Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

On the geometric flows of Kirchhoff elastic rods

(revised version: September 2004)

by

Chun-Chi Lin and Hartmut Schwetlick

Preprint no.: 52 2003

ON THE GEOMETRIC FLOW OF KIRCHHOFF ELASTIC RODS

CHUN-CHI LIN AND HARTMUT R. SCHWETLICK

Abstract. Recently, rod theory has been applied to the mathematical modeling of bacterial fibers and biopolymers (e.g. DNA), to study their mechanical properties and shapes (e.g. supercoiling). In static rod theory, an elastic rod in equilibrium is the critical point of an elastic energy. This induces a natural question of how to find elasticae. In this paper, we focus on how to find the critical points by means of gradient flows. We relate a geometric functional of curves to the isotropic Kirchhoff elastic energy of rods so that the generalized elastic curves are the centerlines of elastic rods in equilibrium. Thus, the variational problem for rods is formulated in curve geometry. This problem turns out to be a generalization of curve-straightening flows, which induce nonlinear fourth-order evolution equations. We establish the long time existence of length preserving gradient flow for the geometric energy. Furthermore, by studying the asymptotic behaviour, we show that the limit curves are the centerlines of the Kirchhoff elastic rods in equilibrium.

1. Introduction

Recently, rod theory has been applied to the mathematical modeling of bacterial fibers and biopolymers (e.g. DNA) to study their mechanical properties and shapes (e.g. supercoiling). In static rod theory, an elastic rod in equilibrium is the critical point of an elastic energy. This induces a natural question of how to find elasticae. In our project, we ask the question: starting from a given rod configuration Γ in $\mathbb{R}^3$, can we find the critical points of a Kirchhoff elastic energy, or the so called elasticae, by means of geometric gradient flows? In order to keep the model problem in this paper simple, we only consider a special isotropic Kirchhoff elastic energy. For more general rod theory, readers are referred to [1].

Suppose $f : I = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is the centerline of a closed rod. Let $\gamma = |\partial_x f|$, $ds = \gamma\, dx$ the arclength element, and $\partial_s = \gamma^{-1}\partial_x$ the arclength differentiation. Denote by $T = \partial_s f$ the unit tangent vector, and $\kappa = \partial_s^2 f$ the curvature vector of f. A rod configuration Γ is a framed curve described by $\{f(s); T(s), M_1(s), M_2(s)\}$, where the material frame $\{T, M_1, M_2\}$ forms an orthonormal frame field along f. Thus, we can write the skew-symmetric system

$$\begin{pmatrix} T' \\ M_1' \\ M_2' \end{pmatrix} = \begin{pmatrix} 0 & m_1 & m_2 \\ -m_1 & 0 & m \\ -m_2 & -m & 0 \end{pmatrix} \begin{pmatrix} T \\ M_1 \\ M_2 \end{pmatrix},$$

Date: June 29, 2003; revised version April 12, 2004.

Key words and phrases. geometric flows, fourth order, Kirchhoff elastic rods.

C.C.L. sincerely thanks Stefan Müller and Angela Stevens for encouragement; Dirk Drasdo for many interesting discussions on biophysics; the Max Planck Institute for Mathematics in the Science, Leipzig/Germany and the Department of Mathematics, National Taiwan Normal University for supporting during this work.

with arbitrary functions $m_1(s)$, $m_2(s)$, and $m(s)$. Consider the Kirchhoff elastic energy $\mathcal{E}$ of an isotropic rod Γ, defined by

$$\mathcal{E}[\Gamma] := \int_I \left(\alpha \cdot (m_1^2 + m_2^2) + \beta \cdot m^2\right)\, ds,$$

with material constants $\alpha > 0$ and $\beta \geq 0$. The term involving α gives the bending energy, while the term involving β gives the twisting energy.

Whenever a smooth curve f has no inflection points, the Frenet frame field $\{T, N, B\}$ along f is well-defined. By using the Frenet frame field, it can be easily verified that

$$\mathcal{E}[\Gamma] = \int_I \left(\alpha |\kappa|^2 + \beta m^2\right)\, ds, \tag{1.1}$$

(e.g., see [7]). A natural frame is an orthonormal frame field along a given curve f, which is uniquely determined by its initial data at a point and the skew-symmetric system,

$$\begin{pmatrix} T' \\ U' \\ V' \end{pmatrix} = \begin{pmatrix} 0 & u & v \\ -u & 0 & 0 \\ -v & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ U \\ V \end{pmatrix},$$

(see [3] or [7] p. 607). A natural frame can be thought as a frame without twisting. As we denote by θ the angle from U to M_1 with $\theta(0) = 0$, one can verify that m is equal to the twisting rate, i.e., $m(s) = \theta'(s)$. Whenever f contains no inflection points, the Frenet frame is well defined along f. Denote by ϕ the angle from U to N, then it is easy to verify that the torsion of the curve satisfies $\tau = \phi'$. Denote by $\Psi := \theta - \phi$ the angle from N to M_1 and let $\triangle\Psi := \Psi(L) - \Psi(0)$, where L is the total length of f. By these notations, we have

$$Tw[\Gamma] = \int_I m\, ds = \triangle\Psi + \int_I \tau\, ds. \tag{1.2}$$

It is worth mentioning here that whenever f contains neither self-intersection nor inflection points, applying the so called Fuller-Calugareanu-White formulae provides another approach to derive Eq.(1.2). This approach is less general and less direct, but reveals the topological meaning of $\triangle\Psi$, although the total twisting number of Γ, $Tw[\Gamma]$, and the total torsion of f, $\int_I \tau\, ds$, are not topological invariants. We thus set up the boundary value problem by prescribing a real number, $\triangle\Psi$, which is called the end point condition of rod configurations in the rest of this paper. From above, we would like to emphasize that the bending energy and twisting energy interact as rod configurations achieving the critical points of the elastic energy. More precisely, the twisting depends on the centerlines of rods as well. Otherwise, the twisting energy and bending energy can be considered separately and the resulting centerlines of rod elasticae would simply be curve elasticae.

In [7], Langer and Singer proposed to study the generalized elastic curves by introducing the geometric functional $\widetilde{\mathcal{F}}$ of curves $f : I \to \mathbb{R}^3$,

$$\widetilde{\mathcal{F}}[f] := \lambda_3 \mathcal{K}[f] + \lambda_2 \mathcal{T}[f] + \lambda_1 \mathcal{L}[f], \tag{1.3}$$

where

$$\mathcal{K}[f] := \int_I \frac{1}{2}|\kappa|^2 \ ds,\ \mathcal{T}[f] := \int_I \tau \ ds,\ \mathcal{L}[f] := \int_I ds,$$

and λ_i in Eq.(1.3) are Lagrange multipliers for $i = 1, 2$. According to their formulation, a generalized elastic curve f in equilibrium is a critical point of the elastic energy $\widetilde{\mathcal{F}}$ among the class of curves with fixed total torsion $\mathcal{T}[f] = T_0$ and length $\mathcal{L}[f] = L$. As long as λ_i together with the fixed total torsion T_0 fit certain relations, they showed that f is the centerline of an isotropic elastic rod in equilibrium. The problems considered in this paper and in Eq. (1.3) is closely related to curve straightening flows. To the authors' knowledge, curve straightening flows have been studied by Wen [9], Polden [8], Koiso[6] and Dziuk, Kuwert, Schätzle [4]. At the beginning, we tried to apply the method used in the problems of curve straightening flows to the geometric functional $\widetilde{\mathcal{F}}$ proposed in [7]. However, an essential difficulty coming from the constraint of fixing the total torsion fails this approach. Namely, after multiplying the term of the first variation of the total torsion $\mathcal{T}[f]$ by its Lagrange multiplier, the method of L^2 curvature estimates combined with Gagliardo-Nirenberg-type interpolation inequalities used in the problems of curve straightening flows fails, because this term has higher power of derivatives in total than those from $\mathcal{K}[f]$.

In order to resolve the difficulty mentioned above, we propose another approach based on Theorem 1 below. We learn from [5] and [7] that a symmetric elastic rod (or, equivalently, an isotropic elastic rod) must have a constant twisting rate. Observe that among all isotropic rod configurations Γ with constant twisting rate $m = \frac{\mathcal{T}[f]+\triangle\Psi}{L}$, fixed length L, but without inflection points, we have the identity,

$$\mathcal{E}[\Gamma] = \mathcal{G}_{\triangle\Psi,L}[f] := 2\alpha\mathcal{K}[f] + \frac{\beta}{L}\left(\mathcal{T}[f] + \triangle\Psi\right)^2.$$

Theorem 1 basically means that the equilibrium elastic rods must stay in the subclass of rod configurations with constant twisting rate and fixed length L. Thus, in order to find closed elastic rods of $\mathcal{E}$, we work with the geometric functional,

$$\mathcal{F}[f] := \mathcal{G}_{\triangle\Psi,L}[f] + \lambda_1 \cdot (\mathcal{L}[f] - L), \tag{1.4}$$

where λ_1 is the Lagrange multiplier. It turns out that working with the functional $\mathcal{G}_{\triangle\Psi,L}$ of curves with fixed length L is more suitable than working directly with the rod energy $\mathcal{E}$ in our geometric approach .

