Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 55 2003

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Abstract

In this paper, we prove the Morse index theorem for the index forms of regular Lagrangian systems with selfadjoint boundary condition coming from variational problems. We then give the relationship of them under two different boundary conditions. The Morse index theorem for the corresponding second order selfadjoint differential operators will be proved in [6].

Key words and phrases: regular Lagrangian system; index form; spectral flow; relative Morse index; Maslov index; Maslov-type index

Mathematical subject codes: 34B05; 53C22; 53C50; 58E10; 70H05

1 Introduction

Let (M,g) be an n-dimensional Riemannian manifold. The classical Morse Index Theorem states that the number of conjugate points along a geodesic $\gamma:[a,b]\to M$ counted with multiplicities is equal to the Morse index of the second variation of the Riemannian action functional $E(c)=\frac{1}{2}\int_a^b g(\dot{c},\dot{c})dt$ at the critical point γ, where $\dot{c}$ denotes $\frac{d}{dt}c$. Such second variation is called the index form for E at γ. The theorem has later been extended in several directions (see [1, 2, 10, 18, 19, 22, 23] for versions of this theorem in different contexts). In [21], J. Robbin and D. Salamon gave a new simple proof of the Morse index theorem. In [10] of 1976, J. J. Duistermaat proved his general Morse index theorem for those Lagrangian systems with positive definite second order terms and selfadjoint boundary conditions coming from variational problems. In [1] of 1996, A. A. Agrachev and A. V. Sarychev studied the Morse indices and rigidity of the abnormal geodesics in sub-Riemannian manifolds. In [18, 19] of 2000, P. Piccione and D. V. Tausk proved a version of the Morse index theorem for geodesics in semi-Riemannian manifolds with both endpoints varies on two submanifolds of M under some nondegenerate conditions (cf. Theorems 5.2 and 5.10 in [19]). After I had got the preprint of their papers, I found that this problem was still not completely solved at that time. In this paper, we will prove a general version of Morse index theorem for regular Lagrangian system with selfadjoint boundary conditions coming from variational problems, and show how the indices varies under different choices of the frames. The relation between the indices for two different boundary conditions is a easy corollary of Proposition 2.4 below. Hence we solve the problem for the index forms by a quite different method from the one by P. Piccione and D. V. Tausk. In order to get such a general theorem for regular Lagrangian system, one has to overcome the following difficulties.

(1) Since the set of the conjugate points is in general not discrete, the multiplicities of conjugate points may be meaningless.

(2) The Morse index of the index form will be infinity when the second order term is not positive definite.

(3) The corresponding second order operators may have different domains.

(4) In the general case, it is a nontrivial problem to show that the problem of the Morse index of index forms and the corresponding second order operators are equivalent.

We overcome these difficulties except the last one by using the notions of Maslov-type indices (see [3, 7, 13, 16]) and the spectral flow (see [4, 5, 9, 21]). We will settle the last one in [6].

Let M be a smooth manifold of dimension n, points in its tangent bundle TM will be denoted by (m, v), with $m \in M$, $v \in T_m M$. Let f be a real-valued C^3 function on an open subset Z of $\mathbf{R} \times TM$. Then

$$E(c) = \int_0^T f\left(t, c(t), \dot{c}(t)\right) dt \tag{1}$$

defines a real-valued C^2 function E on the space of curves

$$\mathcal{C} = \left\{ c \in C^1([0,T], M); (t, c(t), \dot{c}(t)) \in Z \text{ for all } t \in [0,T] \right\}. \tag{2}$$

The set $\mathcal{C}$ is a C^2 Banach manifold modeled on the Banach space $C^1([0,T], \mathbf{R}^n)$ with its usual topology of uniform convergence of the curves and their derivatives.

Boundary conditions will be introduced by restricting E to the set of curves

$$\mathcal{C}_N = \{c \in \mathcal{C}; (c(0), c(T)) \in N\}, \tag{3}$$

where N is a given smooth submanifold of $M \times M$. The most familiar example are $N = \{(m(0), m(T)\}$ and $N = \{(m_1, m_2) \in M \times M; m_1 = m_2\}$. In the general case $\mathcal{C}_N$ is a smooth submanifold of $\mathcal{C}$ with tangent space equal to

$$T_c\mathcal{C}_N = \left\{ \delta c \in C^1([0,T], c^*TM); (\delta c(0), \delta c(T)) \in T_{(c(0),c(T))}N \right\}. \tag{4}$$

$c \in \mathcal{C}_N$ is called **stationary curve** for the boundary condition N if the restriction of E to $\mathcal{C}_N$ has a stationary point at c, that is, if $DE(c)(\delta c) = 0$ for all $\delta c \in T_c\mathcal{C}_N$. For such a curve c is of class C^2.

Let c is of class C^2. Then the second order differential $D^2E(c)$ of E at c is symmetric bilinear form on $T_c\mathcal{C}_N$, which is called the **index form** of E at c with respect to the boundary condition N. In general the Morse index of $D^2E(c)$ will be infinite. In order to get a well-defined integer, we introduce the following concept.

Assume that f is a **regular Lagrangian**, that is,

$$D_v^2 f(t, m, v) \text{ is nondegenerate for all } (t, m, v) \in Z. \tag{5}$$

Here D_v denotes differential of functions on Z with respect to $v \in T_mM$, keeping t and m fixed. The condition (5) is called the **Legendre condition**.

Let $H = H^1(T_c\mathcal{C}_N)$ be the H^1 completion of $T_c\mathcal{C}_N$. By Sobolev embedding theorem, $H \subset C([0,T], c^*TM)$. Then $D^2E(c)$ is well defined on H. In local coordinates, we have

$$\begin{aligned} D^2E(c)(X,Y) &= \int_0^T ((D_v^2 f(\tilde{c}(t))\dot{\alpha} + D_m D_v(\tilde{c}(t))\alpha, \dot{\beta}) \\ &\quad +(D_v D_m(\tilde{c}(t))\dot{\alpha}, \beta) + (D_m D_m(\tilde{c}(t))\alpha, \beta))dt, \end{aligned} \tag{6}$$

where $X, Y \in H$, α, β are the local coordinate expression of X, Y defined by $X = (\alpha, \partial m)$, $Y = (\beta, \partial m)$, ∂m is the natural frame of T_mM, and we have use the abbreviation

$$\tilde{c}(t) = (t, c(t), \dot{c}(t)). \tag{7}$$

In general ∂m and α is not globally well-defined along c. Choosing a C^1 frame e of $T_c\mathcal{C}_N$. Such a frame can be obtained by the parallel transformation of the induced connection on c^*TM of a connection on TM (for example, the Levi-Civita connection with respect to a semi-Riemannian metric on TM). Then in local coordinates, there is a C^1 path $a(t) \in \mathrm{gl}(n, \mathbf{R})$ which is nondegenerate for all t such that $\partial m = (a(t), e(t)) = a(t)^* e(t)$, where $a(t)^*$ denotes the transpose conjugate of $a(t)$. The vector fields $X, Y \in H$ along c can be written as $X = (x, e)$, $Y = (y, e)$, where $x, y \in H^1([0,T], \mathbf{R}^n)$ and $(x(0), x(T)), (y(0), y(T)) \in R$, R is defined by [1]

$$R = \{(x,y) \in \mathbf{R}^{2n}; ((x, e(0)), (y, e(T))) \in T_{(c(0),c(T))}N\}.$$

