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**Hierarchical Quadrature of Singular
Integrals**

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Hierarchical Quadrature for Singular Integrals

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Abstract

We introduce a method for the computation of singular integrals arising in the discretization of integral equations.

The basic method is based on the concept of *admissible subdomains*, known, e.g., from panel clustering techniques and \mathcal{H} -matrices: We split the domain of integration into a hierarchy of subdomains and perform standard quadrature on those subdomains that are amenable to it.

By using additional properties of the integrand, we can significantly reduce the algorithmic complexity of our approach.

The method works also well for hypersingular integrals.

AMS Subject Classification: 65D32, 42B20

Key words: Numerical quadrature, singular integrals

1 Introduction

We consider integrals as they arise in the boundary element method. Let us consider

$$I := \int_0^1 \int_0^1 |x - y|^\alpha dy dx \quad (1.1)$$

as an example. The kernel function is improperly integrable if $\alpha > -1$. However, since also strongly singular integrals appear in the BEM, $\alpha \leq -1$ will not be excluded. In the latter case, I has to be interpreted as *partie finie* integral in the sense of Hadamard [8, pp.184].

The standard quadrature methods apply one of the follow techniques:

1. Use a quadrature rule adapted to the singularity of the kernel (cf. [17, 16]).
2. Apply at least for one of the double integrals an exact integration. Hopefully the remaining integral is regular and standard Gauss quadrature works well (cf. [12, 17]).
3. Apply a transformation that removes the singularity (cf. [2, 10, 11]).
4. Apply a transformation like $\xi = x - y$, $\eta = x + y$, that changes the moving singularity $|x - y|^\alpha$ of (1.1) into the fixed singularity $|\xi|^\alpha$ and apply 1 or 2 (cf. [10]).

5. Adaptive refinement, i.e., the integration region is split into suitable subregions. Subregions that do not contain the singularity are treated by standard quadrature, the remaining subregions are split again (cf. [18]).

Although the techniques 1-4 are quite successful in special situations, they are of limited use. Method 5 might be costly because of many levels of refinement. The mentioned techniques depend strongly on the dimensionality of the integral. In particular, all methods are not easily extended to strongly singular integrals.

In this paper, we describe a technique which combines the adaptive refinement from 5 with further structural properties of the integrand. Eventually, we have to determine only few integrals over smooth integrands and to solve a small system of equations. Moreover, this approach extends to strong singularities.

Our assumptions are explained for

$$I := \int_0^1 \int_0^1 \kappa(x, y) dy dx.$$

We suppose that the integrand $\kappa(x, y)$ — possibly after subtracting a smooth part — satisfies two conditions: *translation invariance* and *homogeneity*. The first condition reads

$$\kappa(x, y) = \kappa(x + c, y + c) \text{ for all } c \in \mathbb{R}, \quad (1.2)$$

while a homogeneous integrand satisfies

$$\kappa(sx, sy) = s^\alpha \kappa(x, y) \text{ for all } s \in \mathbb{R}_{>0}, \quad (1.3)$$

where $\alpha \in \mathbb{R}$ is the degree of homogeneity. A closely related variant of (1.3) occurs, e.g., for $\kappa(x, y) = \log(|x - y|)$, where

$$\kappa(sx, sy) = \beta(s) + \kappa(x, y) \text{ for all } s \in \mathbb{R}_{>0}. \quad (1.4)$$

Furthermore, κ is assumed to be sufficiently smooth outside of a neighborhood of the possible singularity at $x = y$ (A possible qualification of the smoothness of κ is given by the asymptotic smoothness, cf. (2.13)).

In addition to (1.2), a symmetry condition

$$\kappa(y, x) = \kappa(x, y) \quad (1.5)$$

can be exploited (antisymmetry $\kappa(y, x) = -\kappa(x, y)$ would lead immediately to the trivial result $I = 0$).

Obviously, the integrand of example (1.1) satisfies the conditions (1.2) and (1.3) with the same value α as well as (1.5).

In Subsection 2.1 we first apply a recursive additive splitting of the integral which yields an infinite sum of subintegrals. The situation simplifies due to the translation invariance as explained in Subsection 2.2, where we identify equivalence classes of integrals. The essential step is the use of the homogeneity condition in Subsection 2.3, which reduces the infinite sum to a finite linear combination, where the number of terms is the number of different

equivalence classes. The remaining few integrals possess a regular integrand. The arising linear system is discussed in Subsection 2.4.

The use of homogeneity can be illustrated by the one-dimensional integral

$$Q := \int_0^1 \kappa(x, 0) dx.$$

We split Q into the sum

$$Q = \sum_{i=0}^{\infty} Q^{(i)}, \quad Q^{(i)} := \int_{2^{-i-1}}^{2^{-i}} \kappa(x, 0) dx,$$

which has to converge if κ is weakly singular. The homogeneity condition (1.3) together with the transformation $x = 2^{-i}t$ yields

$$Q^{(i)} = \int_{2^{-i-1}}^{2^{-i}} \kappa(x, 0) dx = 2^{-i} \int_{1/2}^1 \kappa(2^{-i}t, 0) dt = 2^{-i} 2^{-\alpha i} \int_{1/2}^1 \kappa(t, 0) dt = 2^{-(1+\alpha)i} Q^{(0)},$$

thus we find

$$Q = Q^{(0)} \sum_{i=0}^{\infty} 2^{-(1+\alpha)i}.$$

The assumed summability implies $\alpha > -1$ and

$$\sum_{i=0}^{\infty} 2^{-(1+\alpha)i} = \frac{1}{1 - 2^{-(1+\alpha)}}.$$

The integral

$$Q^{(0)} = \int_{1/2}^1 \kappa(t, 0) dt$$

can be approximated by standard quadrature techniques, since κ is assumed to be smooth outside the singularity at $t = 0$.

In some applications, the integrand may consist of a product of a kernel function κ satisfying the conditions (1.2) and (1.3) and smooth additional factors that do not satisfy these conditions. If these additional factors are polynomials, we can still use a modified version of our method that is presented in Section 3.

So far, we have assumed that the integral has a weak singularity. The case of a strong singularity of κ is discussed in Section 4. We recall the definition of partie finie integrals and show how this definition can directly be used in our approach. However, to do this we have to split the integration domain $[0, 1]^2$ of (1.1) into triangles. In Subsection 4.4 we show that the simpler splitting into subsquares works as well.

2 Weak singularities

Let us consider the approximate computation of

$$I := \int_0^1 \int_0^1 \kappa(x, y) dy dx$$

for $\alpha \in \mathbb{R}_{>-1}$. In the case of negative α , we have a singularity at the diagonal $x = y$ of the domain $[0, 1]^2$, so using standard quadrature techniques will not lead to acceptable results.

In this Section, we will derive an algorithm (cf. (2.8)) that expresses I in terms of the integrals

$$\begin{aligned} I_2 &= \int_0^1 \int_0^1 \kappa(x+2, y) dy dx, & I_{-2} &= \int_0^1 \int_0^1 \kappa(x, y+2) dy dx, \\ I_3 &= \int_0^1 \int_0^1 \kappa(x+3, y) dy dx, & I_{-3} &= \int_0^1 \int_0^1 \kappa(x, y+3) dy dx. \end{aligned}$$

Since all these integrals are regular, we can approximate them directly by standard quadrature techniques and get an approximation of I .

2.1 Splitting

While standard quadrature is not working on the *entire* domain of integration, there obviously are subdomains in which the integrand is analytic, e.g., $[0, 1/4] \times [1/2, 3/4]$ or $[3/4, 1] \times [1/4, 1/2]$.

Therefore it is straightforward to split the domain into subdomains on which standard quadrature is applicable, to compute the results and to add them up. In order to do this efficiently, we organize the subdomains in a hierarchical way: We split the original domain $\Omega = [0, 1] \times [0, 1]$ into four subdomains

$$\begin{aligned} \Omega_{0,0} &:= [0, 1/2] \times [0, 1/2], & \Omega_{1,0} &:= [1/2, 1] \times [0, 1/2], \\ \Omega_{0,1} &:= [0, 1/2] \times [1/2, 1], & \Omega_{1,1} &:= [1/2, 1] \times [1/2, 1], \end{aligned}$$

and define the corresponding integrals

$$\begin{aligned} I_{0,0} &:= \int_{\Omega_{0,0}} \kappa(x, y) dy dx, & I_{1,0} &:= \int_{\Omega_{1,0}} \kappa(x, y) dy dx, \\ I_{0,1} &:= \int_{\Omega_{0,1}} \kappa(x, y) dy dx, & I_{1,1} &:= \int_{\Omega_{1,1}} \kappa(x, y) dy dx. \end{aligned}$$

Obviously, the computation of $I_{0,0}$ and $I_{1,1}$ is just as hard as that of I : We again have to deal with a singularity along the diagonal of the domain. In the computation of $I_{1,0}$ and $I_{0,1}$, only a pointwise singularity occurs.

