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by

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Asymptotically anti-de Sitter space-times: symmetries and conservation laws revisited

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In this short note, we verify explicitly in static coordinates that the non trivial asymptotic Killing vectors at spatial infinity for anti-de Sitter space-times correspond one to one to the conformal Killing vectors of the conformally flat metric induced on the boundary. The fall-off conditions for the metric perturbations that guarantee finiteness of the associated conserved charges are derived.

1. Exact Killing vectors and their fall-off

Consider the $(n+1)$-dimensional flat embedding space with coordinates X^Δ, $\Delta = 0,\dots,n$ and metric $\eta_{\Delta\Gamma} = \mathrm{diag}(-1,-1,1,\dots,1)$ for $n \geq 3$. Anti-de Sitter space-time can be defined as the hypersurface

$$\eta_{\Delta\Gamma} X^\Delta X^\Gamma = -l^2. \tag{1}$$

Static coordinates x^μ, $\mu = 0,\dots,n-1$ on (universal) anti-de Sitter space-time can be chosen as $x^\mu \equiv \tau, r, y^A$, $A = 2,\dots,n-1$, with

$$\begin{array}{rcl} X^0 &=& l(1+\frac{r^2}{l^2})^{\frac{1}{2}}\sin(\tau/l),\\ X^1 &=& l(1+\frac{r^2}{l^2})^{\frac{1}{2}}\cos(\tau/l),\\ X^2 &=& r\cos y^2,\\ X^3 &=& r\sin y^2\cos y^3,\\ &\vdots& \\ X^{n-1} &=& r\sin y^2\dots\sin y^{n-2}\cos y^{n-1},\\ X^n &=& r\sin y^2\dots\sin y^{n-1}. \end{array} \tag{2}$$

In these coordinates, the metric on anti-de Sitter space-time is given by

$$d\bar{s}^2 \equiv \bar{g}_{\mu\nu}dx^\mu dx^\nu = -(1+\frac{r^2}{l^2})d\tau^2 + +(1+\frac{r^2}{l^2})^{-1}dr^2 + r^2\sum_A f_A (dy^A)^2, \tag{3}$$

with $f_2 = 1$, $f_A = \sin^2 y^2 \sin^2 y^3 \dots \sin^2 y^{A-1}$ for $3 \leq A \leq n-1$.

The Killing vectors for this metric can be obtained as follows: using the metric $\eta_{\Delta\Gamma}$ to lower the indices, the Killing vectors of the flat embedding space are $\xi_\Delta = a_\Delta + b_{\Delta\Gamma}X^\Gamma$, with constant $a_\Delta, b_{\Delta\Gamma} = -b_{\Gamma\Delta}$. Only the vectors corresponding to the Lorentz transformations are tangent to the hypersurface. The Killing vectors of $d\bar{s}^2$ are then given by

$$\xi^\alpha = \bar{g}^{\alpha\beta}\xi_\beta,\ \xi_\beta = b_{\Delta\Gamma}\frac{\partial X^\Delta}{\partial x^\beta}X^\Gamma, \tag{4}$$

or explicitly,

$$\xi^\tau = -b_{01}l - b_{0A}X^A(1+\frac{r^2}{l^2})^{-\frac{1}{2}}\cos(\tau/l) +b_{1A}X^A(1+\frac{r^2}{l^2})^{-\frac{1}{2}}\sin(\tau/l), \tag{5}$$

$$\xi^r = -l(1+\frac{r^2}{l^2})^{\frac{1}{2}}\Big[\sin(\tau/l)(b_{0A}\frac{X^A}{r}) + \cos(\tau/l)(b_{1A}\frac{X^A}{r})\Big], \tag{6}$$

$$\xi^A = \frac{1}{r^2 f_A}\frac{\partial X^B}{\partial y^A}\Big[b_{B0}l(1+\frac{r^2}{l^2})^{\frac{1}{2}}\sin(\tau/l) + b_{B1}l(1+\frac{r^2}{l^2})^{\frac{1}{2}}\cos(\tau/l) + b_{BC}X^C\Big]. \tag{7}$$

