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**The numerical solution of the Plateau
problem for minimal surfaces of higher
topological type.**

Part I: Annulus type surfaces

by

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1 Introduction

The classical Plateau problem consists in finding a minimal surface spanning a given closed Jordan curve in Euclidean space, or more generally, a finite collection of disjoint such curves.

The first existence proofs were found independently by J. Douglas and T. Radó, who obtained minimal surfaces of the topological type of the disk. In order to formulate their result, we let γ be a closed rectifiable Jordan curve in R^3 and $D := \{w = (u, v) \in R^2, |w| < 1\}$ be the open unit disk in R^2 . The desired minimal surface then is represented as a map $F \in C^0(\overline{D}, R^3) \cap C^2(D, R^3)$ satisfying:

i) F maps ∂D bijectively onto γ ;

ii) F is harmonic in D , i.e.

$$\Delta F(w) = 0 \quad \text{for all } w \in D \quad ,$$

where Δ is the usual Euclidean Laplace operator;

iii) F is conformal, i.e.

$$\langle F_u, F_u \rangle = \langle F_v, F_v \rangle \quad \text{and} \quad \langle F_u, F_v \rangle = 0 \quad \text{in } D$$

where a subscript denotes a partial derivative, and $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in R^3 .

Condition ii) means of course that F is a critical point of the Dirichlet functional

$$E(G) := \frac{1}{2} \int_D \langle \nabla G, \nabla G \rangle \, dudv$$

where $G \in C^0(\overline{D}, R^3) \cap C^2(D, R^3)$, and $\nabla = (\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the Euclidean gradient.

Condition iii) implies that $E(F)$ equals the area of $F(D)$ equipped with the metric induced by the Euclidean metric of R^3 .

One knows that such a map F is an immersion with the possible exception of finitely many branch points, and at all immersion points, the mean curvature of $F(D)$ vanishes. Therefore, $F(D)$ is a minimal surface in the differential geometric sense.

Precise references and a thorough discussion of all theoretical aspects can be found in the beautiful monograph of Dierkes-Hildebrandt-Küster-Wohlrab [DHKW]. While the Dirichlet problem for minimal surfaces has been studied numerically by many authors, the more difficult Plateau problem was first investigated numerically by Jarausch [Ja] and Wohlrab [Wo]. The most comprehensive numerical results were obtained by Dziuk - Hutchinson [DH], who used the theoretical analysis of Struwe [St] and were able to perform a numerical version of the Morse theory for minimal surfaces with given boundary γ , i.e. to find all stable and unstable minimal surfaces of disk type bounded by γ .

These authors all construct minimal surfaces of disk type of the form described above. Often, the minimal surface produced in the above setting minimizes area only among surfaces of disk type with boundary γ , but there may exist other surfaces of higher topological type with smaller area. The following picture describes two curves where the area minimizers are not of disk type, but have genus $g = 1$. These area minimizers can also be found experimentally by Plateau's method, namely by dipping a wire in the shape of one of the curves in the picture into some soap fluid and representing the minimal surface by the resulting soap film.



Therefore, we wish to construct minimal surfaces of higher topological type numerically. One way to proceed would be to minimize area over surfaces of varying topological type. This has been done by Pinkall-Polthier [PP], using some kind of discretized mean curvature flow with infinite time step. In their method, in contrast to the numerical studies mentioned above, the surfaces are not parameterized over some fixed reference domain. This, of course, contributes to the flexibility of their method. On the other hand, however, if one wishes to extend the work of Dziuk - Hutchinson to higher topological type, it is necessary to study parameterized minimal surfaces and to keep track of their conformal structure. The conformal structure is an additional parameter that is trivial for disk type surfaces, but has to be taken into account in more general cases. For the general theory, we refer to Jost-Struwe [JS].

The present paper represents a first step in this direction, namely, we construct minimal surfaces of annulus type. We describe the general set-up for a numerical algorithm for producing higher genus minimal surfaces, and we treat the details for the case of annulus type minimal surfaces bounded by two disjoint closed rectifiable Jordan curves in R^3 . The details of the construction of higher genus minimal surfaces will be studied in a subsequent paper.

As already indicated, the main new difficulty in our work is the numerical investigation of the variation of the conformal structure of the underlying reference domain.

Our paper is organized as follows. In section 2, we discuss some theoretical background, in particular the so-called Douglas condition, which is a sufficient condition for the existence of minimal surface of high genus. In section 3, we introduce the set-up of our numerical scheme. The two main steps of the scheme, named "harmonic step" and "conformal step", are treated in section 4 and 5, resp. In section 6 we present some numerical simulation results.

The alternation of a harmonic and a conformal step can already be found in Wohlrab [Wo], based on the existence method of Courant. In Wohlrab's situation, however, the conformal step becomes much easier because there one only needs to vary the boundary parametrization, whereas here the conformal type of the domain becomes an additional variable.

The convergence of our method will be discussed elsewhere. The issue of convergence becomes somewhat subtle in case where the Douglas condition fails, and one needs to compactify the moduli of spaces of annulus by pair of disks.