Theorem 1. *Let $f : I = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be the centerline of a closed rod Γ. Assume f contains no inflection points. Then, subject to variations of fixed length L and end point condition $\triangle\Psi$ in Eq.(1.2), Γ is an equilibrium of the elastic energy $\mathcal{E}$ if and only if f is a critical point of the geometric functional $\mathcal{F}$ and the twisting rate is equal to the constant $\frac{\triangle\Psi+\mathcal{T}[f]}{L}$.*

The inflection points in above theorem simply mean points of zero curvature. We exclude the situation of the limit curves containing inflection points because our argument in Theorem 1 relies on the formulation of Frenet frames, which are ill-defined at an inflection point. By the first variational formulae in Lemma 6, we obtain for the length preserving L^2 gradient flow of $\mathcal{F}$ the evolution equation,

$$\partial_t f = \lambda_3 \cdot (-\nabla_s^2\kappa - \frac{|\kappa|^2}{2}\kappa) + \lambda_2(t) \cdot \nabla_s(T \times \kappa) + \lambda_1(t) \cdot \kappa, \tag{1.5}$$

where $f : [0, \infty) \times I \to \mathbb{R}^3$ has smooth initial data f_0. Here, the covariant derivative $\nabla_s \eta$ denotes the normal component of $\partial_s \eta$, i.e., $\nabla_s \eta = \partial_s \eta - \langle \partial_s \eta, T \rangle T$, and

$$\lambda_1(t) := \frac{2\alpha \int_I \langle \kappa, \nabla_s^2 \kappa + \frac{|\kappa|^2}{2} \kappa \rangle \, ds - \frac{2\beta}{L} \left(\mathcal{T}[f] + \triangle \Psi \right) \int_I \langle \kappa, \nabla_s (T \times \kappa) \rangle \, ds}{\int_I |\kappa|^2 \, ds}, \tag{1.6}$$

$$\lambda_2(t) := \frac{2\beta}{L} \left(\mathcal{T}[f] + \triangle \Psi \right), \; L, \, \beta > 0, \tag{1.7}$$

$$\lambda_3 := 2\alpha, \; \alpha > 0. \tag{1.8}$$

Notice that $\lambda_1(t)$ in Eq.(1.6) is chosen so that $\frac{d}{dt} \mathcal{L}[f_t] = 0$ for all time. The following theorem is the main result of this paper.

Theorem 2. *For any real number $\triangle \Psi$ and any smooth initial closed curve f_0, there exists a smooth solution to the L^2-gradient flow in Eq.(1.5), until the appearance of inflection points. With the assumption of no inflection points appearing during the flow, the curves subconverge to f_∞, an equilibrium of the energy functional $\mathcal{F}$, after reparametrization by arclength and translation. Furthermore, if f_∞ contains no inflection points, then f_∞ is the centerline of an equilibrium Kirchhoff elastic rod with constant total twisting rate $\frac{\mathcal{T}[f_\infty] + \triangle \Psi}{L}$.*

This paper is arranged as follows. In section 2 we introduce further notation and collect the results needed from [4]. Since most of these preliminaries follow the lines in [4], the reader is recommended to consult [4] for further details. In Section 3 we present the proof of the main results. Finally, Section 4 is devoted to the numerical treatment of the problem. We explain the algorithm we have used and show several computational results.

2. Preliminaries

Lemma 1 (Lemma 2.1 in [4])**.** *Suppose ϕ is any normal field along f and $f : [0, \epsilon) \times I \to \mathbb{R}^n$ is a time dependent curve satisfying $\partial_t f = V + \varphi T$, where V is the normal velocity and $\varphi = \langle T, \partial_t f \rangle$. Then the following formulae hold.*

$$\nabla_s \phi = \partial_s \phi + \langle \phi, \kappa \rangle T, \tag{2.1}$$

$$\partial_t (ds) = (\partial_s \varphi - \langle \kappa, V \rangle) \; ds, \tag{2.2}$$

$$\partial_t \partial_s - \partial_s \partial_t = (\langle \kappa, V \rangle - \partial_s \varphi) \; \partial_s, \tag{2.3}$$

$$\partial_t T = \nabla_s V + \varphi \cdot \kappa, \tag{2.4}$$

$$\partial_t \phi = \nabla_t \phi - \langle \nabla_s V + \varphi \; \kappa, \phi \rangle T, \tag{2.5}$$

$$\nabla_t \kappa = \nabla_s^2 V + \langle \kappa, V \rangle \kappa + \varphi \cdot \nabla_s \kappa, \tag{2.6}$$

$$(\nabla_t \nabla_s - \nabla_s \nabla_t) \phi = (\langle \kappa, V \rangle - \partial_s \varphi) \nabla_s \phi + \langle \kappa, \phi \rangle \nabla_s V - \langle \nabla_s V, \phi \rangle \cdot \kappa. \tag{2.7}$$

Lemma 2 (Lemma 2.2 in [4])**.** *Suppose $f : [0, \widehat{T}) \times I \to \mathbb{R}^n$ moves in a normal direction with velocity $\partial_t f = V$, ϕ is a normal vector field along f, and $\nabla_t \phi + \nabla_s^4 \phi = Y$. Then*

$$\frac{d}{dt} \frac{1}{2} \int_I |\phi|^2 \; ds + \int_I \left| \nabla_s^2 \phi \right|^2 \; ds = \int_I \langle Y, \phi \rangle \; ds - \frac{1}{2} \int_I |\phi|^2 \langle \kappa, V \rangle \; ds. \tag{2.8}$$

Furthermore, $\psi = \nabla_s \phi$ satisfies the equation

$$\nabla_t \psi + \nabla_s^4 \psi = \nabla_s Y + \langle \kappa, \phi \rangle \nabla_s V - \langle \nabla_s V, \phi \rangle \kappa + \langle \kappa, V \rangle \psi. \tag{2.9}$$

For normal vector fields $\phi_1, \cdots, \phi_k$ along f, we denote by $\phi_1 * * * \phi_k$ a term of the type

$$\phi_1 * * * \phi_k = \begin{cases} \langle \phi_{i_1}, \phi_{i_2} \rangle \cdots \langle \phi_{i_{k-1}}, \phi_{i_k} \rangle & \text{, for } k \text{ even,} \\ \langle \phi_{i_1}, \phi_{i_2} \rangle \cdots \langle \phi_{i_{k-2}}, \phi_{i_{k-1}} \rangle \cdot \phi_{i_k}, & \text{for } k \text{ odd,} \end{cases}$$

where $i_1, \cdots, i_k$ is any permutation of $1, \cdots, k$. Slightly more generally, we allow some of the ϕ_i to be functions, in which case the $*$-product reduces to multiplication. For a normal vector field ϕ along f, we denote by $P_\nu^\mu(\phi)$ any linear combination of terms of the type $\nabla_s^{i_1} \phi * \cdots * \nabla_s^{i_\nu} \phi$ with universal constant coefficients, where $\mu = i_1 + \cdots + i_\nu$ is the total number of derivatives. Notice that the following formulae hold:

$$\text{(2.10)} \quad \begin{cases} \nabla_s \left(P_b^a(\phi) * P_d^c(\phi) \right) = \nabla_s P_b^a(\phi) * P_d^c(\phi) + P_b^a(\phi) * \nabla_s P_d^c(\phi), \\ P_b^a(\phi) * P_d^c(\phi) = P_{b+d}^{a+c}(\phi), \ \nabla_s P_d^c(\phi) = P_d^{c+1}(\phi). \end{cases}$$

Similarly, we denote by $Q_\nu^\mu(\kappa)$ the linear combination of $\partial_s^{i_1}\kappa * * * \partial_s^{i_\nu}\kappa$, where $i_1 + \cdots + i_\nu = \mu$.

The following lemma states the important interpolation inequality for higher order curvature functionals.