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Let $a = 0, 2, \ldots, n-1$, with $x^0 = \tau, x^A = y^A$. At spatial infinity $r \longrightarrow +\infty$, the asymptotic fall-off conditions for these Killing vectors are

$$\xi^r \longrightarrow O(r),\ \xi^a \longrightarrow O(r^0). \tag{8}$$

2. Asymptotic Killing vectors and conformal Killing vectors on the boundary

2.1. Asymptotic Killing vectors of anti-de Sitter space-times

For arbitrary vector fields ξ^μ that satisfy the fall-off conditions (8) of the exact Killing vectors, as well as $\xi^\mu \longrightarrow O(r^m) \Rightarrow \partial_r \xi^\mu \longrightarrow O(r^{m-1})$, we determine the fall-off of $L_\xi \bar{g}_{\alpha\beta}$. Explicitly,

$$L_\xi \bar{g}_{rr} \longrightarrow O(r^{-2}),\ L_\xi \bar{g}_{ar} \longrightarrow O(r),\quad L_\xi \bar{g}_{ab} \longrightarrow O(r^2). \tag{9}$$

This motivates us to define asymptotic Killing vectors through the fall-off conditions (8) and the constraints

$$L_\xi \bar{g}_{rr} \longrightarrow o(r^{-2}),\ L_\xi \bar{g}_{ar} \longrightarrow o(r),\quad L_\xi \bar{g}_{ab} \longrightarrow o(r^2). \tag{10}$$

We remark that this definition does not use assumptions on fall-off conditions for the metric perturbations. In fact, eventually we shall derive such conditions by requiring the existence of asymptotically conserved $(n-2)$-forms associated to the asymptotic Killing vectors. Regarding this point, we thus reverse the approach used in [10] where we derived fall-off conditions determining asymptotic Killing vectors from fall-off conditions for the fields.

Asymptotic Killing vectors that fall-off as

$$\xi^r \longrightarrow o(r),\ \xi^a \longrightarrow o(r^0) \tag{11}$$

automatically satisfy (10) and are considered trivial. We want to compute the equivalence classes of asymptotic Killing vectors modulo trivial ones.

Non trivial Killing vectors can thus be parameterized by

$$\xi^\tau \longrightarrow lT(\tau, y),\ \xi^r \longrightarrow rR(\tau, y) + o(r),\quad \xi^A \longrightarrow \Phi^A(\tau, y). \tag{12}$$

The constraints (10) then reduce to [2]

$$R = -lT_{,\tau}, \tag{13}$$

$$T_{,A} = l f_{(A)} \Phi^A_{,\tau}, \tag{14}$$

$$lT_{,\tau} = \Phi^A_{,(A)} + \textstyle\sum_{B<A} \Phi^B \cot y^B, \tag{15}$$

$$\Phi^B_{,A} f_{(B)} + \Phi^A_{,B} f_{(A)} = 0,\ A \neq B. \tag{16}$$

[2] An index in parentheses indicates that the summation convention does not apply.

2.2. Conformal Killing vectors on the boundary

For large r, the metric $d\bar{s}^2$ behaves to leading order as $d\bar{s}^2 \longrightarrow \frac{r^2}{l^2}(ds'^2)$ where

$$ds'^2 = g'_{ab} dx^a dx^b = -d\tau^2 + l^2 \sum_{A=2}^{n-1} f_A (dy^A)^2 \tag{17}$$

and is thus asymptotically conformal to this latter metric, which will be referred to as the "metric induced on the boundary" below.