The original integral I can be computed by

$$I = I_{0,0} + I_{1,0} + I_{0,1} + I_{1,1}. \quad (2.1)$$

If we use the integrals $I, I_{0,0}, I_{1,0}, I_{0,1}$ and $I_{1,1}$ as vertices and denote the contributions they give to each other by directed edges, we can express (2.1) in the form of a *splitting graph* given in Figure 2.1.

Let us now consider the integral $I_{1,0}$ corresponding to the domain $\Omega_{1,0} = [1/2, 1] \times [0, 1/2]$. We once more split the domain in order to get

$$\begin{aligned} \Omega_{2,0} &:= [1/2, 3/4] \times [0, 1/4], & \Omega_{3,0} &:= [3/4, 1] \times [0, 1/4], \\ \Omega_{2,1} &:= [1/2, 3/4] \times [1/4, 1/2], & \Omega_{3,1} &:= [3/4, 1] \times [1/4, 1/2]. \end{aligned}$$

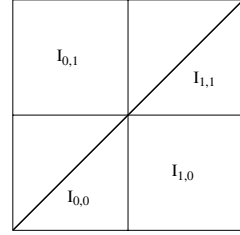
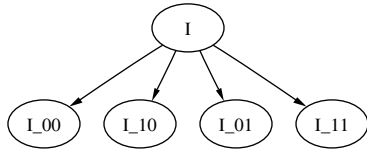


Figure 2.1: Splitting graph corresponding to (2.1)

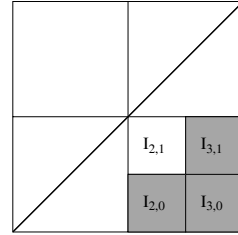
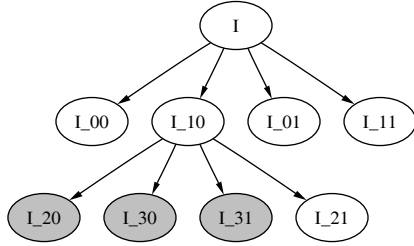


Figure 2.2: Splitting graph and domain decomposition corresponding to (2.1) and (2.2)

We define the integrals $I_{2,0}$, $I_{3,0}$, $I_{2,1}$ and $I_{3,1}$ as above and find

$$I_{1,0} = I_{2,0} + I_{3,0} + I_{2,1} + I_{3,1}. \quad (2.2)$$

Now the situation is changed: The integrals $I_{2,0}$, $I_{3,0}$ and $I_{3,1}$ (see gray squares in Figure 2.2) no longer involve a singularity, therefore we can apply standard quadrature to compute them. The integral $I_{2,1}$ is similar to $I_{1,0}$. By repeating this procedure for all subdomains still containing singularities, we can construct a splitting tree.

If the tree were finite, we could approximate the integrals corresponding to its leaves by standard quadrature and then accumulate them along its edges in order to obtain an approximation for the integral corresponding to the root and all intermediate subdomains.

Since the simple splitting is not able to eliminate singularities, this tree is infinite, so we cannot approximate the integral directly. What we *can* do is to consider a subtree, e.g., consisting of all nodes up to a given level ℓ in the tree, use quadrature on those leaves that admit it, approximate by zero on the remaining leaves and then sum up as before.

Splitting Ω as above leads, among others, to the subdomains $\Omega_{0,0}$ and $\Omega_{1,1}$, and these subdomains contain the same type of singularity as the original Ω . This means that the splitting tree, when cut off at level ℓ , will contain at least 2^ℓ leaves that cannot be treated by standard quadrature. Therefore, the complexity of the computation will grow exponentially in ℓ .

Section 6 of [18] deals with the problem of computing an approximation of an entire integral equation, and in this context the exponential growth of the complexity can be tolerated, since it only occurs for diagonal elements of the stiffness matrix and does not effect the total order of complexity.

We are interested in bounding the complexity of the approximation of single integrals,

so controlling only the average efficiency is not enough. In order to reach higher efficiency, we have to make use of additional properties of the kernel function.

2.2 Equivalence classes

We use the *translation invariance* (1.2) of the kernel κ :

$$\begin{aligned} I_{1,1} &= \int_{\Omega_{1,1}} \kappa(x, y) dy dx = \int_{1/2}^1 \int_{1/2}^1 \kappa(x, y) dy dx = \int_0^{1/2} \int_0^{1/2} \kappa\left(x + \frac{1}{2}, y + \frac{1}{2}\right) dy dx \\ &= \int_0^{1/2} \int_0^{1/2} \kappa(x, y) dy dx = \int_{\Omega_{0,0}} \kappa(x, y) dy dx = I_{0,0}. \end{aligned}$$

This equation holds because the subdomains satisfy

$$\Omega_{1,1} = \Omega_{0,0} + (1/2, 1/2),$$

i.e., because $\Omega_{1,1}$ is a translation of $\Omega_{0,0}$.

This means that the equation (2.1) takes the form

$$I = 2I_{0,0} + I_{1,0} + I_{0,1}. \quad (2.3)$$

Similarly, we find that $I_{2,0} = I_{3,1}$, and therefore

$$I_{1,0} = 2I_{2,0} + I_{3,0} + I_{2,1}. \quad (2.4)$$

We represent the additional factors in our splitting graph by adding them to the edges. The graph from Figure 2.2 then takes the form displayed in Figure 2.3.

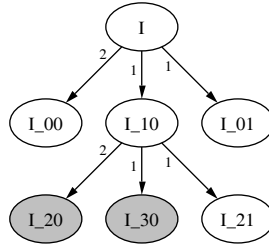


Figure 2.3: Reduced graph corresponding to (2.3) and (2.4)

In order to estimate the impact of the reduction, we consider a domain $[a, b] \times [c, d]$. Let $[a^0, b^0] \times [c^0, d^0]$ be one of the predecessors of this domain in the splitting graph. Since only domains containing singularities are split in the course of our procedure, we must have $[a^0, b^0] \cap [c^0, d^0] \neq \emptyset$. By construction, there are not more than 3 subdomains satisfying this condition. Since each subdomain has in turn exactly 4 successors in the splitting graph, there can be no more than 12 equivalence classes on each level of the splitting graph.

This implies that, by using equivalence of subdomains, we can reduce the number of subdomains, and therefore the complexity of the computation, on a level ℓ from $\mathcal{O}(2^\ell)$ to $\mathcal{O}(1)$.

2.3 Homogeneity

Although we can reduce the complexity of our algorithm by using equivalence classes, we still have to truncate the splitting graph in order to get a finite algorithm, and of course this truncation implies a loss of precision.

One way of avoiding this is to make use of the homogeneity (cf. (1.3)) of the kernel: We have

$$\kappa(sx, sy) = s^\alpha \kappa(x, y)$$

for all $s \in \mathbb{R}_{>0}$, and, due to the choice of our subdomain splitting, this implies

$$\begin{aligned} I_{2,1} &= \int_{1/2}^{3/4} \int_{1/4}^{1/2} \kappa(x, y) dy dx = \int_{1/4}^{1/2} \int_0^{1/4} \kappa(x, y) dy dx \\ &= \frac{1}{4} \int_{1/2}^1 \int_0^{1/2} \kappa\left(\frac{x}{2}, \frac{y}{2}\right) dy dx = \frac{2^{-\alpha}}{4} \int_{1/2}^1 \int_0^{1/2} \kappa(x, y) dy dx = \frac{2^{-\alpha}}{4} I_{1,0}. \end{aligned}$$

By setting $c := 2^{-\alpha}/4$, equation (2.4) takes the form

$$I_{1,0} = 2I_{2,0} + I_{3,0} + cI_{1,0}. \quad (2.5)$$

This means that now we have to introduce *cycles* into the splitting graph, leading to a graph of the form given in Figure 2.4.

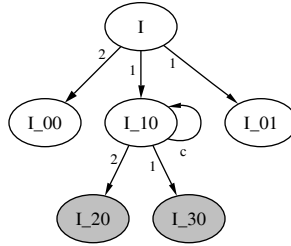


Figure 2.4: Reduced cyclic graph corresponding to (2.3) and (2.5)

Applying the entire procedure to the domain I , we end up with the finite splitting graph shown in Figure 2.5 instead of an infinite graph.