So we have

$$x = a\alpha, \quad \dot{x} = a\dot{\alpha} + \dot{a}\alpha, \quad y = a\beta, \quad \dot{y} = a\dot{\beta} + \dot{a}\beta. \tag{8}$$

Substitute (8) to (6), we get the following form of the the index form:

$$D^2E(c)(X,Y) = \int_0^T ((p\dot{x} + qx, \dot{y}) + (q^*\dot{x}, y) + (rx, y))\, dt, \tag{9}$$

where $p, q, r \in C([0,T], \mathrm{gl}(n, \mathbf{R}))$, p is of class C^1, $p(t) = p(t)^*$, $r(t) = r(t)^*$, $p(t)$ are invertible for all $t \in [0,T]$, and * denotes the conjugate transpose. Now define

$$\mathcal{I}_s(x,y) = \int_0^T ((p\dot{x} + sqx, \dot{y}) + (sq^*\dot{x}, y) + (srx, y))\, dt, \quad s \in [0,1], \tag{10}$$

where $x, y \in H^1([0,T], R^n)$ and $(x(0), x(T)), (y(0), y(T)) \in R$. Since p is of class C^1 and $p(t)$ are nondegenerate, the relative Morse index $I(\mathcal{I}_0, \mathcal{I}_1) \equiv -\mathrm{sf}\{\mathcal{I}_s\}$ is a well-defined finite integer. When p is positive definite, $I(\mathcal{I}_0, \mathcal{I}_1)$ is the Morse index of $D^2E(c)$ (where we only require that p is continuous).

The main results in this paper is the following.

Let $p, q, r \in C([0,1] \times [0,T], \mathrm{gl}(n, \mathbf{C}))$ be families of matrices such that p is of class C^1, $p_s(t) = p_s(t)^*$, $r_s(t) = r_s(t)^*$, and $p_s(t)$ are invertible for all $s \in [0,1]$ and $t \in [0,T]$.

[1] In this paper, all vectors are viewed as column vectors. For a pair of vectors $x, y \in \mathbf{C}^n$, (x,y) has two meanings: one is the standard Hermitian inner product of x, y, the other is the the vector $(x^*, y^*)^* \in \mathbf{C}^{2n}$. The readers can easily see it from the content.

Let R be a subspaces of $\mathbf{C}^{2n}$. Let H_R be the Hilbert space defined by

$$H_R = \left\{ x \in H^1([0,T], \mathbf{C}^n); (x(0), x(T)) \in R \right\}. \tag{11}$$

For each $s \in [0,1]$, let $\mathcal{I}_s$ be the index form defined by

$$\mathcal{I}_s(x,y) = \int_0^T ((p_s\dot{x} + q_s x, \dot{y}) + (q_s^*\dot{x}, y) + (r_s x, y))\, dt, \qquad x, y \in H_R. \tag{12}$$

Let $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix on $\mathbf{R}^n$. When there is no confusion we will omit the subindex of identity matrices. Let

$$b_s(t) = \begin{pmatrix} p_s^{-1}(t) & -p_s^{-1}(t)q_s(t) \\ -q_s^*(t)p_s^{-1}(t) & q_s^*(t)p_s^{-1}(t)q_s(t) - r_s(t) \end{pmatrix}. \tag{13}$$

For each $s \in [0,1]$, let $\gamma_s(t)$ be the fundamental solution of the linear Hamiltonian equation

$$\dot{u} = Jb_s(t)u. \tag{14}$$

Define

$$\begin{aligned} R^b &= \{(x,y) \in \mathbf{C}^{2n}; (x,-y) \in R^\perp\}, \\ W(R) &= \{(x,y,z,u) \in \mathbf{C}^{4n}; (x,-z) \in R^\perp, (y,u) \in R\}. \end{aligned}$$

Then we have

Theorem 1.1 *Let* $\mathrm{sf}\{\mathcal{I}_s; 0 \le s \le 1\}$ *be the spectral flow of* $\mathcal{I}_s$*, and* $i_{W(R)}(\gamma)$ *be the Maslov-type index of* γ *defined below. Then we have*

$$-\mathrm{sf}\{\mathcal{I}_s; 0 \le s \le 1\} = i_{W(R)}(\gamma_1) - i_{W(R)}(\gamma_0). \tag{15}$$

Assume that $q_0(t) = r_0(t) = 0$ for all $t \in [0,T]$. Then we have $b_0(t) = \mathrm{diag}(p_0^{-1}(t), 0)$ and $\gamma_0(t) = \begin{pmatrix} I & 0 \\ \int_0^t p_0^{-1}(s)ds & I \end{pmatrix}$ for all $t \in [0,T]$.

Theorem 1.2 *Let* $P \in C([0,T], \mathrm{gl}(n,\mathbf{C}))$ *be a path of selfadjoint matrices. Define* $\gamma(t) = \begin{pmatrix} I & 0 \\ P(t) & I \end{pmatrix}$ *for all* $t \in [0,T]$*. Then we have*

$$i_{W(R)}(\gamma) = m^+(P(T)|_S) - m^+(P(0)|_S), \tag{16}$$

where m^+ *denotes the Morse positive index, and*

$$S = \{x \in \mathbf{C}^n; (x,x) \in \mathbf{R}^b\}.$$

As a special case, we get the following theorem of J. J. Duistermaat [10].

Corollary 1.1 *Assume that* $p_1(t)$ *are positive definite for all* $t \in [0, T]$. *Then we have*

$$m^-(\mathcal{I}_1) = i_{W(R)}(\gamma_1) - \dim_{\mathbf{C}} S, \tag{17}$$

where m^- *denotes the Morse (negative) index, and*

$$S = \{x \in \mathbf{C}^n; (x, x) \in \mathbf{R}^b\}.$$

Let $a(t) \in \mathrm{Gl}(n, \mathbf{C})$, and

$$R^{'} = \{(x, y) \in \mathbf{C}^{2n}; (a(0)x, a(T)y) \in R\}.$$

After change of frame $x \longmapsto ax$, $\mathcal{I}_1(ax, ay)$ defines a quadratic form on $H_{R'}$ and we can get the corresponding $p_1^{'}, q_1^{'}$ and $r_1^{'}$. By (13) and (14) we get the corresponding $b_1^{'}$ and $\gamma_1^{'}$.

