# Max-Planck-Institut
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**Sharp rigidity estimates for nearly umbilical surfaces**

by

*Camillo De Lellis and Stefan Müller*

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# OPTIMAL RIGIDITY ESTIMATES FOR NEARLY UMBILICAL SURFACES

CAMILLO DE LELLIS, STEFAN MÜLLER

## 1. Introduction

Let $\Sigma \subset \mathbf{R}^3$ be a smooth surface. A point $p$ of $\Sigma$ is called umbilical if the principal curvatures of $\Sigma$ at $p$ are equal. A classical theorem in differential geometry states that if $\Sigma$ is connected and all points of $\Sigma$ are umbilical, then either $\Sigma$ is a subset of a round sphere or it is a subset of a plane. Thus, if $\Sigma$ is a compact surface without boundary, then $\Sigma$ must be a round sphere and therefore its second fundamental form is a constant multiple of the identity.

In this paper, we generalize this well–known rigidity result in the following theorem. Here:

$A$ denotes the second fundamental form of $\Sigma$;
Id denotes the identity $(1,1)$–tensor and the $(0,2)$–tensor naturally associated to it;
$\mathring{A}$ denotes the traceless part of $A$, i.e. the tensor $A - \frac{\operatorname{tr} A}{2}\mathrm{Id}$;
$\mathrm{id} : \mathbf{S}^2 \subset \mathbf{R}^3 \to \mathbf{R}^3$ is the standard isometric embedding of the round sphere.

**Theorem 1.1.** *Let $\Sigma \subset \mathbf{R}^3$ denote a smooth compact connected surface without boundary and for convenience normalize the area of $\Sigma$ by $\mathrm{ar}(\Sigma) = 4\pi$. Then*

$$\|A - \mathrm{Id}\|_{L^2(\Sigma)} \;\leq\; C \left\|\mathring{A}\right\|_{L^2(\Sigma)}\,, \tag{1}$$

*where $C$ is a universal constant. If in addition $\left\|\mathring{A}\right\|^2_{L^2(\Sigma)} \leq 8\pi$, then there exists a conformal parameterization $\psi : \mathbf{S}^2 \to \Sigma$ and a vector $c_\Sigma \in \mathbf{R}^3$ such that*

$$\|\psi - (c_\Sigma + \mathrm{id})\|_{W^{2,2}(\mathbf{S}^2)} \;\leq\; C \left\|\mathring{A}\right\|_{L^2(\Sigma)}\,. \tag{2}$$

**Remark 1.2.** Note that (1) is a very natural estimate, since $\|\mathring{A}\|_{L^2(\Sigma)}$ is scaling invariant. Indeed (1) can be easily converted into the following scale–invariant estimate

$$\|A - r_\Sigma \mathrm{Id}\|_{L^2(\Sigma)} \;\leq\; C \left\|\mathring{A}\right\|_{L^2(\Sigma)} \qquad \text{where } r_\Sigma \;:=\; \sqrt{\frac{\mathrm{ar}(\Sigma)}{4\pi}}\,.$$

In order to have the second estimate of Theorem 1.1 it is sufficient to assume $\left\|\mathring{A}\right\|^2_{L^2} \leq 16\pi - \varepsilon$. In this case $C$ in (2) must be substituted by $C(\varepsilon)$, where $C(\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$.

In Section 7 we show that these estimates are optimal. More precisely we show a sequence of smooth connected compact surfaces $\Sigma_n$ without boundary such that

$$\left\|\mathring{A}\right\|_{L^p} \;\to\; 0 \qquad \text{for every } p < 2,$$
$$\Sigma_n \text{ converges to the union of two spheres with different radii.}$$

The key point for proving Theorem 1.1 is the following remark. Let us fix an orthonormal frame $e_1, e_2$ on $\Sigma$ and denote by $A_{ij}$ the quantities $A(e_i, e_j)$ and by $\nabla A_{ijk}$ the quantities

$[\nabla_{e_i} A] (e_j, e_k)$. The Codazzi equations imply that $\nabla A_{ijk} = \nabla A_{jik}$. Hence the symmetry of $A$ gives that $\nabla A$ is a symmetric tensor. In view of this fact, straightforward algebraic computations give that $\nabla_{e_i} [A_{11} + A_{22}]$ can be written as a linear combination of $\nabla_{e_j} [A_{11} - A_{22}]$ and $\nabla_{e_j} [A_{12}]$ plus some error terms of type $A(\nabla_{e_j} e_k, e_l)$. Moreover, these error terms can be written as nonlinear expressions involving $\mathring{A}$.

If $\mathring{A}$ were identically 0, then $\operatorname{tr} A$ would be constant. Roughly speaking, a control on $\mathring{A}$ gives some control on the oscillation of $\operatorname{tr} A = A_{11} + A_{22}$. Thus, if $\mathring{A}$ is small in a $C^1$ sense, then $\Sigma$ would be close to a round sphere. This remark was used in [HY] to give a definition of center of mass for isolated gravitating systems in General Relativity. If view of our result, one should be able to weaken the hypotheses under which Huisken–Yau's construction is possible.

In our case the difficulties in getting the bound (1) are considerably increased by the weakness of the right hand side of (1) and the nonlinearity of the error terms of type $A(\nabla_{e_j} e_k, e_l)$. The outline of our proof is the following.

- First we show that, when $\|\mathring{A}\|_{L^2}$ is sufficiently small, $\Sigma$ is a sphere and there exists a good parameterization by a conformal map $\psi : \mathbf{S}^2 \to \Sigma$. By "good" we mean that, after a suitable rescaling, the conformal factor $h$ satisfies uniform $L^\infty$ and $W^{1,2}$ bounds (independent of $\Sigma$). In order to get these bounds, we derive Hardy space estimates on the Gauss curvature, using some ideas of [MS]. This is accomplished in Section 3.

- We then perform the computations outlined above in the coordinate charts naturally induced by $\psi$. The control on $\psi$ is sufficient to get an $L^1$ bound on the nonlinear error terms. We use this bound and the regularity theory for the Laplacian to prove the existence of a universal constant $C$ such that
$$\min_{\lambda \in \mathbf{R}} \|\operatorname{tr} A - \lambda\|_{L^{2,\infty}(\Sigma)} \leq C \left\|\mathring{A}\right\|_{L^2(\Sigma)}, \tag{3}$$
where $L^{2,\infty}$ is the weak Marcinkiewicz space (see Appendix B for the precise definition). This estimate is proved in Proposition 4.1.

- In Section 5 we show that the weak estimate (3) can be improved to the desired stronger estimate (1). This improvement heavily relies on some algebraic computations which exploit the special structure of the tensor $A$. The proof uses Hardy space estimates for skew–symmetric quantities and the duality between the Hardy space $\mathcal{H}^1$ and BMO.

- In Section 6 we use (1) and some of the information derived in the previous sections to prove (2). The main difficulty here is due to the action of the conformal group of $\mathbf{S}^2$. The existence of the map $\psi$ is proved into two steps: in the first one we prove that there is a conformal parameterization with conformal factor close to 1. In the second step we use the formalism of moving frames to show that this map $\psi$ is $L^2$ near to a smooth isometric embedding of the standard sphere.

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## 2. Preliminaries

2.1. **Notation.** Throughout this paper we will use the following notational conventions:

| | |
|---|---|
| $\mathbf{S}^2$ | standard sphere |
| $\Sigma$ | compact connected smooth surface in $\mathbf{R}^3$ without boundary |
| $T_p\Sigma$, $T\Sigma$ | tangent space in $p$, tangent bundle |
| $\mathrm{ar}(\Sigma)$, $\mathbf{g}(\Sigma)$ | area of $\Sigma$, genus of $\Sigma$ |
| $D_r(x)$, $\partial D_r(x)$ | distance disk and distance circle of radius $r$ and center $x$ in a 2d Riemannian manifold |
| $\mathcal{D}_1$, $\partial\mathcal{D}_1$ | unit disk and unit circle in $\mathbf{R}^2$ |
| $g$, $\sigma$ | Riemannian metric on $\Sigma$, standard metric on $\mathbf{S}^2$ |
| $\delta_{ij}$, $A$, $N$ | Kronecker symbol, second fundamental form, Gauss map |
| $\mathrm{tr}\, B$, $\det B$, $\lvert B\rvert$, Id | trace of $B$, determinant, Hilbert–Schmidt norm, identity matrix |
| $\kappa_1$, $\kappa_2$, $K_G$ | principal curvatures, Gaussian curvature |
| $\mathrm{Deg}\,(\Gamma,\Sigma,u)$ | topological degree of the map $u:\Gamma\to\Sigma$ |
| $L^p$, $\mathcal{H}^1(\Omega)$ | $L^p$ spaces, Hardy space |
| $\Delta_\Sigma$ | Laplace operator on the Riemannian manifold $\Sigma$ |

Let $\psi:\Sigma\to\Gamma$ be an immersion and $g$ a metric on $\Gamma$. Then we denote by $\psi^*g$ the metric on $\Sigma$ which is the pull back of $g$ via $\psi$. That is

$$(\psi^*g)_p(v,w) \;:=\; g_{\psi(p)}(d\psi(v),d\psi(w)) \qquad \text{for every } v,w\in T_p(\Sigma).$$

A system of coordinates on an open set $U\subset\Sigma$ can be regarded as a smooth diffeomorphism $\psi:\mathbf{R}^2\supset\Omega\to U$. Hence, writing the metric in these coordinates is equivalent to calculate the pull–back metric $\psi^*g$.

In the rest of this paper we assume that $\Sigma$ is compact, connected, and without boundary. Moreover, we assume that $\mathrm{ar}(\Sigma)=4\pi$ and we set

$$\delta^2 \;:=\; \int_\Sigma \left|\mathring{A}\right|^2 . \tag{4}$$

We will make a frequent use of some elementary relations between differential geometric quantities, in particular the identities

$$\left|\mathring{A}\right|^2 \;=\; \kappa_1^2+\kappa_2^2-2\kappa_1\kappa_2 \;=\; |A|^2-2\det A \;=\; |A|^2-2K_G\,, \tag{5}$$

combined with Gauss–Bonnet Theorem:

$$\int_\Sigma |A|^2 \;=\; \int_\Sigma \left|\mathring{A}\right|^2 + 2\int_\Sigma K_G \;=\; \delta^2+2\int_\Sigma K_G \;=\; \delta^2+8\pi(1-\mathbf{g}(\Sigma))\,. \tag{6}$$

**Remark 2.1.** Note that

$$\|A-\mathrm{Id}\|^2_{L^2} \;\le\; 2\int_\Sigma |A|^2 + 2\mathrm{ar}(\Sigma)\,.$$

Since $\mathbf{g}(\Sigma)\ge 0$, by (6) for every $c>0$ there exists $C>0$ such that

$$\|A-\mathrm{Id}\|_{L^2(\Sigma)} \;\le\; C\left\|\mathring{A}\right\|^2_{L^2(\Sigma)} \qquad \text{for every } \Sigma \text{ with } \delta\ge c.$$

Thus it suffices to show (1) for $\delta$ sufficiently small.

2.2. $\Sigma$ **is a sphere.** In the following lemma we show that, when $\delta$ is sufficiently small, $\Sigma$ is a sphere. The proof uses well known elementary facts of differential geometry of surfaces. We report it for the reader's convenience.

**Lemma 2.2.** *If $\delta^2 = 16\pi - \eta < 16\pi$, then $\Sigma$ is a sphere.*

*Proof.* Note that

$$\int_\Sigma |\det A| \;\leq\; \frac{1}{2}\int_\Sigma |A|^2 \stackrel{(6)}{=} \; 8\pi - \frac{\eta}{2} + 4\pi(1-\mathbf{g}(\Sigma)) \;<\; 4\pi(3-\mathbf{g}(\Sigma))\,. \tag{7}$$

Hence $\mathbf{g}(\Sigma)$ is either 0, 1, or 2. Let $N : \Sigma \to \mathbf{S}^2$ be the Gauss map, which to every point $x \in \Sigma$ associates the exterior unit normal to $\Sigma$ in $x$. Since $A = dN$, the area formula gives

$$\int_\Sigma |\det A| \;=\; \int_{\mathbf{S}^2} \# N^{-1}(\{\xi\})\, d\xi\,. \tag{8}$$

Note that $N$ is surjective. Indeed let $\xi \in \mathbf{S}^2$ and consider the largest real number $a$ such that the set $\mathrm{Ex} := \{x \in \Sigma : x \cdot \xi = a\}$ is not empty. For any $y \in \mathrm{Ex}$ we have $N(x) = \xi$.

This implies that $\# N^{-1}(\{\xi\}) \geq 1$ and hence gives $\int |\det A| \geq 4\pi$, which thanks to (7) rules out the possibility $\mathbf{g}(\Sigma) = 2$. Moreover, if $\mathbf{g}(\Sigma) = 1$ (i.e. if $\Sigma$ were a torus), the degree $\mathrm{Deg}\,(\Sigma, \mathbf{S}^2, N)$ would necessarily be 0, which implies $\# N^{-1}(\{\xi\}) \geq 2$. Hence (8) and (7) rule out the possibility $\mathbf{g}(\Sigma) = 1$. This gives $\mathbf{g}(\Sigma) = 0$ and completes the proof. □

## 3. Existence of a good conformal parameterization

In this section we show that, if $\delta$ is sufficiently small, then the surface $\Sigma$ has a conformal parameterization which enjoys good bounds.