The conformal Killing vectors of the metric induced on the boundary satisfy

$$L_\xi g'_{ab} = \frac{2}{n-1} D'_c \xi^c\, g'_{ab}. \tag{18}$$

Using the notation[3] $\xi^\tau = lT(\tau, y)$ and $\xi^A = \Phi^A(\tau, y)$ one obtains

$$D'_c \xi^c = \partial_c \xi^c + \frac{1}{2} \sum_{A=2}^{n-1} \xi^c \partial_c \ln f_A = lT_{,\tau} + \sum_{A=2}^{n-1} (\Phi^A_{,A} + \sum_{B<A} \Phi^B \cot y^B). \tag{19}$$

Explicitly, this gives the conditions

$$lT_{,\tau} = \frac{1}{n-1}\Big(lT_{,\tau} + \sum_{A=2}^{n-1} (\Phi^A_{,A} + \sum_{B<A} \Phi^B \cot y^B)\Big), \tag{20}$$

$$\Phi^A_{,(A)} + \sum_{B<A} \Phi^B \cot y^B = \frac{1}{n-1}\Big(lT_{,\tau} + \sum_{A=2}^{n-1} (\Phi^A_{,A} + \sum_{B<A} \Phi^B \cot y^B)\Big), \tag{21}$$

$$T_{,A} = l f_{(A)} \Phi^A_{,\tau}, \tag{22}$$

$$\Phi^B_{,A} f_{(B)} + \Phi^A_{,B} f_{(A)} = 0,\ A \neq B. \tag{23}$$

[3] At this stage it is actually not yet clear that the functions $T(\tau, y)$ and $\Phi^A(\tau, y)$ coincide with those in (12). Nevertheless we employ the same notation.

2.3. Correspondence

Theorem 1 *The non trivial asymptotic Killing vectors are in one to one correspondence with the conformal Killing vectors of the metric g'_{ab} induced on the boundary.*

Proof: The asymptotic Killing equations imply the conformal ones for the boundary metric. Indeed, (20) is equivalent to

$$lT_{,\tau} = \frac{1}{n-2}\sum_{A=2}^{n-1}(\Phi^A_{,A} + \sum_{B<A}\Phi^B \cot y^B), \qquad (24)$$

which is implied by the sum over A of (15). Using (24), (21) is equivalent to

$$\Phi^A_{,(A)} + \sum_{B<A}\Phi^B \cot y^B = $$
$$= \frac{1}{n-2}\sum_{A=2}^{n-1}(\Phi^A_{,A} + \sum_{B<A}\Phi^B \cot y^B), \qquad (25)$$

which is implied by using (15) in (24). Finally (22) is (14) and (23) is (16).

Conversely, the conformal Killing vectors imply the asymptotic ones. Indeed, we only need to verify that (15) hold (since (13) only determines R and does not impose any constraint). This follows from using (25) in (24).□

2.4. Discussion

The boundary metric is conformal to the flat Minkowski metric η_{ab}, so that the conformal Killing vectors of the boundary metric, and thus also the equivalence classes of asymptotic Killing vectors, correspond one to one to the conformal Killing vectors of Minkowski space in $n-1$ dimensions. For $n = 3$ one thus gets the infinite dimensional pseudo conformal algebra in 2 dimensions. There is thus symmetry enhancement as the exact Killing vectors are given by the 6 dimensional $so(2,2)$ algebra. For $n > 3$, one gets an algebra that is isomorphic to $so(n-1,2)$ of dimension $\frac{(n+1)n}{2}$, and thus a one to one correspondence between exact and non trivial asymptotic Killing vectors.

3. Charges and fall-off conditions

Let $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, $h = g^{\mu\nu}h_{\mu\nu}$, and S^∞_{n-2} the $(n-2)$-dimensional sphere at spatial infinity defined by $\tau =$ constant, $r = R \to \infty$, and

$$(d^{n-p}x)_{\mu_1\ldots\mu_p} = \frac{1}{p!(n-p)!}\epsilon_{\mu_1\ldots\mu_n} dx^{\mu_{p+1}}\ldots dx^{\mu_n},$$

with $\epsilon_{0\ldots n-1} = 1$. We restrict the discussion to functions $h_{\mu\nu}$ which depend on r such that $h_{\mu\nu} \longrightarrow O(r^m)$ implies $\partial_r h_{\mu\nu} \longrightarrow O(r^{m-1})$ and $\partial_r^2 h_{\mu\nu} \longrightarrow O(r^{m-2})$ for $r \longrightarrow \infty$.

