem (1.1)-(1.4). By the smoothness of $\gamma$, $\mu^{\pm}$ and the convergence of $u^{\varepsilon m \pm}$ and $s^{\varepsilon m}$ to $u^{\pm}$ and $s$ respectively in the Hölder spaces $C^{k,\delta}(\Omega_T)$, $(u^{+}, u^{-}, s)$ clearly satisfies (1.1)-(1.3). Since $u^{\varepsilon m \pm}$ and $s^{\varepsilon m}$ converge in $C^\delta$ to $u^{\pm}$ and $s$, $u^{\pm}$ and $s$ satisfy the initial conditions (1.4) and (2.1). A priori estimates imply that $(u^{+}, u^{-}, s) \in [W^{k, p}(\Omega_T)]^3$ for all $k \geq 0$ and all $1 \leq p \leq \infty$.

**Remark 2.** Every estimate works in a similar way for the stationary case with dependence of the turning rates $\mu^{\pm} = \mu^{\pm}(s, s_1, s_2)$ also on the highest derivative $s_{xx}$ and the temporal gradient $s_t$ of $s$. Thus we also have global classical solutions in the case, which covers and is even a stronger result than in [11].

**Remark 3.** So far we have considered the system (1.1)-(1.4) for $\beta > 0$. As mentioned at the beginning, the case $\beta = 0$ can be proved by slight modification of our arguments. One minor change, in case $\beta = 0$, will be the $L^\infty$-estimate of $s$ in Lemma 4, which should be replaced by

(5.1) \[
\|s\|_{L^\infty(\mathbb{R})} \leq C(\alpha) t^\frac{1}{2} \|u_0\|_{L^1(\mathbb{R})}.
\]

Indeed, using the Fourier transform of the heat kernel, we have

\[
\int_{-\infty}^{\infty} |\hat{s}(\xi, t)| d\xi \leq C \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^\infty(\mathbb{R})} \int_0^t \int_{-\infty}^{\infty} e^{-\tau(t-\tau)} 4\xi^2 d\xi d\tau \leq C(\alpha) t^\frac{1}{2} \sup_{0 \leq \tau \leq t} \|\hat{u}\|_{L^\infty(\mathbb{R})},
\]

where we used the change of variables in the last inequality. The above calculation automatically implies (5.1). To prove our main result for the case $\beta = 0$ needs simple modifications, like the one above. But since these are obvious, we omit the details.
6. The Parabolic Limit

In [12] a formal parabolic limit was derived from the general hyperbolic model for chemotaxis. Similar to considerations done for higher dimensional analogues of this model, compare [2, 3, 10], in this section we rigorously derive a parabolic Keller-Segel type system in one dimension from the kinetic model where $\gamma$ is assumed to be constant.

$$u_t^+ + \epsilon^{-1} \gamma u_x^+ = -\mu^+(s, s_t, s_{xx}) u^+ + \mu^-(s, s_t, s_{xx}) u^-$$

$$u_t^- - \epsilon^{-1} \gamma u_x^- = -\mu^+(s, s_t, s_{xx}) u^+ - \mu^-(s, s_t, s_{xx}) u^-$$

coupled with the equation for the chemotactic signal (1.3). Using a diffusive scaling of time and space, the kinetic equations in non-dimensional form become

$$(6.1) \quad u_t^+ + \epsilon^{-1} \gamma u_x^+ = -\mu^+(s, s_t, s_{xx}) u^+ + \epsilon^{-2} \mu^-(s, s_t, s_{xx}) u^{-}\epsilon$$

$$(6.2) \quad u_t^- - \epsilon^{-1} \gamma u_x^- = -\mu^+(s, s_t, s_{xx}) u^+ - \epsilon^{-2} \mu^-(s, s_t, s_{xx}) u^{-}\epsilon$$

$$(6.3) \quad s_t^\epsilon - \Delta s^\epsilon = -\beta s^\epsilon + \alpha u^\epsilon.$$

$$(6.4) \quad u^\epsilon(\cdot, 0) = u_0, \quad s^\epsilon(\cdot, 0) = s_0,$$

where $\epsilon$ is a non-dimensional small parameter. Here we note that $u^\epsilon$ is regular for each fixed $\epsilon > 0$ under the assumptions (A2) and (A3). Let $\xi = \mu^+ - \mu^-$ and $\eta = \mu^+ + \mu^-$. Then by adding and subtracting (6.1) and (6.2), we obtain

$$(6.5) \quad u_t^\epsilon + \epsilon^{-1} \gamma u_x^\epsilon = 0, \quad v_t^\epsilon + \epsilon^{-1} \gamma v_x^\epsilon = -\epsilon^{-2} \xi u^\epsilon - \epsilon^{-2} \eta v^\epsilon,$$

