Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Mappings of finite distortion:The degree of regularity.

by

Daniel Faraco, Pekka Koskela, and Xiao Zhong

Mappings of finite distortion: The degree of regularity

Daniel Faraco   Pekka Koskela   Xiao Zhong

Abstract

Let $K(x)$ be such that $\exp(\beta K(x)) \in L^1_{\text{loc}}$, $\beta > 0$. We show that there exists two universal constants $c_1(n), c_2(n)$ with the following property. Let f be in $W^{1,1}_{\text{loc}}$ with $|Df(x)|^n \leq K(x)J(x,f)$ and the Jacobian determinant $J(x,f)$ in $L^1 \log^{-c_1(n)\beta} L$. Then automatically $J(x,f)$ is in $L^1 \log^{c_2(n)\beta} L$. This result has interesting consequences in the theory of mappings of finite distortion and it constitutes the appropriate analog for the self-improving regularity of quasiregular mappings. For example, we obtain novel results on the size of removable singularities for mappings of finite distortion.

1   Introduction

Let us begin with the definition. We assume that $\Omega \subset \mathbf{R}^n$ is a connected open set. We say that a mapping $f : \Omega \to \mathbf{R}^n$ has finite distortion if:

(FD-1) $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^n)$.

(FD-2)   The Jacobian determinant $J(x,f)$ of f is locally integrable.

(FD-3)   There is a measurable function $K = K(x) \geq 1$, finite almost everywhere, such that f satisfies the distortion inequality

$$|Df(x)|^n \leq K(x)J(x,f) \qquad \text{a.e.}$$

Above, $|Df(x)|$ is the operator norm of the differential matrix $Df(x)$.

We arrive at the usual definition of a mapping of bounded distortion (a quasiregular mapping) when we require above that $K \in L^\infty(\Omega)$. The theory of quasiregular mappings is a central topic in modern analysis with important connections to a variety of topics such as elliptic partial differential equations, complex dynamics, differential geometry and calculus of variations, and is by now well understood, see the monographs [31] by Reshetnyak, [32] by Rickman, and [19] by Iwaniec and Martin.

A remarkable feature of quasiregular mappings is the self-improving regularity. In this case condition (FD-2) assures that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$. This is the natural regularity assumption for quasiregular mappings. In 1973, Gehring [7] showed that the differential of a quasiconformal mapping (homeomorphic quasiregular mapping) of a domain $\Omega \subset \mathbf{R}^n$ is locally integrable with a power strictly greater than n, and proved the celebrated Gehring's Lemma. This fundamental result not only extended an earlier result of Bojarski [3] but also opened up the most direct way to the analytic foundation of quasiregular mappings in $\mathbf{R}^n$. A bit later, Elcrat and Meyers [5] showed that Gehring's ideas can be further exploited to treat quasiregular mappings and partial differential systems, see also [29].

The higher integrability result admits a dual version. In two remarkable papers, Iwaniec and Martin [18] (for even dimensions) and Iwaniec [13] (for all dimensions) proved that there is an exponent $q(n, K) < n$ such that quasiregular mappings *a priori* in $W^{1,q}_{\mathrm{loc}}(\Omega)$ with $q > q(n, K)$ belong to the natural Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega)$.

The fundamentals of the theory of mappings of finite distortion have been recently established in a sequence of papers [15], [2], [17], [22], [23], [24], [16], [11], [25]. For earlier developments see e.g. [8], [21], [10], [28]. The monograph [19] contains discussions on many basic properties of these mappings as well as connections to other topics.

The recent works have established a rich theory of mappings of finite distortion under relaxed conditions on the distortion function K that do not require K to be bounded. Namely, it has been proven under the assumption of exponential integrability of K that f is continuous, sense preserving, either constant or both open and discrete, and maps sets of measure zero to sets of measure zero.

The motivation for relaxing the boundedness of the distortion function partially arises from non-linear elasticity. See the paper [30] by Müller and Spector for a nice introduction to the mappings arising in that theory.

In this paper we consider the regularity of mappings with exponentially integrable finite distortion. It was noticed in [15], [16] that if the distortion function $K(x)$ satisfies $\exp(\beta K(x)) \in L^1_{\mathrm{loc}}$ for some $\beta > 0$, the distortion inequality (FD-3) implies that the differential of our mapping of finite distortion is p-integrable for all $p < n$ and, in fact,

$$\frac{|Df|^n}{\log(e + |Df|)} \in L^1_{\mathrm{loc}}. \tag{1.1}$$

On the other hand, the Jacobian of each mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ that satisfies both (1.1) and $J(x, f) \geq 0$ a.e. is locally integrable by results of Iwaniec and Sbordone [21]. Thus the natural regularity assumption for mappings with exponentially integrable finite distortion is (1.1).

First, we consider the higher regularity of mappings with exponentially integrable finite distortion. As mentioned in [19], one can not expect quite

2

the same sort of results as we have attained for quasiregular mappings. In this setting, we must be satisfied with only a very slight degree of improved regularity. The results in [2], [15] and [16] showed that the scale of improved degree is logarithmic. More precisely, it was shown in [2] ($n = 2$), [15] (n is even), [16] (all dimensions n) that (1.1) can be improved on: for each dimension $n \geq 2$ and all $\alpha \geq 0$ there exists $\beta_\alpha(n) \geq 1$ such that if $f : \Omega \to \mathbf{R}^n$ has finite distortion and the distortion function $K(x)$ of f satisfies

$$\int_\Omega \exp(\beta K(x))dx < \infty$$

for some $\beta \geq \beta_\alpha(n)$, then $|Df(x)|^n \log^\alpha(e + |Df|) \in L^1_{\text{loc}}(\Omega)$. However the techniques developed in these papers do not seem to work in the general case (with arbitrary $\beta > 0$). Thus, this remained as a challenging and interesting open problem. As pointed out in [19], different ideas are needed to treat this situation.

Our first theorem not only shows the higher integrability of the differential of a mapping with exponentially integrable distortion for any $\beta > 0$ but it also indicates exactly how the degree of improved regularity depends on β.

Theorem 1.1. *Let f be a mapping of finite distortion in $\Omega \subset \mathbf{R}^n$, $n \geq 2$. Assume that the distortion $K(x) \geq 1$ satisfies $\exp(\beta K) \in L^1_{\text{loc}}(\Omega)$, for some $\beta > 0$. Then*

$$J(x, f) \log^\alpha(e + J(x, f)) \in L^1_{\text{loc}}(\Omega), \ \ and \ |Df|^n \log^{\alpha-1}(e + |Df|) \in L^1_{\text{loc}}(\Omega),$$

where $\alpha = c_1\beta$ and $c_1 = c_1(n) > 0$. Moreover, for any ball B such that $2B \Subset \Omega$,

$$
\begin{aligned}
&\fint_B J(x, f) \log^\alpha \left(e + \frac{J(x, f)}{J(x, f)_{2B}} \right) dx \\
&\leq c(n, \beta) \left(\fint_{2B} \exp(\beta K(x)) \, dx \right) \left(\fint_{2B} J(x, f) \, dx \right).
\end{aligned}
\tag{1.2}
$$

Here and in the following, $g_E = \fint_E g$ is the average of g over the set E and $2B$ is a ball with the same center as B and with radius twice that of B.