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2 The Douglas condition

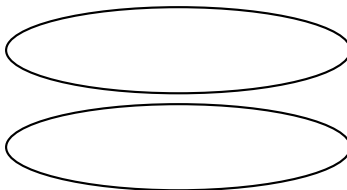
Let $\gamma := (\gamma_1, \dots, \gamma_k)$ be a configuration of disjoint, oriented, closed, rectifiable Jordan curves in R^3 . We let Σ_{m,g_1,\dots,g_m} be a Riemann surface with boundary, consisting of m connected components of genera g_1, \dots, g_m , for which $\partial\Sigma_{m,g_1,\dots,g_m}$ consists of k disjoint closed curves, and for which $\Sigma_{m,g_1,\dots,g_m} \cup \partial\Sigma_{m,g_1,\dots,g_m}$ is compact.

Definition: A minimal surface of topological type m, g_1, \dots, g_m with boundary γ is represented by a map $F \in C^0(\overline{\Sigma}_{m,g_1,\dots,g_m}, R^3) \cap C^2(\Sigma_{m,g_1,\dots,g_m}, R^3)$ for some suitable Riemann surface Σ_{m,g_1,\dots,g_m} as above satisfying:

- i) F maps $\partial\Sigma_{m,g_1,\dots,g_m}$ bijectively onto γ , preserving the orientation.
- ii) F is harmonic in Σ_{m,g_1,\dots,g_m} .
- iii) F is conformal in Σ_{m,g_1,\dots,g_m} .

In order to obtain the existence of such minimal surfaces of a prescribed topological type, we need to impose the so-called Douglas condition on γ . Before giving the formal statement, it might be useful to consider the following simple example.

Let γ consist of two central parallel circles with the distance d and the same radius of r .



Of course, γ bounds a disconnected minimal surface, namely, the one consisting of the two (flat) minimal disks bounded by the two circles, has the area of $2\pi r^2$. If the distance between the two circles is small compared to the radius, however, a cylindrical surface bounded by γ has the area of $\pi r d$, which is smaller than the area of two disks, such that, the Douglas condition is satisfied. In fact, one may find a minimal surface of annulus type in this situation, namely a so-called catenoid that looks like a cylinder that is bent inward in the middle. If we now increase the distance d between the two circles, this neck in the middle of the catenoid will become thinner until it breaks at a certain critical distance, and the catenoid disappears. In fact, beyond this critical distance, the boundary configuration does not bound

any annulus type minimal surface anymore.

This phenomenon suggests the **Douglas Condition**, which is defined as the following.

$$\inf\{E(f)|f : \Sigma_{m,g_1,\dots,g_m} \rightarrow R^3 \text{ mapping } \partial\Sigma_{m,g_1,\dots,g_m} \text{ bijectively onto } \gamma, \\ \text{and preserving the orientation.} \\ \Sigma_{m,g_1,\dots,g_m} \text{ a Riemann surface of topological type } (m, g_1, \dots, g_m)\}$$

<

$$\inf\{E(g)|f : \Sigma'_{n,g'_1,\dots,g'_n} \rightarrow R^3 \text{ mapping } \partial\Sigma'_{n,g'_1,\dots,g'_n} \text{ bijectively onto } \gamma, \\ \text{again with the correct orientation.} \\ \Sigma'_{n,g'_1,\dots,g'_n} \text{ a Riemann surface of topological type } (n, g'_1, \dots, g'_n) \\ \text{with } \sum_{i=1}^n (g'_i - 1) < \sum_{i=1}^m (g_i - 1)\}$$

The Douglas condition thus is the requirement that the minimum of Dirichlet's functional over surfaces of the given topological type (m, g_1, \dots, g_m) is strictly smaller than the minimum over surfaces of smaller topological type.

One then has the following

Theorem: *If γ satisfies the Douglas condition for the topological type (m, g_1, \dots, g_m) , it bounds a minimal surface of topological type (m, g_1, \dots, g_m) .*

Results of this type first occurred in the work of Douglas and Courant. For a modern treatment with a complete proof see Jost [Jo1, Jo2] or Tomi-Tromba [TT].

Let us note that in general γ as in the theorem may bound more than one minimal surface of type (m, g_1, \dots, g_m) .

3 Set up

We discuss the construction here only for annulus type minimal surfaces, but the formulae will be stated in such a way as to be directly generalizable to the higher genus case studied in Part II.

Let $\gamma := (\gamma_1, \gamma_2)$ be a configuration of two closed, disjoint, rectifiable Jordan curves in R^3 . For $r \in R, 0 < r < 1$ we consider the annulus

$$A_r := \{w = (u, v) \in R^2, \quad r < u^2 + v^2 < 1\}$$

The following definition of minimal surface is a special case of the definition presented in **2**:

Definition: *A minimal surface of annulus type with boundary γ is represented by a map $F \in C^0(\overline{A_r}, R^3) \cap C^2(A_r, R^3)$ for some $0 < r < 1$ satisfying:*

- i) F maps ∂A_r bijectively onto γ , preserving the orientation.*
- ii) F is harmonic in A_r .*
- iii) F is conformal in A_r .*

We represent our annulus in an equivalent way by a rectangle

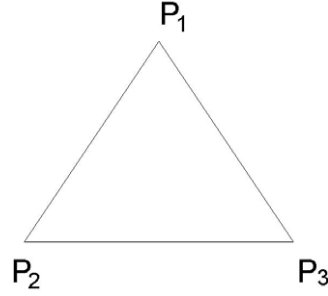
$$R_H := \{0 < u < 1, 0 < v < (m-1)H, (u, v) \in R^2\}$$

with the vertical boundary curves $u = 0$ and $u = 1$ identified. $H \in R^+$ is the parameter describing the conformal type. Because of the conformal invariance of the Dirichlet integral E , we may parameterize our minimal surfaces on R_H instead of A_r , if H is so chosen that R_H and A_r are conformally equivalent.