Lemma 3 (Proposition 2.5 in [4]). *For any term $P_\nu^\mu(\kappa)$ with $\nu \geq 2$ which contains only derivatives of κ of order at most $k-1$, we have*

$$\text{(2.11)} \qquad \int_I \left| P_\nu^\mu(\kappa) \right| \, ds \leq c \, \mathcal{L}[f]^{1-\mu-\nu} \|\kappa\|_2^{\nu-\gamma} \|\kappa\|_{k,2}^{\gamma},$$

where $\gamma = \left(\mu + \frac{1}{2}\nu - 1\right)/k$, $c = c(n, k, \mu, \nu)$, and

$$\|\kappa\|_{k,p} := \sum_{i=0}^{k} \left\| \nabla_s^i \kappa \right\|_p, \ \left\| \nabla_s^i \kappa \right\|_p := \mathcal{L}[f]^{i+1-1/p} \left(\int_I \left| \nabla_s^i \kappa \right|^p \, ds \right)^{1/p}.$$

Moreover, if $\mu + \frac{1}{2}\nu < 2k+1$, then $\gamma < 2$ and we have for any $\varepsilon > 0$,

(2.12)
$$\int_I \left| P_\nu^\mu(\kappa) \right| \, ds \leq \varepsilon \int_I \left| \nabla_s^k \kappa \right|^2 \, ds + c\varepsilon^{\frac{-\gamma}{2-\gamma}} \left(\int_I |\kappa|^2 \, ds \right)^{\frac{\nu-\gamma}{2-\gamma}} + c\left(\int_I |\kappa|^2 \, ds \right)^{\mu+\nu-1}.$$

Lemma 4 (Lemma 2.6 in [4]). *We have the identities*

$$\text{(2.13)} \qquad \nabla_s \kappa - \partial_s \kappa = |\kappa|^2 T,$$

$$\text{(2.14)} \qquad \nabla_s^m \kappa - \partial_s^m \kappa = \sum_{i=1}^{[\frac{m}{2}]} Q_{2i+1}^{m-2i}(\kappa) + \sum_{i=1}^{[\frac{m+1}{2}]} Q_{2i}^{m+1-2i}(\kappa) T.$$

Lemma 5 (Lemma 2.7 in [4]). *Assume the bounds $\|\kappa\|_{L^2} \leq \Lambda_0$ and $\|\nabla_s^m \kappa\|_{L^1} \leq \Lambda_m$ for $m \geq 1$. Then for any $m \geq 1$ one has*

$$\text{(2.15)} \qquad \left\| \partial_s^{m-1} \kappa \right\|_{L^\infty} + \left\| \partial_s^m \kappa \right\|_{L^1} \leq c_m(\Lambda_0, ..., \Lambda_m).$$

3. Proof of the Main Results

Lemma 6. *Let $f : I = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ represent a smooth curve in $\mathbb{R}^3$ without inflection points. Then, for any variation $f_\varepsilon(x) = f(x) + \varepsilon W(x)$, where $f, W \in C^\infty(I)$, one has the followings:*

$$\frac{d}{d\varepsilon}\lfloor_{\varepsilon=0}\mathcal{L}[f_\varepsilon] = -\int_I \langle \kappa, W\rangle \, ds + [\langle T, W\rangle]_0^L,$$

$$\frac{d}{d\varepsilon}\lfloor_{\varepsilon=0}\mathcal{T}[f_\varepsilon] = -\int_I \langle \nabla_s (T \times \kappa), W\rangle \, ds + \left[\langle \nabla_s^2 (W - \langle W, T\rangle T) + \langle W, T\rangle \cdot \nabla_s \kappa, \tfrac{B}{|\kappa|}\rangle + \langle W, T \times \kappa\rangle\right]_0^L,$$

$$\frac{d}{d\varepsilon}\lfloor_{\varepsilon=0}\mathcal{K}[f_\varepsilon] = \int_I \langle \nabla_s^2 \kappa + \tfrac{|\kappa|^2}{2}\kappa, W\rangle \, ds + \left[\langle T, W\rangle \cdot \tfrac{|\kappa|^2}{2} + \langle \kappa, \nabla_s (W - \langle W, T\rangle T)\rangle - \langle \nabla_s \kappa, W\rangle\right]_0^L.$$

Proof. The formulae can be directly verified by applying the general formulae in Lemma 1 below. Thus, we skip the detail of the computation and leave the verification to the reader. □

Proof of Theorem 1. If we perturb the rod configuration Γ of a given elastic rod in equilibrium without perturbing the centerline f, then

$$0 = \delta\mathcal{E}[\Gamma] = \beta \cdot \delta \int_I m^2 \, ds = 2\beta \cdot \int_I m \cdot (\delta m) \, ds. \tag{3.1}$$

By the end point condition in Eq.(1.2) and the formula $m(s) = \theta'(s)$, we conclude that m is a constant and

$$m = \frac{(\triangle\Psi + \mathcal{T}[f])}{L} = L^{-1} \int_I m \, ds.$$

Thus, any closed Kirchhoff elastic rod in equilibrium with end point condition $\triangle\Psi$ and length L belongs to the subclass of rod configurations $\mathcal{A}_{\triangle\Psi,L}$, where

$$\mathcal{A}_{\triangle\Psi,L} := \{\Gamma : m(s) = \frac{\triangle\Psi + \mathcal{T}[f]}{L}, \, \mathcal{L}[f] = L\}.$$

Observe that for any rod configurations $\Gamma \in \mathcal{A}_{\triangle\Psi,L}$, we have

$$\mathcal{E}[\Gamma] = \mathcal{G}_{\triangle\Psi,L}[f]. \tag{3.2}$$

Now, perburbations of Γ preserving the length in the subclass of rod configurations $\mathcal{A}_{\triangle\Psi,L}$ induce the variational equation,

$$\delta(\mathcal{G}_{\triangle\Psi,L}[f] + \lambda_1 \cdot (\mathcal{L}[f] - L)) = 0,$$

where λ_1 is the Lagrange multiplier.

Conversely, by assuming f being the critical point of $\mathcal{F}$, we have

$$\delta_L \mathcal{G}_{\triangle\Psi,L}[f] = 0,$$

where δ_L denotes perturbations of preserving the length. Since the rod configuration Γ has constant twisting rate $\frac{(\triangle\Psi + \mathcal{T}[f])}{L}$, therefore $\Gamma \in \mathcal{A}_{\triangle\Psi,L}$. Thus,

$$\delta_L \mathcal{E}[\Gamma] = \delta_L \mathcal{G}_{\triangle\Psi,L}[f] = 0.$$

□

Proof of Theorem 2. The proof is motivated by the arguments in [4]. Recalling that no inflection point is on the initial curve, the short time existence is a standard argument. We thus skip it here, and focus on the long time existence and asymptotic behaviour.

To prove global bounds we wish to estimate higher Sobolev norms of the curvature. Their evolution is given by

$$\nabla_t \nabla_s^m \kappa = -\nabla_s^4 \nabla_s^m \kappa + \text{ tensors of lesser order.}$$

Therefore we arrive at

$$\frac{d}{dt}\frac{1}{2}\int_I |\nabla_s^m \kappa|^2 \, ds + \int_I |\nabla_s^{m+2}\kappa|^2 \, ds = \text{ terms of lesser order.}$$

It will be not necessary to compute the error terms explicitly. It is sufficient to keep track of their scaling, in other words we have to know the order of the derivatives involved. Using the notation introduced before the next lemma characterizes the error terms coming from the twist term, i.e., dealing with the new situation that the total torsion is included in our energy.

Lemma 7. *For $m \geq 2$, we have the formula,*

$$\begin{aligned}\nabla_s^m (T\times\kappa) &= T\times\nabla_s^m\kappa + \sum_{a_1,b_1,c_1,d_1}\left[P_{b_1}^{a_1}(\kappa)\times P_{d_1}^{c_1}(\kappa)\right]^{\perp} \\ &+ \sum_{i=1,2}\ \sum_{a_2^{(i)},b_2^{(i)},c_2^{(i)},d_2^{(i)},e_2^{(i)},f_2^{(i)}} \left[\left(P_{b_2^i}^{a_2^i}(\kappa)\times P_{d_2^i}^{c_2^i}(\kappa)\right) * P_{f_2^i}^{e_2^i}(\kappa)\right]^{\perp} \\ &+ \sum_{i=1,2}\ \sum_{a_3^{(i)},b_3^{(i)},c_3^{(i)},d_3^{(i)}} \left(\left(T\times P_{b_3^{(i)}}^{a_3^{(i)}}(\kappa)\right) * P_{d_3^{(i)}}^{c_3^{(i)}}(\kappa)\right),\end{aligned}$$

where the sums are taken such that $(a_1+c_1)+(b_1+d_1)/2 = m$, $(a_2^{(i)}+c_2^{(i)}+e_2^{(i)}) + (b_2^{(i)}+d_2^{(i)}+f_2^{(i)})/2 = m-i$, and $(a_3^{(i)}+c_3^{(i)}) + (b_3^{(i)}+d_3^{(i)})/2 = m-i+1/2$ for $i\in\{1,2\}$.

Proof of Lemma 7. We first need the following formulae, which can be easily verified by applying Eq.(2.1). Assume $P_b^a(\phi)$ and $P_d^c(\phi)$ are normal vector fields, then

$$(3.3)\quad \begin{cases} \nabla_s\left(P_b^a(\phi)\times P_d^c(\phi)\right) = & [(\nabla_s P_b^a(\phi) + \langle\kappa, P_b^a(\phi)\rangle\cdot T)\times P_d^c(\phi) \\ & + P_b^a(\phi)\times(\nabla_s P_d^c(\phi)+\langle\kappa,P_d^c(\phi)\rangle\cdot T)]^{\perp}, \\ \nabla_s\left(T\times P_d^c(\phi)\right) = [\kappa\times P_d^c(\phi)]^{\perp} + T\times P_d^{c+1}(\phi), & \end{cases}$$

where $[\cdots]^{\perp}$ denotes its normal component and $\times$ denotes the exterior product in $\mathbb{R}^3$. Notice that in Eq.(3.3), we use $+$ instead of $-$ for our convenience because the sign is meaningless as using universal constant coefficients in those terms, $P_\beta^\alpha(\phi)$.