Theorem 1.3 *We have the following*

$$i_{W(R^{'})}(\gamma_1^{'}) - i_{W(R)}(\gamma_1) = \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R^{'}) - \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R), \tag{18}$$

where $\mathrm{Gr}(I)$ *denotes the graph of* I.

The paper is organized as follows. In §2, we discuss the properties of the spectral flow. In §3, we discuss the properties of the Maslov-type indices. In §4, we prove our main results.

2 Spectral flow

2.1 Definition of the spectral flow

The spectral flow for a one parameter family of linear selfadjoint Fredholm operators is introduced by Atiyah-Patodi-Singer [4] in their study of index theory on manifolds with boundary. Since then other significant applications have been found. In [9] the notion of the spectral flow was generalized to the higher dimensional case by X. Dai and W. Zhang. In [25, 26] it is generalized to more general operators.

Let X be a Banach space. We denote the set of closed operators, bounded linear operators and compact linear operators on X by $\mathcal{C}(X)$, $\mathcal{B}(X)$ and $\mathcal{CL}(X)$ respectively. For $A \in \mathcal{C}(X)$, an operator B is called **A-compact** if $\mathbf{D}(A) \subset \mathbf{D}(B)$, and view B as operator from $\mathbf{D}(A)$ to X, is compact, where $\mathbf{D}(A)$ is the domain of A with the graph norm of A.

According to Atiyah-Patodi-Singer [4], we define

Definition 2.1 *(cf. Definition 1.3.6 of [25] and Definition 2.6 of [26]) (1) Let* l *be a real dimension* 1 *cooriented embedded* C^1 *submanifold of* $\mathbf{C}$ *without boundary. Let* A *be in* $\mathcal{C}(X)$. A *is said to be* **admissible** *with respect to* l *if the spectrum of* A *near* l *lies on a compact subset of* l *and is of finite algebraic multiplicity. If* ∞ *is a limit point of* l, *we require* $A \in \mathcal{B}(X)$. *Let* $P_l(A)$ *be the spectral projection of* A *on* l. *The* **nullity** $\nu_l(A)$ *of* A *with respect to* l *is defined to be* $\nu_l(A) = \dim_{\mathbf{C}} \mathrm{im}\, P_l(A)$. *All such* A *will be denoted by* $\mathcal{A}_l(X)$.

(2) Let A_s, $0 \le s \le 1$ *be a curve in* $\mathcal{A}_l(X)$. *The* **spectral slow** $\mathrm{sf}_l\{A_s\}$ *of* A_s *counts the algebraic multiplicities of the spectral of* A_s *cross the manifold* l, *i.e., the number of the spectral lines of* A_s *cross* l *from the negative part of* $\mathbf{C}$ *near* l *to the non-negative part of* $\mathbf{C}$

near l minus the number of the spectral lines of A_s cross l from the non-negative part of $\mathbf{C}$ *near l to the negative part of* $\mathbf{C}$ *near l. When $l = \sqrt{-1}(-K, K)$ and $A_s \in \mathcal{A}_l(X)$ be such that $\sigma(A_s) \cap \sqrt{-1}\mathbf{R} \subset \sqrt{-1}(-K, K)$ for all s, where $(-K, K)$ $(K > 0)$ denotes the open interval on* $\mathbf{R}$ *and $\sigma(A)$ denotes the spectrum of A, we set $\mathrm{sf}\{A_s\} = \mathrm{sf}_l\{A_s\}$.*

(3) Let $l = \sqrt{-1}\mathbf{R}$, $A \in \mathcal{A}_l(X)$ and $B \in \mathcal{CL}(X)$, or X is a Hilbert space, A is a selfadjoint Fredholm operator with compact resolvent and B is bounded selfadjoint operator, The **relative Morse index** *of the pair A, $A + B$ is defined by*

$$I(A, A + B) = -\mathrm{sf}\{A + sB\}. \tag{19}$$

(4) When $l = \sqrt{-1}\mathbf{R}$ and $A \in \mathcal{A}_l(X)$, or A is selfadjoint Fredholm on Hilbert space X and $l = \sqrt{-1}(-\epsilon, \epsilon)$, we call the algebraic multiplicity of the spectrum of A in the right side and the left side l the **Morse positive index** *and the* **Morse (negative) index***, and denote them by $m^+(A)$ and $m^-(A)$. The* **signature** $\mathrm{sign}(A)$ *is defined by* $\mathrm{sign}(A) = m^+(A) - m^-(A)$.

2.2 Calculation of the spectral flow

Let X be a complex Banach space, γ be a C^1 curve in $\mathbf{C}$ which bounds a bounded open subset Ω of $\mathbf{C}$. Let A_s, $s \in (-\epsilon, \epsilon)$, where $\epsilon > 0$, be a curve in $\mathcal{C}(X)$. Assume that $\gamma \cap \sigma(A_s) = \emptyset$ for all $s \in (-\epsilon, \epsilon)$, where $\sigma(A_s)$ denotes the spectral of A_s. Set $A_0 = A$, and

$$P_s \equiv P_\gamma(A_s) = -\frac{1}{2\pi\sqrt{-1}} \int_\gamma R(\zeta, A_s) d\zeta, \tag{20}$$

where $R(\zeta, A_s) = (A_s - \zeta I)^{-1}$, $\zeta \in \mathbf{C} \setminus \sigma(A_s)$ is the resolvent of A_s. Then $P_s^2 = P_s$. Set $P_0 = P$. Assume that $\mathrm{im}\, P \subset \mathbf{D}(A_s)$, for all $s \in (-\epsilon, \epsilon)$, $\mathrm{im}\, P$ is a finitely dimensional subspace of X, and $\frac{d}{ds}|_{s=0}(A_s P) = B$ (in the bounded operator sence). Let f be a polynomial. Then $P_s f(A_s) P_s$ is uniformly bounded on any compact subsets of $(-\epsilon, \epsilon)$, and

$$P_s f(A_s) P_s = -\frac{1}{2\pi\sqrt{-1}} \int_\gamma f(\zeta) R(\zeta, A_s) d\zeta. \tag{21}$$

Set $R_s = (I - (P_s - P)^2)^{-\frac{1}{2}}$. Then $R_s P = P R_s$ and $R_s P_s = P_s R_s$. Set

$$\begin{aligned} U_s' &= P_s P + (I - P_s)(I - P), & U_s &= U_s' R_s, \\ V_s' &= P P_s + (I - P)(I - P_s), & V_s &= V_s' R_s. \end{aligned}$$