We approximate γ by a polygon $\gamma_{2n} = (\gamma_n^1, \gamma_n^2)$ and triangulate the domain R_H with $n \cdot m$ vertices and $2n(m-1)$ triangles. We shall use piecewise linear finite elements on R_H . This finite element space is spanned by functions ψ_i satisfying $\psi_i(w_j) = \delta_{ij}$ (w_i a vertex). A general reference is the book of Braess [Br]. In order to fix our notation, let Δ_{123} be a reference triangle with vertices denoted by P_1, P_2, P_3 or simply 1, 2, 3. Writing $P_j = (u_j, v_j)$,

$$\psi_i(w) = a^i u + b^i v + c^i$$

we obtain the coefficients a^i, b^i, c^i through the condition $\psi_i(P_j) = \delta_{ij}$ (9 linear equations and 9 variables)



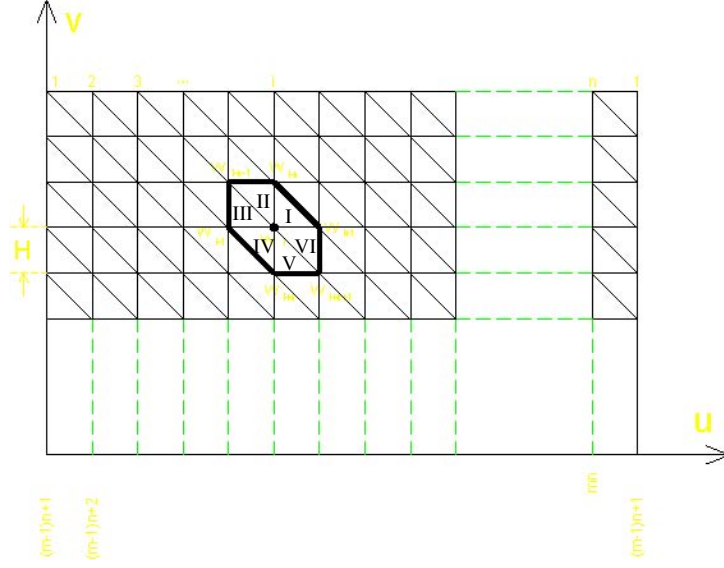
as the following formulas show:

$$\begin{aligned} a^1 &= \frac{v_2 - v_3}{2|\Delta|}; & b^1 &= -\frac{u_2 - u_3}{2|\Delta|}; & c^1 &= \frac{u_2 v_3 - u_3 v_2}{2|\Delta|}; \\ a^2 &= \frac{v_3 - v_1}{2|\Delta|}; & b^2 &= -\frac{u_3 - u_1}{2|\Delta|}; & c^2 &= \frac{u_3 v_1 - u_1 v_3}{2|\Delta|}; \\ a^3 &= \frac{v_1 - v_2}{2|\Delta|}; & b^3 &= -\frac{u_1 - u_2}{2|\Delta|}; & c^3 &= \frac{u_1 v_2 - u_2 v_1}{2|\Delta|}; \end{aligned}$$

where

$$2|\Delta| := \pm \det \begin{pmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{pmatrix}$$

The next figure shows the triangulation of our rectangle R_H consisting of m rows and n columns.



We can see that every vertex has at most six neighbors. This is not always the situation for higher genus. Let us point out that the labelling of the triangles will be important for the efficient organizations of our numerical schemes.

We take any interior point $(u_i, v_i) = w_i$ and try to calculate $a_{ij} := \int_{R_H} \left(\frac{\partial \psi_j}{\partial u} \frac{\partial \psi_i}{\partial u} + \frac{\partial \psi_j}{\partial v} \frac{\partial \psi_i}{\partial v} \right)$. Every such w_i has 6 neighboring vertices and 6 neighboring triangles. Because of the condition $\psi_i(w_j) = \delta_{ij}$ there are only 7 terms of a_{ij} which are not zero for fixed i . We use the reference triangle to determine the derivatives for the finite elements on the 6 triangles. On the reference triangle we fix the index of w_i always as P_3 , or simply 3, we also know $v_{n+k} - v_j = H$ for $j = \pm(k-1, k, k+1)$. from this information, one obtains the following table for the hexagon.