**Definition 3.1.** *Denote by $\sigma$ the metric on the standard sphere $\mathbf{S}^2$ and by $g$ the standard metric on $\Sigma$ as submanifold of $\mathbf{R}^3$. If $\psi : \mathbf{S}^2 \to \Sigma$ is conformal, then $h$ denotes the unique function $h : \mathbf{S}^2 \to \mathbf{R}^+$ with $h^2\sigma = \psi^* g$.*

**Proposition 3.2.** *If $\delta^2 = 8\pi - \eta < 8\pi$, then there exists a constant $C(\eta)$ and a conformal parameterization $\psi : \mathbf{S}^2 \to \Sigma$ such that*

$$(C(\eta))^{-1} \;\leq\; h \;\leq\; C(\eta) \qquad\qquad \|dh\|_{L^2} \;\leq C(\eta) \tag{9}$$

A classical theorem (see for example [Mo]) implies the existence of conformal parameterizations $\psi : \mathbf{S}^2 \to \Sigma$. However, we cannot hope to have the bounds of Proposition 3.2 for *all* such $\psi$ (due to the action of the conformal group). The choice of a good $\psi$ is based on the following remark (cf. [MS]). If $h = e^u$, then

$$\int_{\mathbf{S}^2} e^{2u} \;=\; 4\pi \qquad\qquad -\Delta_{\mathbf{S}^2} u \;=\; Ke^{2u} - 1\,, \tag{10}$$

where $\Delta_{\mathbf{S}^2}$ is the Laplace operator on $\mathbf{S}^2$ and $K(x) = K_\Sigma(\psi(x))$. If we can bound the norm of the right hand side of (10) in the Hardy space $\mathcal{H}^1$, then the proposition follows from the results of Fefferman and Stein [FS] (for the definition of $\mathcal{H}^1$ and for a precise statement of the result of [FS] needed here, see appendix A). Hence it suffices to show the existence of a constant $C(\eta)$ and of a conformal $\psi$ such that $\|Ke^{2u}\|_{\mathcal{H}^1(\mathbf{S}^2)} \leq C(\eta)$. To derive this estimate we will use some ideas of [MS] and the following result of [CLMS]:

**Theorem 3.3.** *Let $u \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^n)$. Then there exists a constant $c$ (depending only on $n$) such that*

$$\|\det du\|_{\mathcal{H}^1(\mathbf{R}^n)} \;\leq\; c\|du\|_{L^n}\,. \tag{11}$$

As already pointed out, in order to get the estimates (9) we have to mod out the action of the conformal group of the sphere. This is accomplished in the following

**Lemma 3.4.** *Assume that $\delta^2 = 8\pi - \eta < 8\pi$. Let $x_1, x_2$, and $x_3$ be standard coordinates in $\mathbf{R}^3$ and set $\mathbf{S}^\pm_i := \{\pm x_i > 0\} \cap \mathbf{S}^2$. Then there exists a conformal $\psi : \mathbf{S}^2 \to \Sigma$ such that*

$$\int_{\psi(\mathbf{S}^j_i)} |A|^2 \;=\; 8\pi - \frac{\eta}{2} \qquad \textit{for all } j \in \{+,-\} \textit{ and every } i \in \{1,2,3\}. \tag{12}$$

*Proof.* Thanks to Lemma 2.2, $\Sigma$ is a sphere. Hence, equation (6) implies

$$\int_\Sigma |A|^2 \;=\; 16\pi - \eta\,.$$

Denote by $e_i$ the vectors of the standard basis of $\mathbf{R}^3$ relative to the system of coordinates $x_i$. For each $i$, we denote by $\mathcal{S}^i : \mathbf{S}^2 \to \mathbb{C} \cup \{\infty\}$ the stereographic projection which maps $e_i$ to the origin and the equator $\{x_i = 0\} \cap \mathbf{S}^2$ onto the unit circle $\{|z| = 1\}$. For each $r > 0$ we define $\mathcal{O}_r : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ by $\mathcal{O}_r(z) = rz$. For every $i \in \{1,2,3\}$ and $r > 0$, we denote by $F^i_r : \mathbf{S}^2 \to \mathbf{S}^2$ the conformal diffeomorphism $(\mathcal{S}^i)^{-1} \circ \mathcal{O}_r \circ \mathcal{S}^i$.

Choose a conformal parameterization $\varphi : \mathbf{S}^2 \to \Sigma$. Note that

$$\lim_{t\uparrow\infty} \int_{\varphi\left(F^1_t(\mathbf{S}^+_1)\right)} |A|^2 \;=\; \int_\Sigma |A|^2 \qquad \text{and} \qquad \lim_{t\downarrow 0} \int_{\varphi\left(F^1_t(\mathbf{S}^+_1)\right)} |A|^2 \;=\; 0\,.$$

By continuity there exists a $t$ such that

$$\int_{\varphi\left((F^1_t(\mathbf{S}^+_1)\right)} |A|^2 \;=\; \frac{1}{2}\int_\Sigma |A|^2 \;=\; 8\pi - \frac{\eta}{2}\,. \tag{13}$$

Define $\psi_1 := \varphi \circ F^1_t$ and again note that for some $\tau$ we have

$$\int_{\varphi_1\left(F^2_\tau(\mathbf{S}^+_2)\right)} |A|^2 \;=\; \frac{1}{2}\int_\Sigma |A|^2 \;=\; 8\pi - \frac{\eta}{2}\,. \tag{14}$$

Note that $F^2_\tau$ maps $\mathbf{S}^+_1$ onto itself. Thus, we have $\int_{\varphi_1\left(F^2_\tau(\mathbf{S}^+_1)\right)} |A|^2 = 8\pi - \eta/2$. A similar choice of $F^3_\sigma$ shows that $\varphi \circ F^1_t \circ F^2_\tau \circ F^3_\sigma$ has the desired properties. □

Below we adopt the following convention. Let $\alpha$ be a 2–form on $\Sigma$, let $\beta$ be the standard volume form of $\Sigma$, and denote by $f$ the function such that $\alpha = f\beta$. If $H$ is any function space, then we write $\|\alpha\|_H$ for $\|f\|_H$. When $H = \mathcal{H}^1$, i.e. the first Hardy space, the maximal function of $f$ will be sometimes called "maximal function of $\alpha$".

*Proof of Proposition 3.2.* Fix $\psi$ as in Lemma 3.4 and let $N : \Sigma \to \mathbf{S}^2$ be the Gauss map. Set $N' := N \circ \psi$ and note that $K' := Ke^{2u}$ is the Jacobian determinant of $dN'$.

The proof of the $\mathcal{H}^1$ estimate is based on some arguments of Section 3 of [MS]. We first fix some notation. We denote by $\omega$ the standard volume form on $\mathbf{S}^2$. Then $K'\omega$ is the pull–back of $\omega$ via the map $N'$, that is $K'\omega = (N')^*\omega$. Moreover, any disk $D_\rho(x) \subset \mathbf{S}^2$ will be identified with a disk $\mathcal{D}_{\rho'} = \mathcal{D}_{\rho'}(0)$ in the complex plane via the standard stereographic projection which maps $x$ onto 0.

We will show that there are constants $r$ and $C(\eta)$ with the following property. For any $x \in \mathbf{S}^2$, there exists a map $M : \mathbb{C} \to \mathbf{S}^2$ such that

(i) $M = N'$ on $\mathcal{D}_{r'}$ ($\approx D_r(x)$);
(ii) $M$ is constant on $\mathbb{C} \setminus \mathcal{D}_{(2r)'}$;
(iii) $\int_{\mathbb{C}} M^*\omega = 0$;
(iv) $\|M^*\omega\|_{W^{-1,2}} + \|dM\|_{L^2} \leq C(\eta)$.

**Step 1** From (i)–(iv) to the $\mathcal{H}^1$ bound.

We first prove that the existence of $M$ as above gives an $\mathcal{H}^1$ bound for $(N')^*\omega$. We make the usual identification $\mathbf{S}^2 = P^1(\mathbb{C})$ and denote by $\pi : \mathbb{C}^2 \supset \mathbf{S}^3 \to P^1(\mathbb{C})$ the Hopf fibration. Then, Proposition 3.4.3 of [MS] implies that $M$ lifts to a map $F : \mathbb{C} \to \mathbf{S}^3 \subset \mathbb{C}^2$ (that is $M = \pi \circ F$) with

$$\|dF\|_{L^2} = \|dM\|_{L^2} + \|M^*\omega\|_{W^{-1,2}} . \tag{15}$$

Note that the existence of liftings is guaranteed by condition (iii). If $F_1$ and $F_2$ denote the components of $F$ in a standard basis of $\mathbb{C}^2$, then $M^*\omega = F^*\pi^*\omega = idF_1 \wedge d\overline{F}_1 + idF_2 \wedge d\overline{F}_2$. Writing $F_j$ as $F_i^{re} + iF_i^{im}$, it is easy to see that $idF_1 \wedge d\overline{F}_1 + idF_2 \wedge d\overline{F}_2$ can be written as linear combination of forms of type $df_1 \wedge df_2$, where $f_1, f_2 \in W^{1,2}(\mathbb{C}) = W^{1,2}(\mathbf{R}^2)$. Clearly, $df_1 \wedge df_2 = (\det df) dx_1 \wedge dx_2$, where $x_1, x_2$ are standard coordinates in $\mathbf{R}^2$. Hence we can apply Theorem 3.3 to derive

$$\|M^*\omega\|_{\mathcal{H}^1} \;\leq\; C\|dF\|_{L^2} \;\stackrel{(15)}{=}\; C\|dM\|_{L^2} + \|M^*\omega\|_{W^{-1,2}} \;\stackrel{(iv)}{\leq}\; C(\eta) .$$

Let $g$ be the maximal function of $M^*\omega$. Then

$$\|g\|_{L^1(D_{r/2}(x))} \;\leq\; \|g\|_{L^1(\mathbf{R}^2)} \;=\; \|M^*\omega\|_{\mathcal{H}^1} \;\leq\; C(\eta) . \tag{16}$$

Let $f$ be the maximal function of $(N')^*\omega$. Since $dN' \in L^2$, clearly $\det dN' \in L^1$ and hence $(N')^*\omega \in L^1$. By the definition of maximal functions we have

$$\|f\|_{L^1(D_{r/2}(x))} \;\leq\; \|g\|_{L^1(D_{r/2}(x))} + C\|(N')^*\omega\|_{L^1} ,$$

where the constant $C$ depends only on $r$. Since $\mathbf{S}^2$ can be covered with finitely many disks of radius $r/2$, we find that $\|(N')^*\omega\|_{\mathcal{H}^1(\mathbf{S}^2)}$ is bounded by a constant depending on $\eta$ and $r$.

**Step 2** Construction of $M$ and $W^{-1,2}$ estimate.

We now come to the proof of the existence of constants $r$ and $C(\eta)$ which satisfy (i)–(iv) above. We first construct an intermediate function $\zeta : \mathbb{C} \to \mathbf{S}^2$. The constant $r$ is chosen so small that the disk $D_{2r}(x)$ is contained in one of the half spheres $\mathbf{S}_i^{\pm}$ of Lemma 3.4. Thus

$$\int_{D_{2r}(x)} |\det dN'| \;\leq\; \frac{1}{2}\int_{\mathbf{S}_i^{\pm}} |dN'|^2 \;=\; 4\pi - \frac{\eta}{4} . \tag{17}$$

Using the Fubini–Tonelli Theorem, we can find a $\rho \in ]r, 2r[$ such that

$$\int_{\partial D_\rho(x)} |dN'|^2 \;\leq\; \frac{4\pi}{r} . \tag{18}$$

We identify $D_\rho(x)$ with $\mathcal{D}_{\rho'} \subset \mathbb{C}$ (using the stereographic projection, see the discussion above) and we define $\zeta : \mathbb{C} \to \mathbf{S}^2$ by setting:

$$\zeta \;=\; N' \quad \text{on } \mathcal{D}_{\rho'} \qquad \text{and} \qquad \zeta(z) \;=\; N'\left(\rho\tfrac{z}{|z|}\right) \quad \text{on } \mathbb{C}\setminus\mathcal{D}_{\rho'}.$$

Clearly, $\zeta$ satisfies (i). We now show that

(iv)' $\|\zeta^*\omega\|_{W^{-1,2}\left(\mathcal{D}_{\rho'+1}\right)}$ and $\|d\zeta\|_{L^2\left(\mathcal{D}_{\rho'+1}\right)}$ are bounded by a constant $C(\eta)$.