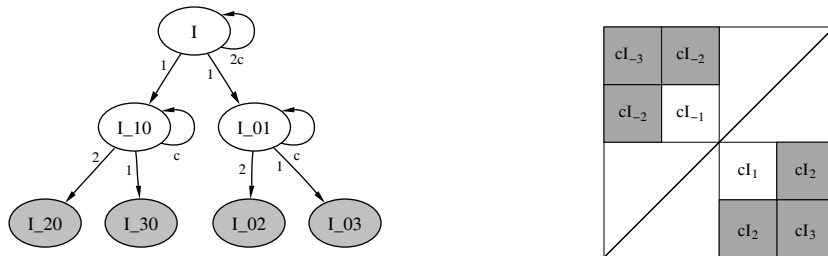


Figure 2.5: Final splitting graph and domain decomposition

As a finishing touch, we transform all integrals to the domain Ω , so we can use the same quadrature rule for all leaves of the splitting graph: We set

$$I_\gamma := \int_0^1 \int_0^1 \kappa(x + \gamma, y) dy dx$$

and rewrite the transfer equations in these new variables:

$$I = I_0 = 2cI_0 + cI_1 + cI_{-1}, \quad I_1 = cI_1 + 2cI_2 + cI_3, \quad I_{-1} = cI_{-1} + 2cI_{-2} + cI_{-3}. \quad (2.6)$$

2.4 System of equations

Since the splitting graph given in Figure 2.5 is no longer acyclic, we can no longer simply compute the value I by traversing the graph. Instead, we interpret the graph again as a system of linear equations for the unknown variables I , I_1 and I_{-1} and find

$$\begin{pmatrix} 2c & c & c \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} I \\ I_1 \\ I_{-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 2cI_2 + cI_3 \\ 2cI_{-2} + cI_{-3} \end{pmatrix} = \begin{pmatrix} I \\ I_1 \\ I_{-1} \end{pmatrix}, \quad (2.7)$$

which can be rewritten as

$$\begin{pmatrix} 1 - 2c & -c & -c \\ 0 & 1 - c & 0 \\ 0 & 0 & 1 - c \end{pmatrix} \begin{pmatrix} I \\ I_1 \\ I_{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 2cI_2 + cI_3 \\ 2cI_{-2} + cI_{-3} \end{pmatrix}. \quad (2.8)$$

Since $c = 2^{-\alpha}/4 < 1/2$, all the diagonal entries are positive, therefore this system has a unique solution.

If κ is symmetric (cf. (1.5)), we have $I_\gamma = I_{-\gamma}$ and can reduce the system to get

$$\begin{pmatrix} 1 - 2c & -2c \\ 0 & 1 - c \end{pmatrix} \begin{pmatrix} I \\ I_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2cI_2 + cI_3 \end{pmatrix}. \quad (2.9)$$

Remark 2.1 (Logarithmic kernel functions) *Let us consider the logarithmic kernel function $\kappa(x, y) = \log|x - y|$. It is invariant under translation, but not homogeneous. Therefore, we replace the homogeneity condition (1.3) by*

$$\kappa(sx, sy) = \beta(s) + \kappa(x, y),$$

i.e., equation (1.4) (for our example, $\beta(s) = \log(s)$ satisfies this equation) and find that

$$I_\gamma = \frac{1}{2}I_{2\gamma} + \frac{1}{4}I_{2\gamma-1} + \frac{1}{4}I_{2\gamma+1} + \beta(1/2)$$

holds. This implies

$$\begin{pmatrix} 1/2 & -1/4 & -1/4 \\ 0 & 3/4 & 0 \\ 0 & 0 & 3/4 \end{pmatrix} \begin{pmatrix} I \\ I_1 \\ I_{-1} \end{pmatrix} = \begin{pmatrix} \beta(1/2) \\ \beta(1/2) + I_2/2 + I_3/4 \\ \beta(1/2) + I_{-2}/2 + I_{-3}/4 \end{pmatrix}.$$

Due to the symmetry of κ , we have $I_1 = I_{-1}$ and find

$$\begin{pmatrix} 1/2 & -1/2 \\ 0 & 3/4 \end{pmatrix} \begin{pmatrix} I \\ I_1 \end{pmatrix} = \begin{pmatrix} \beta(1/2) \\ \beta(1/2) + I_2/2 + I_3/4 \end{pmatrix}. \quad (2.10)$$

A kernel function like $\log^2|x - y|$ leads to a more involved equation than (1.4); however, similar techniques can be applied.

2.5 Approximation

Up to this point, we have used no approximation. The equations (2.8), (2.9) and (2.10) hold for the *exact* integrals. In order to get a numerical approximation scheme, we replace the regular integrals I_2 , I_{-2} , I_3 and I_{-3} by approximations and use the equations (2.8), (2.9) or (2.10) in order to derive approximations of I , I_1 and I_{-1} .

2.5.1 Quadrature rule

The right-hand side of (2.8) contains integrals I_γ corresponding to regular integrals, i.e., with $|\gamma| > 1$. In order to find the solution I , we have to approximate these integrals.

We denote the m -th order Gauss quadrature points for the interval $[0, 1]$ by $(x_k)_{k=1}^m$ and the corresponding weights by $(w_k)_{k=1}^m$ and approximation I_γ by applying quadrature to both integrals:

$$\tilde{I}_\gamma := \sum_{k=1}^m \sum_{\ell=1}^m w_k w_\ell \kappa(x_k + \gamma, x_\ell). \quad (2.11)$$

2.5.2 Quadrature error estimate

Since Gauss quadrature is exact for polynomials of order $2m - 1$, we have

$$|I_\gamma - \tilde{I}_\gamma| \leq 2 \min_{k \in \mathcal{Q}_{2m-1}} \|\kappa - k\|_{\infty, [\gamma, 1+\gamma] \times [0, 1]}, \quad (2.12)$$

where \mathcal{Q}_{2m-1} is the space of tensor product polynomials of order $2m - 1$, i.e.,

$$\mathcal{Q}_{2m-1} = \text{span}\{p \otimes q : p, q \text{ are polynomials of order } 2m - 1\}.$$

We assume that the kernel function κ is *asymptotically smooth* with a singularity of order $g \in \mathbb{R}_{\geq 0}$, i.e., that there are constants $c_0, c_1 \in \mathbb{R}_{>0}$ such that

$$|\partial_x^\nu \partial_y^\mu \kappa(x, y)| \leq c_1 (c_0)^{|\nu+\mu|} (\nu + \mu)! \|x - y\|^{-g-|\nu+\mu|} \quad (2.13)$$

holds for all $\nu, \mu \in \mathbb{N}_0$. The parameter g will typically coincide with $-\alpha$ from (1.3).

The estimate (2.13) implies

$$\|\partial_x^\nu \partial_y^\mu \kappa\|_{\infty, \tau \times \sigma} \leq \frac{c_1}{\text{dist}(\tau, \sigma)^g} \left(\frac{c_0}{\text{dist}(\tau, \sigma)} \right)^{|\nu+\mu|} (\nu + \mu)!$$

for all intervals $\tau, \sigma \subseteq \mathbb{R}$ with $\tau \cap \sigma = \emptyset$, so we can apply [1, Theorem 3.2] to the standard tensor-product Chebyshev interpolation operator in dimension $d = 2$ in order to find a polynomial $k \in \mathcal{Q}_{2m-1}^2$ satisfying

$$\|\kappa - k\|_{\infty, \tau \times \sigma} \leq \frac{8ec_1(2m)^3}{\text{dist}(\tau, \sigma)^g} \left(1 + \frac{c_0 \text{diam}(\tau \times \sigma)}{\text{dist}(\tau, \sigma)} \right) \left(1 + \frac{2 \text{dist}(\tau, \sigma)}{c_0 \text{diam}(\tau \times \sigma)} \right)^{-2m}. \quad (2.14)$$

We apply this to $\tau = [\gamma, 1 + \gamma]$, $\sigma = [0, 1]$ and find

$$\text{diam}(\tau \times \sigma) = \sqrt{2}, \quad \text{dist}(\tau, \sigma) = \max\{0, |\gamma| - 1\},$$

so (2.14) takes the form

$$\|\kappa - k\|_{\infty, [\gamma, 1+\gamma] \times [0, 1]} \leq \frac{8ec_1(2m)^3}{(|\gamma| - 1)^g} \left(1 + \frac{c_0\sqrt{2}}{|\gamma| - 1}\right) \left(\frac{c_0}{c_0 + \sqrt{2}(|\gamma| - 1)}\right)^{2m}.$$

Combining this estimate with (2.12), we get

$$|I_\gamma - \tilde{I}_\gamma| \leq \frac{16ec_1(2m)^3}{(|\gamma| - 1)^g} \left(1 + \frac{c_0\sqrt{2}}{|\gamma| - 1}\right) \left(\frac{c_0}{c_0 + \sqrt{2}(|\gamma| - 1)}\right)^{2m},$$

so the quadrature error converges exponentially in the order m .