	I	II	III	IV	V	VI
area	$H u_i - u_{i-1} $	$\frac{1}{2}H u_{i-n} - u_{i-n-1} $	$\frac{1}{2}H u_{i+1} - u_i $	$\frac{1}{2}H u_{i+1} - u_i $	$\frac{1}{2}H u_{n+i+1} - u_{n+i} $	$\frac{1}{2}H u_i - u_{i-1} $
a^1	$\frac{1}{u_i - u_{i-1}}$	$\frac{1}{u_i - u_{i-1}}$	0	$-\frac{1}{u_{i+1} - u_i}$	$-\frac{1}{u_{n+i+1} - u_{n+i}}$	0
a^2	0	$-\frac{1}{u_{i-n} - u_{i-n-1}}$	$-\frac{1}{u_{i+1} - u_i}$	0	$\frac{1}{u_{n+i+1} - u_{n+i}}$	$\frac{1}{u_i - u_{i-1}}$
a^3	$-\frac{1}{u_i - u_{i-1}}$	0	$\frac{1}{u_{i+1} - u_i}$	$\frac{1}{u_{i+1} - u_i}$	0	$-\frac{1}{u_i - u_{i-1}}$
b^1	$-\frac{1}{H} \frac{u_{i-n-1} - u_i}{u_i - u_{i-1}}$	$-\frac{1}{H} \frac{u_{i-n} - u_i}{u_{i-n} - u_{i-n-1}}$	$-\frac{1}{H}$	$-\frac{1}{H} \frac{u_{n+i+1} - u_i}{u_{i+1} - u_i}$	$-\frac{1}{H} \frac{u_{n+i} - u_i}{u_{n+i+1} - u_{n+i}}$	$\frac{1}{H}$
b^2	$-\frac{1}{H}$	$-\frac{1}{H} \frac{u_i - u_{i-n-1}}{u_{i-n} - u_{i-n-1}}$	$-\frac{1}{H} \frac{u_i - u_{i-n}}{u_{i+1} - u_i}$	$\frac{1}{H}$	$-\frac{1}{H} \frac{u_i - u_{n+i+1}}{u_{n+i+1} - u_{n+i}}$	$-\frac{1}{H} \frac{u_i - u_{i+n}}{u_i - u_{i-1}}$
b^3	$-\frac{1}{H} \frac{u_{i-1} - u_{i-n-1}}{u_i - u_{i-1}}$	$+\frac{1}{H}$	$-\frac{1}{H} \frac{u_{i-n} - u_{i+1}}{u_{i+1} - u_i}$	$-\frac{1}{H} \frac{u_{i+1} - u_{i+n+1}}{u_{i+1} - u_i}$	$-\frac{1}{H}$	$-\frac{1}{H} \frac{u_{n+i} - u_{i-1}}{u_i - u_{i-1}}$
c^1						
c^2						
c^3						

It is then straightforward to calculate the coefficients

$$a_{ij} := \int_{R_H} \langle \nabla \psi_i, \nabla \psi_j \rangle dudv$$

for $j = i - n - 1, i - n, i - 1, i, i + 1, n + i, n + i + 1$, for all indices mod n , of the structure matrix A . The explicit form of the dependence of the a_{ij} on the u_k and on H will be crucial in the sequel.

$$a_{i, i-n-1} = \frac{1}{2H} (u_i - u_{i-n} + u_{i-1} - u_{i-n-1})$$

$$\begin{aligned}
a_{i,i-n} &= \frac{1}{2H}(u_{i-n-1} - u_i + u_{i-n} - u_{i+1}) \\
a_{i,i-1} &= -\frac{H}{u_i - u_{i-1}} + \frac{1}{2H} \frac{1}{u_i - u_{i-1}} [(u_{i-n-1} - u_i)(u_{i-1} - u_{i-n-1}) + (u_{i+n} - u_i)(u_{i-1} - u_{i+n})] \\
a_{i,i} &= \frac{H}{u_{i+1} - u_i} + \frac{H}{u_i - u_{i-1}} \\
&\quad + \frac{1}{2H}(u_{i-n} - u_{i-n-1} + u_{i+n+1} - u_{i+n}) \\
&\quad + \frac{1}{2H} \frac{1}{u_{i+1} - u_i} [(u_{i-n} - u_{i+1})^2 + (u_{i+1} - u_{i+n+1})^2] \\
&\quad + \frac{1}{2H} \frac{1}{u_i - u_{i-1}} [(u_{i+n} - u_{i-1})^2 + (u_{i-1} - u_{i-n-1})^2] \\
a_{i,i+1} &= -\frac{H}{u_{i+1} - u_i} + \frac{1}{2H} \frac{1}{u_{i+1} - u_i} [(u_{i-n} - u_i)(u_{i+1} - u_{i-n}) + (u_{i+n+1} - u_i)(u_{i+1} - u_{i+n+1})] \\
a_{i,i+n} &= \frac{1}{2H}(u_{i-1} - u_{i+n} + u_i - u_{i+n+1}) \\
a_{i,i+n+1} &= \frac{1}{2H}(-u_{i+1} - u_{i+n+1} + u_{i+n} - u_i)
\end{aligned}$$

If w_i is a boundary point, that means $i = 1, 2, \dots, n, (m-1)n+1, \dots, mn$, we have for $i = 1, 2, \dots, n$:

$$\begin{aligned}
a_{i,i-1} &= \frac{1}{2} \frac{H}{u_i - u_{i-1}} + \frac{1}{2H} \frac{1}{u_i - u_{i-1}} (u_{i+n} - u_i)(u_{i-1} - u_{i+n}) \\
a_{i,i} &= \frac{1}{2} \left[\frac{H}{u_{i+1} - u_i} + \frac{H}{u_i - u_{i-1}} \right] \\
&\quad + \frac{1}{2H}(u_{i+n+1} - u_{i+n}) \\
&\quad + \frac{1}{2H} \frac{1}{u_{i+1} - u_i} (u_{i+1} - u_{i+n+1})^2 \\
&\quad + \frac{1}{2H} \frac{1}{u_i - u_{i-1}} (u_{i+n} - u_{i-1})^2 \\
a_{i,i+1} &= \frac{1}{2} \frac{H}{u_{i+1} - u_i} + \frac{1}{2H} \frac{1}{u_{i+1} - u_i} (u_{i+n+1} - u_i)(u_{i+1} - u_{i+n+1}) \\
a_{i,i+n} &= \frac{1}{2H}(u_{i-1} - u_{i+n} + u_i - u_{i+n+1}) \\
a_{i,i+n+1} &= \frac{1}{2H}(-u_{i+1} - u_{i+n+1} + u_{i+n} - u_i)
\end{aligned}$$