Theorem 1.1 is sharp in the sense that, given n, the conclusion fails for $\alpha = n\beta$ for all $\beta > 0$. This is seen by considering the mappings $f(x) = \log^s(e+1/|x|)\frac{x}{|x|}$, defined in the unit ball of $\mathbf{R}^n$, for $s < 0$. In the planar case, the conclusion of Theorem 1.1 follows from results of David [4] provided f is either a homeomorphism or is *a priori* assumed to belong to $W^{1,2}(\Omega, \mathbf{R}^2)$.

As a consequence of Theorem 1.1 we obtain the following sharp measure distortion estimate, in which $|A|$ denotes the Lebesgue measure of $F \subset \mathbf{R}^n$.

Corollary 1.2. *Let f be a mapping of finite distortion in $B(0,2) \subset \mathbf{R}^n$, $n \geq 2$. Assume that the distortion $K(x) \geq 1$ satisfies $\exp(\beta K) \in L^1(B(0,2))$, for some $\beta > 0$. Then*

$$|f(E)| \leq C \log^{-c_1 \beta}(e + \frac{1}{|E|})$$

for each measurable set $E \subset B(0,1)$, where $c_1 = c_1(n)$ and C depends only on n, β, f.

The planar version of the above corollary for homeomorphisms is due to David [4]. The sharpness of our estimate means that one can find a constant c_1 for which the estimate fails. This follows from an example for distortion estimates for generalized Hausdorff measures given in [12]; our corollary also yields a sharp higher dimensional version of the dimension estimates given in [12].

Second, we study the dual version of the higher integrability Theorem 1.1. The following theorem shows that the natural regularity assumption (1.1) can be relaxed. This is the first result in this direction for mappings with exponentially integrable finite distortion. Clearly, it is an analog of Iwaniec and Martin's [18], [13] results on very weak quasiregular mappings.

Theorem 1.3. *Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^n)$, $n \geq 2$, satisfy the distortion inequality*

$$|Df(x)|^n \leq K(x)J(x,f) \qquad a.e. \ in \ \Omega$$

where $K(x) \geq 1$ satisfies $\exp(\beta K) \in L^1_{\text{loc}}(\Omega)$, for some $\beta > 0$. There is a constant $c_2 = c_2(n)$ such that if

$$|Df|^n \log^{-\alpha-1}(e + |Df|) \in L^1_{\text{loc}}(\Omega),$$

with $\alpha = c_2 \beta$, then (1.1) holds and $J(x,f) \in L^1_{\text{loc}}(\Omega)$. In particular, f is then a mapping of finite distortion.

The mapping $f(x) = \log^s(e + 1/|x|)\frac{x}{|x|}$, defined in the unit ball of $\mathbf{R}^n$, for which $J(x,f)$ is not locally integrable when $s > 0$, shows that for each $c_2 < 1$ and any β, there are mappings for which the claim fails. Thus Theorem 1.3 only admits improvement in finding the precise value of $c_2(n)$.

In fact, Iwaniec and Martin [18], [13] not only proved the self improved regularity of quasiregular mappings but also a so-called Caccioppoli type inequality for the exponent $n - \epsilon$. From this they obtained fundamental results in the theory of removable singularities for quasiregular mappings. We also obtain a new Caccioppoli type inequality in the settings of mappings of finite distortion. This yields the following novel result on removable sets for mappings of finite distortion.

Corollary 1.4. *Let $\beta > 0$. Let $E \subset \mathbf{R}^n$ be a closed set of $L^n \log^{n-1-c_2\beta} L$– capacity zero where c_2 is the constant from Theorem 1.3 and let $f : \Omega \setminus E \to \mathbf{R}^n$ be a bounded mapping of finite distortion and assume that the distortion function K satisfies*

$$\int_{\Omega \setminus E} \exp(\beta K(x)) \, dx < \infty.$$

Then f extends to a mapping of finite distortion in Ω with the exponentially integrable distortion function $K(x)$.

It was previously only known that sets of $L^n \log^{n-1} L$–capacity zero are removable [15], [16], [26]. A weaker version of Corollary 1.4 is given in [2] in the planar case. This corollary is essentially sharp, see [2].

The proof of Theorem 1.1 is different from those in [2], [15] and [16]. Those proofs were based on the ideas from the papers [18], [13]. Our proof is inspired by the work of Gehring [7] on the reverse Hölder inequality. Of course, in our case, the differential of the mapping does not satisfy the classical reverse Hölder inequality but satisfies a reverse inequality in the setting of Orlicz spaces, see (2.3). The nice paper [14] by Iwaniec largely extended the Gehring lemma to the setting of Orlicz spaces. Unfortunately, the framework of [14] does not apply to our case. In order to deal with our situation, we need to modify the original idea of Gehring. The proof of Theorem 1.1 indicates that it is possible to extract a non-homogeneous version of Gehring's Lemma in the setting of Orlicz spaces. We believe that this more general version would be of its own interest.

Concerning the proof of Theorem 1.3, it seems that the methods in [18] and [13] can not be adapted to treat the case of unbounded distortion. Our approach follows the lines of [6] where the authors gave a short proof of the Caccioppoli type inequality mentioned above. Their technique is inspired by a paper of Lewis [27]. The basic idea is to truncate the maximal function of the gradient along the level sets to construct a Lipschitz continuous test-function. In the finite distortion setting these ideas work relatively nicely as well but a natural obstruction arises and the method of [6] needs to be modified. More precisely, it is more convenient to obtain the improved regularity directly instead of using the Caccioppoli inequality and Gehring's lemma as in [6].

Finally, we remark that it seems to be a long way to get the exact values of the constants c_1 in Theorem 1.1 and c_2 in Theorem 1.3. Even for quasiregular mappings, only the planar case is known [1].

2   Proofs of Theorem 1.1 and Corollary 1.2

The following proposition is well-known. The proof involves Vitali's lemma and the Calderón-Zygmund decomposition.