where $u^\epsilon = u^+ + u^-$ and $v^\epsilon = u^+ - u^-$. The following analysis is based on an asymptotic expansion of the turning rates $\mu^\pm = \mu^{(0)} \pm \epsilon \mu^{(1)} \pm \epsilon^2 \mu^{(2)} \pm O(\epsilon^3)$. Our goal is, to derive equations for the leading order terms of $u^\epsilon = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + O(\epsilon^3)$, $v^\epsilon = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + O(\epsilon^3)$, and $s^\epsilon = s^{(0)} + \epsilon s^{(1)} + \epsilon^2 s^{(2)} + O(\epsilon^3)$. Due to our definition, we have $\xi = \xi^{(0)} + \epsilon \xi^{(1)} + \epsilon^2 \xi^{(2)} + O(\epsilon^3)$, where $\xi^{(0)} = 0$, $\xi^{(1)} = 2\mu^{(1)+} - \mu^{(2)-}$, and $\eta = \eta^{(0)} + \epsilon \eta^{(1)} + \epsilon^2 \eta^{(2)} + O(\epsilon^3)$, where $\eta^{(0)} = 2\mu^{(0)+}$, $\eta^{(1)} = 0$, $\eta^{(2)} = \mu^{(2)+} + \mu^{(2)-}$. Before proceeding further, we give some structure conditions on the turning rates $\mu^\pm$.

**Assumption 1.** The leading order terms $\mu^{(0)} \pm$ are balanced and strictly positive, and the first order term $\mu^{(1)} \pm$ have opposite sign

$$\mu^{(0)+} = \mu^{(0)-} > C > 0, \quad \mu^{(1)+} = -\mu^{(1)-},$$

where $C$ is a positive constant. Moreover, there exist constants $C_1, C_2 > 0$ such that

$$(6.6) \quad 0 < C_1 \leq \mu^\pm(s, s_t, s_{xx}, s_t) \leq C_2(1 + s(x, t) + s(x + e\gamma, t) + s(x - e\gamma, t) + |s_x(x, t)|)$$

for any $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ and $s \in L_{loc}^\infty(W^{1,\infty}(\mathbb{R}))$. \hfill \Box$

We first derive the Keller-Segel type system formally from (6.1) and (6.2) for $\epsilon \to 0$. For convenience we define $\mu_0 \equiv \mu^{(0)+}$ and $\mu_1 \equiv \mu^{(1)+}$. Comparing the coefficients of $\epsilon^{-2}$ in the second equation of (6.5), we have

$$-\xi^{(0)} u^{(0)} - \eta^{(0)} v^{(0)} = 0.$$

Therefore, since $\xi^{(0)} = 0$ and $\eta^{(0)} = 2\mu_0 > 0$, we have $v^{(0)} = 0$. Comparing the coefficients of $\epsilon^{-1}$ in the second equation of (6.5), we get

$$(6.7) \quad v^{(1)} = -\frac{\xi^{(1)}}{\eta^{(0)}} u^{(0)} - \frac{\gamma}{\eta^{(0)}} u_x^{(0)}.$$
Now we consider the zero order terms in both equations of (6.5). After simple computations, we have

\begin{equation}
0 = u_t^{(0)} + \gamma v_x^{(1)}, \quad \eta^{(0)} v_x^{(2)} = -\xi^{(2)} u^{(0)} - \xi^{(1)} u^{(1)} - \gamma u_x^{(1)}.
\end{equation}

Due to (6.7) and the first equation in (6.8), the diffusion limit reads

\begin{equation}
0 = u_t^{(0)} + \gamma v_x^{(1)} = u_t^{(0)} - \gamma \left( \frac{\xi^{(1)}}{\eta^{(0)}} u^{(0)} + \frac{\gamma}{\eta^{(0)}} u_x^{(0)} \right)_x = u_t^{(0)} - \gamma \left( \frac{\mu_1}{\mu_0} u^{(0)} + \frac{\gamma}{2\mu_0} u_x^{(0)} \right)_x.
\end{equation}

For the second equation in (6.8) we use \( u_t^{(1)} + \gamma v_x^{(2)} = 0 \) from (6.5). By taking the derivative with respect to \( x \) of the second equation of (6.8), we have

\begin{equation}
v_x^{(2)} = - \left( \frac{\xi^{(2)}}{\eta^{(0)}} u^{(0)} \right)_x - \left( \frac{\xi^{(1)}}{\eta^{(0)}} u^{(1)} \right)_x - \left( \frac{\gamma}{\eta^{(0)}} u_x^{(1)} \right).
\end{equation}