Then we have

$$\begin{aligned} U_s V_s &= V_s U_s = I, \\ U_s P &= P_s U_s = P_s R_s P, \\ P V_s &= V_s P_s = P R_s P_s. \end{aligned}$$

Lemma 2.1 *We have*

$$\frac{d}{ds}|_{s=0}(U_s^{-1} P_s A_s P_s U_s) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma R(\zeta, A) B R(\zeta, A) d\zeta. \tag{22}$$

If $(PAP)(PB) = (PB)(PAP)$*, then we have*

$$\frac{d}{ds}|_{s=0}(U_s^{-1}P_sA_sP_sU_s) = PB. \tag{23}$$

Proof. By the definition of U_s and V_s we have

$$U_s^{-1}P_sA_sP_sU_s = V_sP_sP_sU_s = PR_sP_sA_sP_sR_sP.$$

By (21) we have

$$(P_sf(A_s)P_s - Pf(A)P)P = -\frac{1}{2\pi\sqrt{-1}}\int_\gamma f(\zeta)R(\zeta, A_s)(A_sP - AP)R(\zeta, A)d\zeta. \tag{24}$$

Since A_s, $s \in (-\epsilon, \epsilon)$ is a curve in $\mathcal{C}(X)$, and im P has finite dimension, we have

$$\frac{d}{ds}|_{s=0}(P_sf(A_s)P_sP) = -\frac{1}{2\pi\sqrt{-1}}\int_\gamma f(\zeta)R(\zeta, A)BR(\zeta, A)d\zeta. \tag{25}$$

Take $f = 1$, we have $\frac{d}{ds}|_{s=0}P_sP$ exists. By the definition of R_s we have $\frac{d}{ds}|_{s=0}R_sP = 0$. Now formal calculation shows

$$\frac{d}{ds}|_{s=0}(PR_sP_sA_sP_sR_sP) = -\frac{1}{2\pi\sqrt{-1}}\int_\gamma \zeta PR(\zeta, A)BR(\zeta, A)Pd\zeta.$$

The assumption that im P has finite dimension shows the calculation is right.

When $(PAP)(PB) = (PB)(PAP)$, we have

$$\begin{aligned}\frac{d}{ds}|_{s=0}(PR_sP_sA_sP_sR_sP) &= -\frac{1}{2\pi\sqrt{-1}}\int_\gamma \zeta PR(\zeta, A)R(\zeta, A)Bd\zeta \\ &= P^2B \\ &= PB.\end{aligned}$$

Q.E.D.

By the above Lemma 2.1, the proof of Theorem 4,1 in [26] also works for the following more general proposition.

Proposition 2.1 *Let* l *be an open submanifold of* $\sqrt{-1}\mathbf{R}$*. Let* A_s*,* $-\epsilon \leq s \leq \epsilon$ *(*$\epsilon > 0$*),be a curve in* $\mathcal{A}_l(X)$*. Set* $P = P_l(A_0)$ *and* $A = A_0$ *Assume that* im $P \subset \mathbf{D}(A_s)$ *and* $B = \frac{d}{ds}\mid_{s=0} A_sP$ *exists. Assume that*

$$(PAP)(PB) - (PB)(PAP) = 0, \tag{26}$$

where $PAP, PB \in \mathcal{B}(\text{im } P)$*, and* $PB : \text{im } P \to \text{im } P$ *is hyperbolic. Then there is a* $\delta \in (0, \epsilon)$ *such that* $\nu_l(A_s) = 0$ *for all* $s \in [-\delta, 0) \cup (0, \delta]$ *and*

$$\text{sf}_l\{A_s, 0 \leq s \leq \delta\} = -m^-(PB : \text{im } P \to \text{im } P), \tag{27}$$

$$\text{sf}_l\{A_s, -\delta \leq s \leq 0\} = m^+(PB : \text{im } P \to \text{im } P). \tag{28}$$

Q.E.D.

Now we consider two special cases.

Lemma 2.2 *Let X be a Hilbert space. Let $A_s, -\epsilon < s < \epsilon$ be a curve of selfadjoint Fredholm operators with constant domain D such that $A_s \le A_t$ for all $-\epsilon < s < t < \epsilon$. Assume that $\frac{d}{ds}A_s x$ exist for all $s \in (-\epsilon, \epsilon)$, $x \in D$, or A_s, $s \in (-\epsilon, \epsilon)$ is a curve of bounded operators. Then for $s < 0$ with $|s|$ small, $\dim_{\mathbf{C}} \ker A_s$ is constant and A_s has no positive small eigenvalue. For $s > 0$ small, $\dim_{\mathbf{C}} \ker A_s$ is constant and A_s has no negative eigenvalue whose absolute value is small.*

Proof. Firstly assume that $\frac{d}{ds}A_s x$ exist for all $s \in (-\epsilon, \epsilon)$, $x \in D$. Let $\lambda_1(s) \le \ldots \le \lambda_k(s)$ be the spectral lines of A_s for $|s|$ small such that $\lambda_1(0) = \ldots = \lambda_k(0) = 0$. Fix $j = 1, \ldots k$ and t with $|t|$ small. Pick $x(s) \in \ker(A_s - \lambda_j(s)I)$ such that $\|x_s\| = 1$. Then every subsequence of x_s, $s \to t$ has a convergent subsequence. Let s_n be the subsequence of s, $s \to t$ such that

$$a_j(t) \equiv \liminf_{s\to t} \frac{\lambda_j(s) - \lambda_j(t)}{s - t} = \lim_{n\to\infty} \frac{\lambda_j(s_n) - \lambda_j(t)}{s_n - t}$$

and $\lim_{n\to\infty} x(s_n) = x$. Then $x \in \ker(A_t - \lambda_j(t)I)$. So we have

$$\begin{aligned} a_j(t) &= \lim_{n\to\infty} \frac{\lambda_j(s_n) - \lambda_j(t)}{s_n - t} \\ &= \lim_{n\to\infty} \frac{\lambda_j(s_n) - \lambda_j(t)}{s_n - t}(x(s_n), x) \\ &= \lim_{n\to\infty} \frac{1}{s_n - t}((A_{s_n}x(s_n), x) - (x_{s_n}, A_t x)) \\ &= \lim_{n\to\infty} \left(\frac{(A_{s_n} - A_t)x}{s_n - t}, x(s_n)\right) \\ &= \left(\frac{d}{ds}|_{s=t} A_s x, x\right) \\ &\ge 0. \end{aligned}$$

Hence $\lambda_j(s) \le \lambda_j(t)$ for $s < t$ and $|s|, |t|$ small, and our results follows.