and for $i = (m-1)n+1, \dots, mn$,

$$\begin{aligned}
a_{i,i-n-1} &= \frac{1}{2H}(u_i - u_{i-n} + u_{i-1} - u_{i-n-1}) \\
a_{i,i-n} &= \frac{1}{2H}(u_{i-n-1} - u_i + u_{i-n} - u_{i+1}) \\
a_{i,i-1} &= \frac{1}{2} \frac{H}{u_i - u_{i-1}} + \frac{1}{2H} \frac{1}{u_i - u_{i-1}} (u_{i-n-1} - u_i)(u_{i-1} - u_{i-n-1})
\end{aligned}$$

$$\begin{aligned}
a_{i,i} &= \frac{1}{2} \frac{H}{u_{i+1} - u_i} + \frac{H}{u_i - u_{i-1}} \\
&\quad + \frac{1}{2H} (u_{i-n} - u_{i-n-1}) \\
&\quad + \frac{1}{2H} \frac{1}{u_{i+1} - u_i} (u_{i-n} - u_{i+1})^2 \\
&\quad + \frac{1}{2H} \frac{1}{u_i - u_{i-1}} (u_{i-1} - u_{i-n-1})^2 \\
a_{i,i+1} &= \frac{1}{2} \frac{H}{u_{i+1} - u_i} + \frac{1}{2H} \frac{1}{u_{i+1} - u_i} (u_{i-n} - u_i)(u_{i+1} - u_{i-n})
\end{aligned}$$

4 Harmonic step

This is a standard step. Defining

$$V := \left\{ \sum_1^{mn} (x_i \psi_i, y_i \psi_i, z_i \psi_i), \quad x_i, y_i, z_i \in R \right\} \subset H^{1,2}(R_H)$$

and

$$V_0 := \left\{ \sum_{n+1}^{(m-1)n} (x_i \psi_i, y_i \psi_i, z_i \psi_i), \quad x_i, y_i, z_i \in R \right\} \subset H_0^{1,2}(R_H)$$

which are approximation spaces for $H^{1,2}$ and $H_0^{1,2}$, we want to find approximating solutions in V . In the present step, we keep the boundary values fixed. Thus, the values of (x_i, y_i, z_i) are fixed for $i = 1, \dots, n$ and for $i = (m-1)n + 1, \dots, mn$. A solution