2.5.3 Approximation of singular integrals

In the right-hand side of (2.8) we have I_2 , I_3 , I_{-2} and I_{-3} . If we approximate these integrals by the quadrature rule (2.11), we have

$$\begin{aligned} \epsilon_{\text{rhs}} &:= \max\{|I_2 - \tilde{I}_2|, |I_{-2} - \tilde{I}_{-2}|, |I_3 - \tilde{I}_3|, |I_{-3} - \tilde{I}_{-3}|\} \\ &\leq \frac{C(m)}{(|\gamma| - 1)^g} \left(\frac{c_0}{c_0 + \sqrt{2}}\right)^{2m} \quad \text{with} \quad C(m) := 16ec_1(2m)^3 \left(1 + \frac{c_0\sqrt{2}}{|\gamma| - 1}\right). \end{aligned}$$

Solving (2.8) with these approximated right-hand side values, we get

$$\tilde{I}_1 := \frac{2c\tilde{I}_2 + c\tilde{I}_3}{1 - c}, \quad \tilde{I}_{-1} := \frac{2c\tilde{I}_{-2} + c\tilde{I}_{-3}}{1 - c}, \quad \tilde{I} := \frac{c\tilde{I}_1 + c\tilde{I}_{-1}}{1 - 2c}$$

and find

$$|I_1 - \tilde{I}_1| \leq \frac{c}{|1 - c|} \epsilon_{\text{rhs}}, \quad |I_{-1} - \tilde{I}_{-1}| \leq \frac{c}{|1 - c|} \epsilon_{\text{rhs}}, \quad |I - \tilde{I}| \leq \frac{6c^2}{|1 - c| |1 - 2c|} \epsilon_{\text{rhs}},$$

so the approximation of the integrals improves exponentially if the quadrature order is increased.

3 Integrals with polynomials

3.1 General case

If an integral equation is discretized by other than piecewise constant functions, we have to compute integrals of the form

$$I_{i,j}^\gamma := \int_0^1 \int_0^1 |x - y + \gamma|^\alpha \varphi_i(x) \varphi_j(y) dy dx,$$

where φ_i, φ_j are n -th order polynomials and $\gamma \in \mathbb{R}$. If we split this integral and transform as before, we find

$$\begin{aligned} I_{i,j}^\gamma &= c \int_0^1 \int_0^1 |x - y + 2\gamma|^\alpha \varphi_i\left(\frac{x}{2}\right) \varphi_j\left(\frac{y}{2}\right) dy dx + c \int_0^1 \int_0^1 |x - y + 2\gamma - 1|^\alpha \varphi_i\left(\frac{x}{2}\right) \varphi_j\left(\frac{y+1}{2}\right) dy dx \\ &\quad + c \int_0^1 \int_0^1 |x - y + 2\gamma + 1|^\alpha \varphi_i\left(\frac{x+1}{2}\right) \varphi_j\left(\frac{y}{2}\right) dy dx + c \int_0^1 \int_0^1 |x - y + 2\gamma|^\alpha \varphi_i\left(\frac{x+1}{2}\right) \varphi_j\left(\frac{y+1}{2}\right) dy dx \end{aligned} \quad (3.1)$$

Since $(x, y) \mapsto \varphi_i(x)\varphi_j(y)$ is in general neither invariant under translation nor homogeneous, we cannot apply our method directly.

Although a single polynomial is not invariant under translation and scaling, the space of *all* polynomials of a certain order is. Therefore we treat an entire basis of a polynomial space simultaneously: Let $(\varphi_i)_{i=0}^n$ be a Lagrange basis of the space of n -th order polynomials, i.e., we assume that there are interpolation points $(\xi_i)_{i=0}^n$ satisfying

$$p(x) = \sum_{i=0}^n p(\xi_i)\varphi_i(x) \quad (3.2)$$

for all n -th order polynomials p .

Since the integrals depend on two polynomials, we introduce the index set

$$P := \{0, \dots, n\} \times \{0, \dots, n\}$$

and collect the integrals for all combinations of basis functions in vectors $I^\gamma \in \mathbb{R}^P$ defined by

$$I_{i,j}^\gamma := \int_0^1 \int_0^1 \kappa(x + \gamma, y)\varphi_i(x)\varphi_j(y) dy dx \quad (3.3)$$

for $i, j \in \{0, \dots, n\}$.

We define the *transfer matrices* $T^{00}, T^{01}, T^{10}, T^{11} \in \mathbb{R}^{P \times P}$ by setting

$$T_{(i,j)(k,l)}^{\alpha\beta} := \varphi_i\left(\frac{\xi_k + \alpha}{2}\right)\varphi_j\left(\frac{\xi_l + \beta}{2}\right)$$

for $\alpha, \beta \in \{0, 1\}$ and $(i, j), (k, l) \in P$ and deduce from (3.2) that

$$\sum_{(k,l) \in P} T_{(i,j)(k,l)}^{\alpha\beta} \varphi_k(x)\varphi_l(y) = \varphi_i\left(\frac{x + \alpha}{2}\right)\varphi_j\left(\frac{y + \beta}{2}\right).$$

Remark 3.1 *Since $(\varphi_i)_{i=0}^n$ is a Lagrange basis, we have $\sum_{i=0}^n \varphi_i \equiv 1$. This implies*

$$\sum_{(i,j) \in P} T_{(i,j)(k,l)}^{\alpha\beta} = \left(\sum_{i=0}^n \varphi_i\right)\left(\frac{\xi_k + \alpha}{2}\right)\left(\sum_{j=0}^n \varphi_j\right)\left(\frac{\xi_l + \beta}{2}\right) = 1,$$

so the column sum of any column of a transfer matrix $T^{\alpha\beta}$ will be 1. Therefore the constant vector is a left eigenvector of any transfer matrix, and the corresponding eigenvalue is 1.

Using these new notations, we can write equation (3.1) in the form

$$I^\gamma = cT^{00}I^{2\gamma} + cT^{01}I^{2\gamma-1} + cT^{10}I^{2\gamma+1} + cT^{11}I^{2\gamma}$$

and use it to replace the scalar-valued recurrence relation (2.7) by

$$c \underbrace{\begin{pmatrix} T^{00} + T^{11} & T^{10} & T^{01} \\ & T^{01} & \\ & & T^{10} \end{pmatrix}}_{=:M} \begin{pmatrix} I^0 \\ I^1 \\ I^{-1} \end{pmatrix} + c \begin{pmatrix} 0 \\ (T^{00} + T^{11})I^2 + T^{10}I^3 \\ (T^{00} + T^{11})I^{-2} + T^{01}I^{-3} \end{pmatrix} = \begin{pmatrix} I^0 \\ I^1 \\ I^{-1} \end{pmatrix}. \quad (3.4)$$

Now we can proceed as before and solve the resulting system of linear equations.

3.2 Linear polynomials

Let us now consider a simple example: The Lagrange basis

$$\varphi_0(t) = 1 - t \quad \text{and} \quad \varphi_1(t) = t$$

of affine functions corresponding to the interpolation points

$$\xi_0 = 0 \quad \text{and} \quad \xi_1 = 1.$$

We identify $P = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ with the set $\{1, 2, 3, 4\}$ and find

$$T^{00} = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/4 \\ 0 & 1/2 & 0 & 1/4 \\ 0 & 0 & 1/2 & 1/4 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}, \quad T^{01} = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 1/2 & 1/4 & 1 & 1/2 \\ 0 & 1/4 & 0 & 1/2 \end{pmatrix},$$

$$T^{10} = \begin{pmatrix} 1/2 & 0 & 1/4 & 0 \\ 1/2 & 1 & 1/4 & 1/2 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1/2 \end{pmatrix}, \quad T^{11} = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 1/4 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/2 & 0 \\ 1/4 & 1/2 & 1/2 & 1 \end{pmatrix},$$

so the matrix M from (3.4) takes the form

$$M = c \left(\begin{array}{cccc|cccc|cccc} 5/4 & 1/2 & 1/2 & 1/4 & 1/2 & 0 & 1/4 & 0 & 1/2 & 1/4 & 0 & 0 \\ 1/4 & 1 & 0 & 1/4 & 1/2 & 1 & 1/4 & 1/2 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1 & 1/4 & 0 & 0 & 1/4 & 0 & 1/2 & 1/4 & 1 & 1/2 \\ 1/4 & 1/2 & 1/2 & 5/4 & 0 & 0 & 1/4 & 1/2 & 0 & 1/4 & 0 & 1/2 \\ \hline & & & & 1/2 & 1/4 & 0 & 0 & & & & \\ & & & & 0 & 1/4 & 0 & 0 & & & & \\ & & & & 1/2 & 1/4 & 1 & 1/2 & & & & \\ & & & & 0 & 1/4 & 0 & 1/2 & & & & \\ \hline & & & & & & & & 1/2 & 0 & 1/4 & 0 \\ & & & & & & & & 1/2 & 1 & 1/4 & 1/2 \\ & & & & & & & & 0 & 0 & 1/4 & 0 \\ & & & & & & & & 0 & 0 & 1/4 & 1/2 \end{array} \right).$$