Proposition 2.1. *Let $h \in L^1(\mathbf{R}^n)$. We have for any $t > 0$ that*

$$\frac{1}{2^n t} \int_{|h|>t} |h(x)| \, dx \leq |\{x \in \mathbf{R}^n : Mh(x) > t\}| \leq \frac{2 \cdot 3^n}{t} \int_{|h|>t/2} |h(x)| \, dx.$$

Proof of Theorem 1.1. We rely on a result from [20] according to which

$$\int_\Omega \phi J(x, (f_1, f_2, ..., f_n)) \, dx = \int_\Omega f_1 J(x, (\phi, f_2, ..., f_n)) \, dx$$

whenever $\phi \in C_0^\infty(\Omega)$ and $f = (f_1, f_2, ..., f_n) \in W_{\mathrm{loc}}^{1,1}(\Omega, \mathbf{R}^n)$ is a Sobolev mapping with $J(x, f) \geq 0$ a.e. and

$$\frac{|Df|^n}{\log(e + |Df|)} \in L_{\mathrm{loc}}^1(\Omega).$$

Thus

$$\int_\Omega \phi J(x, f) \, dx \leq \int_\Omega |f||Df|^{n-1}|\nabla \phi| \, dx. \tag{2.1}$$

This is a reverse inequality, from which the higher integrability result is derived. Actually, it can be proved in the same way as in the proof of Theorem 1.3 in the next section, see (3.16). Let $\phi \in C_0^\infty(B(y, 2r))$ satisfy $\phi = 1$ in $B(y, r)$, $0 \leq \phi \leq 1$ in $\mathbf{R}^n$ and $|\nabla \phi| \leq 2/r$, where $B(y, 2r)$ is a ball lying in Ω. With this choice of ϕ, (2.1) leads to the following inequality with $q = n^2/(n+1)$:

$$\int_{B(y,r)} J(x, f) \, dx \leq \frac{2}{r} \int_{B(y,2r)} |f||Df|^{n-1} \, dx$$

$$\leq \frac{2}{r} \left(\int_{B(y,2r)} |Df|^q \right)^{\frac{n-1}{q}} \left(\int_{B(y,2r)} |f|^{n^2} \, dx \right)^{\frac{1}{n^2}}.$$

Note that this inequality remains valid if we substract from f any constant vector. In particular, it holds for $f - f_{B(y,2r)}$ replacing f. Here $f_{B(y,2r)}$ is the average of f over $B(y, 2r)$. Applying the Poincaré-Sobolev inequality yields

$$\left(\int_{B(y,2r)} |f - f_{B(y,2r)}|^{n^2} \, dx \right)^{\frac{1}{n^2}} \leq c(n) \left(\int_{B(y,2r)} |Df|^q \, dx \right)^{\frac{1}{q}}.$$

Combining these last two inequalities, we finally obtain

$$\frac{1}{|B(y,r)|} \int_{B(y,r)} J(x, f) \, dx \leq c(n) \left(\frac{1}{|B(y,2r)|} \int_{B(y,2r)} |Df|^q \, dx \right)^{\frac{n}{q}} \tag{2.3}$$

whenever $B(y, 2r) \Subset \Omega$. Here $|E|$ is the Lebesgue measure of $E \subset \mathbf{R}^n$. The above argument is standard, see Lemma 7.6.1 in [19].

6

Now we fix a ball $B_0 = B(x_0, r_0) \Subset \Omega$. Assume that

$$\int_{B_0} J(x, f)\, dx = 1. \tag{2.4}$$

This assumption involves no loss of generality for us as the distortion inequality and (1.2) are homogeneous with respect to f. Let us introduce the auxiliary functions defined in $\mathbf{R}^n$ by

$$
\begin{aligned}
h_1(x) &= d(x)^n J(x, f), \\
h_2(x) &= d(x)|Df(x)|, \\
h_3(x) &= \chi_{B_0}(x),
\end{aligned} \tag{2.5}
$$

where $d(x) = \operatorname{dist}(x, \mathbf{R}^n \setminus B_0)$ and χ_E is the characteristic function of the set E. We claim that

$$\left(\frac{1}{|B|}\int_B h_1\, dx\right)^{\frac{1}{n}} \le c(n)\left(\frac{1}{|2B|}\int_{2B} h_2^q\, dx\right)^{\frac{1}{q}} + c(n)\left(\frac{1}{|2B|}\int_{2B} h_3\, dx\right)^{\frac{1}{n}} \tag{2.6}$$

for all balls $B \subset \mathbf{R}^n$. Indeed, we may assume that B meets B_0; otherwise (2.6) is trivial. Our derivation of (2.6) falls naturally into two cases.

Case 1. We assume that $3B \subset B_0$. By an elementary geometric consideration we find that

$$\max_{x \in B} d(x) \le 4 \min_{x \in 2B} d(x).$$

Applying (2.3) yields

$$
\begin{aligned}
\left(\frac{1}{|B|}\int_B h_1\, dx\right)^{\frac{1}{n}} &\le \max_B d(x)\left(\frac{1}{|B|}\int_B J(x, f)\, dx\right)^{\frac{1}{n}} \\
&\le c(n) \min_{2B} d(x)\left(\frac{1}{|2B|}\int_{2B}|Df|^q\, dx\right)^{\frac{1}{q}} \\
&\le c(n)\left(\frac{1}{|2B|}\int_{2B} h_2^q\, dx\right)^{\frac{1}{q}}.
\end{aligned}
$$

Case 2. We assume that $3B$ is not contained in B_0 and recall that B meets B_0. We have that

$$\max_{x \in B} d(x) \le \max_{x \in 2B} d(x) \le c(n)|2B \cap B_0|^{\frac{1}{n}}.$$

Hence we conclude that

$$
\begin{aligned}
\left(\frac{1}{|B|}\int_B h_1\, dx\right)^{\frac{1}{n}} &\le \max_B d(x)\left(\frac{1}{|B|}\int_{B \cap B_0} J(x, f)\, dx\right)^{\frac{1}{n}} \\
&\le c(n)\left(\frac{|2B \cap B_0|}{|B|}\int_{B_0} J(x, f)\, dx\right)^{\frac{1}{n}} \\
&\le c(n)\left(\frac{1}{|2B|}\int_{2B} h_3\, dx\right)^{\frac{1}{n}},
\end{aligned}
$$

7

where we used (2.4). Combining these two cases proves inequality (2.6).