$$f = \sum_1^{mn} (x_i \psi_i, y_i \psi_i, z_i \psi_i)$$

has to satisfy the linear system

$$\begin{aligned}
Ax &= b^1, \\
Ay &= b^2, \\
Az &= b^3,
\end{aligned}$$

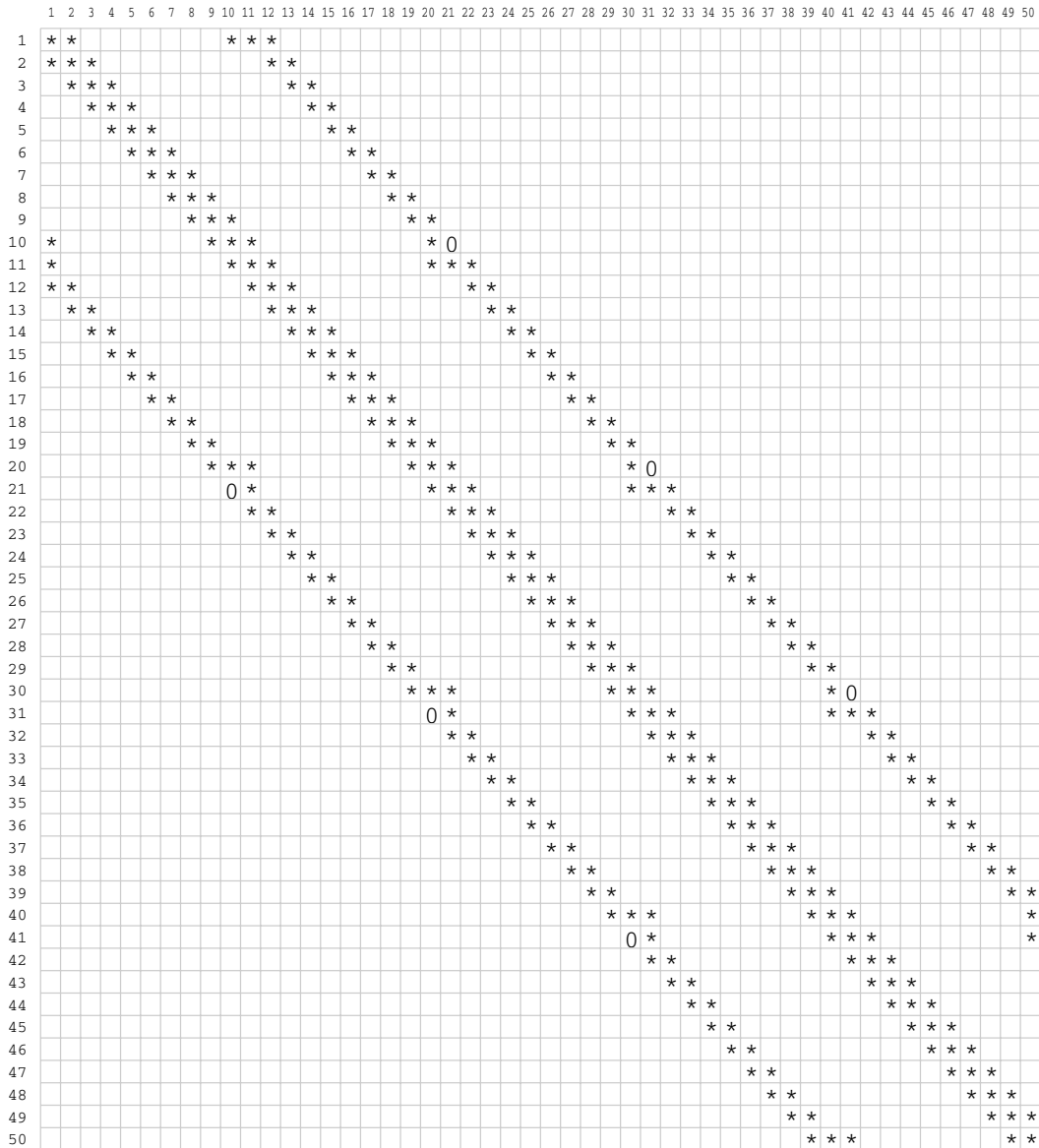
where

$$\begin{aligned}
A &= \begin{pmatrix} a_{n+1,1} & \cdots & a_{n+1,n} \\ a_{n+2,1} & \cdots & a_{n+2,n} \\ \vdots & \ddots & \vdots \\ a_{(m-1)n,1} & \cdots & a_{(m-1)n,n} \end{pmatrix}, \\
x &= \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{(m-1)n} \end{pmatrix}, \quad y = \begin{pmatrix} y_{n+1} \\ \vdots \\ y_{(m-1)n} \end{pmatrix}, \quad z = \begin{pmatrix} z_{n+1} \\ \vdots \\ z_{(m-1)n} \end{pmatrix}, \\
b^1 &= \begin{pmatrix} \sum_{i=1}^n a_{n+1,i} x_i + \sum_{i=(m-1)n+1}^m n a_{n+1,i} x_i \\ \vdots \\ \sum_{i=1}^n a_{(m-1)n,i} x_i + \sum_{i=(m-1)n+1}^m n a_{(m-1)n,i} x_i \end{pmatrix},
\end{aligned}$$

$$b^2 = \begin{pmatrix} \sum_{i=1}^n a_{n+1,i} y_i + \sum_{i=(m-1)n+1}^m n a_{n+1,i} y_i \\ \vdots \\ \sum_{i=1}^n a_{(m-1)n,i} y_i + \sum_{i=(m-1)n+1}^m n a_{(m-1)n,i} y_i \end{pmatrix},$$

$$b^3 = \begin{pmatrix} \sum_{i=1}^n a_{n+1,i} z_i + \sum_{i=(m-1)n+1}^m n a_{n+1,i} z_i \\ \vdots \\ \sum_{i=1}^n a_{(m-1)n,i} z_i + \sum_{i=(m-1)n+1}^m n a_{(m-1)n,i} z_i \end{pmatrix}.$$

Here, according to **3** and the definition of E , A is symmetric, positive definite, and is a sparse matrix with a special band structure as shown in the following. For the numerical solution of this system, we have used a conjugate gradient method.



5 Conformal step

The map f produced in the preceding step satisfied the discretized versions of conditions i) and ii) of the definition of an annulus type minimal surface with boundary γ , but in general not condition iii). In order to achieve condition iii), we need to vary both the boundary parametrization and the parameter H . The resulting map will then satisfy i) and iii), but in general not ii). Therefore, one has to apply the harmonic and the conformal step alternately, until either the process stabilizes and produces an approximation of a minimal surface, or it degenerates in the sense that the conformal parameter H tends to infinity. The latter cannot happen if the Douglas condition is satisfied. Since, however, one may not know a priori whether the Douglas condition is satisfied, the numerical scheme can also be used to test its validity. (Note, however, that there may exist a locally minimizing annulus type surface even if the Douglas condition is not fulfilled, and for suitable starting values, such a metastable solution might also be produced by our method.)

The conformal step will consist of two substeps that have to be applied alternatively until the solution stabilizes. The first substep changes the boundary values whereas the second one varies the conformal type of the domain.

1st substep:

Fixing the boundary values of f means fixing the boundary points of R_H , i.e. $(u_i, 1), i = 1, \dots, n$, and $(u_i, 0), i = (m-1)n + 1, \dots, mn$, because the values of f at these points are determined by the discretization of γ . Namely, each such boundary point has to be mapped to some preassigned vertex of the discretization of γ . However, the position of the vertices on ∂R_H is variable, and a variation of these vertices will affect $E(f)$, because some of the a_{ij} depend on these boundary points. We therefore write

$$E(f) = \sum_{i,j=1,\dots,mn} a_{ij}(u_1, \dots, u_n)(x_i x_j + y_i y_j + z_i z_j)$$

($f = \sum_1^{mn} (x_i \psi_i, y_i \psi_i, z_i \psi_i)$), where we write u_i instead of $(u_i, 0)$. (A similar consideration has to be performed for the points $(u_j, 1), j = (m-1)n + 1, \dots, mn$).