Now assume that A_s is a bounded continuous curve. For $t > 0$ small, consider the curve $A_0 + \frac{s}{t}(A_t - A_0)$, $0 \le s \le t$. By the above arguments and the definition of the spectral flow, the curve has nonnegative spectral flow. Since local spectral flow only depends on end points, the curve A_s, $0 \le s \le t$ also has nonnegative spectral flow. Hence A_t has no negative eigenvalue near 0 for $t > 0$ small. Q.E.D.

By Lemmas 2.2 and the definition of the spectral flow we have

Proposition 2.2 *Let X be a Hilbert space.*

(1) Let $A_s, 0 \le s \le 1$ be a curve of selfadjoint Fredholm operators with constant domain D such that $A_s \le A_t$ for all $0 \le s < t \le 1$. Assume that $\frac{d}{ds}A_s x$ exist for all $s \in [0, 1]$, $x \in D$, or A_s, $s \in [0, 1]$ is a continuous curve of bounded operators. Then we have

$$\mathrm{sf}\{A_s\} = \sum_{0<s\le 1} \left(\dim_{\mathbf{C}} \ker A_s - \lim_{t\to s^-} \dim_{\mathbf{C}} \ker A_t\right) \ge 0. \qquad (29)$$

(2) Let $A \in \mathcal{C}(X)$ be a selfadjoint operator with compact resolvant and $B_s \in \mathcal{B}(X)$, $0 \le s \le 1$ be a curve of bounded selfadjoint operators such that $B_s \le B_t$ for all $0 \le s < t \le 1$. Then we have

$$\mathrm{sf}\{A+B_s\} = \sum_{0<s\leq 1}\Big(\dim_{\mathbf{C}}\ker(A+B_s) - \lim_{t\to s^-}\dim_{\mathbf{C}}\ker(A+B_t)\Big) \geq 0. \tag{30}$$

Q.E.D.

Similarly we have

Proposition 2.3 *Let X be a Hilbert space and $l=(1-\epsilon,1+\epsilon)$ ($\epsilon>0$ small). Let $A_s\in\mathcal{B}(X)$, $0\leq s\leq 1$ be a curve of unitary operators. Assume that A_s-I is Fredholm and $\sqrt{-1}A_s^{-1}\dot{A}_s\leq 0$ for all $s\in[0,1]$. Then we have*

$$\mathrm{sf}_l\{A_s\} = \sum_{0<s\leq 1}\Big(\dim_{\mathbf{C}}\ker(A_s-I) - \lim_{t\to s^-}\dim_{\mathbf{C}}\ker(A_t-I)\Big) \geq 0. \tag{31}$$

Q.E.D.

2.3 Spectral flow for curves of quadratic forms

The spectral flow for curves of selfadjoint Fredholm operators has some special properties as finite dimensional case.

The following lemma is Corollary 2.2 of [26].

Lemma 2.3 *Let X be a Hilbert space. Let A be a selfadjoint Fredholm operator on X with compact resolvent, and B be a bounded selfadjoint operator on X. Set $K=(|A|+I)^{-1}$. Then we have*

$$I(AK, AK+KB\} = I(A, A+B), \tag{32}$$

where AK, $AK+KB$ are linear operators defined on the Hilbert space $V=D(|A|^{\frac{1}{2}})$ with graph norm

$$\|x\|_V = (\||A|^{\frac{1}{2}}x\|_X^2 + \|x\|_X^2)^{\frac{1}{2}}.$$

Q.E.D.

Lemma 2.4 *Let X be a Hilbert space. Let A_s, $0\leq s\leq 1$ be a curve of closed selfadjoint Fredholm operators. Then for any curve $P_s\in\mathcal{B}(X)$ of invertible operators, we have*

$$\mathrm{sf}\{P_sP_s^*A_s\} = \mathrm{sf}\{P_s^*A_sP_s\} = \mathrm{sf}\{A_s\}. \tag{33}$$

Proof. Since A_s is a curve of closed selfadjoint Fredholm operators and P_s is a curve of bounded invertible operators, the family $P_s^*A_sP_s$, $0\leq s\leq 1$, is a curve of closed selfadjoint Fredholm operators. By the definition of the spectral flow we have

$$\begin{aligned}\mathrm{sf}\{P_sP_s^*A_s\} &= \mathrm{sf}\{P_s(P_s^*A_sP_s)P_s^{-1}\}\\ &= \mathrm{sf}\{P_s^*A_sP_s\}.\end{aligned} \tag{34}$$

Since $P_s^*A_tP_s$ are selfadjoint Fredholm operators and $\dim_{\mathbf{C}}\ker(P_s^*A_tP_s) = \dim_{\mathbf{C}}\ker A_t$, we have

$$\begin{aligned}\mathrm{sf}\{P_s^*A_sP_s\} &= \mathrm{sf}\{P_0^*A_sP_0\} + \mathrm{sf}\{P_s^*A_1P_s\}\\ &= \mathrm{sf}\{P_0^*A_sP_0\}\\ &= \mathrm{sf}\{P_1^*A_sP_1\}.\end{aligned} \tag{35}$$

Let Q_s be curves of bounded positive operators on X satisfying $Q_0 = I$, $Q_1^2 = P_0 P_0^*$. By (34) and (35) we have

$$\begin{aligned}
\mathrm{sf}\{P_s^* A_s P_s\} &= \mathrm{sf}\{P_0^* A_s P_0\} \\
&= \mathrm{sf}\{P_0 P_0^* A_s\} \\
&= \mathrm{sf}\{Q_1 A_s Q_1\} \\
&= \mathrm{sf}\{Q_0 A_s Q_0\} \\
&= \mathrm{sf}\{A_s\}.
\end{aligned}$$

Q.E.D.

The above lemma leads the following definition.

Definition 2.2 *Let X be a Hilbert space. Let $\mathcal{I}_s$, $0 \le s \le 1$ be a family of quadratic forms. Assume that $\mathcal{I}_s(x,y) = (A_s x, y)$ for all $x, y \in X$, where A_s is a curve of bounded selfadjoint Fredholm operators.*

(1) The **spectral flow** $\mathrm{sf}\{\mathcal{I}_s\}$ *of $\mathcal{I}_s$ is defined to be the spectral flow $\mathrm{sf}\{A_s\}$.*

(2) If $A_1 - A_0 \in \mathcal{CL}(X)$, the **relative Morse index** $I(\mathcal{I}_0, \mathcal{I}_1)$ *is defined to be the relative Morse index $I(A_0, A_1)$.*

Based on this observation we can prove the following lemma.