The bound on $\|d\zeta\|_{L^2\left(\mathcal{D}_{\rho'+1}\right)}$ is given by the fact that $\|dN'\|_{L^2(\mathbf{S}^\pm_i)}$ is uniformly bounded and by the choice (18). We retain

$$\|d\zeta\|_{L^2\left(\mathcal{D}_{\rho'+1}\right)} \;\leq\; C(\eta)\,. \tag{19}$$

We now come to the $W^{-1,2}$ bound. Note that

$$\mathrm{ar}(\zeta(\mathbb{C})) \;\leq\; \int_{D_{2r}(x)} |\det dN'| \;\leq\; 4\pi - \frac{\eta}{4}\,.$$

Thus $\mathbf{S}^2\setminus\zeta(\mathbb{C})$ has area at least $\eta/4$. This means that we can find a closed set $E\subset\mathbf{S}^2\setminus\zeta(\mathbb{C})$, with area $\eta/8$. Arguing as in the proof of Proposition 3.5.5 of [MS] we can find a 1–form $\alpha_E$ on $\mathbf{S}^2\setminus E$ such that

$$\|\alpha_E\|_{L^\infty(\mathbf{S}^2)} \;\leq\; \frac{C}{\mathrm{ar}(E)} \qquad \text{and} \qquad d\alpha_E = \omega \;\text{ on } \mathbf{S}^2\setminus E\;, \tag{20}$$

where $C$ is a universal constant. Using $\alpha_E$ one finds $\zeta^*\omega = d(\zeta^*\alpha_E)$. Let $\varphi\in W^{1,2}(\mathcal{D}_{\rho'+1})$. Then, since $\zeta$ takes values in $\mathbf{S}^2\setminus E$, we have

$$\int_{\mathcal{D}_{\rho'+1}} \varphi\,\zeta^*\omega \;=\; \int_{\partial\mathcal{D}_{\rho'+1}} \varphi\zeta^*\alpha - \int_{\mathcal{D}_{\rho'+1}} d\varphi\wedge\zeta^*\alpha\,.$$

Recall that $\zeta|_{\partial\mathcal{D}_{\rho'+1}} = N'|_{\partial D_{\rho'}}$. Thus, by (20), we get

$$\left|\int_{\partial\mathcal{D}_{\rho'+1}} \varphi\zeta^*\alpha\right| \;\leq\; \frac{C}{\mathrm{ar}(E)}\|\varphi\|_{L^2\left(\partial\mathcal{D}_{\rho'+1}\right)}\|d\zeta\|_{L^2\left(\partial\mathcal{D}_{\rho'+1}\right)} \;\overset{(18)}{\leq}\; C(\eta)\|\varphi\|_{W^{1,2}\left(\mathcal{D}_{\rho'+1}\right)}\,.$$

Analogously,

$$\left|\int_{\mathcal{D}_{\rho'+1}} d\varphi\wedge\zeta^*\alpha\right| \;\leq\; \frac{C}{\mathrm{ar}(E)}\|d\varphi\|_{L^2\left(\mathcal{D}_{\rho'+1}\right)}\|d\zeta\|_{L^2\left(\mathcal{D}_{\rho'+1}\right)} \;\overset{(19)}{\leq}\; C\|\varphi\|_{W^{1,2}\left(\mathcal{D}_{\rho'+1}\right)}\,.$$

This establishes the $W^{-1,2}$ bound of (iv)'

**Step 3** The existence of $M$.

In this step we modify $\zeta$ so to reach (ii) and (iii), while keeping (i) and upgrading (iv)' to (iv). Consider the restriction of $\zeta$ to $\mathcal{D}_{\rho'}$ and define for every regular value $x\in\mathbf{S}^2$ its degree $\deg(\zeta,x)$. Standard arguments give that $\deg(\zeta,x)$ is constant on the connected components of $\mathbf{S}^2\setminus\zeta(\partial\mathcal{D}_{\rho'})$. Thus, by continuity it can be extended to an integer valued piecewise constant function on $\mathbf{S}^2\setminus\zeta(\partial\mathcal{D}_{\rho'})$. Define

$$U_0 \;:=\; \{x\in\mathbf{S}^2|\,\deg(\zeta,x)=0\}\,. \tag{21}$$

Then $U_0$ is an open set contained in $\mathbf{S}^2 \setminus \zeta(\partial \mathcal{D}_{\rho'})$. The idea is to choose $y \in U_0$ and to take a retraction of $R : [0,1] \times \mathbf{S}^2 \setminus \{y\} \to \mathbf{S}^2$ onto the antipodal of $y$. Then we define $M = \zeta$ on $\mathcal{D}_{\rho'}$ and on $D_{\rho'+1} \setminus \mathcal{D}_\rho$ we put

$$M(z) \;=\; R(\rho' + 1 - |z|, \zeta(z))\,.$$

Since $\zeta(\mathcal{D}_{\rho'+1} \setminus D_{\rho'}) = \zeta(\partial D_{\rho'})$, we have $U_0 \cap \zeta(\mathcal{D}_{\rho'+1} \setminus D_{\rho'}) = \emptyset$. Thus $M$ is well defined. From the definition of (21), we clearly have $\deg(\mathbb{C}, \mathbf{S}^2, M) \equiv 0$, and thus $M$ satisfies (iii). Moreover $M|_{\mathcal{D}_{\rho'}} = \zeta$ and $M|_{\mathbb{C} \setminus \mathcal{D}_{\rho'+1}}$ is constant; hence, $M$ satisfies (i) and (ii). The only difficulty is to choose $y$ and the retraction $R$ so to achieve the bounds (iv).

Clearly, $U_0$ contains $\mathbf{S}^2 \setminus \zeta(\mathbb{C})$ and thus $\mathrm{ar}(U_0) \geq \eta$. Moreover $U_0$ is an open set bounded by a subset of the curve $\gamma = \zeta(\partial \mathcal{D}'_\rho) = N'(\partial D_\rho(x))$, which, in view of (18) has bounded length. Thanks to Lemma C.1, there exists a $\delta$, depending on $\mathrm{ar}(U_0)$ and $\mathrm{length}(\gamma)$, such that $U_0$ contains a ball $D_\delta(y)$. Thus $\delta$ can be chosen bigger than a constant which depends only on $\eta$.

Fix such a $y$ and such a $\delta$ and define a $C^1$ map $R : [0,1] \times (\mathbf{S}^2 \setminus D_\delta(y)) \to \mathbf{S}^2$ which retracts on the antipode $\overline{y}$ of $y$. This can be done so that $\|R\|_{C^1}$ depends only on $\eta$. Thus

$$\|M^*\omega\|_{W^{-1,2}(\mathbb{C})} \;\leq\; C_1(\eta) \|\zeta^*\omega\|_{W^{-1,2}(D_{\rho'+2}(0))} \;\stackrel{(iv)'}{\leq}\; C_2(\eta)\,.$$

An analogous estimate holds for $\|dM\|_{L^2}$. This gives (iv) and completes the proof. □

## 4. An $L^{2,\infty}$ estimate for $(A - \overline{H}\,\mathrm{Id})$

**Proposition 4.1.** *There exists $C > 0$ such that, if*

$$\mathrm{ar}(\Sigma) = 4\pi\,, \qquad \textit{and} \qquad \int_\Sigma |\mathring{A}|^2 \;\leq\; \delta^2\,, \tag{22}$$

*then*

$$\left\| A - \left( \fint_\Sigma \frac{\mathrm{tr}\,A}{2} \right) \mathrm{Id} \right\|_{L^{2,\infty}(\Sigma)} \;\leq\; C\delta \tag{23}$$

For the definition and properties of the Marcinkiewicz space $L^{2,\infty}$ we refer to Appendix B.

*Proof.* Below we will prove the existence of a universal constant $C$ such that, for for every $\Sigma$ with $\delta^2 \leq 4\pi$, there exist two conformal parameterizations $\varphi^+, \varphi^- : \mathcal{D}_1 \to \Sigma$ with the following properties:

(a) $\varphi^+(\mathcal{D}_1) \cup \varphi^-(\mathcal{D}_1) = \Sigma$;

(b) $\mathrm{ar}(\varphi^+(\mathcal{D}_1) \cap \varphi^-(\mathcal{D}_1)) \geq C^{-1}$;

(c) $\|\,\mathrm{tr}\,A - \lambda^\pm\|_{L^{2,\infty}(\varphi^\pm(\mathcal{D}_1))} \leq C\delta$ for some constants $\lambda^\pm$.

We first show how this would give (23). Note that

$$\begin{aligned} C^{-1}|\lambda^+-\lambda^-| &\overset{(b)}{\leq} \int_{\varphi^+(\mathcal{D}_1)\cap\varphi^-(\mathcal{D}_1)} |\lambda^+-\lambda^-| \;\leq\; \int_{\varphi^+(\mathcal{D}_1)} |\operatorname{tr} A-\lambda^+| + \int_{\varphi^-(\mathcal{D}_1)} |\operatorname{tr} A-\lambda^-| \\ &\leq C_1\|\operatorname{tr} A-\lambda^+\|_{L^{2,\infty}(\varphi^+(\mathcal{D}_1))} + C_1\|\operatorname{tr} A-\lambda^-\|_{L^{2,\infty}(\varphi^-(\mathcal{D}_1))} \\ &\overset{(c)}{\leq} 2C_1C\delta\,. \end{aligned}$$

Hence $|\lambda^+-\lambda^-|\leq 2C_1C^2\delta$. This means that $\|\operatorname{tr} A-\lambda^+\|_{L^{2,\infty}(\Sigma)}\leq C_2\delta$, where $C_2$ is another constant. Let us set $2\overline{H} := \int_\Sigma \operatorname{tr} A$. Then

$$4\pi|2\overline{H}-\lambda^+| \;\leq\; \int_\Sigma |\operatorname{tr} A-\lambda^+| \;\leq\; C_1\|\operatorname{tr} A-\lambda^+\|_{L^{2,\infty}(\Sigma)} \;\leq\; C_3\delta\,.$$

This gives $\|\operatorname{tr} A-2\overline{H}\|_{L^{2,\infty}(\Omega)}\leq C_4\delta$. Then

$$\|A-\overline{H}\mathrm{Id}\|_{L^{2,\infty}(\Omega)} \;\leq\; \left(\int_\Sigma |\mathring{A}|^2\right)^{1/2} + \sqrt{2}\left\|\tfrac{\operatorname{tr} A}{2}-\overline{H}\right\|_{L^{2,\infty}(\Omega)} \;\leq\; C_6\delta\,.$$

Subsections 4.1 and 4.2 are devoted to prove the existence of $\varphi^\pm$ as above. To explain the underlying key idea, we have to set some notation. Let $\varphi:\mathcal{D}_1\to\Sigma$ be a conformal parameterization of $\varphi(\mathcal{D}_1)$. We denote by $x_1,x_2$ a system of orthonormal coordinates in $\mathbf{R}^2$. Thus, in these conformal coordinates, the metric of $\Sigma$ is given by $h^2\delta_{ij}$. We denote by $e_i\in T\Sigma$ the unit vectors $\frac{1}{h}\frac{\partial}{\partial x_i}$ and we set $A_{ij}:=A(e_i,e_j)$.

Set $f:=\operatorname{tr} A$, $f_d:=A_{11}-A_{22}$, and $f_m:=2A_{12}$. In Subsection 4.1 we use Codazzi–Mainardi equations to control $\nabla f$ in terms of $f_m$, $f_d$, $\nabla f_m$, and $\nabla f_d$ (here, if $w:\Sigma\to\mathbf{R}$, then $\nabla w$ denotes the gradient of $g$ in the Riemannian manifold $\Sigma$; that is, for any vector field $X:\Sigma\to T\Sigma$, we have $g(\nabla w,X)=dw(X)$).

Potentially this control will depend in a rather subtle way on the conformal parameterization $\varphi$. This is not a surprise, since the functions $f_d$ and $f_m$ *depend* on $\varphi$ (whereas $\operatorname{tr} A$ depends only on the immersion of $\Sigma$ in $\mathbf{R}^3$). In Subsection 4.2 we use the results of Section 2 in order to choose $\varphi^\pm$ which satisfy (a) and (b) and enjoy good bounds. We then show that these bounds and the relation derived in Subsection 4.1 are sufficient to prove (c).

4.1. **Key calculation.** Let $\varphi$, $e_i$, $A_{ij}$, $f$, $f_d$ and $f_m$ be as above. When $w$ is a function, $D_{e_i}w$ denotes the Lie derivative of $w$ with respect to $e_i$, whereas we will use the notations $\partial_{x_i}w$ and $w_i$ for $\frac{\partial}{\partial x_i}[w\circ\varphi]=D_{\frac{\partial}{\partial x_i}}w=hD_{e_i}w$.

If $X$ is a vector field on $\Sigma$, then we denote by $\nabla_{e_i}X$ the covariant derivative of $X$ with respect to $e_i$. For every $(2,0)$–tensor $B$ on $\Sigma$, $\nabla B$ denotes the usual $(3,0)$–tensor given by

$$\nabla B(X,Y,Z) \;:=\; D_X(B(Y,Z)) - B(\nabla_X Y,Z) - B(Y,\nabla_X Z)\,.$$

We set $\nabla B_{ijk}=\nabla B(e_i,e_j,e_k)$ and recall Codazzi–Mainardi equations:

$$\nabla A_{ijk} \;=\; \nabla A_{jik}\,. \tag{24}$$

To compute $\nabla f$, recall that $\nabla f=(D_{e_1}f)\,e_1+(D_{e_2}f)\,e_2$. Straightforward calculations give

$$\begin{aligned} D_{e_1}f &= D_{e_1}(A_{11}+A_{22}) = \nabla A_{111}+\nabla A_{122} + 2A(\nabla_{e_1}e_1,e_1)+2A(\nabla_{e_1}e_2,e_2) \\ D_{e_1}f_d &= D_{e_1}(A_{11}-A_{22}) = \nabla A_{111}-\nabla A_{122} + 2A(\nabla_{e_1}e_1,e_1)-2A(\nabla_{e_1}e_2,e_2) \\ D_{e_2}f_m &= 2D_{e_2}A_{12} = 2\nabla A_{212} + 2A(\nabla_{e_2}e_1,e_2)+2A(e_1,\nabla_{e_2}e_2) \end{aligned}$$

Thus $D_{e_1}f = D_{e_1}f_d + D_{e_2}f_m + 2\tilde{R}_1$, where

$$\tilde{R}_1 \;=\; A(\nabla_{e_1}e_1, e_1) + A(\nabla_{e_1}e_2, e_2) - A(\nabla_{e_2}e_1, e_2) - A(e_1, \nabla_{e_2}e_2)\,. \tag{25}$$