Since (2.6) is true for all balls $B \subset \mathbf{R}^n$, we have the following point-wise inequality for the maximal functions. For all $y \in \mathbf{R}^n$,

$$M(h_1)(y)^{\frac{1}{n}} \le c(n)M(h_2^q)(y)^{\frac{1}{q}} + c(n)M(h_3)(y)^{\frac{1}{n}},$$

from which it follows that for $\lambda > 0$

$$\begin{aligned}|\{x \in \mathbf{R}^n : M(h_1)(x) > \lambda^n\}| \le &|\{x \in \mathbf{R}^n : c(n)M(h_2^q) > \lambda^q\}| \\ &+ |\{x \in \mathbf{R}^n : c(n)M(h_3)(x) > \lambda^n\}|.\end{aligned} \quad (2.9)$$

We recall that $h_3(x) = \chi_{B_0}(x)$. So $M(h_3)(x) \le 1$ in $\mathbf{R}^n$, and then the set $\{x \in \mathbf{R}^n : c(n)M(h_3)(x) > \lambda^n\}$ is empty for $\lambda > \lambda_1 = \lambda_1(n)$. Hence

$$|\{x \in \mathbf{R}^n : M(h_1)(x) > \lambda^n\}| \le |\{x \in \mathbf{R}^n : c(n)M(h_2^q) > \lambda^q\}|$$

for all $\lambda > \lambda_1$. Now applying Proposition 2.1 yields

$$\int_{h_1 > \lambda^n} h_1 \, dx \le c(n)\lambda^{n-q} \int_{c(n)h_2 > \lambda} h_2^q \, dx \quad (2.10)$$

for all $\lambda > \lambda_1$. We may assume that the constant $c(n)$ in (2.10) is bigger than one.

Let $\alpha > 0$ be a constant, which will be chosen later and set

$$\Psi(\lambda) = \frac{n-q}{\alpha} \log^\alpha \lambda + \log^{\alpha-1} \lambda,$$

where $q = n^2/(n+1)$ as above. Notice that

$$\Phi(\lambda) := \frac{d}{d\lambda}\Psi(\lambda) = \frac{n-q}{\lambda} \log^{\alpha-1} \lambda + \frac{\alpha-1}{\lambda} \log^{\alpha-2} \lambda > 0$$

for all $\lambda > \lambda_2 = \exp((n+1)/n)$, and that

$$\lambda^{n-q}\Phi(\lambda) = \frac{d}{d\lambda}\left(\lambda^{n-q} \log^{\alpha-1} \lambda\right).$$

We multiply both sides of (2.10) by $\Phi(\lambda)$, and integrate with respect to λ over (λ_0, ∞) for $\lambda_0 = \max(\lambda_1, \lambda_2)$, and finally change the order of the integration to obtain that

$$\int_{h_1 > \lambda_0^n} h_1 \int_{\lambda_0}^{h_1^{\frac{1}{n}}} \Phi(\lambda) \, d\lambda dx \le c(n) \int_{c(n)h_2 > \lambda_0} h_2^q \int_{\lambda_0}^{c(n)h_2} \lambda^{n-q}\Phi(\lambda) \, d\lambda dx,$$

that is,

$$\int_{h_1 > \lambda_0^n} \left(\Psi(h_1^{\frac{1}{n}}) - \Psi(\lambda_0)\right) h_1 \, dx \le c(n) \int_{c(n)h_2 > \lambda_0} h_2^n \log^{\alpha-1}(c(n)h_2) \, dx,$$

8

Hence, taking into account the normalization (2.4),

$$\frac{1}{\alpha} \int_{h_1 > \lambda_0^n} h_1 \log^\alpha h_1^{\frac{1}{n}} \, dx \leq c(n) \int_{c(n)h_2 > \lambda_0} h_2^n \log^{\alpha-1}(c(n)h_2) \, dx + c(n,\alpha)|B_0|,$$
(2.11)

where $c(n) \geq 1$.

In the remaining part of the proof, we will choose a suitable constant $\alpha > 0$ such that the integral in the right hand side of (2.11) can be absorbed in the left, by using the distortion inequality. Actually, this only works if we have *a priori* $|Df| \in L^n \log^{\alpha-1} L_{\text{loc}}(\Omega)$. We cannot assume this. To overcome this, in the above argument we integrate with respect to λ over (λ_0, j) for j large, instead of over (λ_0, ∞). Then we proceed in the same way as above and get a similar inequality as (2.11). The proof will be then eventually be concluded, by letting $j \to \infty$ and using the monotone convergence theorem. For simplicity, we only write down the proof for $j = \infty$.

We need the following elementary inequality; the proof is in the appendix. Let $a, b, c(n) \geq 1$. Then

$$ab \log^{\alpha-1}(c(n)(ab)^{\frac{1}{n}}) \leq \frac{C(n)}{\beta} a \log^\alpha(a^{\frac{1}{n}}) + C(\alpha, \beta, n) \exp(\beta b). \quad (2.12)$$

We recall the definitions of h_1 and h_2 in (2.5) and notice that the distortion inequality reads as

$$h_1(x) \leq h_2^n(x) \leq h_1(x)K(x).$$

Put $h_1(x) = a$, $K(x) = b$ in the inequality (2.12). We infer from (2.11) that

$$\frac{1}{\alpha} \int_{h_1 > \lambda_0^n} h_1 \log^\alpha h_1^{\frac{1}{n}} \, dx \leq \frac{c(n)}{\beta} \int_{h_1 > \lambda_0^n} h_1 \log^\alpha h_1^{\frac{1}{n}} \, dx +$$

$$+ c(n,\alpha,\beta) \int_{B_0} \exp(\beta K(x) \, dx$$

$$+ c(n,\alpha)|B_0| \qquad\qquad (2.13)$$

$$\leq \frac{c(n)}{\beta} \int_{h_1 > \lambda_0^n} h_1 \log^\alpha h_1^{\frac{1}{n}} \, dx$$

$$+ c(n,\alpha,\beta) \int_{B_0} \exp(\beta K(x)) \, dx.$$

Now letting $\alpha = \beta/(2c(n))$, (2.13) becomes

$$\int_{h_1 > \lambda_0^n} h_1 \log^\alpha h_1^{\frac{1}{n}} \, dx \leq c(n,\beta) \int_{B_0} \exp(\beta K(x)) \, dx.$$

That is,

$$\int_{B_0} d(x)^n J(x,f) \log^\alpha(e + d(x)^n J(x,f)) \, dx = \int_{B_0} h_1 \log^\alpha(e + h_1) \, dx$$

$$\leq c(n,\beta) \int_{B_0} \exp(\beta K(x)) \, dx.$$
(2.14)

9

Noticing that in $\sigma B_0 = B(x_0, \sigma r_0)$ with $0 < \sigma < 1$ we have $d(x)^n \geq (1 - \sigma)^n r_0^n \geq c(n, \sigma)|B_0|$, and taking account of the normalization (2.4), we arrive at

$$
\fint_{\sigma B_0} J(x, f) \log^\alpha \left(e + \frac{J(x, f)}{\fint_{B_0} J(x, f)\, dx} \right) dx
$$
$$
\leq c(n, \beta, \sigma) \left(\fint_{B_0} \exp(\beta K(x))\, dx \right) \left(\fint_{B_0} J(x, f)\, dx \right), \tag{2.15}
$$

which proves (1.2), and the higher integrability of $|Df|$ follows from this, the distortion inequality, inequality (2.12), and the exponential integrability of K. This concludes the proof of Theorem 1.1.