Lemma 2.5 *Let X be a Hilbert space. Let $A_s \in \mathcal{B}(X)$, $0 \le s \le 1$ be a curve of selfadjoint Fredholm operators and $\mathcal{I}_s$ be quadratic forms defined by $\mathcal{I}_s(x,y) = (A_s x, y)$ for all $x, y \in X$. Assume that $P_s \in \mathcal{B}(X)$, $0 \le s \le 1$ is a curve of operators such that $P_s^2 = P_s$ and $\mathcal{I}_s(x,y) = 0$ for all $x \in \mathrm{im}\ P_s$, $y \in \mathrm{im}\ Q_s$, where $Q_s = I - P_s$.*

$$\mathrm{sf}\{\mathcal{I}_s\} = \mathrm{sf}\{\mathcal{I}_s|_{\mathrm{im}\ P_s}\} + \mathrm{sf}\{\mathcal{I}_s|_{\mathrm{im}\ Q_s}\}. \tag{36}$$

Proof. Let $R_s = P_s^* P_s + Q_s^* Q_s$. Since $P_s + Q_s = I$ and $P_s^2 = I$, we have

$$R_s = \frac{I}{2} + 2(\frac{I}{2} - P_s^*)(\frac{I}{2} - P_s) > 0.$$

So $R_s^{-1} A_s$ are Fredholm operators. Now consider the new inner product $(R_s X, y)$, $x, y \in X$ on X. For this inner product P_s is an orthogonal projection. Let $B_s \in \mathcal{B}(\mathrm{im}\ P)$ and $C_s \in \mathcal{B}(\mathrm{im}\ Q_s)$ satisfy $\mathcal{I}_s(x,y) = (B_s x, y)$ for all $x, y \in \mathrm{im}\ P_s$ and $\mathcal{I}_s(x,y) = (C_s x, y)$ for all $x, y \in \mathrm{im}\ Q_s$ respectively. By the fact that $\mathrm{im}\ P_s$ and $\mathrm{im}\ Q_s$ are $\mathcal{I}_s$-orthogonal, we have

$$(A_s x, y) = (R_s(B_s \oplus C_s)x, y), \quad \forall x, y \in X.$$

So $R_s^{-1} A_s = B_s \oplus C_s$, and B_s, C_s are Fredholm operators. By Lemma 2.4 and the definition of the spectral flow we have

$$\begin{aligned}
\mathrm{sf}\{\mathcal{I}_s\} &= \mathrm{sf}\{R_s^{-1} A_s\} \\
&= \mathrm{sf}\{B_s\} + \mathrm{sf}\{C_s\} \\
&= \mathrm{sf}\{\mathcal{I}_s|_{\mathrm{im}\ P_s}\} + \mathrm{sf}\{\mathcal{I}_s|_{\mathrm{im}\ Q_s}\}.
\end{aligned}$$

Q.E.D.

Remark 2.1 *Here we allow the Hilbert space* $\mathrm{im}\, P_s$ *continuous varying. By Lemma I.4.10 in [12], for* $t \in [0,1]$ *being close enough to* s*, there are invertible operators* $U_{s,t} \in \mathcal{B}(X)$ *such that*

$$P_t U_{s,t} = U_{s,t} P_s, \qquad U_{s,t} \to I, \quad \text{as } t \to s.$$

So locally we can define the spectral flow of B_t *as that of* $U_{s,t}^{-1} B_t U_{s,t} : \mathrm{im}\, P_s \to \mathrm{im}\, P_s$ *(*s *fixed), and globally add them up.*

Lemma 2.6 *Let* X *be a Hilbert space, and* M *be a closed subspace with finite codimension. Let* $A \in \mathcal{B}(M)$ *be a selfadjoint Fredholm operator and* $\mathcal{I}(x,y) = (Ax,y)$ *for all* $x, y \in M$*. Let* N_0 *and* N_1 *be subspace of* X *such that* $X = M \oplus N_0 = M \oplus N_1$*. Define* $\mathcal{I}_k$ *on* X*,* $k = 0,1$ *by*

$$\mathcal{I}_k(x+u, y+v) = (Ax, y), \quad \forall x, y \in M, \forall u, v \in N_k.$$

Then we have $I(\mathcal{I}_0, \mathcal{I}_1) = 0$.

Proof. Without loss of generality, we assume that N_0 is the orthogonal complement of M. Set $A_0 = \mathrm{diag}(A, 0)$. Let $B : N_1 \to N_0$ be a linear isomorphism. Define $P_1 \in \mathcal{B}(X)$ by $P_1(x+y) = x + By$ for all $x \in M$, $y \in N_1$. Set $A_1 = P_1^* A_0 P_1$. Then P_1 is invertible, $P_1 - I$ is compact, and $\mathcal{I}_k(x,y) = (A_k x, y)$ for all $x, y \in X$ and $k = 0,1$. Let $P_s \in \mathcal{B}(X)$, $0 \le s \le 1$ be a curve of invertible operators such that $P_0 = I$ and $P_s - I$ are compact. By the definition of the spectral flow we have

$$\begin{aligned} I(\mathcal{I}_0, \mathcal{I}_1) &= I(A_0, A_1) \\ &= -\mathrm{sf}\{P_s^* A_0 P_s) \\ &= 0. \end{aligned}$$

Q.E.D.

The following proposition gives a generalization of Proposition 5.3 in [1] and a formula of M. Morse.

Proposition 2.4 *Let* X *be a Hilbert space and* $A \in \mathcal{B}(X)$ *be a selfadjoint Fredholm operator. Let* P *be an orthogonal projection such that* $\ker P$ *is of finite dimensional. Let* $\mathcal{I}$ *be quadratic form on* X *defined by* $\mathcal{I}(x,y) = (Ax,y)$*,* $x, y \in X$*. Set* $M = \mathrm{im}\, P$ *and* N *be the* $\mathcal{I}$*-orthogonal complement of* M*:* $N = \{x \in X; \mathcal{I}(x,y) = 0, \forall y \in M\}$*. Then we have*

$$I(PAP, A) = m^-(\mathcal{I}|_N) + \dim_{\mathbf{C}} \ker \mathcal{I}|_N - \dim_{\mathbf{C}} \ker \mathcal{I}. \tag{37}$$

Proof. Since $PAP - A$ is a finite rank operator, $sPAP + (1-s)A$ are selfadjoint Fredholm operators. We divide our proof into three steps.

Step 1. Equation (37) holds when $\ker A = 0$ and $N \subset M$.