Now, the proof is an induction argument. As $m=2$,

$$\nabla_s^2(T\times\kappa) = T\times\nabla_s^2\kappa + (\kappa\times\nabla_s\kappa)^{\perp} = T\times\nabla_s^2\kappa + \left(P_1^0(\kappa)\times P_1^1(\kappa)\right)^{\perp}.$$

As $m \geq 3$, we apply Eqs. (2.1), (2.10) and (3.3) in the following calculation:

$$\begin{aligned}\nabla_s^m(T\times\kappa) &= \nabla_s\{T\times\nabla_s^{m-1}\kappa+\sum_{a_1,b_1,c_1,d_1}\left[P_{b_1}^{a_1}(\kappa)\times P_{d_1}^{c_1}(\kappa)\right]^{\perp}\\ &\quad+\sum_{i=1}^{2}\sum_{a_2^{(i)},b_2^{(i)},c_2^{(i)},d_2^{(i)},e_2^{(i)},f_2^{(i)}}\left[\left(P_{b_2^i}^{a_2^i}(\kappa)\times P_{d_2^i}^{c_2^i}(\kappa)\right)*P_{f_2^i}^{e_2^i}(\kappa)\right]^{\perp}\\ &\quad+\sum_{i=1}^{2}\sum_{a_3^{(i)},b_3^{(i)},c_3^{(i)},d_3^{(i)}}\left(\left(T\times P_{b_3^{(i)}}^{a_3^{(i)}}(\kappa)\right)*P_{d_3^{(i)}}^{c_3^{(i)}}(\kappa)\right)\}\\ &= \nabla_s\{I_1+I_2+\sum_{i=1}^{2}I_3^{(i)}+\sum_{i=1}^{2}I_4^{(i)}\}.\end{aligned}$$

1^0 :

$$\nabla_s I_1=\nabla_s\left[T\times\nabla_s^{m-1}\kappa\right]=T\times\nabla_s^m\kappa+\left[P_1^0(\kappa)\times P_1^{m-1}(\kappa)\right]^{\perp},$$

2^0 :

$$\begin{aligned}\nabla_s I_2 &= \sum_{a_1,b_1,c_1,d_1}\nabla_s\left[P_{b_1}^{a_1}(\kappa)\times P_{d_1}^{c_1}(\kappa)\right]^{\perp}\\ &= \sum_{a_1,b_1,c_1,d_1}\nabla_s[P_{b_1}^{a_1}(\kappa)\times P_{d_1}^{c_1}(\kappa)]-\langle P_{b_1}^{a_1}(\kappa)\times P_{d_1}^{c_1}(\kappa),T\rangle\cdot\kappa\\ &= \sum_{a_1,b_1,c_1,d_1}[((P_{b_1}^{a_1+1}(\kappa)+P_{b_1+1}^{a_1}(\kappa)T)\times P_{d_1}^{c_1}(\kappa))\\ &\quad+(P_{b_1}^{a_1}(\kappa)\times\left(P_{d_1}^{c_1+1}(\kappa)+P_{b_1+1}^{a_1}(\kappa)T\right)]^{\perp}+\left(T\times P_{b_1}^{a_1}(\kappa)\right)*P_{d_1+1}^{c_1}(\kappa)\\ &= \sum_{a,b,c,d}[P_b^a(\kappa)\times P_d^c(\kappa)]^{\perp}+\sum_{A,B,C,D}\left(T\times P_B^A(\kappa)\right)*P_D^C(\kappa),\end{aligned}$$

where $(a+c)+(b+d)/2=(a_1+c_1)+(b_1+d_1)/2+1$ and $(A+C)+(B+D)/2=(a_1+c_1)+(b_1+d_1)/2+1/2$.

3^0 :

$$\begin{aligned}\nabla_s I_3^{(i)} &= \sum_{a_2^{(i)},b_2^{(i)},c_2^{(i)},d_2^{(i)},e_2^{(i)},f_2^{(i)}}\nabla_s\left[\left(P_{b_2^i}^{a_2^i}(\kappa)\times P_{d_2^i}^{c_2^i}(\kappa)\right)*P_{f_2^i}^{e_2^i}(\kappa)\right]^{\perp}\\ &= \sum_{a_2^{(i)},b_2^{(i)},c_2^{(i)},d_2^{(i)},e_2^{(i)},f_2^{(i)}}\nabla_s[(P_{b_2^{(i)}}^{a_2^{(i)}}(\kappa)\times P_{d_2^{(i)}}^{c_2^{(i)}}(\kappa))*P_{f_2^{(i)}}^{e_2^{(i)}}(\kappa)]\\ &\qquad-\langle(P_{b_2^{(i)}}^{a_2^{(i)}}(\kappa)\times P_{d_2^{(i)}}^{c_2^{(i)}}(\kappa))*P_{f_2^{(i)}}^{e_2^{(i)}}(\kappa),T\rangle\cdot\kappa\\ &= \sum_{a_2^{(i)},b_2^{(i)},c_2^{(i)},d_2^{(i)},e_2^{(i)},f_2^{(i)}}\nabla_s[(P_{b_2^{(i)}}^{a_2^{(i)}}(\kappa)\times P_{d_2^{(i)}}^{c_2^{(i)}}(\kappa))]*P_{f_2^{(i)}}^{e_2^{(i)}}(\kappa)\\ &\qquad+(P_{b_2^{(i)}}^{a_2^{(i)}}(\kappa)\times P_{d_2^{(i)}}^{c_2^{(i)}}(\kappa))*\nabla_s P_{f_2^{(i)}}^{e_2^{(i)}}(\kappa)\\ &\qquad-\langle(P_{b_2^{(i)}}^{a_2^{(i)}}(\kappa)\times P_{d_2^{(i)}}^{c_2^{(i)}}(\kappa))*P_{f_2^{(i)}}^{e_2^{(i)}}(\kappa),T\rangle\cdot\kappa\\ &= \sum_{a,b,c,d,e,f}[(P_b^a(\kappa)\times P_d^c(\kappa))*P_f^e(\kappa)]^{\perp}+\sum_{A,B,C,D}\left(T\times P_B^A(\kappa)\right)*P_D^C(\kappa)\end{aligned}$$

where $(a+c+e)+(b+d+f)/2=(a_2^{(i)}+c_2^{(i)}+e_2^{(i)})+(b_2^{(i)}+d_2^{(i)}+f_2^{(i)})/2+1$ and $(A+C)+(B+D)/2=(a_2^{(i)}+c_2^{(i)})+(b_2^{(i)}+d_2^{(i)})/2+1/2$.

4^0 :

$$\begin{aligned}
\nabla_s I_4^{(i)} &= \sum_{a_3^{(i)},b_3^{(i)},c_3^{(i)},d_3^{(i)}} \nabla_s[(T\times P_{b_3^{(i)}}^{a_3^{(i)}}(\kappa))*P_{d_3^{(i)}}^{c_3^{(i)}}(\kappa)] \\
&= \sum_{a_3^{(i)},b_3^{(i)},c_3^{(i)},d_3^{(i)}} \{\partial_s[(T\times P_{b_3^{(i)}}^{a_3^{(i)}}(\kappa))*P_{d_3^{(i)}}^{c_3^{(i)}}(\kappa)]\}^{\perp} \\
&= \sum_{a_3^{(i)},b_3^{(i)},c_3^{(i)},d_3^{(i)}} \{(P_1^0(\kappa)\times P_{b_3^{(i)}}^{a_3^{(i)}}(\kappa)+(T\times P_{b_3^{(i)}}^{a_3^{(i)}+1}(\kappa)))*P_{d_3^{(i)}}^{c_3^{(i)}}(\kappa)\}^{\perp} \\
&\qquad + (T\times P_{b_3^{(i)}}^{a_3^{(i)}}(\kappa))*P_{d_3^{(i)}}^{c_3^{(i)}+1}(\kappa) \\
&= \sum_{a,b,c,d}[(P_b^a(\kappa)\times P_d^c(\kappa))*P_f^e(\kappa)]^{\perp} + \sum_{A,B,C,D}\left(T\times P_B^A(\kappa)\right)*P_D^C(\kappa)
\end{aligned}$$

where $(a+c+e)+(b+d+f)/2=(a_3^{(i)}+c_3^{(i)})+(b_3^{(i)}+d_3^{(i)})/2+1/2$ and $(A+C)+(B+D)/2=(a_3^{(i)}+c_3^{(i)})+(b_3^{(i)}+d_3^{(i)})/2+1$.