Proof of Corollary 1.2. We refer to [23] for the fact that f maps sets of measure zero to sets of measure zero. Thus the volume of $f(E)$ can be estimated from above by $\int_E J(x, f)\, dx$. The desired volume distortion estimate thus follows from the Orlicz-Hölder inequality and the higher integrability of the Jacobian given in Theorem 1.1.

3 Proofs of Theorem 1.3 and Corollary 1.4

Central building blocks of the proof of Theorem 1.3 are the following well-known point-wise estimates for the Sobolev function, whose proofs rely on an argument due to Hedberg [9].

Lemma 3.1 (Point-wise inequalities for the Sobolev functions). *Let $u \in W^{1,p}(\mathbf{R}^n), 1 < q < \infty$, and let x and y be Lebesgue points of u such that $x \in B_0 = B(x_0, r)$. Then*

$$
|u(x) - u_{B_0}| \leq crM(|\nabla u|\chi_{2B_0}))(x_0) \tag{3.1}
$$
$$
|u(x) - u(y)| \leq c|x - y|(M(|\nabla u|)(x) + M(|\nabla u|)(y)), \tag{3.2}
$$

where $c = c(n) > 0$, χ_E is the characteristic function of the set E, v_{B_0} is the average of v over $B_0 = B(x_0, r)$ and Mh is the Hardy-Littlewood maximal function of h.

Proof of Theorem 1.3. We start along the lines of the proof in [6]. Let $\varphi \in C_0^\infty(B_0)$, $B_0 = B(x_0, r) \Subset \Omega$, and $\varphi \geq 0$. Let $u = f_1\varphi$ and extend it to be zero in $\mathbf{R}^n \setminus B_0$. Then $u \in W^{1,q}(\mathbf{R}^n)$ for all $q < n$, by the assumption on f in the theorem. Denote for $\lambda > 0$,

$$
F_\lambda = \{x \in B(x_0, r) : M(g)(x) \leq \lambda \text{ and } x \text{ is a Lebesgue point of } u\},
$$

where $g = |\varphi Df| + |f \otimes \nabla \varphi|$ in B_0 and $g = 0$ in $\mathbf{R}^n \setminus B_0$. It is easy to show that u is $c\lambda$-Lipschitz continuous on the set $F_\lambda \cup (\mathbf{R}^n \setminus B_0)$ for $c = c(n) \geq 1$.

Indeed, suppose that $x, y \in F_\lambda$. Since $|\nabla u| \le c(n)g$, then it follows from (3.2) that

$$
\begin{aligned}
|u(x) - u(y)| &\le c|x - y|(M(|\nabla u|)(x) + M(|\nabla u|)(y)) \\
&\le c|x - y|(M(g)(x) + M(g)(y)) \\
&\le c\lambda|x - y|.
\end{aligned}
\tag{3.3}
$$

If $x \in F_\lambda$ and $y \in \mathbf{R}^n \setminus B_0$, set $\rho = 2\operatorname{dist}(x, \mathbf{R}^n \setminus B(x_0, r))$. Since $|\{x \in B(x, \rho) : u(x) = 0\}| \ge |B(x, \rho) \cap (\mathbf{R}^n \setminus B_0)| \ge c(n)|B(x, \rho)|$, the Poincaré inequality yields

$$
|u_{B(x,\rho)}| \le c(n)\rho|\nabla u|_{B(x,\rho)} \le c\rho M g(x) \le c\lambda|x - y|
$$

Thus by (3.1),

$$
\begin{aligned}
|u(x) - u(y)| = |u(x)| &\le |u(x) - u_{B(x,\rho)}| + |u_{B(x,\rho)}| \\
&\le c\rho M(|\nabla u|)(x) + c\lambda|x - y| \\
&\le c\rho M g(x) + c\lambda|x - y| \le c\lambda|x - y|.
\end{aligned}
\tag{3.4}
$$

If $x, y \in \mathbf{R}^n \setminus B_0$, then the claim is clear. Since all the other cases follow by symmetry, it follows that $u|_{F_\lambda \cup (\mathbf{R}^n \setminus B_0)}$ is Lipschitz continuous with the constant $c\lambda$.

We extend $u|_{F_\lambda \cup (\mathbf{R}^n \setminus B_0)}$ to a Lipschitz continuous function u_λ in $\mathbf{R}^n$ with the same constant by the classical McShane extension theorem. Then we consider the mapping $f_\lambda = (u_\lambda, \varphi f_2, \varphi f_3, ..., \varphi f_n)$. Since $f \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ for all $q < n$ and u_λ is Lipschitz we have that

$$
\int_{B_0} J(x, f_\lambda)\, dx = 0,
$$

and hence,

$$
\int_{F_\lambda} J(x, \varphi f)\, dx \le -\int_{B_0 \setminus F_\lambda} J(x, f_\lambda)\, dx.
\tag{3.5}
$$

Now $|f_i \nabla \varphi| \le C(n)|f \otimes \nabla \varphi|$ and $|\nabla(\varphi f_i)| \le c(n)g$. Putting these estimates together with (3.5) and expressing the Jacobian as a wedge product of differential forms, we obtain that

$$
\int_{F_\lambda} \varphi^n J(x, f)\, dx \le c(n)\left(\int_{F_\lambda} |f \otimes \nabla \varphi| g^{n-1}\, dx + \lambda \int_{B_0 \setminus F_\lambda} g^{n-1}\, dx\right).
\tag{3.6}
$$

Here we digress from [6]. We claim that

$$
\int_{g \le \lambda} \varphi^n J(x, f)\, dx \le c(n) \int_{g \le 2\lambda} |f \otimes \nabla \varphi| g^{n-1}\, dx + c(n)\lambda \int_{g > \lambda} g^{n-1}\, dx.
\tag{3.7}
$$

Indeed, by Proposition 2.1,

$$\int_{B_0\setminus F_\lambda} g^{n-1}\,dx \le \int_{g>\lambda} g^{n-1}\,dx + \lambda^{n-1}|\{x\in\mathbf{R}^n : Mg(x) > \lambda\}|$$
$$\le c(n)\int_{g>\lambda/2} g^{n-1}\,dx, \tag{3.8}$$

and

$$\int_{g\le\lambda} \varphi^n J(x,f)\,dx \le \int_{Mg\le\lambda} \varphi^n J(x,f)\,dx + \lambda^n|\{x\in\mathbf{R}^n : Mg(x) > \lambda\}|$$
$$\le \int_{Mg\le\lambda} \varphi^n J(x,f)\,dx + c(n)\lambda\int_{g>\lambda/2} g^{n-1}\,dx. \tag{3.9}$$

Combining (3.6), (3.8) and (3.9), we obtain that

$$\int_{g\le\lambda} \varphi^n J(x,f)\,dx \le c(n)\int_{g\le\lambda} |f\otimes\nabla\varphi|g^{n-1}\,dx + c(n)\lambda\int_{g>\lambda/2} g^{n-1}\,dx.$$

Then (3.7) follows by replacing $\lambda/2$ by λ.