Using \( \gamma v_x^{(2)} = -u_t^{(1)} \), we obtain

\begin{equation}
u_t^{(1)} = \gamma \left( \frac{\xi^{(2)}}{\eta^{(0)}} u^{(0)} \right)_x + \gamma \left( \frac{\xi^{(1)}}{\eta^{(0)}} u^{(1)} \right)_x + \gamma^2 \left( \frac{u_x^{(1)}}{\eta^{(0)}} \right).
\end{equation}

Note that this equation is non-degenerate second order parabolic equation with smooth coefficients, because we proved that \( u^\epsilon \) and \( s^\epsilon \) are regular for each \( \epsilon > 0 \) in the previous sections and \( \eta^{(0)} > 0 \), see Assumption 1. Therefore, \( u_t^{(1)} \) can be solved, which implies that \( u^{(2)} \) can be automatically recovered from (6.8). Equation (6) compares exactly to (6) in case \( \xi^{(2)} = 0 \).

To sum up, the formal parabolic limit leads to

\begin{equation}
u_t^{(0)} = (Du_x^{(0)} + \mathcal{H} u^{(0)})_x,
\end{equation}

where the diffusion coefficient \( D \) and the drift coefficient \( \mathcal{H} \) are

\begin{align}
D & = \frac{\gamma^2}{\eta^{(0)}(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})} = \frac{\gamma^2}{2\mu_0(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})}, \\
\mathcal{H} & = \gamma \frac{\xi^{(1)}}{\eta^{(0)}(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})} = \gamma \frac{\mu_1(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})}{\mu_0(s^{(0)}, s_t^{(0)}, s_x^{(0)}, s_{xx}^{(0)})},
\end{align}

which compares to the formulations given in [12], for \( \gamma = \text{const.} \). How \( \mathcal{H} \) relates to the chemotactic sensitivity \( \chi \) times \( s_x \), which is the classical parabolic formulation used for chemotaxis, we will see later in an example. The formal limit of (6.1) and (6.2) is (6.9), coupled to chemo-attractant equation for \( s^{(0)} \),

\begin{equation}s_t^{(0)} - \Delta s^{(0)} = \alpha u^{(0)} - \beta s^{(0)}.
\end{equation}

Now we rigorously prove the convergence. First, under Assumption 1, we show uniform estimates, independently of \( \epsilon \).

**Lemma 18.** Let \( \Psi \in L^\infty_{\text{loc}}([0, \infty)) \) be a measurable function satisfying the linear growth condition at infinity, e.g. \( |\Psi(x)| \leq C(1 + |x|) \). Suppose Assumption 1 holds and \( u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Assume further that there exists \( C \), independent of \( \epsilon \), such that

\begin{equation}
\mu^+ + \mu^- = \eta \geq C(1 - \epsilon \Psi(||\xi||_{W^{1,\infty}})), \quad |\mu^+ - \mu^-|^2 = |\xi|^2 \leq C \epsilon^2 \Psi(||\xi||_{W^{1,\infty}}).
\end{equation}

Then the solution \((u^\epsilon, s^\epsilon)\) in (6.1)-(6.4) satisfies, uniformly in \( \epsilon \),

\begin{equation}
u^\epsilon \in L^\infty_{\text{loc}}((0, \infty) ; L^2(\mathbb{R})), \quad s^\epsilon \in L^\infty_{\text{loc}}((0, \infty) ; L^p(\mathbb{R}) \cap C^{1,\alpha}(\mathbb{R})),
\end{equation}

where \( 1 \leq p < \infty \) and \( 0 < \alpha \leq 1/2 \).
Proof. First we note that mass is conserved

\[ \|u^\epsilon(\cdot, t)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}. \]

Multiply \( u^\epsilon \) and \( v^\epsilon \) to the first and second equation in (6.5), respectively, then we have

\[ \frac{1}{2} \int_\mathbb{R} (|u^\epsilon|^2 + |v^\epsilon|^2) dx + \int_0^t \int_\mathbb{R} \epsilon^{-1} \gamma u^\epsilon v^\epsilon dx dt = \frac{1}{2} \int_\mathbb{R} (|u_0|^2 + |v_0|^2) dx - \int_0^t \int_\mathbb{R} \epsilon^{-2} \xi u^\epsilon v^\epsilon dx dt. \]

Adding together and integrating in space and time, we obtain

\[ \int_\mathbb{R} (|u^\epsilon|^2 + |v^\epsilon|^2) dx + \int_0^t \int_\mathbb{R} \epsilon^{-2} \eta |v^\epsilon|^2 dx dt = \frac{1}{2} \int_\mathbb{R} (|u_0|^2 + |v_0|^2) dx - \int_0^t \int_\mathbb{R} \epsilon^{-2} \xi u^\epsilon v^\epsilon dx dt. \]