We shall now discuss the charges associated with the asymptotic Killing vectors, see, e.g., [1–6,9,10]. Our discussion is based on results in the last of these references where the following expression for these charges was derived:

$$Q_\xi = -\int_{S^\infty_{n-2}} (d^{n-2}x)_{\nu\mu} k^{\nu\mu}_\xi =$$
$$= -\lim_{r\longrightarrow\infty}\int \prod_{A=2}^{n-1} dy^A k^{[\tau r]}_\xi \qquad (26)$$

with

$$k^{\nu\mu}_\xi[h;\bar{g}] = -\frac{\sqrt{-\bar{g}}}{16\pi}\Big[\bar{D}^\nu(h\xi^\mu) + \bar{D}_\sigma(h^{\mu\sigma}\xi^\nu) + \bar{D}^\mu(h^{\nu\sigma}\xi_\sigma) + \frac{3}{2}h\bar{D}^\mu\xi^\nu + \frac{3}{2}h^{\sigma\mu}\bar{D}^\nu\xi_\sigma + \frac{3}{2}h^{\nu\sigma}\bar{D}_\sigma\xi^\mu - (\mu\leftrightarrow\nu)\Big]. \qquad (27)$$

As shown in [10], in order that these charges are meaningful, the $(n-2)$-forms $(d^{n-2}x)_{\nu\mu}k^{\nu\mu}_\xi$ must be asymptotically conserved at S^∞_{n-2} which imposes

$$d^n x\, \mathcal{H}^{\mu\nu}\, L_\xi \bar{g}_{\mu\nu} \longrightarrow 0, \qquad (28)$$

where $\mathcal{H}^{\mu\nu}$ are the left hand sides of the linearized field equations associated to the action $S = \int d^n x\sqrt{-g}(R - 2\Lambda)$, with $\Lambda = -\frac{(n-1)(n-2)}{l^2}$. In our present approach, (28) determines the fall-off conditions for the functions $h_{\mu\nu}$ to which the results apply. To derive these conditions, we use that (10) and (28) impose

$$\mathcal{H}^{rr} \longrightarrow O(r),\ \mathcal{H}^{ar} \longrightarrow O(r^{-2}),$$
$$\mathcal{H}^{ab} \longrightarrow O(r^{-3}). \qquad (29)$$

Using now the explicit expression for $\mathcal{H}^{\mu\nu}$,

$$\mathcal{H}^{\mu\nu}[h;\bar{g}] = \frac{\sqrt{-\bar{g}}}{32\pi}\Big(\frac{2\Lambda}{n-2}(2h^{\mu\nu} - \bar{g}^{\mu\nu}h) + \bar{D}^\mu\bar{D}^\nu h + \bar{D}^\lambda\bar{D}_\lambda h^{\mu\nu} - 2\bar{D}_\lambda\bar{D}^{(\mu}h^{\nu)\lambda} - \bar{g}^{\mu\nu}(\bar{D}^\lambda\bar{D}_\lambda h - \bar{D}_\lambda\bar{D}_\rho h^{\rho\lambda})\Big), \qquad (30)$$

we can eventually derive the fall-off behaviour that generic functions $h_{\mu\nu}$ must have in order that (29) holds. The result is

$$h_{rr} \longrightarrow O(r^{-n-1}),\ h_{ar} \longrightarrow O(r^{-n}),\\ h_{ab} \longrightarrow O(r^{-n+3}). \qquad (31)$$

These conditions agree with those imposed in [3–5] and imply the finiteness of the charges (26).

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