The matrix is upper block triangular, and all but the upper left block are again upper or lower block triangular, so we can find its eigenvalues by computing the Schur form of the upper left block: We have

$$Q^T \begin{pmatrix} 5/4 & 1/2 & 1/2 & 1/4 \\ 1/4 & 1 & 0 & 1/4 \\ 1/4 & 0 & 1 & 1/4 \\ 1/4 & 1/2 & 1/2 & 5/4 \end{pmatrix} Q = \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ 1/2 & & & 1/2 \end{pmatrix},$$

where the unitary transformation Q is given by

$$Q := \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

so we find $\sigma(M) = \{2c, c, c/2, c/4\}$, i.e., the matrix $I - M$ is singular if $2^{-\alpha}/4 = c \in \{1/2, 1, 2, 4\}$ holds, and this is the case if the order α of homogeneity is in $\{-1, -2, -3, -4\}$.

4 Strong singularities

4.1 Finite part integrals

In the case of $\alpha \leq -1$, the integral (1.1) does not exist in the standard sense. Instead we consider the family I_ϵ of integrals

$$I_\epsilon := \int_0^1 \int_0^1 \kappa(x, y) dy dx = \int_0^{1-\epsilon} \int_{x+\epsilon}^1 \kappa(x, y) dy dx + \int_\epsilon^1 \int_0^{x-\epsilon} \kappa(x, y) dy dx \quad (4.1)$$

for $\epsilon > 0$. The integration domain is $[0, 1]^2$ minus a symmetric strip around $x = y$ as depicted in Figure 4.1.

For instance, in the case of (1.1), i.e., $\kappa(x, y) = |x - y|^\alpha$, with $\alpha < -1$ and $\alpha \neq -2$, we have

$$I_\epsilon = c_0 + c_{\alpha+1}\epsilon^{\alpha+1} + c_{\alpha+2}\epsilon^{\alpha+2} \quad \text{with}$$

$$c_0 = \frac{2}{(\alpha+1)(\alpha+2)}, \quad c_{\alpha+1} = -\frac{2}{\alpha+1}, \quad c_{\alpha+2} = \frac{2}{\alpha+2}.$$

In the case $-2 < \alpha < -1$, the term $c_{\alpha+1}\epsilon^{\alpha+1}$ diverges as $\epsilon \rightarrow 0$, while $c_{\alpha+2}\epsilon^{\alpha+2} \rightarrow 0$. In the case $\alpha < -2$, both terms $c_{\alpha+2}\epsilon^{\alpha+2}$, $c_{\alpha+1}\epsilon^{\alpha+1}$ diverge. Omitting the divergent terms, we define the partie finie integral (cf. [8]) by the constant term c_0 :

$$\text{p.f.} \int_0^1 \int_0^1 |x - y|^\alpha dy dx := \frac{2}{(\alpha+1)(\alpha+2)}.$$

In the case of $\alpha = -1$ or $\alpha = -2$, logarithmic terms appear and result in

$$\text{p.f.} \int_0^1 \int_0^1 |x - y|^{-1} dy dx = -2 = \text{p.f.} \int_0^1 \int_0^1 |x - y|^{-2}.$$

4.2 Triangular splitting

The system (2.8) can be solved if $c \notin \{1/2, 1\}$, i.e., if $\alpha \notin \{-1, -2\}$. In order to find an interpretation for its solution, we introduce an auxiliary splitting strategy: Different from before, we no longer split the domain $\Omega = [0, 1] \times [0, 1]$ into square subdomains, but we split it first into the triangles

$$\Omega^l := \{(x, y) \in \Omega : y < x\}, \quad \Omega^u := \{(x, y) \in \Omega : y \geq x\}$$

and then split these triangles into four similar subtriangles by applying the standard ‘‘red’’ refinement strategy.

As before, we split only those subdomains that touch the singularity and get the sequence of splittings given in Figure 4.1.

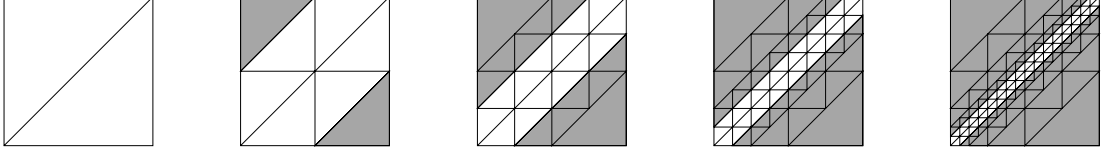


Figure 4.1: Triangular splitting strategy

For the case of the domain Ω^l , we end up with four equivalence classes:

$$\begin{aligned}
 I_0^l &:= \int_0^1 \int_0^x \kappa(x, y) dy dx, & I_1^l &:= \int_0^1 \int_x^1 \kappa(x+1, y) dy dx, \\
 I_2^l &:= \int_0^1 \int_0^x \kappa(x+1, y) dy dx, & I_3^l &:= \int_0^1 \int_x^1 \kappa(x+2, y) dy dx,
 \end{aligned}$$

(cf. Figure 4.2) since we have

$$\begin{aligned}
 I_0^l &= \int_0^1 \int_0^x \kappa(x, y) dy dx \\
 &= \int_0^{1/2} \int_0^x \kappa(x, y) dy dx + \int_{1/2}^1 \int_0^{x-1/2} \kappa(x, y) dy dx + \int_{1/2}^1 \int_{x-1/2}^{1/2} \kappa(x, y) dy dx + \int_{1/2}^1 \int_{1/2}^x \kappa(x, y) dy dx \\
 &= \int_0^{1/2} \int_0^x \kappa(x, y) dy dx + \int_0^{1/2} \int_0^x \kappa(x + \frac{1}{2}, y) dy dx + \int_0^{1/2} \int_x^{1/2} \kappa(x + \frac{1}{2}, y) dy dx + \int_0^{1/2} \int_0^x \kappa(x, y) dy dx \\
 &= cI_0^l + cI_2^l + cI_1^l + cI_0^l = 2cI_0^l + cI_1^l + cI_2^l,
 \end{aligned}$$

$$\begin{aligned}
 I_1^l &= \int_0^1 \int_x^1 \kappa(x+1, y) dy dx \\
 &= \int_0^{1/2} \int_x^{1/2} \kappa(x+1, y) dy dx + \int_0^{1/2} \int_{1/2}^{1/2+x} \kappa(x+1, y) dy dx \\
 &\quad + \int_0^{1/2} \int_{1/2+x}^1 \kappa(x+1, y) dy dx + \int_{1/2}^1 \int_x^1 \kappa(x+1, y) dy dx \\
 &= \int_0^{1/2} \int_x^{1/2} \kappa(x+1, y) dy dx + \int_0^{1/2} \int_0^x \kappa(x + \frac{1}{2}, y) dy dx \\
 &\quad + \int_0^{1/2} \int_x^{1/2} \kappa(x + \frac{1}{2}, y) dy dx + \int_0^{1/2} \int_x^{1/2} \kappa(x+1, y) dy dx \\
 &= cI_3^l + cI_2^l + cI_1^l + cI_3^l = cI_1^l + cI_2^l + 2cI_3^l
 \end{aligned}$$

and since I_2^l and I_3^l are nonsingular integrals.

For the domain Ω^u , we get the four equivalence classes

$$\begin{aligned}
 I_0^u &:= \int_0^1 \int_x^1 \kappa(x, y) dy dx, & I_1^u &:= \int_0^1 \int_0^x \kappa(x, y+1) dy dx, \\
 I_2^u &:= \int_0^1 \int_x^1 \kappa(x, y+1) dy dx, & I_3^u &:= \int_0^1 \int_0^x \kappa(x, y+2) dy dx
 \end{aligned}$$

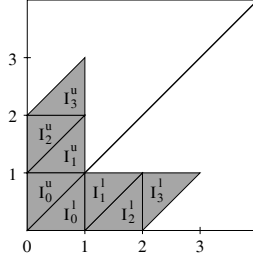


Figure 4.2: Equivalence classes for the triangular case.