The proof is finished by summing up all these terms from 1^0 to 4^0. □

Lemma 8 (Corresponding to Lemma 2.3 in [4])**.** *Suppose*

$$\partial_t f = -\nabla_s^2\kappa + \sigma|\kappa|^2\kappa + \lambda_1\kappa + \lambda_2\nabla_s(T\times\kappa),$$

where σ, $\lambda_i\in\mathbb{R}$. *Then,*

For $m\geq 0$, *the derivatives of the curvature* $\phi_m=\nabla_s^m\kappa$ *satisfy*

(3.4)

$$\begin{aligned}
&\nabla_t\phi_m + \nabla_s^4\phi_m \\
&= P_3^{m+2}(\kappa) + \sigma\cdot\left(P_3^{m+2}(\kappa)+P_5^m(\kappa)\right) + \lambda_1\cdot\left(\nabla_s^{m+2}\kappa + P_3^m(\kappa)\right) \\
&\quad + \lambda_2\cdot\left(\nabla_s^{m+3}(T\times\kappa) + \nabla_s^{m+1}(T\times\kappa)*P_2^0(\kappa)+\cdots+\nabla_s^1(T\times\kappa)*P_2^m(\kappa)\right).
\end{aligned}$$

The statement is still true when $\lambda_i=\lambda_i(t)$ *depends on time.*

Proof of Lemma 8. The case of $m=0$ follows from Eq.(2.6) and the definition of $\partial_t f$,

$$\begin{aligned}
\nabla_t\kappa &= -\nabla_s^4\kappa + \sigma\cdot(\nabla_s^2(|\kappa|^2\kappa)+|\kappa|^4\kappa) + \lambda_1\cdot(\nabla_s^2\kappa+|\kappa|^2\kappa) \\
&\quad +\lambda_2\cdot\left(\nabla_s^3(T\times\kappa)+\kappa\langle\kappa,\nabla_s(T\times\kappa)\rangle\right).
\end{aligned}$$

For $m\geq 1$, Eq.(3.4) can be inductively derived by using Eq.(2.9),

$$\begin{aligned}
&\nabla_t\phi_m + \nabla_s^4\phi_m \\
&= \nabla_s[P_3^{m+1}(\kappa) + \sigma\cdot\left(P_3^{m+1}(\kappa)+P_5^{m-1}(\kappa)\right) + \lambda_1\cdot\left(\nabla_s^{m+1}\kappa+P_3^{m-1}(\kappa)\right) \\
&\qquad + \lambda_2\cdot(\nabla_s^{m+2}(T\times\kappa)+\nabla_s^m(T\times\kappa)*P_2^0(\kappa)+\cdots+\nabla_s^1(T\times\kappa)*P_2^{m-1}(\kappa))] \\
&+\langle\kappa,\phi_{m-1}\rangle\cdot\nabla_s[-\nabla_s^2\kappa+\sigma|\kappa|^2\kappa+\lambda_1\kappa+\lambda_2\nabla_s(T\times\kappa)] \\
&-\langle\nabla_s[-\nabla_s^2\kappa+\sigma|\kappa|^2\kappa+\lambda_1\kappa+\lambda_2\nabla_s(T\times\kappa)],\phi_{m-1}\rangle\cdot\kappa \\
&+\langle\kappa,-\nabla_s^2\kappa+\sigma|\kappa|^2\kappa+\lambda_1\kappa+\lambda_2\nabla_s(T\times\kappa)\rangle\cdot\phi_m \\
&\\
&= P_3^{m+2}(\kappa)+\sigma\cdot\left(P_3^{m+2}(\kappa)+P_5^m(\kappa)\right)+\lambda_1\cdot\left(\nabla_s^{m+2}\kappa+P_3^m(\kappa)\right) \\
&+\lambda_2\cdot(\nabla_s^{m+3}(T\times\kappa)+\nabla_s^{m+1}(T\times\kappa)*P_2^0(\kappa)+\cdots+\nabla_s^1(T\times\kappa)*P_2^m(\kappa)).
\end{aligned}$$

□

By Eqs.(2.8) and (3.4), we have

$$\begin{aligned}(3.5)\quad &\frac{d}{dt}\frac{1}{2}\int_I \mid \nabla_s^m \kappa \mid^2 \ ds + \int_I \mid \nabla_s^{m+2}\kappa \mid^2 \ ds + \lambda_1(t) \int_I \mid \nabla_s^{m+1}\kappa \mid^2 \ ds \\ &= \lambda_1(t) \int_I \langle \nabla_s^m \kappa, P_3^m(\kappa)\rangle \, ds + \int_I \langle \nabla_s^m \kappa, P_3^{m+2}(\kappa) + P_5^m(\kappa)\rangle \, ds \\ &\quad + \lambda_2(t) \int_I \langle \nabla_s^m \kappa, \nabla_s^{m+3}(T\times\kappa) + \nabla_s^{m+1}(T\times\kappa) * P_2^0(\kappa) \\ &\qquad + \cdots + \nabla_s^1(T\times\kappa) * P_2^m(\kappa)\rangle \, ds.\end{aligned}$$

Notice that estimating terms in Eq.(3.5) is the key argument of this paper. One can verify from Lemma 6 that

$$\begin{aligned}(3.6)\quad \frac{d}{dt}\mathcal{F}[f_t] &= 2\alpha \frac{d}{dt}\mathcal{K}[f] + \frac{2\beta}{L}(\mathcal{T}[f] + \triangle\Psi)\frac{d}{dt}\mathcal{T}[f] + \lambda_1(t)\cdot \frac{d}{dt}\mathcal{L}[f] \\ &= \int_I \langle 2\alpha(\nabla_s^2\kappa + \frac{|\kappa|^2}{2}\kappa) - \lambda_2(t)\nabla_s(T\times\kappa) - \lambda_1(t)\kappa, \partial_t f\rangle \, ds \\ &= -\int_I \mid 2\alpha(-\nabla_s^2\kappa - \frac{|\kappa|^2}{2}\kappa) + \lambda_2(t)\nabla_s(T\times\kappa) + \lambda_1(t)\kappa \mid^2 \ ds \\ &\le 0.\end{aligned}$$

Note that $\lambda_1(t)$ is chosen to fulfill $\mathcal{L}[f_t] \equiv L$. From Eq.(3.6), $\mathcal{F}[f_t]$ is nonincreasing as t is increasing. Thus,

$$\begin{aligned}\frac{\beta}{L}(\mathcal{T}[f_t] + \triangle\Psi)^2 &\le 2\alpha\mathcal{K}[f_t] + \frac{\beta}{L}(\mathcal{T}[f_t] + \triangle\Psi)^2 \\ &= \mathcal{G}_{\triangle\Psi,L}[f_t] = \mathcal{F}[f_t] \le \mathcal{F}[f_0] \\ &= 2\alpha\mathcal{K}[f_0] + \frac{\beta}{L}(\mathcal{T}[f_0] + \triangle\Psi)^2.\end{aligned}$$

Therefore,

$$(3.7)\quad |\lambda_2(t)| = \frac{2\beta}{L}|\mathcal{T}[f] + \triangle\Psi| \le C(f_0, \triangle\Psi, \alpha, \beta, L),$$

is uniformly bounded. Furthermore, by Eq.(3.6),

$$(3.8)\quad \|\kappa\|_{L^2}^2 = 2\mathcal{K}[f_t] \le C(f_0, \alpha).$$

Thus $\|\kappa\|_{L^2}^2$ is uniformly bounded for any $t \ge 0$.

By applying Eqs.(2.12), (3.7), (3.8) and Lemma 7, the sum of the last two terms in Eq.(3.5) satisfies the inequality,

$$\begin{aligned}(3.9)\quad &\int_I \langle \nabla_s^m \kappa, P_3^{m+2}(\kappa) + P_5^m(\kappa)\rangle \, ds \\ &+\lambda_2(t)\int_I \langle \nabla_s^m \kappa, \nabla_s^{m+3}(T\times\kappa) + \nabla_s^{m+1}(T\times\kappa) * P_2^0(\kappa) \\ &\qquad + \cdots + \nabla_s^1(T\times\kappa) * P_2^m(\kappa)\rangle \, ds \\ &\le C(f_0, \triangle\Psi, \alpha, \beta, L)\cdot(\varepsilon\int_I \left|\nabla_s^{m+2}\kappa\right|^2 \ ds + C(f_0, m, \varepsilon)),\end{aligned}$$