Recall that $hD_{e_1}h = D_{\frac{\partial}{\partial x_i}}h = h_i$. Straightforward computations give:

$$\nabla_{e_1}e_1 \;=\; -\frac{h_2}{h^2}e_2 \qquad \nabla_{e_2}e_1 \;=\; \frac{h_1}{h^2}e_2 \qquad \nabla_{e_1}e_2 \;=\; \frac{h_2}{h^2}e_1 \qquad \nabla_{e_2}e_2 \;=\; -\frac{h_1}{h^2}e_1\,. \tag{26}$$

Plugging these relations into (25) we get

$$\tilde{R}_1 \;:=\; -\frac{2h_2}{h^2}A_{12} + \frac{h_1}{h^2}(A_{22} - A_{11}) \;=\; -\frac{h_2}{h^2}f_m - \frac{h_1}{h^2}f_d\,. \tag{27}$$

A similar computation for $D_{e_2}f$ yields $D_{e_2}f = -D_{e_2}f_d + D_{e_1}f_m + 2\tilde{R}_2$, where $\tilde{R}_2$ is given by an expression similar to the one of (27). Recall that $h_i = D_{\frac{\partial}{\partial x_i}}f = \partial_{x_i}f$. Hence

$$\begin{cases} \partial_{x_1}f \;=\; \partial_{x_1}f_d + \partial_{x_2}f_m + 2h\tilde{R}_1 \\ \partial_{x_2}f \;=\; -\partial_{x_2}f_d + \partial_{x_1}f_m + 2h\tilde{R}_2\,. \end{cases} \tag{28}$$

Denote by $R$ the vector

$$R \;:=\; (R_1, R_2) \;:=\; (2h\tilde{R}_1, 2h\tilde{R}_2)\,, \tag{29}$$

by $\mathrm{div}_E R$ the "Euclidean" divergence $\partial_{x_1}R_1 + \partial_{x_2}R_2$ and by $\Delta_E f$ the "Euclidean laplacian" $\partial^2_{x_1}f + \partial^2_{x_2}f$. Then

$$\Delta_E f \;=\; \partial^2_{x_1}f_d - \partial^2_{x_2}f_d + 2\partial_{x_1}\partial_{x_2}f_m + \mathrm{div}_E R\,. \tag{30}$$

4.2. **Choice of $\varphi^\pm$.** Thanks to Lemma 2.2 and Proposition 3.2, $\Sigma$ is a sphere and there exist a universal constant $C$ and a conformal parameterization $\psi : \mathbf{S}^2 \to \Sigma$ such that

$$\psi^* g \;=\; \overline{h}^2\sigma \qquad\qquad C^{-1} \;\le\; \overline{h} \;\le\; C \qquad\qquad \|d\overline{h}\|_{L^2} \;\le C\,. \tag{31}$$

Clearly, there exist a universal constant $C_1$ and two conformal parameterizations $\varphi_1, \varphi_2$ : $\mathbf{R}^2 \to \mathbf{S}^2$ such that

(a') $\varphi_1(\mathcal{D}_1) \cup \varphi_2(\mathcal{D}_1) = \mathbf{S}^2$;

(b') $\mathrm{ar}(\varphi_1(\mathcal{D}_1) \cap \varphi_2(\mathcal{D}_1)) \ge 1$;

(c') $\|\varphi_i\|_{C^0(K)} + \|\varphi_i\|_{C^1(K)} + \|\varphi_i\|_{C^2(K)} \le C_1(K)$ for every compact set $K$.

Let us define $\varphi^+ := \psi \circ \varphi_1$ and $\varphi^- := \psi \circ \varphi_2$. Clearly, $\varphi^\pm$ are conformal and for some universal constant $C$, they satisfy (a) and (b). It remains to show (c). Without loss of generality we show it for $\varphi = \varphi^+$. We fix a system of orthonormal coordinates $x_1, x_2$ in $\mathbf{R}^2 \supset \mathcal{D}_1$ and we adopt the notation of Subsection 4.1. Thus, in this system of conformal coordinates, the metric $g$ on $\Sigma$ is given by $h^2\delta_{ij}$. Set $f := \mathrm{tr}\ A$ as in Subsection 4.1.

Our goal is to bound $\|f - \lambda\|_{L^{2,\infty}(\varphi(\mathcal{D}_1))}$ for some $\lambda \in \mathbf{R}$. Since the conformal factor enjoys $L^\infty$ estimates from above and from below, this is equivalent to show that $\|f - \lambda\|_{L^{2,\infty}(\mathcal{D}_1)} \le C\delta$. Thus, from now on we work in the Euclidean disk $\mathcal{D}_1$: in order to achieve our estimate we use equation (30).

**First estimate** Let us denote by $\hat{w}$ the Fourier transform of $w$ and by $\check{w}$ the inverse Fourier transform. Moreover let $\xi$ be the frequency variables. Recall that since $\varphi : \mathbf{R}^2 \to \mathbf{S}^2$, the functions $f$, $f_m$ and $f_d$ are defined everywhere on $\mathbf{R}^2$. Let $\zeta$ be a smooth cut–off function supported on $\mathcal{D}_{3/2}$ and such that $\varphi = 1$ on $\mathcal{D}_1$. Define $f'$ as

$$f_1 \;:=\; \frac{(\xi_1^2 - \xi_2^2)}{|\xi|^2}\widehat{\zeta f}_d + 2\frac{\xi_1\xi_2}{|\xi|^2}\widehat{\zeta f}_m \qquad f' \;:=\; \check{f}_1 \,.$$

By Plancherel theorem, there exists a constant $C$ (which depends on the cut–off function $\varphi$) such that

$$\|f'\|_{L^2} \;\leq\; C\left(\|f_d\|_{L^2(\mathcal{D}_2)} + \|f_m\|_{L^2(\mathcal{D}_2)}\right) \;\leq\; C_1\delta \,.$$

Moreover, on the set $\mathcal{D}_{3/2}$ we have

$$\Delta_E f' \;=\; \partial^2_{x_1} f_d - \partial^2_{x_2} f_d + 2\partial_{x_1}\partial_{x_2} f_m \,. \tag{32}$$

**Second estimate** Let $K(x) = \frac{1}{2\pi}\log(|x|)$ be the fundamental solution of the Laplacian in $\mathbf{R}^2$ and set $f'' = K * \mathrm{div_E}\,\mathrm{R}$. Thus $f'' = (\partial_{x_1}K) * R_1 + (\partial_{x_2}K) * R_2$. Recall the definition of $R$ in (29). By (27) we have

$$R_1 \;=\; -\frac{\overline{h}_2}{\overline{h}} f_m - \frac{\overline{h}_1}{\overline{h}} f_d \,.$$

Hence, the estimate (31) gives that $\|R_1\|_{L^1} \leq C\delta$. An analogous estimate holds for $R_2$. The locality of convolution, Lemma B.1 and Lemma B.2 give that $\|f''\|_{L^{2,\infty}(\mathcal{D}_2)} \leq C\delta$. Moreover,

$$\Delta_E f'' \;=\; \mathrm{div} R \,. \tag{33}$$

**Third estimate** Let $\alpha := f - f'' - f'$. Then, thanks to (30), (32), and (33), $\alpha$ is harmonic on $\mathcal{D}_{3/2}$. Moreover, the relations (28) give

$$\begin{cases} \partial_{x_1}\alpha &= \quad \partial_{x_1} f_d + \partial_{x_2} f_m + R_1 - \partial_{x_1}(f' + f'') \\ \partial_{x_2}\alpha &= -\partial_{x_2} f_d + \partial_{x_1} f_m + R_2 - \partial_{x_2}(f' + f'') \end{cases} .$$

Let $|||\cdot|||_{\mathcal{D}_{3/2}}$ be a norm which is controlled by both the $L^1(\mathcal{D}_{3/2})$ norm and the $W_0^{-1,2}(\mathcal{D}_{3/2})$ norm. Then, the various estimates give that $|||\nabla\alpha|||_{\mathcal{D}_{3/2}} \leq C\delta$. Since $\alpha$ is harmonic and $\mathcal{D}_1 \subset\subset \mathcal{D}_{3/2}$, there is a universal constant $C_1$ such that $\|\nabla\alpha\|_{L^\infty} \leq C_1\delta$. Thus, for some $\lambda > 0$ and for some universal constant $C_2$, we have $\|\alpha - \lambda\|_{L^\infty(\mathcal{D}_1)} \leq C_2\delta$. Since $f = f' + f'' + \alpha$, we get

$$\|f - \lambda\|_{L^{2,\infty}(\mathcal{D}_1)} \;\leq\; C_3\|f'\|_{L^2(\mathcal{D}_1)} + C_4\|f''\|_{L^{2,\infty}(\mathcal{D}_1)} + C_5\|\alpha - \lambda\|_{L^\infty(\mathcal{D}_1)} \;\leq\; C_6\delta \,. \tag{34}$$

## 5. Proof of the $L^2$ estimate for $A - \mathrm{Id}$

In the previous section we have achieved the following: If we define $2\overline{H} := \int_\Sigma \mathrm{tr}\; A$, then $\|A - \overline{H}\mathrm{Id}\|_{L^{2,\infty}} \leq C\delta$. The goal of this section is to use this information to prove

$$\int_\Sigma |A - \mathrm{Id}|^2 \;\leq\; C\delta^2 \,. \tag{35}$$

In order to do this we will show that $|1-\overline{H}^2| \leq C\delta^2$. This is sufficient to get (35). Indeed

$$\begin{aligned} |\operatorname{tr} A - 2\overline{H}|^2 &= \kappa_1^2 + \kappa_2^2 + 4\overline{H}^2 + 2\kappa_1\kappa_2 - 4\overline{H}\kappa_1 - 4\overline{H}\kappa_2 \\ &= |\kappa_1 - \kappa_2|^2 + 4\overline{H}^2 - 4\overline{H}\operatorname{tr} A + 4\det A\,. \end{aligned} \tag{36}$$

Integrating (36) and taking into account $\int_\Sigma \det A = 4\pi = \operatorname{ar}(\Sigma)$ and $\int_\Sigma \operatorname{tr} A = 2\overline{H}\operatorname{ar}(\Sigma)$, we have

$$\int_\Sigma |\operatorname{tr} A - 2\overline{H}\mathrm{Id}|^2 = \frac{1}{2}\int_\Sigma \big|\mathring{A}\big|^2 + 16\pi(1-\overline{H}^2)\,.$$

Thus, $|1-\overline{H}^2| \leq C\delta^2$ would imply $\int_\Sigma |A - \overline{H}\mathrm{Id}|^2 \leq C\delta^2$. Moreover, for $\delta$ small enough, $|1-\overline{H}^2| \leq C\delta^2$ implies $(1-\overline{H})^2 \leq C\delta^2$. Easy calculations give

$$|A - \mathrm{Id}|^2 \leq 2|A - \overline{H}\mathrm{Id}|^2 + 2(1-\overline{H})^2,$$

which would give (35).

For later purposes, we collect the inequality

$$\|A - \overline{H}\mathrm{Id}\|^2_{L^2} \leq C\delta^2 + C_1|1-\overline{H}^2|\,, \tag{37}$$

which is a direct consequence of the computations above. Moreover, we will make use of the following generalization of Wente's estimate:

**Lemma 5.1.** *Let $f, g, h \in C^\infty(\mathbf{S}^2)$. Then there exists a universal constant $C$ such that*

$$\int_{\mathbf{S}^2} h dg \wedge df \leq C\|dh\|_{L^{2,\infty}}\|dg\|_{L^2}\|df\|_{L^2}\,. \tag{38}$$

*Proof.* In local charts, thanks to Theorem 3.3, we have the $\mathcal{H}^1$ estimate

$$\|dg \wedge df\|_{\mathcal{H}^1(\mathcal{D}_1)} \leq C\|dg\|_{L^2(\mathcal{D}_1)}\|df\|_{L^2(\mathcal{D}_1)}$$

in the Euclidean disk $\mathcal{D}_1$. A finite covering of $\mathbf{S}^2$ with smooth coordinate patches yields

$$\|dg \wedge df\|_{\mathcal{H}^1(\mathbf{S}^2)} \leq C\|dg\|_{L^2(\mathbf{S}^2)}\|df\|_{L^2(\mathbf{S}^2)}$$

The duality between Hardy and BMO (see Theorem A.6 and Corollary A.7) gives

$$\int_{\mathbf{S}^2} h dg \wedge df \leq C|h|_{BMO}\|dg\|_{L^2}\|df\|_{L^2}\,. \tag{39}$$

Thanks to Lemma B.3 we have $|h|_{BMO} \leq C\|dh\|_{L^{2,\infty}}$. □

5.1. **Setting.** Using the Gauss–Bonnet formula and the fact that $8\pi\overline{H} = \int_\Sigma \operatorname{tr} A$ we get that

$$4\pi(1-\overline{H}^2) = \int_\Sigma \det A - \overline{H}\int_\Sigma \operatorname{tr} A + \overline{H}^2\int_\Sigma 1\,. \tag{40}$$

We denote by $N : \Sigma \to \mathbf{S}^2 \subset \mathbf{R}^3$ the Gauss map. Fix a conformal map $\psi : \mathbf{S}^2 \to \Sigma \subset \mathbf{R}^3$ satisfying the requirements of Proposition 3.2 and a conformal map $\varphi : \mathbf{R}^2 \supset \mathcal{D}_1 \to \mathbf{S}^2$. Denote by

- $\Psi : \mathcal{D}_1 \to \Sigma \subset \mathbf{R}^3$ the conformal map $\psi \circ \varphi$;
- $\tilde{h}^2$ and $h^2$ the conformal factors of $\Psi$ and $\psi$;
- $M$ and $N'$ the maps $N \circ \Psi$ and $N \circ \psi$.