(cf. Figure 4.2) with the recurrence relations

$$I_0^u = 2cI_0^u + cI_1^u + cI_2^u, \quad I_1^u = cI_1^u + cI_2^u + 2cI_3^u.$$

Combining the equations, we find

$$\underbrace{\begin{pmatrix} 2c & c & & \\ & c & & \\ & & 2c & c \\ & & & c \end{pmatrix}}{=:M} \underbrace{\begin{pmatrix} I_0^l \\ I_1^l \\ I_0^u \\ I_1^u \end{pmatrix}}{=:x} + \underbrace{\begin{pmatrix} cI_2^l \\ cI_2^l + 2cI_3^l \\ cI_2^u \\ cI_2^u + 2cI_3^u \end{pmatrix}}{=:b} = \underbrace{\begin{pmatrix} I_0^l \\ I_1^l \\ I_0^u \\ I_1^u \end{pmatrix}}{=:x}. \quad (4.2)$$

By repeated application of this equation, we find

$$x = Mx + b = M(Mx + b) + b = \dots = M^k x + \sum_{l=0}^{k-1} M^l b.$$

The partial sums

$$x^k := \sum_{l=0}^{k-1} M^l b \quad (4.3)$$

correspond to integrals over subdomains: x^k represents the integrals over those subdomains that do not contain singularities after k splitting operations.

In Figure 4.1, the subdomains marked in grey correspond to x^0, x^1, \dots, x^4 (from left to right). Since our new splitting strategy is based on triangles aligned with the diagonal of the domain, we find that for x^k , a strip of width 2^{-k} around the diagonal has been removed from the domain of integration, so we have

$$x_1^k = \int_{2^{-k}}^1 \int_0^{x-2^{-k}} |x-y|^\alpha dy dx, \quad x_3^k = \int_0^{1-2^{-k}} \int_{x+2^{-k}}^1 |x-y|^\alpha dy dx, \quad I^k := x_1^k + x_3^k = I_{2^{-k}}. \quad (4.4)$$

The latter equation is the reason for our construction: Due to the triangular splitting, the partial sums I^k correspond to the integrals $I_{2^{-k}}$ defined in (4.1). Therefore the convergence properties of I^k are closely related to the quantities occurring in the definition of the partie finie integral.

Remark 4.1 (Symmetric case) *If κ is symmetric, i.e., if the kernel function satisfies (1.5), we have $I_0^u = I_0^l$ and $I_1^u = I_1^l$, so it is sufficient to solve*

$$\begin{pmatrix} 1-2c & -c \\ 0 & 1-c \end{pmatrix} \begin{pmatrix} I_0^l \\ I_1^l \end{pmatrix} = \begin{pmatrix} cI_2^l \\ cI_2^l + 2cI_3^l \end{pmatrix}$$

and set $I := 2I_0^l$.

4.3 Convergence analysis

If $\rho(M)$, the spectral radius of the matrix M from equation (4.2), is smaller than 1, the sum $x^k = \sum_{\ell=0}^{k-1} M^\ell b$ converges to $(I - M)^{-1}b$, so we can compute $x = \lim_{k \rightarrow \infty} x^k = (I - M)^{-1}b$ directly by solving (2.8) (in the case of integrals with polynomials, equation (3.4) can be used).

Now let us consider the general case. Obviously, the spectrum $\sigma(M)$ of M is given by $\sigma(M) = \{c, 2c\}$. Since the eigenspace for the eigenvalue $2c$ is spanned by the vectors $(1, 0, 0, 0)^\top$ and $(0, 0, 1, 0)^\top$ and the one for the eigenvalue c is spanned by $(-1, 1, 0, 0)^\top$ and $(0, 0, -1, 1)^\top$, we can diagonalize M :

$$M = \underbrace{\begin{pmatrix} 1 & -1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}}_{=T} \underbrace{\begin{pmatrix} 2c & & & \\ & c & & \\ & & 2c & \\ & & & c \end{pmatrix}}_{=D} \underbrace{\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}}_{=T^{-1}}.$$

We can apply this to the sum $\sum_{\ell=0}^{k-1} M^\ell$ and get

$$T^{-1}x = \sum_{\ell=0}^{\infty} D^\ell(T^{-1}b), \quad T^{-1}x^k = \sum_{\ell=0}^{k-1} D^\ell(T^{-1}b).$$

Therefore I^k from (4.4) is given by

$$I^k = x_1^k + x_3^k = \left(\sum_{\ell=0}^{k-1} (2c)^\ell \right) \underbrace{(b_1 + b_2 + b_3 + b_4)}_{=:b_+} - \left(\sum_{\ell=0}^{k-1} c^\ell \right) \underbrace{(b_2 + b_4)}_{=:b_-}.$$

In the exceptional cases $c = 1/2$ and $c = 1$, we get the respective expressions

$$I^k = kb_+ - \frac{1-2^{-k}}{1-1/2}b_- = -2b_- + kb_+ + 2^{1-k}b_-, \quad (\text{case } c = 1/2) \quad (4.5a)$$

$$I^k = \frac{1-2^k}{1-2}b_+ - kb_- = -b_+ - kb_- + 2^kb_+, \quad (\text{case } c = 1) \quad (4.5b)$$

and otherwise

$$I^k = \frac{1-(2c)^k}{1-2c}b_+ - \frac{1-c^k}{1-c}b_- = \left(\frac{b_+}{1-2c} - \frac{b_-}{1-c} \right) - \frac{b_+}{1-2c}(2c)^k + \frac{b_-}{1-c}c^k. \quad (4.5c)$$

As we can see in Figure 4.1, we have

$$I^k = I_{\epsilon_k} := \int_{\epsilon_k}^1 \int_0^{x-\epsilon_k} |x-y|^\alpha dy dx + \int_0^{1-\epsilon_k} \int_{x+\epsilon_k}^1 |x-y|^\alpha dy dx$$

for $\epsilon_k := 2^{-k}$, and we find

$$(2c)^k = 2^{-k(\alpha+1)} = \epsilon_k^{\alpha+1}, \quad c^k = 2^{-k(\alpha+2)} = \epsilon_k^{\alpha+2} \quad \text{and} \quad k = -\log_2 \epsilon_k.$$

We can use this to rewrite equations (4.5a-c) in the form

$$I_{\epsilon_k} = \begin{cases} -2b_- - b_+ \log_2 \epsilon_k + 2b_- \epsilon_k & \text{if } \alpha = -1, \\ -b_+ + b_- \log_2 \epsilon_k + b_+ \epsilon_k^{-1} & \text{if } \alpha = -2, \\ \left(\frac{b_+}{1-2c} - \frac{b_-}{1-c} \right) - \frac{b_+}{1-2c} \epsilon_k^{\alpha+1} + \frac{b_-}{1-c} \epsilon_k^{\alpha+2} & \text{otherwise.} \end{cases} \quad (4.6)$$

We are interested in the case $k \rightarrow \infty$, i.e., $\epsilon_k \rightarrow 0$. For $\alpha > -1$, the integral is only weakly singular, and $\epsilon_k^{\alpha+1}$ and $\epsilon_k^{\alpha+2}$ converge to zero. For $\alpha \leq -1$, the integral is not weakly integrable, so we use the *Hadamard* integral, i.e., we take only the finite part of the expansion (4.6) of I_{ϵ_k} :

$$I_\infty := \begin{cases} -2b_- & \text{if } \alpha = -1, \\ -b_+ & \text{if } \alpha = -2, \\ \frac{b_+}{1-2c} - \frac{b_-}{1-c} & \text{otherwise.} \end{cases}$$

Now let us consider a simpler way of finding the Hadamard integral I_∞ . If $c \notin \{1/2, 1\}$, i.e., if $\alpha \notin \{-1, -2\}$, the partial sums (4.3) can be written in the form

$$x^k = (I - M)^{-1}(I - M^k)b.$$

In the weakly singular case, we have $\rho(M) < 1$, and M^k converges to zero if $k \rightarrow \infty$, giving us

$$x^\infty := (I - M)^{-1}b. \quad (4.7)$$

Diagonalization of this equation yields

$$x^\infty = \begin{pmatrix} 1 & -1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-2c} & & & \\ & \frac{1}{1-c} & & \\ & & \frac{1}{1-2c} & \\ & & & \frac{1}{1-c} \end{pmatrix} \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} b = \begin{pmatrix} (b_1 + b_2)/(1-2c) - b_2/(1-c) \\ b_2/(1-c) \\ (b_3 + b_4)/(1-2c) - b_4/(1-c) \\ b_4/(1-c) \end{pmatrix}$$

and we find that

$$x_1^\infty + x_3^\infty = I_\infty$$

holds, so we can compute I_∞ even in the strongly singular case by equation (4.7).