Now we estimate the term involving $\lambda_1(t)$ on the right hand side of Eq.(3.5). Since $\kappa = \partial_s^2 f$, by applying Poincare inequality to $\partial_s f$, we have the estimate

$$L\|\kappa\|_{L^2}^2 \ge 4\pi^2.$$

Thus, by applying Eq.(2.11) to the right hand side of Eq.(1.6) involving $\lambda_1(t)$, we have the estimates,

$$\begin{aligned}&|\lambda_1(t)| \\ &\le C(f_0, \triangle\Psi, \alpha, \beta, L)\cdot \int_I \left(\left|P_2^2(\kappa)\right| + \left|P_4^0(\kappa)\right| + \left|P_2^1(\kappa)\right|\right) \, ds \\ &\le C\cdot(\|\kappa\|_{m+2,2}^{\frac{2}{m+2}}\cdot\|\kappa\|_2^{2-\frac{2}{m+2}} + \|\kappa\|_{m+2,2}^{\frac{1}{m+2}}\cdot\|\kappa\|_2^{4-\frac{2}{m+2}} + \|\kappa\|_{m+2,2}^{\frac{1}{m+2}}\cdot\|\kappa\|_2^{2-\frac{1}{m+2}}),\end{aligned}$$

and

$$|\int_I \langle \nabla_s^m \kappa, P_3^m(\kappa)\rangle\ ds| \leq \int_I \left|P_4^{2m}(\kappa)\right|\ ds \leq c(m,L)\cdot \|\kappa\|_{m+2,2}^{2-\frac{3}{m+2}}\cdot \|\kappa\|_2^{2+\frac{3}{m+2}}.$$

Therefore,

$$\begin{aligned}(3.10)\qquad & |\ \lambda_1(t)\int_I \langle \nabla_s^m \kappa, P_3^m(\kappa)\rangle\ ds\ | \\ & \leq C(f_0,\triangle\Psi,\alpha,\beta,L,m)\cdot(\|\kappa\|_{m+2,2}^{2-\frac{1}{m+2}}+\|\kappa\|_{m+2,2}^{2-\frac{2}{m+2}}) \\ & \leq \varepsilon\int_I \left|\nabla_s^{m+2}\kappa\right|^2\ ds + C(f_0,\triangle\Psi,\alpha,\beta,L,m,\varepsilon),\end{aligned}$$

where the last inequality comes form applying Young's inequality, and the inequality,

$$\|\kappa\|_{k,2}^2 \leq c(k)\left(\left\|\nabla_s^k\kappa\right\|_2^2+\|\kappa\|_2^2\right),$$

(can be yielded by a standard interpolation inequality, see [2]).

The remaining term in Eq.(3.5) to be estimated is $\lambda_1(t)\cdot\int_I \left|\nabla_s^{m+1}\kappa\right|^2 ds$, which is the borderline case as applying the above estimates. In other words, the interpolation technique fails now. Instead, we use the observation that the total torsion is invariant under the rescaling, therefore the rescaling argument in [4] still works. More precisely, it can be verified that as we rescale f by $f^{(\rho)} = p+\rho(f-p)$, we have the properties: $\mathcal{K}\left[f^{(\rho)}\right] = \frac{1}{\rho}\mathcal{K}[f]$, $\mathcal{T}\left[f^{(\rho)}\right] = \mathcal{T}[f]$ and $\mathcal{L}\left[f^{(\rho)}\right] = \rho\mathcal{L}[f]$. Taking the derivative of $\mathcal{F}\left[f^{(\rho)}\right]$ at $\rho = 1$ and using Eq.(1.5), we have

$$2\alpha\mathcal{K}[f]-\lambda_1\mathcal{L}[f] = -\frac{d}{d\rho}\mathcal{F}\left[f^{(\rho)}\right]\lfloor_{\rho=1} = \int_I \langle \partial_t f, f-p\rangle\ ds.$$

Thus, as long as $p = p(t)$ is properly chosen, e.g., $p = L^{-1}\int_I f\ ds$, and by the energy identity,

$$\frac{d}{dt}\mathcal{F}[f_t] = -\int_I |\partial_t f|^2\ ds, \tag{3.11}$$

one has the inequality,

$$-\lambda_1(t) \leq L^{1/2}\|\partial_t f\|_{L^2},$$

which implies the estimate,

$$\int_0^t \left(\lambda_1^-(\tau)\right)^2\ d\tau \leq C(f_0,\triangle\Psi,\alpha,\beta,L),$$

where $\lambda_1^-(t) = -\min\{0,\lambda_1(t)\}$. By applying integration by parts and Hölder inequality, we have

$$-\lambda_1\int_I \left|\nabla_s^{m+1}\kappa\right|^2\ ds \leq \varepsilon\cdot\int_I \left|\nabla_s^{m+2}\kappa\right|^2\ ds + c(\varepsilon)\cdot\left(\lambda_1^-\right)^2\cdot\int_I |\nabla_s^m\kappa|^2\ ds. \tag{3.12}$$

Note that by applying Poincare inequality twice, we have

$$\int_I \left|\nabla_s^{m+2}\kappa\right|^2 \ ds \geq \left(\frac{2\pi}{L}\right)^4 \int_I \left|\nabla_s^m\kappa\right|^2 \ ds. \tag{3.13}$$

Now, by Eqs.(3.5), (3.9), (3.10), (3.12), (3.13), and a small enough number $\varepsilon = \varepsilon(f_0, \triangle\Psi, \alpha, \beta, L, m) > 0$, we have

(3.14)

$$\frac{d}{dt}\int_I \left|\nabla_s^m\kappa\right|^2 \ ds + C_1 \cdot \int_I \left|\nabla_s^m\kappa\right|^2 \ ds \leq C_2 \cdot (1 + (\lambda_1^-(t)^2) \cdot \int_I \left|\nabla_s^m\kappa\right|^2 \ ds),$$

where we let $C_i = C_i(f_0, \triangle\Psi, \alpha, \beta, L, m) > 0$, $\forall\ i \in \mathbb{Z}$, from now on. Let

$$u_m(t) := \exp(C_1 \cdot t) \cdot \int_I \left|\nabla_s^m\kappa\right|^2 \ ds.$$

By applying Gronwall inequality to Eq.(3.14), we have

$$u_m(t) \leq e^{a(t)} \cdot (u_m(0) + C_3 \cdot \int_0^t e^{C_1 \cdot \tau} \ d\tau),$$

where

$$a(t) = \int_0^t C_4 \cdot (\lambda_1^-(\tau))^2 \ d\tau \leq C(f_0, \triangle\Psi, \alpha, \beta, L, m).$$

Therefore, we obtain

$$\begin{aligned} \left\|\nabla_s^m\kappa\right\|_{L^2}^2(t) &\leq C(f_0, \triangle\Psi, \alpha, \beta, L, m) \cdot (1 + e^{-C_1 \cdot t} \cdot \left\|\nabla_s^m\kappa\right\|_{L^2}^2(0)) \\ &\leq C(f_0, \triangle\Psi, \alpha, \beta, L, m), \end{aligned} \tag{3.15}$$

for all $m \geq 0$. In addition, from the definition of λ_1 in Eq.(1.6), we conclude that $|\lambda_1| \leq C(f_0, \triangle\Psi, \alpha, \beta, L)$. Notice that one has the estimate,

$$\left\|\partial_s^{m-1}\kappa\right\|_{L^\infty} \leq c \cdot \left\|\partial_s^m\kappa\right\|_{L^1}, \quad \forall\ m \geq 1. \tag{3.16}$$

Now, by applying induction argument on m, and using Lemma 4, 5, Eq.(3.15), (3.16) and Hölder inequality, we derive the inequalities,

$$\left\|\nabla_s^m\kappa\right\|_{L^\infty} + \left\|\partial_s^m\kappa\right\|_{L^\infty} \leq C(f_0, \triangle\Psi, \alpha, \beta, L, m),\ \forall\ m \geq 0. \tag{3.17}$$

On the asymptotic behaviour of the flow, we choose a subsequence of curves $f(t, \cdot)$ which converges smoothly to a curve f_∞, after reparametrizations of arclength and translations. Lemma 8 and Eq.(3.17) imply

$$\left\|\nabla_t(\nabla_s^m\kappa)\right\|_{L^\infty} \leq C(f_0, \triangle\Psi, \alpha, \beta, L, m),\ \forall\ m \geq 0. \tag{3.18}$$

From Eq.(3.17) and (3.18), one sees that for $u(t) := \int_I |\ \partial_t f\ |^2 \ ds$, the inequality

$$|u'(t)| \leq C(f_0, \triangle\Psi, \alpha, \beta, L),$$

holds. On the other hand, the energy identity, Eq.(3.11), implies $u(t) \in L^1([0, \infty))$. Therefore, $u(t) \to 0$ as $t \to \infty$. In other words, f_∞ is independent of t and thus, by Eq.(1.5), is an equilibrium of $\mathcal{F}$. Now, by Theorem 1, the proof is finished. □

4. Numerical algorithm

We base our numerical treatment on the algorithm proposed in [4] and implement the new nonlinear term $\lambda_2 \nabla_s (T \times \kappa)$ explicitly in time.