For $\alpha = 1, \dots, n$, we put

$$\begin{aligned} P_1 &:= u_\alpha, \\ P_2 &:= u_\alpha + 0.618(u_{\alpha+1} - u_\alpha), \\ P_3 &:= \frac{1}{2}(P_1 + P_2) \quad . \end{aligned}$$

We put

$$\phi_i := \sum_{i,j=1,\dots,mn} a_{ij}(u_1, \dots, u_{\alpha-1}, P_i, u_{\alpha+1}, \dots, n)(x_i x_j + y_i y_j + z_i z_j).$$

We then find a unique parabola through the three points $(P_i, \phi_i) \in \mathbb{R}^2, i = 1, 2, 3$. We then replace u_α by the minimum point of this parabola.

Thus, in this step, the image of the map f is not changed, but only its parametrization through a variation of the boundary vertices of R_H . It is essential to note that each such boundary variation changes the whole triangulation of R_H .

2nd substep:

In this step, we keep the map f and the boundary parametrization fixed, but we vary the conformal type of the reference domain R_H (in order to make the statement that f and the boundary parametrization are kept fixed precise, we need to identify the original and the varied domain in some manner). We recall that the a_{ij} also depend on H and write now

$$E(f) = \sum_{i,j=1,\dots,mn} a_{ij}(H)(x_i x_j + y_i y_j + z_i z_j)$$

Analogously to the 1st substep, we consider the parabola defined by the values of E for three different values of H and determine the new value of H as the minimum point of this parabola.

In the higher genus case, the conformal type of the domain will depend on more than one real parameter (the dimension of the corresponding moduli space is determined from Teichmüller theory), but the principal strategy will be the same.

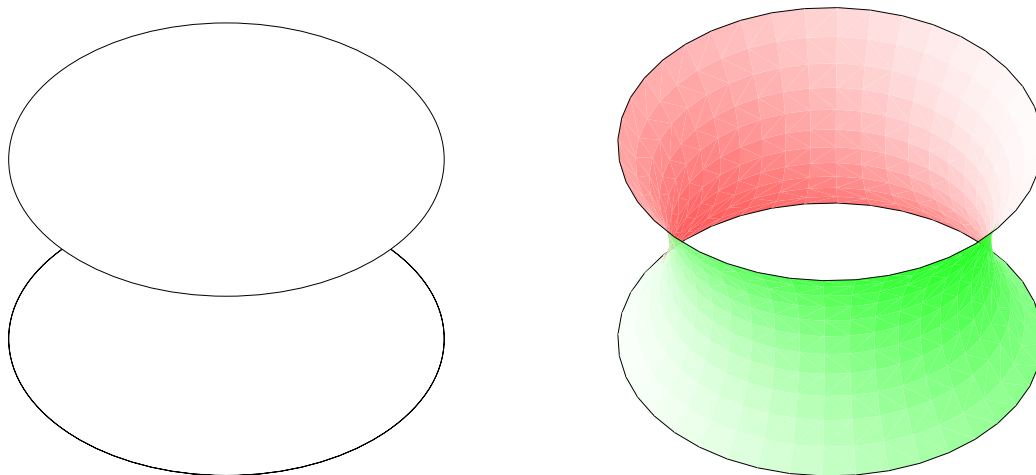
We apply the preceding two substeps alternately until the process stabilizes, or else H tends to infinity. The latter is excluded by the Douglas condition, however, the resulting map will satisfy the discrete version of the conformality condition, but it will no longer be harmonic. Therefore, we apply the harmonic and the conformal step alternately until they stabilize and produce a map f which is both harmonic and conformal in the discrete sense, hence a discrete minimal surface.

6 Some numerical simulation results

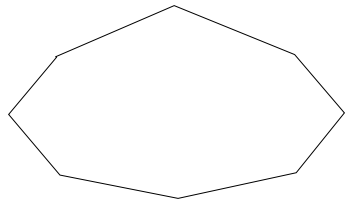
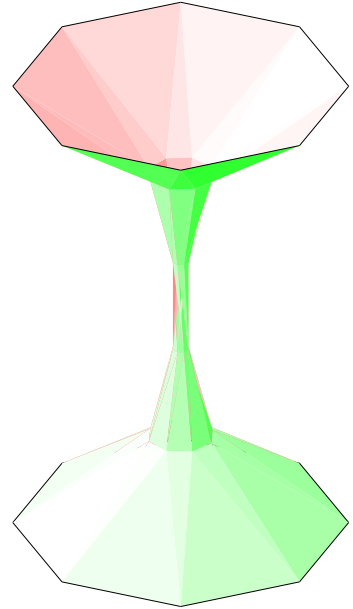
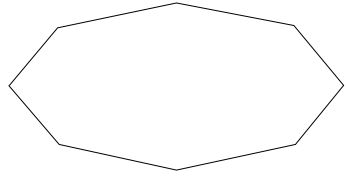
As mentioned in the beginning, we now display some annulus type minimal surfaces with prescribed boundary curves.

6.1

Two parallel circles as boundary configuration. The distance between these two circles is changed from small (a) to large (b).