Now let $\alpha > 0$ be a constant, which will be chosen later. Note that

$$\Phi(\lambda) = \frac{1}{\lambda}\left(\log^{-(1+\alpha)}\lambda - (1+\alpha)\log^{-(2+\alpha)}\lambda\right) \ge 0$$

for $\lambda \ge e^{1+\alpha}$. We multiply both sides of (3.7) by $\Phi(\lambda)$, and integrate with respect to λ over (t,∞) for $t \ge \lambda_0 = \max(e^{1+\alpha}, e^{2\alpha})$, and finally change the order of the integration to obtain that

$$\int_{B_0} \varphi^n J(x,f)\int_{\max(g,t)}^\infty \Phi(\lambda)\,d\lambda dx$$
$$\le c(n)\int_{B_0} |f\otimes\nabla\varphi|g^{n-1}\int_{\max(g/2,t)}^\infty \Phi(\lambda)\,d\lambda dx \tag{3.10}$$
$$+ c(n)\int_{g>t} g^{n-1}\int_t^g \lambda\Phi(\lambda)\,d\lambda dx.$$

Thus

$$\frac{1}{2\alpha}\int_{g<t}\frac{\varphi^n J(x,f)}{\log^\alpha t} + \frac{1}{2\alpha}\int_{g>t}\frac{\varphi^n J(x,f)}{\log^\alpha g}\,dx$$
$$\le \int_{B_0}\varphi^n J(x,f)\left(\frac{1}{\alpha\log^\alpha\max(g,t)} - \frac{1}{\log^{1+\alpha}\max(g,t)}\right)\,dx \tag{3.11}$$
$$\le \frac{c(n)}{\alpha}\int_{B_0}\frac{|f\otimes\nabla\varphi|g^{n-1}}{\log^\alpha\max(g/2,t)}\,dx + c(n)\int_{g>t}\frac{g^n}{\log^{1+\alpha}g}\,dx.$$

Observe that the first inequality holds because $t \geq e^{2\alpha}$. We remark here that the assumption on the regularity of f in the theorem,

$$\frac{|Df|^n}{\log^{1+\alpha}(e + |Df|)} \in L^1_{\text{loc}}(\Omega),$$

implies that the integrals on the right hand side of (3.11) are finite.

Now we use the distortion inequality, which so far has not been used. The distortion inequality

$$|Df|^n \leq K(x)J(x,f)$$

and the inequality

$$ab \leq a\log(1+\alpha) + e^b - 1 \tag{3.12}$$

for non-negative real numbers, imply that in the set where $g(x) \geq \lambda_0$ we have

$$\begin{aligned}
\frac{\varphi^n|Df|^n}{\log^{1+\alpha} g} &\leq \frac{K(x)\varphi^n J(x,f)}{\log^{1+\alpha} g} \\
&\leq \frac{2}{\beta}\left(\exp(\frac{\beta}{2}K(x)) + \frac{3n\varphi^n J(x,f)}{\log^\alpha g}\right).
\end{aligned} \tag{3.13}$$

Therefore, recalling the definition of g and using (3.13),

$$\begin{aligned}
\int_{g>t} \frac{g^n}{\log^{1+\alpha} g}\,dx &\leq 2^n\int_{g>t}\frac{\varphi^n|Df|^n}{\log^{1+\alpha} g}\,dx + 2^n\int_{g>t}\frac{|f\otimes\nabla\varphi|^n}{\log^{1+\alpha} g}\,dx \\
&\leq \frac{c(n)}{\beta}\int_{g>t}\frac{\varphi^n J(x,f)}{\log^\alpha g}\,dx + \frac{c(n)}{\beta}\int_{g>t}\exp(\frac{\beta}{2}K(x))\,dx \\
&\quad + c(n)\int_{g>t}\frac{|f\otimes\nabla\varphi|^n}{\log^{1+\alpha} g}\,dx.
\end{aligned} \tag{3.14}$$

Inserting (3.14) in (3.11), and rearranging, it follows that

$$\begin{aligned}
\frac{1}{2\alpha\log^\alpha t}\int_{g<t}\varphi^n J(x,f) &\leq (\frac{c(n)}{\beta} - \frac{1}{2\alpha})\int_{g>t}\frac{\varphi^n J(x,f)}{\log^\alpha g}\,dx \\
&\quad + \frac{c(n)}{\beta}\int_{g>t}\exp(\frac{\beta}{2}K(x))\,dx + c(n)\int_{g>t}\frac{|f\otimes\nabla\varphi|^n}{\log^{1+\alpha} g}\,dx \\
&\quad + \frac{c(n)}{\alpha}\int_{B_0}\frac{|f\otimes\nabla\varphi|g^{n-1}}{\log^\alpha\max(g/2,t)}\,dx
\end{aligned} \tag{3.15}$$

Now let us fix $\alpha = \beta/4c(n)$ to be the constant in the theorem. We multiply both sides of (3.15) by $\log^\alpha t$ and let $t \to \infty$. We obtain by monotone convergence theorem and Lebesgue dominated convergence theorem that

$$\int_{B_0}\varphi^n J(x,f)\,dx \leq c(n)\int_{B_0}|f\otimes\nabla\varphi|g^{n-1}\,dx. \tag{3.16}$$

13

Here we used the integrability of $|f \otimes \nabla\varphi|g^{n-1}, |f \otimes \nabla\varphi|^n$, and $\exp(\beta K(x))$ to pass to the limit. Hence, (3.16) shows that $J(x, f) \in L^1_{\text{loc}}(\Omega)$, and then (1.1) follows from the distortion inequality using (3.12) and the integrability of $\exp(\beta(K(x)))$. Thus, Theorem 1.3 is proved.

Next we shall prove the following Caccioppoli type inequality, which is critical for the proof of Corollary 1.4:

$$\int_{B_0} \frac{|Df|^n \varphi^n}{\log^{1+\alpha}(e + |Df|\varphi)}\, dx \leq c(n, \beta) \int_{B_0} \frac{|f \otimes \nabla\varphi|^n}{\log^{\alpha+1-n}(e + |f \otimes \nabla\varphi|)}\, dx \tag{3.17}$$
$$+ c(n, \beta) \int_{B_0} \exp(\frac{\beta}{2} K(x))\, dx.$$

To this end, we insert (3.11) into (3.14). We obtain that

$$\int_{g>t} \frac{g^n}{\log^{1+\alpha} g}\, dx \leq c(n, \beta) \left(\int_{B_0} \frac{|f \otimes \nabla\varphi|g^{n-1}}{\log^{\alpha} \max(g/2, t)}\, dx \right.$$
$$+ \int_{g>t} \frac{|f \otimes \nabla\varphi|^n}{\log^{1+\alpha} g}\, dx$$
$$+ \int_{g>t} \exp(\frac{\beta}{2} K(x))\, dx \Bigg) \tag{3.18}$$
$$+ \frac{2\alpha c(n)}{\beta} \int_{g>t} \frac{g^n}{\log^{1+\alpha} g}.$$