First observe that the divergence form of the main part in the evolution equation admits a weak formulation of the flow. In fact, we have

$$\nabla_s^2 \kappa + \tfrac{1}{2} |\kappa|^2 \kappa - \lambda_2 \nabla_s (T \times \kappa) = \partial_s \left(\partial_s \kappa + \tfrac{3}{2} |\kappa|^2 T - \lambda_2 T \times \kappa \right).$$

Secondly, the common way of avoiding higher order elements for the discretization is to rewrite the equation as a second order system for position vector f and the mean curvature vector κ

$$\partial_t f + \partial_s \left(\partial_s \kappa + \tfrac{3}{2} |\kappa|^2 T - \lambda_2 T \times \kappa \right) = \lambda_1 \kappa, \tag{4.1}$$

$$\partial_s^2 f = \kappa. \tag{4.2}$$

The weak form of the problem leads in one space dimension to a difference scheme. Decompose $I = \mathbb{R}/\mathbb{Z} = \cup_1^N I_j$ into intervals $I_j = [x_{j-1}, x_j)$, where x_j are the nodal points. We discretize the space $H^1(I,^n)$ by the space

$$X_h = \left\{ g \in C^0(I,^n) : g|_{I_j} \in_1 (I_j) \right\} = (\mathrm{span}\,\{\phi_1, \dots, \phi_N\})^n$$

of periodic piecewise affine functions spanned by the nodal basis functions $\phi_j \in X_h$ satisfying $\phi_j(x_i) = \delta_{ij}$. The discretization parameter is given by $h = \max_j h_j$, $h_j = |I_j|$. We use the pointwise interpolation $I_h g$, $g \in C^0(I,^n)$ uniquely defined by $I_h g \in X_h$ and $I_h g(x_j) = g(x_j)$ for all $j = 1, \dots, N$. A discrete (weak) solution to (4.1) is then a pair of functions $(f_h, \kappa_h) : [0,T] \to X_h \times X_h$,

$$f_h(x,t) = \sum_{j=1}^N f_j(t) \phi_j(x), \quad \kappa_h(x,t) = \sum_{j=1}^N \kappa_j(t) \phi_j(x)$$

satisfying for all $\phi_h, \psi_h \in X_h$ the weak problem

$$\begin{aligned} &\int_I \left(I_h(\partial_t f_h \phi_h) |\partial_x f_h| - \frac{\partial_x \kappa_h}{|\partial_x f_h|} \partial_x \phi_h - \frac{3}{2} |\kappa_h|^2 \frac{\partial_x f_h}{|\partial_x f_h|} \partial_x \phi_h \right) dx \\ &= \int_I \left(\lambda_1 \frac{\partial_x f_h}{|\partial_x f_h|} \times \kappa_h \partial_x \phi_h + \lambda_2 I_h(\kappa_h \phi_h) |\partial_x f_h| \right) dx = 0, \end{aligned} \tag{4.3}$$

$$-\int_I \frac{\partial_x f_h}{|\partial_x f_h|} \partial_x \psi_h dx = \int_I I_h(\kappa_h \psi_h) |\partial_x f_h| dx \tag{4.4}$$

In the time direction we discretize semi-implicitly. In particular, the new nonlinear term $\lambda_2 \nabla_s (T \times \kappa)$ in our flow equation is treated explicitly. For functions defined on the time interval $[0,T]$ we use the notation $g^m = g(\cdot, mk)$, $kM = T$.

Algorithm. For given initial data $f_0(x)$ and nodal points of the parameterization x_j, $j = 1, \dots, N$ let $f_j^0 = f_0(x_j)$, $h_j^0 = |f_j^0 - f_{j-1}^0|$, and

$$\kappa_j^0 = \frac{2}{h_{j+1}^0 (h_j^0 + h_{j+1}^0)} f_{j+1}^0 - \frac{2}{h_j^0 h_{j+1}^0} f_j^0 + \frac{2}{h_j^0 (h_j^0 + h_{j+1}^0)} f_{j-1}^0,$$

where we use the extensions $f_0^0 = f_N^0$, $f_{N+1}^0 = f_1^0$, $h_0^0 = h_N^0$, $h_{N+1}^0 = h_1^0$.

For $m = 0, \dots, M-1$ we set

$$\begin{aligned} h_j^m &= |f_j^m - f_{j-1}^m|, \\ \beta_j^m &= |\kappa_{j-1}^m|^2 + \kappa_{j-1}^m \kappa_j^m + |\kappa_j^m|^2, \\ \gamma_j^m &= \frac{f_j^m - f_{j-1}^m}{h_j^m} \times \frac{\kappa_j^m + \kappa_{j-1}^m}{2}, \end{aligned}$$

and solve for $f_j^{m+1}, \kappa_j^{m+1}$ in

$$\begin{aligned} &\frac{\beta_j^m}{2h_j^m} f_{j-1}^{m+1} + \left(\frac{h_j^m + h_{j+1}^m}{2k} - \frac{\beta_j^m}{2h_j^m} - \frac{\beta_{j+1}^m}{2h_{j+1}^m} \right) f_j^{m+1} + \frac{\beta_{j+1}^m}{2h_{j+1}^m} f_{j+1}^{m+1} \\ &+ \frac{1}{h_j^m} \kappa_{j-1}^{m+1} - \left(\frac{1}{h_j^m} + \frac{1}{h_{j+1}^m} + \frac{\lambda_1^m}{2}(h_j^m + h_{j+1}^m) + \lambda_2^m * \right) \kappa_j^{m+1} + \frac{1}{h_{j+1}^m} \kappa_{j+1}^{m+1} \\ &\qquad = \frac{h_j^m + h_{j+1}^m}{2k} f_j^m + \lambda_2^m \left(\gamma_{j+1}^m - \gamma_j^m \right), \\ &\frac{1}{h_j^m} f_{j-1}^{m+1} - \left(\frac{1}{h_j^m} + \frac{1}{h_{j+1}^m} \right) f_j^{m+1} + \frac{1}{h_{j+1}^m} f_{j+1}^{m+1} = \frac{h_j^m + h_{j+1}^m}{2} \kappa_j^m. \end{aligned}$$

Hereby, the Lagrange multipliers are computed according to

$$\begin{aligned} \lambda_2^m &= \frac{2\beta}{L^m}(\tau^m + \Delta\Psi). \\ \lambda_1^m &= -\frac{\sum_{j=1}^N \left(|\kappa_j^m - \kappa_{j-1}^m|^2 / h_j^m + (f_j^m - f_{j-1}^m) \cdot (\kappa_j^m - \kappa_{j-1}^m) \beta_j^m / 2h_j^m + \lambda_2^m \Gamma_j^m \right)}{\sum_{j=1}^N h_j^m \beta_j^m / 3} \end{aligned}$$

where

$$\begin{aligned} \Gamma_j^m &= \kappa_j^m \cdot (\gamma_{j+1}^m - \gamma_j^m), \\ \tau^m &= -3 \sum_{j=1}^N \Gamma_j^m / \beta_j^m, \\ L^m &= \sum_{j=1}^N h_j^m. \end{aligned}$$

The algorithm is intrinsic in the sense that it does not explicitly depend on the grid parameter $h = \max_j h_j$. Nevertheless, during time evolution the distribution of nodes drift away from the equidistant grid. Thus, we redistribute the nodes tangentially according to arclength if the ratio $\max_j h_j / \min_j h_j$ exceeds 2.

We also mention that the linear system for $f_j^{m+1}, \kappa_j^{m+1}$ can be decoupled giving a linear system for f_j^{m+1} alone. The tridiagonal structure of the matrices is then replaced by a five-diagonal structure, where the periodicity of the curve implies non-zero elements in the upper right and lower left corners. The implementation of a fifth-diagonal linear solver can easily be generalized to such a situation.

Computations and figures. Let us first note that numerical computations show that the flat circle is a stationary solution which continues to stay stable for small values of β. For increasing values of β the circle loses stability and we observe non-trivial equilibria of non-zero total torsion.

Figure 1 shows a table of stationary states for given values of $\Delta\Psi$ in the interval $(-\pi, \pi)$. Observe that equilibria corresponding to the same absolute value of $\Delta\Psi$ bend into the opposite direction leading to a reflection symmetry wrt. the horizontal plane.

It is interesting to see that the issue of inflection points comes into play if $|\Delta\Psi|$ approaches the value π. Then the corresponding asymptotic stationary curve contains points having a very small magnitude of the curvature vector κ. We mentioned before that the flow equation and the computation of the torsion τ gets ill-defined in such a situation. We observe this problem also in our computations in the sense that the flow gets numerically unstable if curves with points of small $|\kappa|$ evolve.

The next table in Figure 2 presents the evolution of a strongly bended initial curve unfolding along our flow to a stationary curve. Recalling the different lengths the curve is similar but not identical to the one from Figure 1.

References

[1] S. S. Antman, *Nonlinear problems of elasticity*, Applied Mathematical Sciences, 107. Springer-Verlag, New York, 1995.

[2] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grundlehren der Mathematischen Wissenschaften, 252. Springer-Verlag, New York, 1982.

[3] R. Bishop, *There is more than one way to frame a curve*, Amer. Math. Monthly, 82 (1975), 118-133.

[4] G. Dziuk, E. Kuwert, R. Schätzle, *Evolution of elastic curves in $\mathbb{R}^n$, existence and computation,* SIAM J. Math. Anal. 33 (2002), no. 5, 1228–1245.

[5] T. Ivey, D. Singer, *Knot types, homotopies and stability of closed elastic rods*, Proc. London Math. Soc. (3) 79 (1999), no. 2, 429–450.