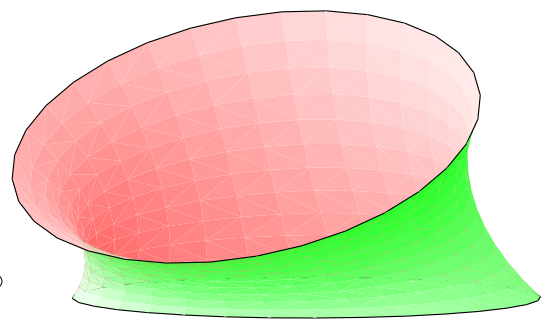
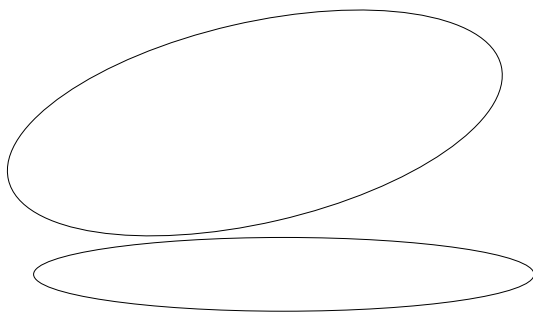


(a)



(b)

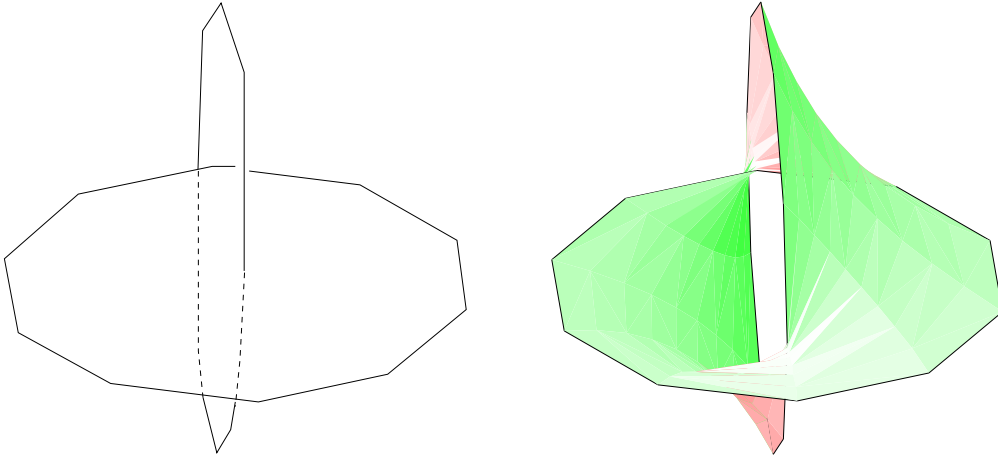
In case (c), the top circle is rotated about an axis in the plane containing the bottom circle.



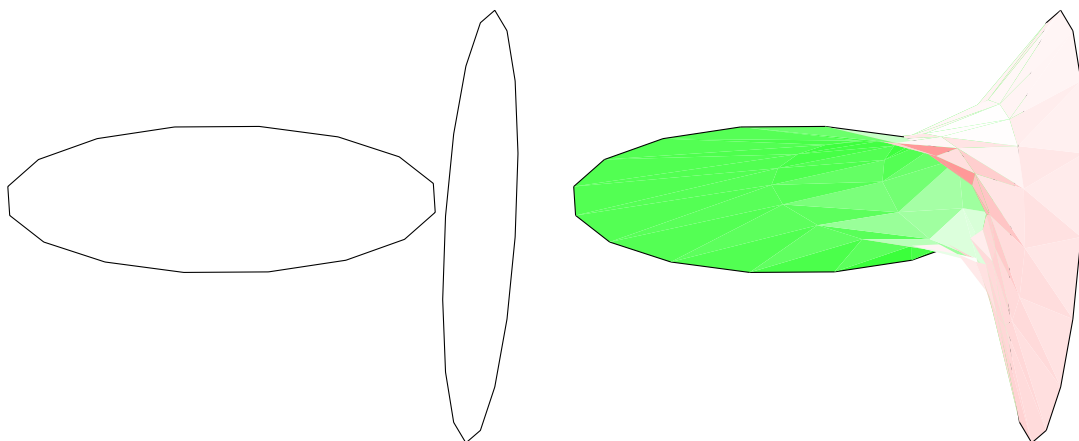
(c)

6.2

Two orthogonal linked circles as boundary.

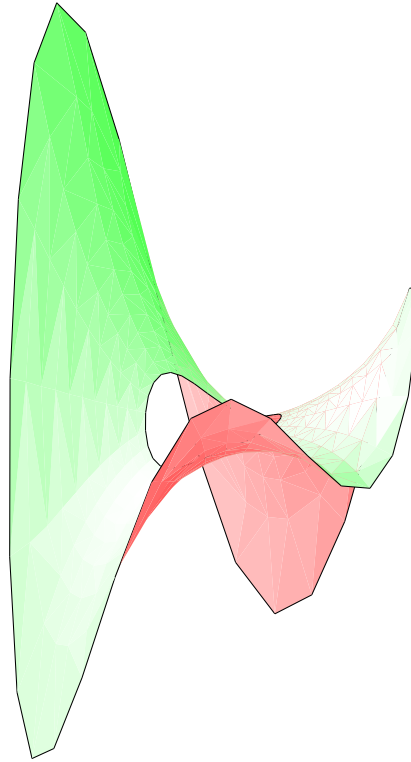


Two orthogonal circles that are not linked.



6.3

Polynomial surface $\operatorname{Re}(z^3 - i)$ with a hole.



As set up here, our method does not change the topological type of the surface. If the Douglas condition is not satisfied, the numerical scheme degenerates as seen in Fig. 6.1.b where the unique minimal surface consists of two plane disks.

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