By the choice of α made before, the last term in the right hand side may be absorbed to the left and therefore ignored. Thus, letting $t = \lambda_0$ in (3.18) and noting that $\exp(\frac{\beta}{2} K(x)) \geq 1$, results in

$$\int_{B_0} \frac{g^n}{\log^{1+\alpha}(e + g)}\, dx \leq c(n, \beta) \left(\int_{B_0} \frac{|f \otimes \nabla\varphi|g^{n-1}}{\log^{\alpha}(e + g)}\, dx \right.$$
$$+ \int_{B_0} \frac{|f \otimes \nabla\varphi|^n}{\log^{1+\alpha}(e + g)}\, dx \tag{3.19}$$
$$+ \int_{B_0} \exp(\frac{\beta}{2} K(x))\, dx \Bigg).$$

To estimate the first integral in the right hand side of (3.19), we use the inequality

$$ab^{n-1} \leq \varepsilon \frac{b^n}{\log(e + b)} + c(\varepsilon, n) a^n \log^{n-1}(e + a)$$

for non-negative numbers a and b, and obtain that

$$\frac{|f \otimes \nabla\varphi|g^{n-1}}{\log^{\alpha}(e + g)} \leq \frac{|f \otimes \nabla\varphi|}{\log^{\frac{\alpha}{n}}(e + |f \otimes \nabla\varphi|)} \times \frac{g^{n-1}}{\log^{\alpha \frac{n-1}{n}}(e + g)}$$
$$\leq \varepsilon C(\alpha, n) \frac{g^n}{\log^{1+\alpha}(e + g)} + c(\varepsilon, n) \frac{|f \otimes \nabla\varphi|^n}{\log^{\alpha+1-n}(e + |f \otimes \nabla\varphi|)}.$$

14

By taking $\varepsilon = c(n,\beta)/2C(\alpha,n)$, where $c(n,\beta)$ is the constant in (3.18), we infer from (3.18) that

$$\int_{B_0} \frac{|Df|^n \varphi^n}{\log^{1+\alpha}(e + |Df|\varphi)}\, dx \leq \int_{B_0} \frac{g^n}{\log^{1+\alpha}(e + g)}\, dx$$

$$\leq c(n,\beta) \int_{B_0} \frac{|f \otimes \nabla\varphi|^n}{\log^{\alpha+1-n}(e + |f \otimes \nabla\varphi|)}\, dx$$

$$+ c(n,\beta) \int_{B_0} \frac{|f \otimes \nabla\varphi|^n}{\log^{\alpha+1}(e + |f \otimes \nabla\varphi|)}\, dx$$

$$+ c(n,\beta) \int_{B_0} \exp(\frac{\beta}{2}K(x))\, dx$$

$$\leq c(n,\beta) \int_{B_0} \frac{|f \otimes \nabla\varphi|^n}{\log^{\alpha+1-n}(e + |f \otimes \nabla\varphi|)}\, dx$$

$$+ c(n,\beta) \int_{B_0} \exp(\frac{\beta}{2}K(x))\, dx,$$

which proves (3.17).

Proof of Corollary 1.4. We note that E has vanishing $(n-1)$-dimensional Hausdorff measure. By Theorem 1.3, it thus suffices to show that

$$\frac{|Df|^n}{\log^{1+\alpha}(e + |Df|)} \in L^1_{\text{loc}}(\Omega), \tag{3.22}$$

where $\alpha = c_2\beta$ is as in Theorem 1.3. To this end, let $\eta \in C_0^\infty(\Omega)$ be an arbitrary non-negative test function. We denote by E' the intersection of E with the support of η. There exists a sequence of functions $\{\phi_j\}_{j=1}^\infty$ such that for each j we have

1. $\phi_j \in C_0^\infty(\Omega)$,
2. $0 \leq \phi_j \leq 1$,
3. $\phi_j = 1$ on some neighborhood U_j of E',
4. $\lim_{j\to\infty} \phi_j(x) = 0$ for almost all $x \in \mathbf{R}^n$,
5. $\lim_{j\to\infty} \int_\Omega |\nabla\phi_j|^n \log^{n-1-\alpha}(e + |\nabla\phi_j|) = 0.$

We set

$$\varphi_j = (1 - \phi_j)\eta \in C_0^\infty(\operatorname{supp}\eta \setminus E').$$

Let $\varphi = \varphi_j$ in (3.17). Recall that $|f|$ is assumed to be bounded in Ω and also that $\nabla\varphi_j = (1 - \phi_j)\nabla\eta - \eta\nabla\phi_j$. It follows from the conditions defining ϕ_j that we can pass to the limit (as $j \to \infty$) in (3.17) to obtain that

$$\int_{B_0} \frac{|Df|^n \eta^n}{\log^{1+\alpha}(e + |Df|^n\eta^n)}\, dx \leq c(n,\beta) \int_{B_0} \frac{|f \otimes \nabla\eta|^n}{\log^{\alpha+1-n}(e + |f \otimes \nabla\eta|)}\, dx$$

$$+ c(n,\beta) \int_{B_0} \exp(\frac{\beta}{2}K(x))\, dx,$$

15

which implies (3.22). This then proves Corollary 1.4.

4   Appendix

Proof of the inequality

$$ab\log^{\alpha-1}(c(n)(ab)^{\frac{1}{n}}) \leq \frac{C(n)}{\beta}a\log^{\alpha}(a^{\frac{1}{n}}) + C(\alpha,\beta,n)\exp(\beta b). \qquad (4.1)$$

The case $a \leq e$ is easy. Suppose $a > e$. We have that

$$\begin{aligned}
ab\log^{-1}(c(n)(ab)^{\frac{1}{n}}) &\leq ab\log^{-1}(a^{\frac{1}{n}}) \\
&\leq \frac{4}{\beta}\left(a\log a + \exp(\frac{\beta}{2}b))\right)\log^{-1}a^{\frac{1}{n}} \\
&\leq \frac{c(n)}{\beta}\left(a + \exp(\frac{\beta}{2}b)\right),
\end{aligned} \qquad (4.2)$$

where we used the elementary inequality

$$ab \leq a\log a + \exp(2b)$$

for $a \geq 1, b \geq 1$. We also have that

$$\log^{\alpha}(c(n)(ab)^{\frac{1}{n}}) \quad \leq 2\log^{\alpha}a^{\frac{1}{n}} + c(n,\alpha)\log^{\alpha}(c(n)b), \qquad (4.3)$$

which follows from

$$(x+y)^{\alpha} \leq 2x^{\alpha} + c(\alpha)y^{\alpha}$$

for $x > 0, y > 0, \alpha > 0$.