Since \(|\xi| \leq 2^{-1}\eta + 2^{-1}|\xi|^2\eta^{-1} \), we have

\[ \int_\mathbb{R} (|u^\epsilon|^2 + |v^\epsilon|^2) dx \leq \int_\mathbb{R} (|u_0|^2 + |v_0|^2) dx + \int_0^t \int_\mathbb{R} \epsilon^{-2} |\xi|^2 |v^\epsilon|^2 dx dt. \]

Due to (6.12), we have

\[ \int_\mathbb{R} |u^\epsilon(\cdot, t)|^2 dx \leq 4 \int_\mathbb{R} |u_0|^2 dx + C \Psi(||s^\epsilon||_{W^{1, \infty}}) \int_0^t \int_\mathbb{R} |u^\epsilon|^2 dx dt. \]

Finally, according to Lemma 4, we have

\[ \int_\mathbb{R} |u^\epsilon(\cdot, t)|^2 dx \leq 4 \int_\mathbb{R} |u_0|^2 dx + C(1 + \log(||u^\epsilon||_{L^2})) \int_0^t \int_\mathbb{R} |u^\epsilon|^2 dx dt, \]

where we used the linear growth condition of \( \Psi \). Since the above estimate is independent of \( \epsilon \), we have an \( L^2 \)-estimate of \( u^\epsilon \) independently of \( \epsilon \) by Gronwall’s inequality. For the chemo-attractant we also have, uniformly in \( \epsilon \)

\[ ||s^\epsilon||_{L^2(\mathbb{R})} + ||s^\epsilon||_{W^{2, 2}(\mathbb{R})} \leq C||u^\epsilon||_{L^2(\mathbb{R})}. \]

Therefore, combining potential estimate and embedding argument, we obtain

\[ ||s^\epsilon||_{L^p(\mathbb{R})} + ||s^\epsilon||_{C^{1, \alpha}(\mathbb{R})} \leq C \sup_{0 \leq \tau \leq t} ||u^\epsilon||_{L^2(\mathbb{R})}, \quad 1 \leq p < \infty, \quad 0 < \alpha \leq \frac{1}{2}. \]

This completes the proof. □

Theorem 2. Let the assumption of Lemma 18 hold. Assume that

\[ \mu^\pm(s^\epsilon, s^\epsilon_t, s^\epsilon_{xx}, s^\epsilon_{xxx}) \to \mu^\pm(s^{(0)}, s^{(0)}_t, s^{(0)}_{xx}, s^{(0)}_{xxx}) \quad \text{as} \quad \epsilon \to 0. \]

Then the solution \( (u^\epsilon, s^\epsilon) \) (6.1)-(6.4) satisfies, after choosing appropriate subsequences

\[ u^\epsilon \rightharpoonup u^{(0)} \quad \text{in} \quad L^\infty_{\text{loc}}([0, \infty), L^1(\mathbb{R}) \cap L^2(\mathbb{R})) \quad \text{weakly}, \]

\[ s^{\epsilon_{xx}} \rightharpoonup s^{(0)}_{xx} \quad \text{in} \quad L^p_{\text{loc}}([0, \infty) \times \mathbb{R}) \quad 1 \leq p < \infty \quad \text{weakly}, \]

\[ s^\epsilon, s^{\epsilon_{xx}} \rightharpoonup s^{(0)}_t, s^{(0)}_{xx} \quad \text{in} \quad L^\infty_{\text{loc}}([0, \infty); L^2(\mathbb{R})) \quad \text{weakly}. \]

In addition,

\[ s^\epsilon \rightharpoonup s^{(0)} \quad \text{in} \quad L^p_{\text{loc}}([0, \infty) \times \mathbb{R}) \quad 1 \leq p < \infty, \]

\[ s^\epsilon \rightharpoonup s^{(0)} \quad \text{in} \quad L^\infty_{\text{loc}}([0, \infty); C^{1, \alpha}(\mathbb{R})) \quad 0 < \alpha < \frac{1}{2}. \]
Proof. Mass conservation and uniform boundedness of the $L^2$-norm of $u^\epsilon$ confirm the weak convergence of $u^\epsilon$ to $u^{(0)}$. Moreover, the estimate (6.13) immediately implies the third statement. In addition, due to the potential estimate, one can see that $s_x \in L^1(\mathbb{R})$, again independently of $\epsilon$, and therefore $s_x \in L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$, uniformly in $\epsilon$. Here we used (6.14) and an interpolation argument. The last two assertions can be achieved by compactness results due to standard embedding arguments. This completes the proof.