Fix an orthonormal system of coordinates $y_1, y_2, y_3$ on $\mathbf{R}^3$ and an orthonormal system $x_1, x_2$ on $\mathcal{D}_1$. If $a$ and $b$ are two vectors of $\mathbf{R}^3$, then $a \wedge b$ denotes the vector of $\mathbf{R}^3$ which is the standard wedge product of $a$ and $b$.

5.2. **Algebraic computations.** As a first step we give give some formulas for $\tilde{h}^2$, $\tilde{h}^2(\det dN)\circ\Psi$ and $\tilde{h}^2(\mathrm{tr}\ dN)\circ\Psi$.

**First Computation** Since $\Psi$ is conformal, we have

$$\det d\Psi \;=\; |\Psi_{,x_1} \wedge \Psi_{,x_2}|\ , \tag{41}$$

where $\Psi_{,x_i}$ denotes the map $\frac{\partial\Psi}{\partial x_i} : \mathcal{D}_1 \to \mathbf{R}^3$. In equation (41) we make a slight abuse of notation. Indeed

- On the left hand side, we consider $\Psi$ as a map taking values on $\Sigma$. Thus $\det d\Psi$ has the usual meaning, since $d\Psi_p$ is a linear map from $T_p\mathbf{R}^2 \to T_{\Psi(p)}\Sigma$.
- On the right hand side, we consider $\Psi$ as a map taking values on $\mathbf{R}^3$.

We now fix the convention on the wedge product of vectors of $\mathbf{R}^3$ in such a way that

$$M \cdot \Psi_{,x_1} \wedge \Psi_{,x_2} \;=\; |\Psi_{,x_1} \wedge \Psi_{,x_2}|\ . \tag{42}$$

Hence we can write

$$\tilde{h}^2 \;=\; M \cdot \Psi_{,x_1} \wedge \Psi_{,x_2}\ . \tag{43}$$

**Second Computation** The normal $M$ is perpendicular to both $M_{,x_1}$ and $M_{,x_2}$. Moreover, the orientation convention which yields (42) gives

$$\det dM \;:=\; M \cdot M_{,x_1} \wedge M_{,x_2}\ . \tag{44}$$

Similarly to (41), equation (44) must be understood in the following way:

- On the left hand side, we consider $M$ as a map taking values on $\mathbf{S}^2$. Thus $\det dM$ has the usual meaning;
- On the right hand side, we consider $M$ as a map taking values on $\mathbf{R}^3$.

The discussion above gives the equality

$$\tilde{h}^2(\det dN)\circ\Psi \;=\; \det dM \;=\; M \cdot M_{,x_1} \wedge M_{,x_2}\ . \tag{45}$$

**Third Computation** Note that $M_{,x_i} = [dN\circ\Psi](\Psi_{,x_i})$. Thus, thanks to the conformality of $\Psi$, we have

$$\begin{aligned}(\mathrm{tr}\ dN)\circ\Psi \;&=\; \left[dN\circ\Psi\left(\frac{\Psi_{,x_1}}{|\Psi_{,x_1}|}\right)\right]\cdot\frac{\Psi_{,x_1}}{|\Psi_{,x_1}|} + \left[dN\circ\Psi\left(\frac{\Psi_{,x_2}}{|\Psi_{,x_2}|}\right)\right]\cdot\frac{\Psi_{,x_2}}{|\Psi_{,x_2}|}\\ &=\; \frac{1}{\tilde{h}^2}\left[M_{,x_1}\cdot\Psi_{,x_1} + M_{,x_2}\cdot\Psi_{,x_2}\right]\ .\end{aligned}$$

Since $\Psi$ is conformal we have

$$M_{,x_1}\cdot\Psi_{,x_1} \;=\; M_{,x_1}\cdot(\Psi_{,x_2}\wedge M) \;=\; M\cdot M_{,x_1}\wedge\Psi_{,x_2}\ .$$

Thus, we get

$$\tilde{h}^2(\mathrm{tr}\ dN)\circ\Psi \;=\; (M\cdot M_{,x_1}\wedge\Psi_{,x_2} + M\cdot\Psi_{,x_1}\wedge M_{,x_2})\ . \tag{46}$$

Combining (43), (45), and (46) we get

$$\begin{aligned}\int_{\Psi(\mathcal{D}_1)} \left(\det A - \overline{H}\,\mathrm{tr}\ A + \overline{H}^2\right)\zeta &= \int_{\mathcal{D}_1} \tilde{h}^2 \left((\det dN)\circ\Psi - \overline{H}(\mathrm{tr}\ dN)\circ\Psi + \overline{H}^2\right)\zeta\circ\Psi \\ &= \int_{\mathcal{D}_1} \left(M\cdot(M-\overline{H}\Psi)_{,x_1}\wedge(M-\overline{H}\Psi)_{,x_2}\right)\zeta\circ\Psi\,, \quad (47)\end{aligned}$$

for every $\zeta\in C^\infty_c(\Psi(\mathcal{D}_1))$.

5.3. **Skew–symmetric quantities.** Let $f,g:\mathcal{D}_1\to\mathbf{R}^3$ be two vector–valued maps. Denote by $f_i, g_i$, $i\in\{1,2,3\}$ the components of $f$ and $g$ in a system of orthonormal coordinates of $\mathbf{R}^3$. Then, straightforward computations give the following identity:

$$f\cdot g_{,x_1}\wedge g_{,x_2} = \sum_{i,j,k=1}^3 \varepsilon_{ijk} f_i\, dg_j\wedge dg_k\,. \quad (48)$$

where $\varepsilon_{ijk}$ is the totally antisymmetric tensor given by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3)\\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3)\\ 0 & \text{otherwise}\,.\end{cases}$$

Thus, equations (47) and (48) give

$$\begin{aligned}&\int_{\Psi(\mathcal{D}_1)} \left(\det A - \overline{H}\,\mathrm{tr}\ A + \overline{H}^2\right)\zeta \\ &= \sum_{i,j,k=1}^3 \varepsilon_{ijk}\int_{\mathcal{D}_1}\left(M_i\, d\left[(M-\overline{H}\Psi)_j\right]\wedge d\left[(M-\overline{H}\Psi)_k\right]\right)\zeta\circ\Psi\,, \quad (49)\end{aligned}$$

for every $\zeta\in C^\infty_c(\Psi(D_1))$. Since $\varphi:\mathcal{D}_1\to\varphi(\mathcal{D}_1)\subset\mathbf{S}^2$ is a diffeomorphism, we can use $\varphi^{-1}$ to pull back the forms on the right hand side of (49) on $\varphi(\mathcal{D}_1)$. Recalling that $N'=M\circ\varphi^{-1}$ and $\psi=\Psi\circ\varphi^{-1}$, we get

$$\begin{aligned}&\int_{\psi(\varphi(\mathcal{D}_1))} \left(\det A - \overline{H}\,\mathrm{tr}\ A + \overline{H}^2\right)\zeta \\ &= \sum_{i,j,k=1}^3 \varepsilon_{ijk}\int_{\varphi(\mathcal{D}_1)}\left(N'_i\, d\left[(N'-\overline{H}\psi)_j\right]\wedge d\left[(N'-\overline{H}\psi)_k\right]\right)\zeta\circ\psi\,. \quad (50)\end{aligned}$$

Hence, thanks to the arbitrariness of the conformal map $\varphi$, the previous equation gives that, for every $\zeta\in C^\infty(\mathbf{S}^2)$ which is supported in a set of diameter strictly less than $4\pi$, we have

$$\begin{aligned}&\int_{\psi(\mathbf{S}^2)} \left(\det A - \overline{H}\,\mathrm{tr}\ A + \overline{H}^2\right)\zeta\circ\psi^{-1} \\ &= \sum_{i,j,k=1}^3 \varepsilon_{ijk}\int_{\mathbf{S}^2}\left(N'_i\, d\left[(N'-\overline{H}\psi)_j\right]\wedge d\left[(N'-\overline{H}\psi)_k\right]\right)\zeta\,. \quad (51)\end{aligned}$$

A partition of unity on $\mathbf{S}^2$ gives

$$\int_\Sigma \left(\det A - \overline{H}\,\mathrm{tr}\ A + \overline{H}^2\right) \;=\; \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \int_{\mathbf{S}^2} N_i'\, d\,[(N' - \overline{H}\psi)_j] \wedge d\,[(N' - \overline{H}\psi)_k]\,. \tag{52}$$

Integrating by parts we can write

$$\int_{\mathbf{S}^2} N_i'\, d\,[(N' - \overline{H}\psi)_j] \wedge d\,[(N' - \overline{H}\psi)_k] \;=\; \int_{\mathbf{S}^2} -(N' - \overline{H}\psi)_j\, dN_i' \wedge d\,[(N' - \overline{H}\psi)_k]\,.$$

5.4. **Final estimates.** Thanks to Lemma 5.1 we have

$$\left| \int_{\mathbf{S}^2} [(N' - \overline{H}\psi)_j]\, dN_i' \wedge d\,[(N' - \overline{H}\psi)_k] \right| \;\leq\; \|N' - \overline{H}\psi\|_{L^{2,\infty}} \|dN'\|_{L^2} \|d(N' - \overline{H}\psi)\|_{L^2}\,. \tag{53}$$

Thus we conclude that

$$\left| \int_\Sigma (\det A - \overline{H}\,\mathrm{tr}\ A + \overline{H}^2) \right| \;\leq\; C \|dN'\|_{L^2(\mathbf{S}^2)} \|d(N' - \overline{H}\psi)\|_{L^2(\mathbf{S}^2)} \|d(N' - \overline{H}\psi)\|_{L^{2,\infty}(\mathbf{S}^2)}\,, \tag{54}$$

for some universal constant $C$. Since $\psi$ is conformal and satisfies the bounds given by Proposition 3.2, we have that there exist universal constants $C_1, C_2$ such that

$$\|dN'\|_{L^2(\mathbf{S}^2)} \;\leq\; C_1 \|dN\|_{L^2(\Sigma)} \;\leq\; C_2$$

$$\|d(N' - \overline{H}\psi)\|_{L^2(\mathbf{S}^2)} \;\leq\; C_1 \|dN - \overline{H}\mathrm{Id}\|_{L^2(\Sigma)}$$

$$\|d(N' - \overline{H}\psi)\|_{L^{2,\infty}(\mathbf{S}^2)} \;\leq\; C_1 \|dN - \overline{H}\mathrm{Id}\|_{L^{2,\infty}(\Sigma)} \;\leq\; C_2\delta\,.$$

Thus, taking into account (40) and (54) we get

$$|1 - \overline{H}^2| \;\leq\; C_3 \delta \|A - \overline{H}\mathrm{Id}\|_{L^2(\Sigma)}\,. \tag{55}$$

Recalling (37) we conclude

$$\|A - \overline{H}\mathrm{Id}\|^2_{L^2(\Sigma)} \;\leq\; C\delta^2 + C_4\delta \|A - \overline{H}\mathrm{Id}\|_{L^2(\Sigma)}\,,$$

which, by Young's inequality, yields

$$\|A - \overline{H}\mathrm{Id}\|^2_{L^2(\Sigma)} \;\leq\; C\delta^2 + \frac{C_4^2\delta^2}{2} + \frac{\|A - \overline{H}\mathrm{Id}\|^2_{L^2(\Sigma)}}{2}\,.$$

Hence,

$$\|A - \overline{H}\mathrm{Id}\|^2_{L^2(\Sigma)} \;\leq\; C_5\delta^2$$

and plugging this into (55) we get $|1 - \overline{H}^2| \;\leq\; C_6\delta^2$, which completes the proof.

## 6. $\Sigma$ is $W^{2,2}$ close to a round sphere

To complete the proof of Theorem 1.1 it remains to show the estimate (2), under the assumption that $\|\mathring{A}\|^2_{L^2} < 8\pi$. The difficulties in getting a conformal $\psi$ satisfying (2) are considerably increased by the action of the conformal group of the sphere. In order to choose $\psi$, as a first step we impose the normalization conditions of Lemma 3.4 and we show that these conditions imply that the conformal factor of $\psi$ is close to 1 (see Subsection 6.1). In a second step, we prove that this, together with the bound on $\|A - \mathrm{Id}\|_{L^2(D_\rho)}$ implies that $\psi$ is close to a smooth isometric embedding of $\mathbf{S}^2$ (see Subsections 6.2, 6.3).

6.1. **The conformal factor of $\psi$ is close to 1.** Fix $\psi$ as in Lemma 3.4 and Proposition 3.2 and denote by $h = e^u$ its conformal factor. The goal of this subsection is to show the existence of a universal constant $C$ such that

$$\|e^u - 1\|_{W^{1,2}} + \|u\|_{W^{1,2}} \;\leq\; C\delta\,. \tag{56}$$

To do so we first show that for $\delta \downarrow 0$, the map $\psi$ must converge to a conformal map, in fact a rigid motion in view of the normalizations. Then we use a linearization of the equation $-\Delta_{\mathbf{S}^2} u = Ke^{2u} - 1$ to get the optimal estimate on $\delta$.