In this case, set $M_0 = \ker \mathcal{I}|_M$, M_1 be the orthogonal complement of M_0 in M, and P_0, P_1 be the orthogonal projection onto M_0, M_1 respectively. Then P_0 is of finite rank and $P = P_0 + P_1$. Let N_1 be the $\mathcal{I}$-orthogonal complement of M_1. Since $\mathcal{I}|_{M_1}$ is nondegenrate, $M_1 \cap N_1 = \{0\}$. Moreover we have

$$\begin{aligned}\dim_{\mathbf{C}} N_1 &= \dim_{\mathbf{C}} \ker(AP_1) - \mathrm{ind}(AP_1)\\ &= \dim_{\mathbf{C}} \ker P_1 - \mathrm{ind} A - \mathrm{ind} P_1\\ &= \dim_{\mathbf{C}} \ker P_1 < \infty.\end{aligned}$$

So $X = M_1 \oplus N_1$. By the fact that $\mathcal{I}$ is nondegenrate, $\mathcal{I}|_{N_1}$ is nondegenrate. Clearly $M_0 \subset N_1$ and M_0 is the $\mathcal{I}|_{N_1}$-orthogonal complement of M_0. N_1 has an orthogonal decomposition $N_1 = N^+ \oplus N^-$ such that N^+ and N^- is $\mathcal{I}$-orthogonal, $\mathcal{I}|_{N^+} > 0$ and $\mathcal{I}|_{N^-} < 0$. Let $P^{\pm}$ be the orthogonal projection onto $N^{\pm}$. Then $P^{\pm}|_{M_0}$ is isomorphism. So we have

$$\dim_{\mathbf{C}} N_1 = 2 \dim_{\mathbf{C}} M_0 = 2m^-(\mathcal{I}|_{N_1}).$$

Let $\mathcal{I}_1$ be defined by $\mathcal{I}_1(x+u, y+v) = \mathcal{I}(x,y)$ for all $x, y \in M_1$, $u, v \in N_1$. By Lemma 2.5 and 2.6 we have

$$\begin{aligned}I(PAP, A) &= I(PAP, P_1AP_1) + I(P_1AP_1, A)\\ &= I(P_1AP_1, A)\\ &= I(\mathcal{I}_1, \mathcal{I})\\ &= I(\mathcal{I}_1|_{M_1}, \mathcal{I}|_{M_1}) + I(\mathcal{I}_1|_{N_1}, \mathcal{I}|_{N_1})\\ &= m^-(\mathcal{I}|_{N_1})\\ &= \dim_{\mathbf{C}} \ker M_0.\end{aligned}$$

Step 2. Equation (37) holds if $M + N = X$.
In this case we have

$$\ker \mathcal{I}|_N = \ker \mathcal{I} = M \cap N.$$

Firstly we assume that $\ker A = \{0\}$. Then $M \cap N = \{0\}$. Let $\mathcal{I}_1$ be defined by $\mathcal{I}_1(x+u, y+v) = \mathcal{I}(x,y)$ for all $x, y \in M$, $u, v \in N$. By Lemma 2.5 and 2.6 we have

$$\begin{aligned}I(PAP, A) &= I(\mathcal{I}_1, \mathcal{I})\\ &= I(\mathcal{I}_1|_M), \mathcal{I}|_M) + I(\mathcal{I}_1|_N), \mathcal{I}|_N)\\ &= m^-(\mathcal{I}|_N).\end{aligned}$$

In the general case, we apply the above special case by taking quotient space with $\ker A$ and get $I(PAP, A) = m^-(\mathcal{I}|_N)$.

Step3. Equation (37) holds.

Firstly we assume that $\ker A = \{0\}$. Let Q be the orthogonal projection onto $M + N$. Then the $\mathcal{I}$-orthogonal complement is $\ker_{\mathbf{C}}(\mathcal{I}|_N)$. By the above two steps we have

$$\begin{aligned}I(PAP, A) &= I(PAP, QAQ) + I(QAQ, A)\\ &= m^-(\mathcal{I}|_N) + \dim_{\mathbf{C}} \ker \mathcal{I}|_N.\end{aligned}$$

In the general case, we apply the above special case by taking quotient space with $\ker A$ and get (37). Q.E.D.

At the end of this subsection we gives the following formula which will be used below.

Lemma 2.7 *Let X be a Hilbert space and $A_s \in \mathcal{C}(X)$, $0 \le s \le 1$ be a curve of Fredholm operators. Let $H = X \oplus X$ and $B_s \in \mathcal{C}(H)$ be defined by $B_s = \begin{pmatrix} 0 & A_s^* \\ A_s & 0 \end{pmatrix}$. Then we have*

$$\mathrm{sf}\{B_s\} = \dim_{\mathbf{C}} \ker A_1 - \dim_{\mathbf{C}} \ker A_0. \tag{38}$$

Proof. Note that for $\lambda \in \mathbf{R}$, $\lambda \in \sigma(B_s)$ if and only if $\lambda^2 \in \sigma(A^*A)$, and the algebraic multiplicity of them are the same if $|\lambda| \neq 0$ is small. Moreover we have

$$\begin{aligned} \dim_{\mathbf{C}} \ker B_s &= \dim_{\mathbf{C}} \ker A_s + \dim_{\mathbf{C}} \ker A_s^* \\ \mathrm{ind} A_s = \mathrm{ind} A_0 &= \dim_{\mathbf{C}} \ker A_s - \dim_{\mathbf{C}} \ker A_s^*. \end{aligned}$$

By the definition of the spectral flow we have

$$\begin{aligned} \mathrm{sf}\{B_s\} &= \frac{1}{2}(\dim_{\mathbf{C}} \ker B_1 - \dim_{\mathbf{C}} \ker B_0)) \\ &= \dim_{\mathbf{C}} \ker A_1 - \dim_{\mathbf{C}} \ker A_0. \end{aligned}$$

Q.E.D.

3 Maslov-type index theory

3.1 Introduction to Maslov index

We begin with the definition of the Lagrangian Grassmannian of a symplectic Hilbert space.

Definition 3.1 *Let X be a Hilbert space. Let $j \in \mathcal{B}(X)$ be an invertible skew selfadjoint operator. Set $\omega(x,y) = (jx,y)$ for all $x,y \in X$. The form ω is called the* **(strong) symplectic structure** *on X, and the space (X,ω) is called* **symplectic Hilbert space**. *The* **linear symplectic group** $\mathrm{Sp}(X,\omega)$ *is defined to be*

$$\mathrm{Sp}(X,\omega) = \{M \in \mathcal{B}(X); M^*jM = j\}.$$

Let (X_l,ω_l), $l = 1,2$ be two symplectic Hilbert spaces. The space of linear symplectic maps $\mathrm{Sp}(X_1,X_2)$ is defined to be

$$\mathrm{Sp}(X_1,X_2) = \{M \in \mathcal{B}(X_1,X_2); \omega_2(Mx,My) = \omega_1(x,y)\}.$$

Set $A = (-j^2)^{\frac{1}{2}}$ and $J = A^{-1}j$. Then (Ax,y), $x,y \in X$ is an equivalent Hermitian metric on X and J is a complex structure on X compatible with ω, i.e., $J^2 = -I$ and $\omega(x,Jy) = (Ax,y)$, $x,y \in X$ is an equivalent Hermitian metric. All such J forms a contractable space. So we can replace the original metric on X with A.