Unfortunately, this equation cannot be applied if $I - M$ is singular, i.e., if $c \in \{1/2, 1\}$. We can fix this by replacing $(I - M)^{-1}$ by the proper pseudo-inverse: We use the function

$$\text{pinv}(\alpha) := \begin{cases} 1/\alpha & \text{if } \alpha \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

to define the pseudo-inverse $(I - M)^\dagger$ by

$$(I - M)^\dagger := T \begin{pmatrix} \text{pinv}(1 - 2c) & & & \\ & \text{pinv}(1 - c) & & \\ & & \text{pinv}(1 - 2c) & \\ & & & \text{pinv}(1 - c) \end{pmatrix} T^{-1}.$$

Obviously, we have $(I - M)^\dagger = (I - M)^{-1}$ if $I - M$ is regular. If it is singular, we use

$$x^\infty := (I - M)^\dagger b$$

and find

$$x^\infty = \begin{pmatrix} (b_1 + b_2) \text{pinv}(1 - 2c) - b_2 \text{pinv}(1 - c) \\ b_2 \text{pinv}(1 - c) \\ (b_3 + b_4) \text{pinv}(1 - 2c) - b_4 \text{pinv}(1 - c) \\ b_4 \text{pinv}(1 - c) \end{pmatrix}.$$

In the cases $c = 1/2$ (i.e., $\alpha = -1$) and $c = 1$ (i.e., $\alpha = -2$), we find

$$\begin{aligned} x_1^\infty + x_3^\infty &= -\frac{b_2 + b_4}{1 - c} = -2b_- = I_\infty && \text{(case } c = 1/2, \alpha = -1), \\ x_1^\infty + x_3^\infty &= \frac{b_1 + b_2 + b_3 + b_4}{1 - 2c} = -b_+ = I_\infty && \text{(case } c = 1, \alpha = -2), \end{aligned}$$

so now we have found a general procedure for the computation of I_∞ : We compute $x^\infty := (I - M)^\dagger b$ and get $I_\infty = x_1^\infty + x_3^\infty$. This will work for all values of α , i.e., for arbitrary degrees of singularity, and it will give us the desired Hadamard integral.

4.4 Reduction to rectangular splitting

We have seen that the solution of (2.8) corresponds to the Hadamard integral if $c \notin \{1/2, 1\}$, i.e., if $\alpha \notin \{-1, -2\}$.

The relation to the Hadamard integral is due to the fact that we use triangular subdomains and approach the diagonal $\{(x, y) \in \Omega : x = y\}$ uniformly. While this works in the one-dimensional setting, its generalization to higher dimensions is not straightforward (it would require the use of curved subdomains).

Let $I_0^l, I_1^l, I_0^u, I_1^u \in \mathbb{R}$ satisfy equation (4.2) and let the regular integrals $I_2^l, I_3^l, I_2^u, I_3^u \in \mathbb{R}$ and $I_2, I_3, I_{-2}, I_{-3} \in \mathbb{R}$ be defined as before.

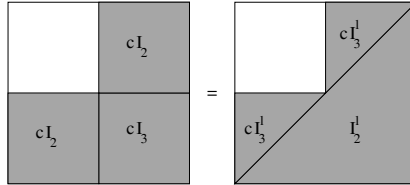


Figure 4.3: Relation of square and triangular equivalence classes

The regular integrals satisfy the equation

$$\begin{aligned}
2cI_2 + cI_3 &= c \int_0^1 \int_0^1 |x - y + 2|^\alpha dy dx + c \int_0^1 \int_0^1 |x - y + 2|^\alpha dy dx + c \int_0^1 \int_0^1 |x - y + 3|^\alpha dy dx \\
&= \int_0^{1/2} \int_0^{1/2} |x - y + 1|^\alpha dy dx + \int_{1/2}^1 \int_{1/2}^1 |x - y + 1|^\alpha dy dx + \int_0^{1/2} \int_0^{1/2} |x - y + 3/2|^\alpha dy dx \\
&= \int_0^{1/2} \int_0^1 |x - y + 1|^\alpha dy dx + \int_{1/2}^1 \int_0^{1/2} |x - y + 1|^\alpha dy dx \\
&= \int_0^1 \int_0^1 |x - y + 1|^\alpha dy dx - \int_0^{1/2} \int_{1/2}^1 |x - y + 1|^\alpha dy dx \\
&= \int_0^1 \int_0^x |x - y + 1|^\alpha dy dx + \int_0^{1/2} \int_x^{1/2} |x - y + 1|^\alpha dy dx + \int_{1/2}^1 \int_x^1 |x - y + 1|^\alpha dy dx \\
&= I_2^l + 2cI_3^l.
\end{aligned}$$

By similar arguments, we find $2cI_{-2} + cI_{-3} = I_2^r + 2cI_3^r$. By setting

$$\hat{I}_0 := I_0^l + I_0^u, \hat{I}_1 := I_1^l + I_2^l, \hat{I}_{-1} := I_1^u + I_2^u,$$

we find that

$$\begin{aligned}
\hat{I}_0 &= I_0^l + I_0^u = 2c(I_0^l + I_0^u) + c(I_1^l + I_2^l) + c(I_1^u + I_2^u) = 2c\hat{I}_0 + c\hat{I}_1 + \hat{I}_{-1}, \\
\hat{I}_1 &= I_1^l + I_2^l = cI_1^l + cI_2^l + 2cI_3^l + I_2^l = c\hat{I}_1 + 2cI_3^l + I_2^l = c\hat{I}_1 + 2cI_2 + cI_3, \\
\hat{I}_{-1} &= c\hat{I}_{-1} + 2cI_{-2} + cI_{-3}.
\end{aligned}$$

holds. This implies

$$\begin{pmatrix} \hat{I}_0 \\ \hat{I}_1 \\ \hat{I}_{-1} \end{pmatrix} = \begin{pmatrix} 2c & c & c \\ & c & \\ & & c \end{pmatrix} \begin{pmatrix} \hat{I}_0 \\ \hat{I}_1 \\ \hat{I}_{-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 2cI_2 + cI_3 \\ 2cI_{-2} + cI_{-3} \end{pmatrix},$$

so each solution of (4.2) corresponds to a solution of (2.7). Since the solutions are unique, the triangular splitting scheme leads to the same result as the rectangular scheme.

5 Implementation

In this paper, we have studied only the one-dimensional case, where only a small number of equivalence classes (4, namely $(I_\gamma)_{\gamma=0}^3$, in the case of a rectangular splitting, and 8, namely $(I_\gamma^u)_{\gamma=0}^3$ and $(I_\gamma^l)_{\gamma=0}^3$, in the case of a triangular splitting) has to be computed. In this simple setting, we can compute the solution of the systems of linear equations explicitly and use them to derive quadrature rules. In the symmetric case for example, we can use (2.9) to derive the quadrature formula

$$\begin{aligned}
I &\approx \frac{4c^2}{(1-2c)(1-c)} \tilde{I}_2 + \frac{2c^2}{(1-2c)(1-c)} \tilde{I}_3 \\
&= \frac{4c^2}{(1-2c)(1-c)} \sum_{k=1}^m \sum_{\ell=1}^m w_k w_\ell \kappa(x_k + 2, x_\ell) + \frac{2c^2}{(1-2c)(1-c)} \sum_{k=1}^m \sum_{\ell=1}^m w_k w_\ell \kappa(x_k + 3, x_\ell),
\end{aligned}$$

where $(x_k)_{k=1}^m$ are the Gauss quadrature points for the interval $[0, 1]$ and $(w_k)_{k=1}^m$ are the corresponding weights. For the logarithmic kernel function $\kappa(x, y) = \log|x - y|$, we have to add an appropriate multiple of the correction term $\beta(1/2)$.

In order to treat integrals with polynomials efficiently, we note that the computation of the vector I^γ from (3.3) requires only one kernel evaluation per quadrature point, since only the multiplicative factors corresponding to the basis functions φ_i and φ_j change. This means that the number of kernel evaluations for the case of integrals with polynomials is the same as for the case of integrals without additional factors.