First we gather all the information acquired in the previous sections (see (57) and Proposition 3.2):

$$u \text{ satisfies } -\Delta_{\mathbf{S}^2} = Ke^{2u} - 1 \text{ and } \textstyle\int e^{2u} = 1 \tag{57}$$

$$\|u\|_{L^\infty} + \|u\|_{W^{1,2}} \leq C \text{ for some universal constant } C \tag{58}$$

$$\text{Let } \mathbf{S}^\pm_i \text{ be as in Lemma 3.4. Then } \int_{\mathbf{S}^\pm_i} |A|^2 e^{2u} = 4\pi + \delta^2/2\,. \tag{59}$$

$$\int_{\mathbf{S}^2} |A - \mathrm{Id}|^2 e^{2u} \;\leq\; C\delta^2 \tag{60}$$

**Step 1** We begin by proving the following statement

$$\text{Fix } p < \infty \text{ and } \eta > 0. \text{ If } \delta > 0 \text{ is sufficiently small, then } \|e^{2u} - 1\|_{L^p} + \|u\|_{L^p} \leq \eta. \tag{61}$$

Since $e^{2v}$ is a locally Lipschitz function, thanks to (58) there exists a constant $C$, independent of $u$, such that

$$\left|e^{2u} - 1\right| \;\leq\; C|u|\,. \tag{62}$$

Thus we have $\|e^{2u} - 1\|_{L^p} \leq C\|u\|_{L^p}$. Assume, by contradiction, that (61) is false. Then there exist $\eta > 0$ and sequences $\delta_n \downarrow 0$, $\{u_n\} \subset C^\infty(\mathbf{S}^2)$ such that

- (57), (58), (59), and (60) hold (with $u_n$ and $\delta_n$ in place of $u$ and $\delta$);
- $\|u_n - 1\|_{L^p} \geq \eta > 0$.

Thanks to these assumptions, $\Delta_{\mathbf{S}^2} u_n$ is a bounded sequence in $L^1$. Let $D(\Delta)$ be the set of functions $f \in L^1(\mathbf{S}^2)$ with 0 average. Recall that $\Delta^{-1}_{\mathbf{S}^2} : D(\Delta) \to W^{1,q}$ is a compact operator for every $q < 2$. Thus a subsequence of $u_n$, not relabeled, converges strongly in $W^{1,q}$ to some $u_\infty$. Equations (60) and (59) give that $K_n - 1$ converges to 0 strongly in $L^1$. Since $e^{2u}$ is bounded and converges strongly in $L^q$ to 1, by the dominated convergence Theorem we conclude that $Ke^{2u}$ converges strongly in $L^1$ to 1. Passing to the limit in (57), (58), (59), and (60) we get

$$-\Delta_{\mathbf{S}^2} u_\infty = e^{2u_\infty} - 1 \qquad \int_{\mathbf{S}_i^\pm} e^{2u_\infty} = 2\pi\,.$$

The first identity implies that $e^{u_\infty}$ is the conformal factor of a map $\psi_\infty : \mathbf{S}^2 \to \mathbf{S}^2$. The second identity implies that $u_\infty = 0$. Hence $u_n \to 0$ in $W^{1,q}$ for every $q < 2$. Since $u_n$ is bounded, we get that $u_n \to 0$ strongly in every $L^p$. This gives a contradiction.

**Step 2** Let $\mathcal{S}$ be the space of functions $v$ such that

$$-\Delta_{\mathbf{S}^2} v \;=\; e^{2v} - 1 \qquad \int_{\mathbf{S}^2} e^{2v} \;=\; 4\pi\,.$$

As noticed above, $\mathcal{S}$ is given by the logarithms of conformal factors of elements of the conformal group of the sphere. Hence, $\mathcal{S}$ is a finite dimensional submanifold of $C^\infty(\mathbf{S}^2)$. Let $C > 0$ and set $\mathcal{S}_C := \mathcal{S} \cap \{v : \|v\|_{L^\infty} \le C\}$. We now show that:

$$\text{If } \delta \text{ is sufficiently small, there exists } v \in \mathcal{S}_C \text{ with } \int_{\mathbf{S}^2} (u-v)\left(e^{2u} - e^{2v}\right) \;=\; 0\,. \tag{63}$$

To see this define the map

$$\Phi : \mathcal{S}_C \ni v \;\to\; \int_{\mathbf{S}^2} v\left(e^{2v} - 1\right)\,.$$

It is easy to see that $\Phi(\mathcal{S}_C)$ contains an interval $[-\varepsilon, \varepsilon]$ and that $\Phi$ is continuous. Note that

$$\int (u-v)\left(e^{2u} - e^{2v}\right) \;=\; \Phi(v) + \int_{\mathbf{S}^2} u\left(e^{2u} - e^{2v}\right) + \int_{\mathbf{S}^2} v\left(1 - e^{2u}\right)\,. \tag{64}$$

Thanks to (61), we have, for $\delta$ is sufficiently small,

$$\left|\int_{\mathbf{S}^2} u\left(e^{2u} - e^{2v}\right) + \int_{\mathbf{S}^2} v\left(1 - e^{2u}\right)\right| \;<\; \varepsilon\,.$$

Hence there exists $v \in \mathcal{S}_C$ such that the right hand side of (64) vanishes.

**Step 3** We now show that for $v$ as in (63), we have $\|d(u-v)\|_{L^2} \le C\delta$. Note that

$$-\Delta_{\mathbf{S}^2}(u-v) = \left(e^{2u} - e^{2v}\right) + (\det A - 1)e^{2u}\,.$$

Multiplying by $u - v$, integrating by parts and using (63) we get

$$\begin{aligned} \|d(u-v)\|_{L^2} \;&=\; \int_{\mathbf{S}^2} |\nabla_{\mathbf{S}^2}(u-v)|^2 \;=\; \int_{\mathbf{S}^2} (u-v)(\det A - 1)e^{2u} \\ &=\; \int_\Sigma (u-v)(\det A - 1)\,. \end{aligned} \tag{65}$$

Recall that $\mathrm{ar}(\Sigma) = 4\pi$. Thus by the Gauss–Bonnet formula $\int_\Sigma (\det A - 1) = 0$. Hence, if we denote by $a$ the average of $u - v$ on $\Sigma$, we get

$$\int_\Sigma (u-v)(\det A - 1) \;=\; \int_\Sigma (u - v - a)(\det A - 1)$$

Recall that $\det A - 1 = \det(A - \mathrm{Id}) + (\mathrm{tr}\, A - 2)$. Moreover:

$$\left|\int_\Sigma (u-v-a)\det(A-\mathrm{Id})\right| \;\leq\; (2\|u\|_{L^\infty}+2\|v\|_{L^\infty})\|\det(A-\mathrm{Id})\|_{L^1} \leq\; C\|A-\mathrm{Id}\|^2_{L^2} \;\leq\; C\delta^2\,, \tag{66}$$

whereas

$$\left|\int_\Sigma (u-v-a)(\mathrm{tr}\, A-2)\right| \;\leq\; \|u-v-a\|_{L^2}\|A-\mathrm{Id}\|_{L^2} \;\leq\; C\delta\|d(u-v)\|_{L^2} \leq\; \frac{C^2}{2}\delta^2+\frac{\|d(u-v)\|^2_{L^2}}{2}\,. \tag{67}$$

Thus, plugging (66) and (67) into (65) we get

$$\|d(u-v)\|^2_{L^2} \;\leq\; C\delta^2+\frac{\|d(u-v)\|^2_{L^2}}{2}\,,$$

which yields the desired estimate.

**Step 4** Conclusion.

By the Poincaré inequality, $\|d(u-v)\|_{L^2} \leq C\delta$ yields $\|u-v-a\|_{L^2} \leq C\delta$ for some $a \in \mathbf{R}$. Thus, from the uniform bounds on $\|v\|_{L^\infty} \leq C$ and $\|u\|_{L^\infty}$ and from (62) we get $\left|\int_{\mathbf{S}^2}\int e^{2(u-a)} - \int_{\mathbf{S}^2} e^{2v}\right| \leq C\delta$. Recall that $\int e^{2u} = \int e^{2v} = 4\pi$. Thus we get $|e^{-a}-1| \leq C\delta$, which implies $|a| \leq C\delta$. Thus $\|u-v\|_{L^2} \leq C\delta$. Again, by the uniform bounds on $\|v\|_{L^\infty} \leq C$ and $\|u\|_{L^\infty}$ and from (62), we get

$$\left|2\pi - \int_{\mathbf{S}^i_j} e^{2v}\right| \;=\; \left|\int_{\mathbf{S}^i_j} e^{2u} - \int_{S^i_j} e^{2v}\right| \;\leq\; C\delta\,. \tag{68}$$

Recall that the only $v \in \mathcal{S}$ such that $\int_{\mathbf{S}^i_j} e^{2v} = 2\pi$ is the trivial solution $v \equiv 0$. Since $\mathcal{S}$ is a finite dimensional manifold contained in $C^\infty$, the inequality (68) implies $\|v\|_{C^2} \leq C'\delta$, for some universal constant $C'$. This gives $\|u\|_{W^{1,2}} \leq C'\delta$. Recalling (62) we get (56).

6.2. **Cartan Formalism.** Let $D_\rho$ be a disk of $\mathbf{S}^2$ and let $(e_1, e_2)$ be an orthonormal frame on $D_\rho$. We assume that this orthonormal frame is generated by a conformal map $\varphi : \mathcal{D}_r \to D_\rho$ via the relations $e_i = \partial_{x_i}\varphi/|\partial_{x_i}\varphi|$. Moreover, we assume that $\|\varphi\|_{C^1}$ is bounded by a universal constant (which is certainly possible if, for instance, $\rho \leq \pi$). We define

$$\Phi : \mathbf{S}^2 \to (e_1, e_2, e_1\wedge e_2) \in SO(3)\,. \tag{69}$$

$$\Psi : \mathbf{S}^2 \;\to\; \Big(e^{-u}d\psi(e_1)\,,\; e^{-u}d\psi(e_2)\,,\; e^{-2u}d\psi(e_1)\wedge d\psi(e_2)\Big) \;\in SO(3)\,. \tag{70}$$

Note that $e^{-2u}d\psi(e_1)\wedge d\psi(e_2) = N\circ\Psi$. Hereby we fix a system of coordinates in $\mathbf{R}^3$ and we regard the elements of SO(3) as matrices: Thus, according to definition (69), for $x \in \mathbf{S}^2$, $\Phi(x)$ is the matrix which has $e_1(x)$, $e_2(x)$, and $e_1(x)\wedge e_2(x)$ as row vectors. We endow SO(3) with the operator norm and we denote by $B\cdot F$ and by $B^{-1}$ respectively the matrix product of $B$ and $F$ and the inverse of $B$.

We want to show that there exist constants $\rho > 0$ and $C > 0$ such that

$$\min_{R\in \mathrm{SO}(3)} \|\Phi - R\cdot\Psi\|_{L^2(D_\rho)} \leq C\delta\,. \tag{71}$$

Note that the left hand side of (71) is actually independent of the choice of the frame. Thus, though the estimate is derived for the particular frame of $TD_\rho$ chosen above, we conclude:

- Let $(e_1, e_2)$ be *any orthonormal frame* and $\Phi$, $\Psi$ as in (69), (70). Then (71) holds.

An easy covering argument yields a constant $C'$ such that, for some $R \in SO(3)$:

$$\text{For } \textit{every } V \text{ and for every frame } (e_1, e_2) \text{ on } TV, \text{ we have } \|\Phi - \Psi \cdot R\|_{L^2(V)} \leq C'\delta \tag{72}$$

One basic property of moving frames (see for instance vol. 3 of [Sp]) is the existence of unique 1–forms with values in skew–symmetric matrices $U$ and $W$ such that

$$\begin{aligned} d\Phi &= \Phi \cdot U \\ d\Psi &= \Psi \cdot W\,. \end{aligned}$$

Alternatively, $U$ and $W$ can be regarded as matrices of 1–forms on $\mathbf{S}^2$. We define the norm of $|U_x|$ (for $x \in D_\rho$) as

$$|U_x| := \sup_{v \in T_x\mathbf{S}^2, |v|=1} |U_x(v)|\,,$$

where $|U_x(v)|$ is the operator norm of the matrix $U_x(v) \in \mathbb{M}^{3\times 3}$.

We now come to the proof of (71). Consider $\Lambda := \Phi \cdot \Psi^{-1}$ and compute

$$\begin{aligned} d\Lambda &= d\Phi \cdot \Psi^{-1} - \Psi^{-1} \cdot d\Psi \cdot \Psi^{-1} \cdot \Phi \\ &= \Phi \cdot U \cdot \Psi^{-1} - \Phi \cdot \Psi^{-1} \cdot \Psi \cdot W \cdot \Psi^{-1} \;=\; \Phi \cdot (U - W) \cdot \Psi^{-1}\,. \end{aligned}$$

The following Lemma is a standard Poincaré inequality (for the reader's convenience we report its proof in Appendix D):

**Lemma 6.1.** *There exists a universal constant $C$ such that for some $R \in SO(3)$ we have*

$$\|\Lambda - R\|_{L^2(D_\rho)} \leq C\rho \|d\Lambda\|_{L^2(D_\rho)}\,.$$

Thus, since $\rho \leq \pi$, there is a constant $C$ such that

$$\|\Lambda - R\|_{L^2(D_\rho)} \leq C\|U - W\|_{L^2(D_\rho)}\,.$$

To complete the proof of (71) it is sufficient to show that there is a universal constant $C$ such that

$$\|U - W\|_{L^2(D_\rho)} \leq C\delta\,. \tag{73}$$

Let $\theta_1, \theta_2$ the basis of the cotangent space $T^*M$ which is dual to $(e_1, e_2)$. Moreover, recall that

- $e^v$ is the conformal factor of $\varphi : \mathcal{D}_r \to D_\rho$;
- $x_1, x_2$ is an orthonormal basis for $\mathcal{D}_r$;
- $e_i = \partial_{x_i}\varphi / |\partial_{x_i}\varphi| = e^{-v}\partial_{x_i}\varphi$.