Combining (4.2) and (4.3) yields

$$\begin{aligned}
&ab\log^{\alpha-1}(c(n)(ab)^{\frac{1}{n}}) \\
&\leq \frac{c(n)}{\beta}\left(a + \exp(\frac{\beta}{2}b)\right)\left(2\log^{\alpha}a^{\frac{1}{n}} + c(n,\alpha)\log^{\alpha}(c(n)b)\right) \\
&\leq \frac{c(n)}{\beta}\left(a\log^{\alpha}a^{\frac{1}{n}} + c(n,\alpha,\beta)\exp(\beta b)\right),
\end{aligned} \qquad (4.4)$$

as desired. The last inequality holds because of the estimates

$$c(n,\alpha)a\log^{\alpha}(c(n)b) \leq a\log^{\alpha}a^{\frac{1}{n}} + c(n,\alpha,\beta)\exp(\beta b),$$

and

$$\exp(\frac{\beta}{2}b)\log^{\alpha}a^{\frac{1}{n}} \leq a\log^{\alpha}a^{\frac{1}{n}} + c(n,\alpha,\beta)\exp(\beta b(x)),$$

for the lower order terms which can be proved easily.

Acknowledgements

References

[1] Astala, K.: Area distortion of quasiconformal mappings. Acta Math. **173** (1994), 37–60.

[2] Astala, K., Iwaniec, T., Koskela, P. and Martin, G.: Mappings of BMO-bounded distortion. Math. Ann. **317** (2000) 4, 703–726.

[3] Bojarski, B.: Homeomorphic solutions of Beltrami systems. Dokl. Akad. Nauk. SSSR. **102** (1955), 661–664.

[4] David, G.: Solutions de l'equation de Beltrami avec $||\mu||_\infty = 1$. Ann Acad. Sci. Fenn. Ser. A I, Math. **13** (1988) No.1, 25–70.

[5] Elcrat, A. and Meyers, N.: Some results on regularity for solutions of non-linear elliptic systems and quasiregular functions. Duke Math. J. **42** (1975), 121–136.

[6] Faraco, D. and Zhong, X.: A short proof of the self-improving regularity of quasiregular mappings. Preprint no. 106, Max-Planck Institute, Leipzig, 2002.

[7] Gehring, F.W.: The L^p-integrability of the partial derivatives of a Quasiconformal mapping. Acta Math., **130**, (1973), 265-277.

[8] Gol'dstein V. and Vodop'yanov S.: Quasiconformal mappings and spaces of functions with generalized first derivatives. Sibirsk. Mat. Z. **17** (1976), 515–531.

[9] Hedberg, L. I. On certain convolution inequalities. Proc. Amer. Math. Soc. **36** (1972), 505–510.

[10] Heinonen, J. and Koskela, P.: Sobolev mappings with integrable dilatations. Arch. Rational Mech. Anal. **125** (1993) no. 1, 81–97.

[11] Hencl, S. and Malý, J.: Mappings of finite distortion: Hausdorff measure of zero sets. Math. Ann. **324** (2002), 451–464..

[12] Herron, D. A. and Koskela, P.: Mappings of finite distortion: The dimension of generalized quasicircles. Illinois J. Math. to appear.

[13] Iwaniec, T.: p–Harmonic tensors and quasiregular mappings. Annals of Math. **136**, (1992), 651–685.

[14] Iwaniec, T.: The Gehring lemma, in P. Duren, J. Heinonen, B. Osgood and B. Palka, Quasiconformal Mappings and Analysis, Springer-Verlag, New-York, 1998.

[15] Iwaniec, T., Koskela P. and Martin, G.: Mappings of BMO-distortion and Beltrami type operators. J. Analyse Math. **88** (2002), 337–381.

[16] Iwaniec, T., Koskela P., Martin, G. and Sbordone, C.: Mappings of exponentially integrable distortion: $L^n \log^\alpha L$-integrability. J. London Math. Soc. (2) **67** (2003), no. 1, 123-136.

[17] Iwaniec, T., Koskela P. and Onninen, J.: Mappings of finite distortion: Monotonicity and continuity. Invent. Math. **144** (2001) 3, 507-531.

[18] T. Iwaniec and Martin, G.: Quasiregular mappings in even dimensions. Acta Math., **170**, (1993), 29–81.

[19] Iwaniec, T. and Martin, G.: *Geometric Function Theory and Nonlinear Analysis.* Oxford Univ. Press, 2001.

[20] Iwaniec, T. and Sbordone, C.: On the integrability of the Jacobian under minimal hypotheses. Arch. Rational Mech. Anal. **119**, (1992), 129–143.

[21] Iwaniec, T. and Šverák, V.: On mappings with integrable dilatation. Proc. Amer. Math. Soc. **118** (1993), 181–188.

[22] Kauhanen, J., Koskela, P. and Malý, J.: Mappings of finite distortion: Discreteness and openness. Arch. Rational Mech. Anal. **160** (2001), 135–151.

[23] Kauhanen, J., Koskela, P. and Malý, J.: Mappings of finite distortion: Condition N. Michigan Math. J. **49** (2001), 169–181.

[24] Kauhanen, J., Koskela, P., Malý, J., Onninen, J. and Zhong, X.: Mappings of finite distortion: Sharp Orlicz-conditions. Rev. Mat. Iberoamericana, to appear.

[25] Koskela, P. and Malý, J.: Mappings of finite distortion: The zero set of the Jacobian. J. Eur. Math. Soc. (JEMS) to appear.

[26] Koskela, P. and Rajala, K.: Mappings of finite distortion: Removable singularities. Israel J. Math. to appear.

[27] Lewis, J. L.: On very weak solutions of certain elliptic systems. Comm. Partial Differential Equations. **18** (1993), 1515–1537.

[28] Manfredi, J. and Villamor, E.: Mappings with integrable dilatation in higher dimensions. Bull. Amer. Math. Soc. (N.S.) **32** (1995), no. 2, 235–240.

[29] Martio, O.: On the integrability of the derivative of a quasiregular mapping. Math. Scand. **35** (1974), 43–48.

[30] Müller, S. and Spector, S.: An existence theory for nonlinear elasticity that allows for cavitation. Arch. Rational Mech. Anal. **131** no. 1 (1995), 1–66.

[31] Reshetnyak, Yu. G.: *Space Mappings with Bounded Distortion.* Trans. of Mathematical Monographs, Amer. Math. Soc, vol. 73, 1989.

[32] Rickman, S.: *Quasiregular mappings.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 26. Springer-Verlag, Berlin, 1993.

Daniel Faraco, Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany

Pekka Koskela and Xiao Zhong, University of Jyväskylä , Department of Mathematics and Statistics, P.O. Box 35, FIN-40351 Jyväskylä, Finland.