Applying our quadrature algorithm to boundary integral operators on curves Γ is straightforward: The curve is approximated by a polygon through a sequence of points $(a_i)_{i=0}^n$, and the entries of the Galerkin stiffness matrix are approximated by

$$A_{\nu\mu} = \sum_{i=1}^n \sum_{j=1}^n \int_{a_{i-1}}^{a_i} \varphi_\nu(x) \int_{a_{j-1}}^{a_j} \kappa(x, y) \varphi_\mu(y) dy dx$$

for polynomial (or piecewise polynomial) basis functions (φ_ν). In order to apply our quadrature scheme, we have to transform the line integrals to integrals over $[0, 1]^2$. If $i = j$, this is straightforward: We use the parametrization

$$\Phi(s, t) = \begin{pmatrix} a_i s + a_{i-1}(1-s) \\ a_i t + a_{i-1}(1-t) \end{pmatrix}$$

and find

$$\int_{a_{i-1}}^{a_i} \int_{a_{j-1}}^{a_j} \kappa(x, y) \varphi_\nu(x) \varphi_\mu(y) dy dx = \|a_i - a_{i-1}\| \|a_j - a_{j-1}\| \int_0^1 \int_0^1 \hat{\kappa}(s, t) \hat{\varphi}_\nu(s) \hat{\varphi}_\mu(t) dt ds$$

with

$$\begin{aligned} \hat{\kappa}(s, t) &= \kappa(a_i s + a_{i-1}(1-s), a_i t + a_{i-1}(1-t)), \\ \hat{\varphi}_\nu(s) &= \varphi_\nu(a_i s + a_{i-1}(1-s)), \quad \hat{\varphi}_\mu(t) = \varphi_\mu(a_i t + a_{i-1}(1-t)). \end{aligned}$$

Obviously, if κ is shift-invariant, so is $\hat{\kappa}$, and if φ_ν and φ_μ are polynomial, so are $\hat{\varphi}_\nu$ and $\hat{\varphi}_\mu$. Therefore we can now apply our quadrature rule to approximate this integral.

Let us consider the case of two adjacent lines, i.e., $a_{i-1} = a_j$ or $a_{j-1} = a_i$. Since both cases are symmetric, we will only give a construction for the first one. Again, we introduce a parametrization

$$\Phi(s, t) = \begin{pmatrix} a_i s + a_{i-1}(1-s) \\ a_j(1-t) + a_{j-1}(2-t) \end{pmatrix}$$

that maps $[0, 1] \times [1, 2]$ to $[a_{i-1}, a_i] \times [a_{j-1}, a_j]$ and ensures $\Phi(1, 1) = (a_{i-1}, a_j) = (a_j, a_j)$. Applying the corresponding transformation, we get

$$\int_{a_{i-1}}^{a_i} \int_{a_{j-1}}^{a_j} \kappa(x, y) \varphi_\nu(x) \varphi_\mu(y) dy dx = \|a_i - a_{i-1}\| \|a_j - a_{j-1}\| \int_0^1 \int_1^2 \hat{\kappa}(s, t) \hat{\varphi}_\nu(s) \hat{\varphi}_\mu(t) dt ds$$

with

$$\begin{aligned} \hat{\kappa}(s, t) &= \kappa(a_i s + a_{i-1}(1-s), a_j(1-t) + a_{j-1}(2-t)), \\ \hat{\varphi}_\nu(s) &= \varphi_\nu(a_i s + a_{i-1}(1-s)), \quad \hat{\varphi}_\mu(t) = \varphi_\mu(a_j(1-t) + a_{j-1}(2-t)). \end{aligned}$$

	$m = 1$ $n = 2$	$m = 2$ $n = 8$	$m = 3$ $n = 18$	$m = 4$ $n = 32$	$m = 5$ $n = 50$
	$ I - \tilde{I} / I $	$ I - \tilde{I} / I $	$ I - \tilde{I} / I $	$ I - \tilde{I} / I $	$ I - \tilde{I} / I $
log	1.19 ₋₂	1.10 ₋₄	1.48 ₋₆	2.44 ₋₈	4.54 ₋₁₀
$\alpha = -0.5$	1.39 ₋₂	1.98 ₋₄	3.42 ₋₆	6.67 ₋₈	1.41 ₋₉
$\alpha = -1.0$	—	—	—	—	—
$\alpha = -1.5$	7.32 ₋₂	2.10 ₋₃	5.53 ₋₅	1.46 ₋₆	3.92 ₋₈
$\alpha = -2.0$	—	—	—	—	—
$\alpha = -2.5$	1.73 ₋₁	8.62 ₋₃	3.26 ₋₄	1.13 ₋₅	3.72 ₋₇
$\alpha = -3.0$	2.35 ₋₁	1.49 ₋₂	6.62 ₋₄	2.58 ₋₅	9.40 ₋₇
$\alpha = -3.5$	3.02 ₋₁	2.39 ₋₂	1.24 ₋₃	5.39 ₋₅	2.16 ₋₆
$\alpha = -4.0$	3.72 ₋₁	3.61 ₋₂	2.15 ₋₃	1.05 ₋₄	4.58 ₋₆
$\alpha = -10.0$	9.29 ₋₁	4.28 ₋₁	8.79 ₋₂	1.14 ₋₂	1.14 ₋₃

Table 6.1: Relative errors and kernel evaluations for different degrees of homogeneity and quadrature orders, rectangular splitting

The transformed integral corresponds to the value I_1 of (2.6), so we can compute it using (2.8) even though the transformed kernel function $\hat{\kappa}$ is no longer invariant under translations.

Our approach is not limited to the one-dimensional case: Let us consider the example of triangular domains. Instead of splitting intervals into two halves, we split triangles into four similar subtriangles and proceed as before. The resulting splitting graph contains 14 non-admissible and 60 admissible equivalence classes. We can treat products of rectangles or quadrilaterals by similar techniques.

6 Numerical experiments

Since we are mainly interested in the behaviour of the quadrature rule for high-order singularities, we consider the simple one-dimensional case with the kernel function

$$\kappa(x, y) = |x - y|^\alpha$$

from (1.1) and test the performance of our method for different values of α . We have

$$I = \int_0^1 \int_0^1 \kappa(x, y) dy dx = 2 \int_0^1 \int_0^x |x - y|^\alpha dy dx = \frac{2}{(\alpha + 1)(\alpha + 2)}.$$

We approximate I as described in Subsection 2.5 for different values of α and collect the relative errors and the number of kernel evaluations in Table 6.1. Here, m denotes the order of the quadrature rule (2.11), n denotes the number of kernel evaluations required, and the rows of the table give the relative error of the integral. The cases $\alpha = -1$ and $\alpha = -2$ are omitted because the regularization described in Subsection 4.3 only works for triangular approximations.

We compare the results with those derived for the triangular approximation introduced in Subsection 4.2. The regular integrals are approximated by applying the Duffy transformation

	$m = 1$ $n = 2$	$m = 2$ $n = 8$	$m = 3$ $n = 18$	$m = 4$ $n = 32$	$m = 5$ $n = 50$
	$ I - \tilde{I} / I $	$ I - \tilde{I} / I $	$ I - \tilde{I} / I $	$ I - \tilde{I} / I $	$ I - \tilde{I} / I $
log	8.73_{-3}	3.86_{-6}	3.26_{-7}	7.75_{-9}	1.75_{-10}
$\alpha = -0.5$	6.02_{-4}	6.18_{-5}	1.43_{-6}	3.19_{-8}	7.36_{-10}
$\alpha = -1.0$	2.86_{-2}	6.50_{-4}	1.48_{-5}	3.49_{-7}	8.56_{-9}
$\alpha = -1.5$	3.36_{-2}	1.23_{-3}	3.47_{-5}	9.54_{-7}	2.62_{-8}
$\alpha = -2.0$	3.35_{-2}	1.88_{-3}	6.40_{-5}	2.01_{-6}	6.15_{-8}
$\alpha = -2.5$	1.10_{-1}	5.99_{-3}	2.33_{-4}	8.16_{-6}	2.72_{-7}
$\alpha = -3.0$	1.61_{-1}	1.07_{-2}	4.88_{-4}	1.92_{-5}	7.04_{-7}
$\alpha = -3.5$	2.18_{-1}	1.77_{-2}	9.31_{-4}	4.10_{-5}	1.64_{-6}
$\alpha = -4.0$	2.79_{-1}	2.72_{-2}	1.65_{-3}	8.06_{-5}	3.53_{-6}
$\alpha = -10.0$	8.58_{-1}	3.62_{-1}	7.34_{-2}	9.53_{-3}	9.47_{-4}

Table 6.2: Relative errors and kernel evaluations for different degrees of homogeneity and quadrature orders, triangular splitting

$(s, t) \mapsto (s, (1-s)t)$ to tensor Gauss quadrature rules. The corresponding relative errors can be found in Table 6.2. Due to the better performance of the quadrature rule for triangular subdomains, they are slightly smaller than those for the case of rectangular splittings.

As predicted by theory, the quadrature error converges exponentially in all cases, while the algorithmic complexity, expressed as the number n of kernel evaluations, increases only quadratically: Since we compute the integrals I_2, I_3 (or I_2^u, I_3^u) by using tensor quadrature of order m , we have $n = 2m^2$.

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