Since the second fundamental form of the sphere is the identity, we have (see e.g. page 97 of Volume III of [Sp])

$$\begin{array}{rcccll} -W_{31} &=& W_{13} &=& A\big(e^{-u}d\psi(e_1), e^{-u}d\psi(e_1)\big)\theta_1 & +A\big(e^{-u}d\psi(e_1), e^{-u}d\psi(e_2)\big)\theta_2 \\ -W_{32} &=& W_{23} &=& A\big(e^{-u}d\psi(e_1), e^{-u}d\psi(e_2)\big)\theta_1 & +A\big(e^{-u}d\psi(e_2), e^{-u}d\psi(e_2)\big)\theta_2 \\ -U_{31} &=& U_{13} &=& \theta_1 & \\ -U_{32} &=& U_{23} &=& & \theta_2 \quad . \end{array}$$

Since $\|A-\mathrm{Id}\|_{L^2}\leq C\delta$, the previous equations give $\|W_{i3}-U_{i3}\|\leq C\delta$. Thus it only remains to show that $\|U_{12}-W_{12}\|\leq C\delta$. Recall that

$$\begin{aligned} W_{12}(e_j) &= g\big(\nabla^{\Sigma}_{e^{-u}d\psi(e_j)}(e^{-u}d\psi(e_2)), e^{-u}d\psi(e_1)\big) \\ U_{12}(e_j) &= \theta^1\big(\nabla^{\mathbf{S}^2}_{e_j}e_2\big)\,, \end{aligned}$$

where $g$ is the Riemannian metric on $\Sigma$. Thus

$$\begin{aligned} U_{12} &= e^{-v}\Big\{\big[\partial_{x_2}v\big]\,\theta_1-\big[\partial_{x_1}v\big]\,\theta_2\Big\} \\ W_{12} &= e^{-u\circ\varphi-v}\left\{\Big[\partial_{x_2}\big(v+u\circ\varphi\big)\Big]\,\theta_1-\Big[\partial_{x_1}(v+u\circ\varphi\big)\Big]\,\theta_2\right\} \end{aligned}$$

Recall that $\|\varphi\|_{C^1}$ is bounded by a universal constant and that $\|e^{-u}-1\|_{L^2}+\|u\|_{W^{1,2}}\leq C\delta$. Hence we conclude that

$$\|U_{12}-W_{12}\|_{L^2(D_\rho)}\;\leq\;C\delta\,.$$

6.3. **Conclusion.** Let us compose $\psi$ with the inverse of the rotation $R$ appearing in (72). By abuse of notation, we denote this map by $\psi$ as well. Then the previous subsection shows the existence of constants $C$ and $\rho$ such that:

- For every disk $D$ of radius $\rho$ in $\mathbf{S}^2$ there exists a conformal map $\varphi$ such that $\|\varphi\|_{C^2}\leq C$ and, if we define $e_i:=\partial_{x_i}\varphi/|\partial_{x_i}\varphi|$ and $\Phi$, $\Psi$ as in (69), (70), then:

$$\begin{aligned} &d\Psi=\Psi\cdot W \qquad d\Phi=\Phi\cdot U \\ &\|\Psi-\Phi\|_{L^2(D)}\;\leq\;C\delta \\ &\|U-W\|_{L^2(D)}\;\leq\;C\delta \end{aligned} \tag{74}$$

Hence, we easily get that

$$\|d\Psi-d\Phi\|_{L^2(D)}\;\leq\;\|\Psi\cdot(U-W)\|_{L^2(D_\rho)}+\|(\Phi-\Psi)\cdot U\|_{L^2(D_\rho)}\;\leq\;C\delta\,, \tag{75}$$

where we have also used the fact that $\|U\|_{L^\infty}$ depends on $\|\varphi\|_{C^1}$, which is bounded by a uniform constant (recall the choice of $\varphi$). Denote by $\mathrm{id}:\mathbf{S}^2\to\mathbf{R}^3$ the standard embedding of the round sphere in the Euclidean space. Note that (74) gives that $\|d\psi-d(\mathrm{id})\|_{L^2(D)}\leq C\delta$. Thus (since $\rho$ is a fixed constant), by an easy covering argument we get $\|d\psi-d(id)\|_{L^2(\mathbf{S}^2)}\leq C_1\delta$ for some universal constant $C_1$. By the Poincaré inequality, there is a vector $c_\Sigma\in\mathbf{R}^3$ such that

$$\|\psi-(c_\Sigma+\mathrm{id})\|_{W^{1,2}(\mathbf{S}^2)}\leq C_2\delta\,.$$

It is not difficult to see that (75) and (74) give an estimate on the second derivatives of $\psi-(c_\Sigma+\mathrm{id})$, yielding the desired bound

$$\|\psi-(c_\Sigma+\mathrm{id})\|_{W^{2,2}(\mathbf{S}^2)}\;\leq\;C_3\delta\,.$$

Indeed fix a system coordinates on $\mathbf{R}^3$ and call $\psi_k$, $\mathrm{id}_k$ the components of $\psi,\mathrm{id}$. Since $\|\varphi\|_{C^2}$ is bounded by a universal constant, it is sufficient to check

$$\left\|\partial^2_{x_ix_j}(\psi_k-\mathrm{id}_k)\right\|_{L^2(D)}\;\leq\;C_4\delta\,. \tag{76}$$

Note that

$$\partial_{x_j}\psi_k\;=\;\big|\partial_{x_j}\varphi\big|\,\big[d\psi(e_j)\big]_k\;=\;h\,\Psi_{jk}$$

where $\Psi_{jk}$ denotes the $jk$ entry of the matrix $\Psi$ and $h$ is the conformal factor of $\varphi$. Thus,

$$\partial^2_{x_i x_j}\psi_k \;=\; \big(h\,\partial_{x_i}h\big)\,\Psi_{jk} + h^2\,d\Psi_{jk}(e_i)\,.$$

Analogously

$$\partial^2_{x_i x_j}\mathrm{id}_k \;=\; \big(h\,\partial_{x_i}\,h\big)\,\Phi_{jk} + h^2\,d\Phi_{jk}(e_i)\,.$$

Hence, thanks to the uniform bounds on $\|h\|_{L^\infty}$ and $\|\partial_{x_j}h\|_{L^\infty}$, the estimates (75) and (74) give (76).

## 7. Optimality

In this section we prove the optimality of Theorem 1.1.

**Proposition 7.1.** *There exists a family of smooth connected compact surfaces $\Sigma_r \subset \mathbf{R}^3$ without boundary such that:*

$$C \geq \mathrm{ar}(\Sigma_r) \geq c > 0 \text{ for every } r \tag{77}$$

$$\lim_{r\downarrow 0}\int_{\Sigma_r}|\mathring{A}|^p \;=\; 0 \qquad \text{for every } p<2 \tag{78}$$

*$\Sigma_r$ converges, in the Hausdorff topology, to the union of two round spheres* (79)

$$\lim_{r\downarrow 0}\left(\inf_\lambda\int_{\Sigma_r}|A-\lambda\mathrm{Id}|^p\right) \;>\; 0\,. \tag{80}$$

*Proof.* The idea of the construction is the following. Let us take two round spheres $\Sigma_1$ and $\Sigma_2$ of radii 1 and 1/2. Then we can glue them with a small hyperbolic neck $\Gamma$ so that the integral $\int_\Gamma|A|^p$ is as small as we want. We now give the details of this construction. The estimate of the quantity $\int_\Gamma|A|^p$ will be simplified by using catenoid necks.

**Detailed construction.** Consider the family of curves $\{\gamma_r\}$ known as catenaries, i.e. the graphs of the functions $f_r:\mathbf{R}\to\mathbf{R}$ given by

$$f_r(x) \;:=\; r\cosh\left(\frac{x}{r}\right)\,.$$

The surface generated by a revolution of $\gamma_r$ around the $x$–axis is called a catenoid and will be denoted by $\Gamma_r$. It is well known that catenoids are minimal surfaces (see for instance page 202 of [DC]). Thus $\mathrm{tr}\,A=\kappa_1+\kappa_2=0$ everywhere on $\Gamma_r$.

Let $x,y,z$ be a system of coordinates in $\mathbf{R}^3$ and assume that the catenoid $\Gamma_r$ is given by $|(x,y)|=r\cosh\left(\frac{z}{r}\right)$. For every $r>0$ we take:

- A round sphere of radius 1/2 centered in a point $(0,0,z_1)$ with $z_1>0$ and tangent to $\Gamma_r$ in a circle $\gamma^1_r$.
- A round sphere of radius 1 centered in a point $(0,0,z_2)$ with $z_2<0$ and tangent to $\Gamma_r$ in a circle $\gamma^2_r$.

Consider the closed surface $\Sigma_r$ which is made of:

- The part of the sphere $\Sigma_1$ lying above $\gamma^1$ (which we denote by $S^2_r$);
- The part of the sphere $\Sigma_2$ lying below $\gamma^2$ (which we denote by $S^1_r$);
- The portion of catenoid lying between $\gamma^1$ and $\gamma^2$ (which we denote by $T_r$).

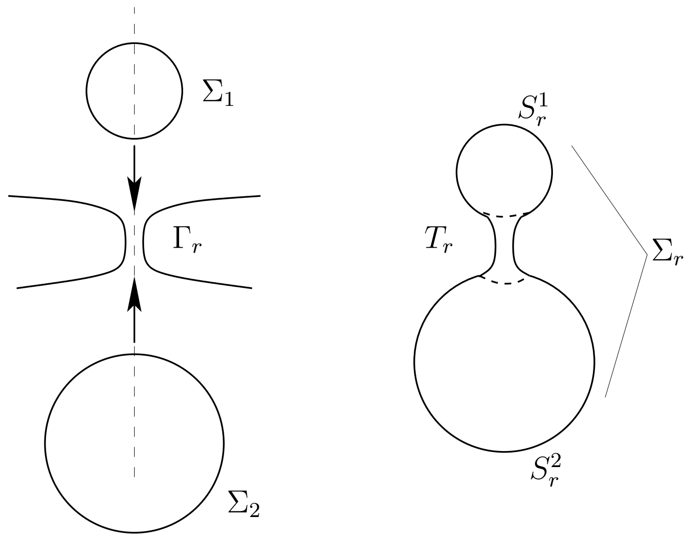


FIGURE 1. Construction of the surface $\Sigma_r$

See Fig. 1 below.

**Step 1** Behavior of $\Sigma_r$ for $r \downarrow 0$.

The circles $\gamma_r^i$ are given by

$$\Gamma_r \cap \{z = z_i(r)\}$$

and straightforward computations give that

$$z_1(r) \text{ is the unique positive solution of } \cosh\left(\frac{z_1(r)}{r}\right) = \frac{1}{\sqrt{2r}}$$

$$z_2(r) \text{ is the unique negative solution of } \cosh\left(\frac{z_2(r)}{r}\right) = \frac{1}{\sqrt{r}}.$$

Hence $z_i(r) \downarrow 0$ as $r \downarrow 0$. Moreover, the radius of $\gamma_r^1$ is $\sqrt{r/2}$, whereas the radius of $\gamma_r^2$ is $\sqrt{r}$. Hence we conclude that

> The surfaces $S_r^1$ and $S_r^2$ converge, respectively, to a sphere $S_\infty^1$ of radius $1/2$ and to a sphere $S_\infty^2$ of radius 1, which are tangent at $(0,0,0)$. (81)
>
> The area of the neck $T_r$ converges to 0. (82)

**Step 2** Estimates.

We now prove that

$$\lim_{r\downarrow 0} \int_{T_r} |\mathring{A}|^p = 0. \tag{83}$$

Since $T_r$ is a portion of a minimal surface, tr $A = 0$ on $T_r$. Thus (83) is equivalent to

$$\lim_{r\downarrow 0} \int_{T_r} |A|^p = 0. \tag{84}$$

Again because of the minimal surface equation, $2\det A = -|A|^2$ on $T_r$. Thus, by Gauss–Bonnet Theorem:

$$8\pi = \int_{\Sigma_r} 2\det A = \int_{S_r^1\cup S_r^2} 2\det A - \int_{T_r} |A|^2 . \tag{85}$$

Since $S_r^1$ and $S_r^2$ are both portions of round spheres, we have

$$\int_{S_r^1\cup S_r^2} 2\det A \le 16\pi .$$

Thus, $\int_{T_r} |A|^2 \le 8\pi$ and, by Hölder inequality,

$$\int_{T_r} |A|^p \le (\mathrm{ar}(T_r))^{\frac{2-p}{2}} \left(\int_{T_r} |A|^2\right)^{\frac{p}{2}} \le (8\pi)^{\frac{p}{2}} (\mathrm{ar}(T_r))^{\frac{2-p}{2}} . \tag{86}$$

By (81), the inequality (86) yields (84). Thus:

- The bound (77) is trivially satisfied.
- Since $S_r^1$ and $S_r^2$ are subsets of round spheres, we have
$$\int_{\Sigma_r} |\mathring{A}|^p = \int_{T_r} |\mathring{A}|^p ,$$
and (78) follows from (83).
- Thanks to (82) and (84)
$$\begin{aligned}\lim_{r\downarrow 0}\left(\inf_\lambda \int_{\Sigma_r} |A-\lambda \mathrm{Id}|^p\right) &= \inf_\lambda \left(\int_{S^1_\infty} |A-\lambda \mathrm{Id}|^p + \int_{S^2_\infty} |A-\lambda \mathrm{Id}|^p\right)\\ &= \inf_\lambda \left[2\pi\left(\frac{1}{2}-\lambda\right)^2 + 8\pi(1-\lambda)^2\right] > 0,\end{aligned}$$
which gives (80).