Example:

We consider the following class of turning rates which is similar to those suggested in [2] and [3].

\begin{equation}
\mu_{\pm}^\epsilon = \phi(s(x,t), s(x \pm \epsilon \gamma, t), s(x \mp \epsilon \gamma, t), s_1(x,t), s_2(x,t)) \in L^2(\mathbb{R}) \cup L^p(\mathbb{R})
\end{equation}

Note, that $\phi$ is an even function with respect to the variable $s_x$. Additionally $\phi$ is strictly positive, decreasing in the second and increasing in the third argument, and we assume the structure condition (6.6) for $\mu_{\pm}^\epsilon$. Biological experiments for positive chemotaxis reveal that individuals moving up gradients of the chemical signal do turn less often than individuals moving down gradients. This fits exactly to our growth conditions assumptions on the growth of $\phi$ with respect to its second and third argument. We can easily see that the turning rates $\mu_{\pm}^\epsilon$ have the asymptotic expansion

\begin{equation}
\mu_{\pm}^\epsilon = \mu_{(0)}^{(0)} \pm \epsilon \mu_{(1)}^{(0)} \pm \epsilon^2 \mu_{(2)}^{(0)} + O(\epsilon^3)
\end{equation}

where

\begin{align*}
\mu_{(0)}^{(0)} &= \phi(s, s, s, s, t, s_x, s_{xx}) \\
\mu_{(1)}^{(0)} &= \pm \partial_2 \phi(s, s, s, s, t, s_x, s_{xx}) \gamma s_x \mp \partial_3 \phi(s, s, s, s, t, s_x, s_{xx}) \gamma s_x, \\
\mu_{(2)}^{(0)} &= \frac{1}{2} (\partial_2 \partial_2 \phi(s, s, s, s, t, s_x, s_{xx}) \pm 2 \partial_2 \partial_3 \phi(s, s, s, s, t, s_x, s_{xx}) \mp \partial_3 \partial_3 \phi(s, s, s, s, t, s_x, s_{xx})) \gamma^2 s_{xx}.
\end{align*}

Here $\partial_2$ and $\partial_3$ indicate differentiation with respect to the second and third argument. As before, we set $\mu_0 = \mu_{(0)}^{(0)}$ and $\mu_1 = \mu_{(1)}^{(0)}$. One can easily see that the turning rates (6.15) satisfy Assumption 1. Substituting the expansions $u_{\pm}^\epsilon = u^{(0)} \pm \epsilon u^{(1)} \pm \epsilon^2 u^{(2)} + O(\epsilon^3)$ and $s_x = s^{(0)} + \epsilon s^{(1)} + \epsilon^2 s^{(2)} + O(\epsilon^3)$ into (6.1) and (6.2) and comparing coefficients of $\epsilon^{-1}$ and $\epsilon^{-2}$, we have $u^{(0)} = u^{(0)}$ and obtain as before

\begin{align*}
0 &= u_t + \gamma (u_{1}^\epsilon - u_1^\epsilon)_x = u_t + \gamma (\frac{-\mu_1}{\mu_0} u - \frac{\gamma}{2\mu_0} u_{xx})_x = u_t - (D u_x)_x + (\chi s_x u)_x,
\end{align*}

where the diffusion coefficient $D$ and the chemotactic sensitivity $\chi$ are

\begin{align*}
D &= \frac{\gamma^2}{2 \phi(s, s, s, s, t, s_x, s_{xx})}, \\
\chi &= -2 [\partial_2 \phi(s, s, s, s, t, s_x, s_{xx}) - \partial_3 \phi(s, s, s, s, t, s_x, s_{xx})] D.
\end{align*}

So $H = \chi s_x$ in this case. In particular, if we take as a specific $\phi$

\begin{align*}
\phi(s(x,t), s(x \pm \epsilon \gamma, t), s(x \mp \epsilon \gamma, t), s_1(x,t), s_2(x,t), s_{xx}(x,t)) &= \varphi(s(x \pm \epsilon \gamma, t) - s(x \mp \epsilon \gamma, t))
\end{align*}

where

\begin{equation}
\varphi(x) = -C_1 \frac{x}{\sqrt{1 + x^2}} + C_2, \quad C_2 > C_1 > 0,
\end{equation}

then we obtain both constant diffusion coefficient $D = \gamma^2 / 2 C_2$ and chemotactic sensitivity $\chi = 4 C_1 D$, which is the classical version of the Keller-Segel model in one space dimension.
References


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