[6] N. Koiso, *On the motion of a curve towards elastica*, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), 403–436, Sémin. Congr., 1, Soc. Math. France, Paris, 1996.

[7] J. Langer, D. Singer, *Lagrangian aspects of the Kirchhoff elastic rod*, SIAM Rev. 38 (1996), no. 4, 605–618.

[8] A. Polden, *Curves and surfaces of least total curvature and fourth-order flows*, Ph.D. dissertation, Universität Tübingen, Tübingen, Germany, 1996.

[9] Y. Wen, *Curve straightening flow deforms closed plane curves with nonzero rotation number to circles*, J. Differential Equations 120 (1995), no. 1, 89–107.

Department of Mathematics, National Taiwan Normal University, Taipei 116, Taiwan, Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, Leipzig D-04103, Germany

E-mail address: chunlin@math.ntnu.edu.tw, schwetlick@mis.mpg.de

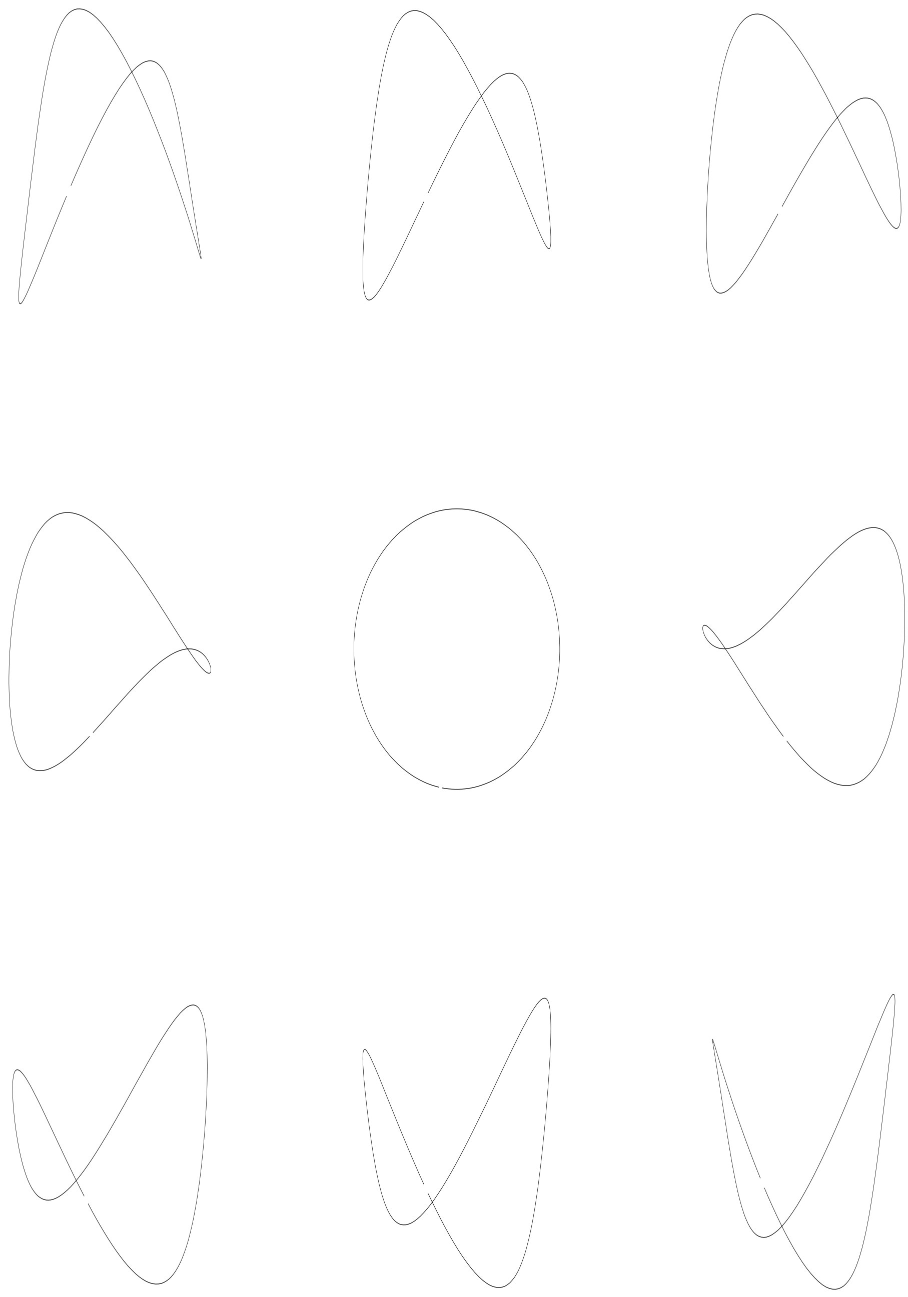

FIGURE 1. Stationary curves of length 6.31 for $\beta = 35$ and $\Delta\Psi = -3.1, -2, -1, -0.5, 0, 0.5, 1, 2, 3.1$

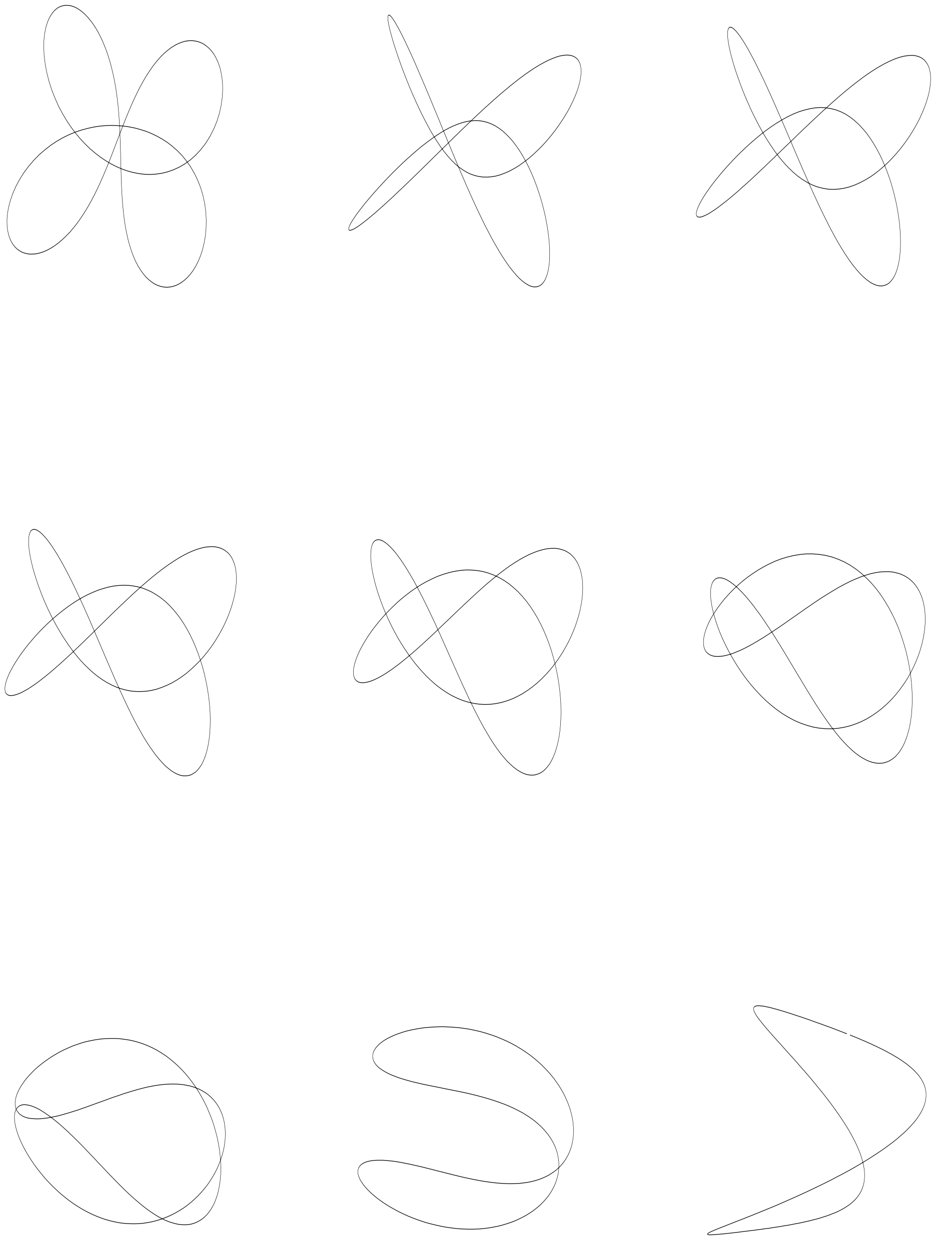

FIGURE 2. Evolution of a curve of length 18.9 under the flow with $\beta = 60$, $\Delta\Psi = -2$ at times $t = 0, 1, 2, 3, 4, 6, 8, 13, 52$