Note that the surfaces just constructed are $C^1$ and piecewise $C^2$. However, they are all surfaces of revolution: The curves which generate them are $C^1$ and piecewise $C^\infty$, whereas the higher derivatives have four points of jump discontinuity. Hence, a standard smoothing argument yields a family of surfaces of revolution which are $C^\infty$ and satisfy all the requirements of the Proposition. □

## Appendix A. Hardy and BMO spaces

We recall here the definitions of Hardy and BMO spaces (see for example [St], sections 1,2,3 and 4). First of all, if $f : \mathbf{R}^n \to \mathbf{R}$, the maximal function $Mf$ is defined as

$$Mf(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \left|\int_{B_r(x)} f\right| .$$

When $f : \Sigma \to \mathbf{R}$ and $\Sigma \subset \mathbf{R}^3$ is a compact surface, we let $d(\Sigma)$ be the *intrinsic* diameter of $\Sigma$ (i.e. the maximum of $\mathrm{dist}_\Sigma(x,y)$ for $x, y \in \Sigma$) and we define

$$Mf(x) := \sup_{0<r<d(\Sigma)} \frac{1}{\mathrm{ar}(D_r(x))} \left|\int_{D_r(x)} f\right| .$$

**Definition A.1.** *The Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ (resp. $\mathcal{H}^1(\Sigma)$) consists of the functions $f \in L^1(\mathbf{R}^n)$ (resp. $f \in L^1(\Sigma)$) such that $Mf \in L^1(\mathbf{R}^n)$ (resp. $Mf \in L^1(\Sigma)$). The norm $\|f\|_{\mathcal{H}^1}$ is given by $\|Mf\|_{L^1}$.*

The following result follows from [FS]

**Theorem A.2.** *Let $g \in \mathcal{H}^1(\mathbf{R}^2)$. Then the equation $\Delta_{\mathbf{R}^2} u = g$ admits a solution $u : \mathbf{R}^2 \to \mathbf{R}$ which is continuous, belongs to $W^{2,1}$ and satisfies*

$$\|du\|_{L^2} + \|u\|_{L^\infty} \;\leq\; C\|g\|_{\mathcal{H}^1}\,,$$

*for some universal constant $C$.*

Using local charts a partition of unity, Theorem A.2 yields the following

**Corollary A.3.** *Let $g \in \mathcal{H}^1(\mathbf{S}^2)$. Then the equation $\Delta_{\mathbf{S}^2} u = g$ admits a solution $u_0$ which is continuous, belongs to $W^{2,1}$ and satisfies*

$$\|du_0\|_{L^2(\mathbf{S}^2)} + \|u_0\|_{L^\infty} \;\leq\; C\left(1 + \|g\|_{\mathcal{H}^1(\mathbf{S}^2)}\right)\,. \tag{87}$$

**Remark A.4.** *Since harmonic functions on $\mathbf{S}^2$ are constant, the general solution of $\Delta_{\mathbf{S}^2} u = g$ can be written as $u = u_0 + c$. Thus the normalization condition*

$$\int_{\mathbf{S}^2} e^{2u} \;=\; 4\pi\,,$$

*yields an estimate like* (87) *also for $u$.*

In section 5 we use the duality between BMO and Hardy, due to Fefferman.

**Definition A.5.** *Let $f \in L^1_{loc}(\mathbf{R}^n)$. We say that $f \in BMO$ if*

$$|f|_{BMO} := \sup_{x\in\mathbf{R}^n}\ \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f - f^{x,r}| \qquad \textit{is finite,}$$

*where $f^{x,r}$ denotes the average of $f$ on $B_r(x)$. As above, we can extend the definition to compact surfaces by taking the second supremum among disks of radius smaller than $d(\Sigma)$.*

**Theorem A.6.** *Let $f, g \in C^\infty_c(\mathbf{R}^n)$. Then*

$$\left|\int fg\right| \;\leq\; \|f\|_{\mathcal{H}^1}|g|_{BMO}\,.$$

Again, using local charts and a partition of unity, we get

**Corollary A.7.** *Let $f, g \in C^\infty(\mathbf{S}^2)$. Then there exists a universal constant $C$ such that*

$$\left|\int_{\mathbf{S}^2} fg\right| \;\leq\; C\|f\|_{\mathcal{H}^1(\mathbf{S}^2)}|g|_{BMO(\mathbf{S}^2)}\,.$$

## Appendix B. The space $L^{2,\infty}$

Given a measure space $(\Omega,\mu)$ with a $\sigma$–finite measure $\mu$, the Marcinkiewicz space $L^{2,\infty}(\Omega,\mu)$ is defined as the set of functions

$$\left\{ f \,\middle|\, \text{there exists } C>0\text{:}\;\; \mu\big(\{f^2\geq k\}\big)\leq \frac{C}{k}\;\;\text{for every } k>0 \right\}.$$

For every $f\in L^{2,\infty}$ is natural to define

$$|f|_{L^{2,\infty}} \;:=\; \inf\left\{C:\mu\big(\{f^2\geq k\}\big)\leq\frac{C}{k}\text{ for every } k>0\right\}. \tag{88}$$

$|\cdot|$ is not a norm. However, it is possible to define a norm $\|\cdot\|_{L^{2,\infty}}$ which endows $L^{2,\infty}$ of a Banach space structure and such that

$$\frac{1}{k}\|\cdot\|_{L^{2,\infty}} \;\leq\; |\cdot|_{L^{2,\infty}} \;\leq\; k\|\cdot\|_{L^{2,\infty}}\,, \tag{89}$$

see e.g. Section 1.8 of [Z]. For the Proof of Proposition 4.1 we need the following two lemmas:

**Lemma B.1.** *If $f\in L^{2,\infty}(\mathbf{R}^n)$, $g\in L^1(\mathbf{R}^n)$, then*

$$\|f*g\|_{L^{2,\infty}} \;\leq\; \|f\|_{L^{2,\infty}}\|g\|_{L^1}\,. \tag{90}$$

**Lemma B.2.** *Let $K$ be the fundamental solution of the Laplacian in $\mathbf{R}^2$ given by $K(x)=\frac{1}{2\pi}\log(|x|)$. Then $\nabla K\in L^{2,\infty}(U)$ for every bounded set $U\subset\mathbf{R}^2$.*

Lemma B.1 follows easily from the fact that $\|\cdot\|_{L^{2,\infty}}$ is a norm, while Lemma B.2 is obtained directly from the definition of $|\cdot|_{L^{2,\infty}}$. Finally, in the proof of Theorem 1.1 we need the following

**Lemma B.3.** *Let $u\in C^\infty(\mathbf{S}^2,\mathbf{R})$. Then there exists a universal constant $C$ such that*

$$|u|_{BMO(\mathbf{S}^2)} \;\leq\; C\|du\|_{L^{2,\infty}(\mathbf{S}^2)}\,.$$

*Proof.* Lemma B.3 follows from the Sobolev embedding $W^{1,1}(\mathbf{S}^2)\hookrightarrow L^2(\mathbf{S}^2)$ and the fact that $|u|_{\mathbf{R}^2}$ and $|u|_{L^{2,\infty}(\mathbf{R}^2)}$ are both invariant under the rescalings $x\to rx$. Indeed, using local charts, it suffices to prove

$$|u|_{BMO(\mathcal{D}_1)} \;\leq\; C\|du\|_{L^{2,\infty}(\mathcal{D}_1)} \tag{91}$$

where $\mathcal{D}_1$ is the Euclidean unit disk. Recall that

$$|u|_{BMO(\mathcal{D}_1)} \;:=\; \sup_{y\in\mathcal{D}_1}\left[\sup_{r<\operatorname{dist}(y,\partial\mathcal{D}_1)}\frac{1}{\operatorname{ar}(\mathcal{D}_r(y))}\int_{\mathcal{D}_r(y)}|u-u^{y,r}|\right], \tag{92}$$

In view of the definition of $|u|_{BMO(\mathcal{D}_1)}$, it is sufficient to prove would be sufficient to prove

$$\frac{1}{\operatorname{ar}(\mathcal{D}_r(y))}\int_{\mathcal{D}_r(y)}|u-u^{y,r}| \;\leq\; C\|du\|_{L^{2,\infty}(\mathcal{D}_r(y))} \qquad \text{for all } r<1.$$

By invariance under translations, we can assume $y=0$. Moreover, we can assume that $r=1$. Indeed, define $u_r(x):=u(rx)$. Then,

$$\frac{1}{\operatorname{ar}(\mathcal{D}_r)}\int_{\mathcal{D}_r}\left|u-u^{0,r}\right| \;=\; \frac{1}{\operatorname{ar}(\mathcal{D}_1)}\int_{\mathcal{D}_1}\left|u_r-u_r^{0,1}\right|$$

and

$$\|u\|_{L^{2,\infty}(\mathcal{D}_r)} \;\leq\; k|u|_{L^{2,\infty}(\mathcal{D}_r)} \;=\; k|u_r|_{L^{2,\infty}(\mathcal{D}_1)} \;\leq\; k^2\|u\|_{L^{2,\infty}(\mathcal{D}_1)}\,.$$

Thus, the proof reduces to the inequality

$$\int_{\mathcal{D}_1} \left|u-u^{0,1}\right| \;\leq\; C\|du\|_{L^{2,\infty}(\mathcal{D}_1)}\,.$$

Clearly, for some universal constant $C$, we have $\|du\|_{L^1(\mathcal{D}_1)} \leq C\|du\|_{L^{2,\infty}(\mathcal{D}_1)}$. Moreover, the Poincaré and Schwartz inequalities give

$$\int_{\mathcal{D}_1} \left|u-u^{0,1}\right| \;\leq\; \pi^{1/2}\|u-u^{0,1}\|_{L^2(\mathcal{D}_1)} \;\leq\; C_1\pi^{1/2}\|du\|_{L^1(\mathcal{D}_1)} \;\leq\; C_1C\pi^{1/2}\|du\|_{L^{2,\infty}(\mathcal{D}_1)}\,.$$

This completes the proof. □

## Appendix C. Lemma on open sets

**Lemma C.1.** *Let $U\subset \mathbf{S}^2$ be an open set and assume that $\partial U\subset\gamma$, where $\gamma$ is a closed curve. Then there exists a constant $\delta>0$, depending only on* $\mathrm{ar}(U)$ *and* $\mathrm{len}\,(\gamma)$ *such that $U$ contains an open disk of radius $\delta$.*

*Proof.* We argue by contradiction. Then there exist a sequence of open sets $U_n$ and a sequence of closed curves $\gamma_n$ such that:

1. $\lim_n \mathrm{len}\,(\gamma_n) = C_1 > 0$ and $\lim_n \mathrm{ar}(U_n) = C_2 > 0$;
2. For every $\delta > 0$ there exists $N$ such that, for every $n > N$, $U_n$ does not contain any disk of radius $\delta$.

Let parameterize $\gamma_n$ by arc–length. Then there is a subsequence, not relabeled, which converges uniformly to a Lipschitz curve $\gamma_\infty$. Hence up to subsequences, $\overline{U}_n$ converges, in the Hausdorff topology, to a closed set $\overline{U}_\infty$ whose boundary is contained in $\gamma_\infty$. Due to 2., the set $\overline{U}_\infty$ has empty interior and thus $\mathrm{ar}(\overline{U}_\infty) = \mathrm{ar}(\partial\overline{U}_\infty) = 0$. But 1. implies that $\mathrm{ar}(\overline{U}_\infty) = C_2 > 0$. This is the desired contradiction. □

## Appendix D. Poincaré inequality for SO(3)–valued maps

Here we give a proof of Lemma 6.1. We embed $SO(3)\subset\mathbb{M}^{3\times 3} = \mathbf{R}^9$ and we set

$$\overline{\Lambda} \;=\; \frac{1}{\mathrm{ar}(D_\rho)}\int_{D_\rho}\Lambda,$$

Since the operator norm on $\mathbb{M}^{3\times3}$ is equivalent to the Euclidean norm on $\mathbf{R}^9$, the Poincaré inequality yields a constant $C$ such that

$$\|\Lambda-\overline{\Lambda}\|_{L^2(D_\rho)} \;\leq\; C\rho\|d\Lambda\|_{L^2(D\rho)}\,.$$

Note that

$$\begin{aligned}
\mathrm{dist}\,(\overline{\Lambda},\mathrm{SO}(3))^2 \;&=\; \frac{1}{\mathrm{ar}(D_\rho)}\int_{D_\rho}\mathrm{dist}\,(\overline{\Lambda},SO(3))^2\\
&\leq\; \frac{1}{\mathrm{ar}(D_\rho)}\int_{D_\rho}\big(|\Lambda-\overline{\Lambda}| + \mathrm{dist}\,(\Lambda,SO(3))\big)^2\\
&=\; \frac{1}{\mathrm{ar}(D_\rho)}\|\Lambda-\overline{\Lambda}\|^2_{L^2(D_\rho)}\,.
\end{aligned}$$

Thus there exists a map $R \in SO(3)$ such that

$$\|\Lambda - R\|_{L^2(D_\rho)} \;\leq\; \sqrt{2}C\rho\|d\Lambda\|_{L^2(D\rho)}\,.$$

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Camillo De Lellis, Max–Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany

*E-mail address*: delellis@mis.mpg.de

Stefan Müller, Max–Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany

*E-mail address*: sm@mis.mpg.de