Definition 3.2 *Let (X,ω) be a symplectic Hilbert space.*

(1) For any subspace Λ of X, the symplectic complement Λ^ω is defined to be

$$\Lambda^\omega = \{y \in X; \omega(x,y) = 0, \forall x \in \Lambda\}.$$

(2) A subspace Λ of X is called **Lagrange** *if $\Lambda^\omega = \Lambda$. The* **Lagrangian Grassmannian** *$\mathcal{L}(X,\omega)$ consists of all the Lagrange subspaces of (X,ω).*

(3) Let $\Lambda, \Lambda' \in \mathcal{L}(X,\omega)$ be two Lagrange subspace. The pair (Λ,Λ') is called a **Fredholm pair** *if $\Lambda \cap \Lambda'$ is of finite dimension and $\Lambda + \Lambda'$ is a finite codimensional subspace of X. The* **Fredholm Lagrangian Grassmannian** *$\mathcal{FL}_\Lambda(X,\omega)$ consists of all the Lagrange subspace Λ' of X such that (Λ,Λ') is a Fredholm pair.*

The following lemma is well-known.

Lemma 3.1 *Let (X,ω) be a symplectic Hilbert space. Assume that there is an invertible skew selfadjoint operator $J \in \mathcal{B}(X)$ such that $J^2 = -I$ and $\omega(x,y) = (Jx,y)$. Let $X_1 = \ker(J - \sqrt{-1}I)$ and $X_2 = \ker(J + \sqrt{-1}I)$. Let Λ_0, Λ are two subspaces of X. Then we have*

(1) *$\Lambda \in \mathcal{L}(X,\omega)$ if and only if there is an linear isometric $U \in \mathrm{U}(X_1,X_2)$ such that Λ is the graph of $\mathrm{Gr}(U)$ of U, where $\mathrm{U}(X_1,X_2)$ denotes the set of linear isometric between X_1 and X_2, $\mathrm{U}(X)$ denotes the unitary group of X and $\mathrm{Gr}(U)$ denotes the graph of U. Specially, $\mathcal{L}(X,\omega) \neq \emptyset$ if and only if $\mathrm{U}(X_1,X_2) \neq \emptyset$.*

(2) *Let $U, U' \in \mathrm{U}(X_1,X_2)$. Set $\Lambda = \mathrm{Gr}(U)$, $\Lambda' = \mathrm{Gr}(U')$. Then (Λ,Λ') is a Fredholm pair if and only if $U - U'$ is Fredholm.*

Q.E.D.

Following [5] we give the following definition.

Definition 3.3 *Let (X,ω) be a symplectic Hilbert space. Assume that there is an invertible skew selfadjoint operator $J \in \mathcal{B}(X)$ such that $J^2 = -I$ and $\omega(x,y) = (Jx,y)$. Let $X_1 = \ker(J - \sqrt{-1}I)$ and $X_2 = \ker(J + \sqrt{-1}I)$. Let $(\Lambda(s),\Lambda'(s))$, $a \le s \le b$ be a curve of Fredholm pairs of Lagrange subspaces of X such that $\Lambda(s) = \mathrm{Gr}(U(s))$, $\Lambda'(s) = \mathrm{Gr}(U'(s))$, where $U(s), U'(s) \in \mathrm{U}(X_1,X_2)$. Let $l = (1-\epsilon, 1+\epsilon) \subset \mathbf{C}$, be a interval on $\mathbf{R}$, where $\epsilon > 0$ is small. The coorientation of l is defined to be from the down half complex plane to the up half complex plane. The* **Maslov index** *$i(\Lambda,\Lambda')$ is defined to be the spectral flow $-\mathrm{sf}_l\{U(s)'^{-1}U(s)\}$. It is independent of the compatible complex structure J.*

To calculate the Maslov indices, the standard method is the crossing form (cf. [10] and [21]).

Let $\Lambda(s)$, $a \le s \le b$ be a curve of Lagrange subspaces of X. Let W be a fixed Lagrangian complement of $\Lambda(t)$. For $v \in \Lambda(t)$ and $|s-t|$ small, define $w(s) \in W$ by $v + w(s) \in \Lambda(s)$. The form

$$Q(\Lambda,t) \equiv Q(\Lambda,W,t)(u,v) = \frac{d}{ds}|_{s=t}\omega(u,w(s)), \quad \forall u,v \in \Lambda(t)$$

is independent of the choice of W. Let $(\Lambda(s),\Lambda'(s))$, $a \le s \le b$ be a curve of Fredholm pairs of Lagrange subspaces of X. For $t \in [a,b]$, the **crossing form** $\Gamma(\Lambda,\Lambda',t)$ is defined on $\Lambda(t) \cap \Lambda'(t)$ by

$$\Gamma(\Lambda,\Lambda',t)(u,v) = Q(\Lambda,t)(u,v) - Q(\Lambda',t)(u,v), \quad \forall u,v \in \Lambda(t) \cap \Lambda'(t).$$

A **crossing** is a time $t \in [a,b]$ such that $\Lambda(t) \cap \Lambda^{'}(t) \neq \{0\}$. A crossing is called **regular** if $\Gamma(\Lambda, \Lambda^{'}, t)$ is nondegenerate. It is called **simple** if in addition $\Lambda(t) \cap \Lambda^{'}(t)$ is one dimensional.

Now let (X, ω) be a symplectic Hilbert space with $\omega(x,y) = (jx, y)$, $x, y \in X$, $j \in \mathcal{B}(X)$ and $j^* = -j$. Then we have a symplectic Hilbert space $(H = X \oplus X, (-\omega) \oplus \omega)$. For any $M \in \mathrm{Sp}(X, \omega)$, its graph $\mathrm{Gr}(M)$ is a Lagrange subspace of H. The following lemma is Lemma 3.1 in [10].

Lemma 3.2 *Let $M(s) \in \mathrm{Sp}(X, \omega)$, $a \leq s \leq b$ be a curve of linear symplectic maps. Assume that $M(s)$ is differentiable at $t \in [a,b]$. Set $B_1(t) = -j\dot{M}(t)M(t)^{-1}$ and $B_2(t) = -jM(t)^{-1}\dot{M}(t)$. Then $B_1(t)$, $B_2(t)$ are selfadjoint, $B_2(t) = M(t)^* B_1(t) M(t)$ and we have*

$$Q(\mathrm{Gr}(M), t)((x, M(t)x), (y, M(t)y)) = (B_2(t)x, y). \tag{39}$$

Q.E.D.

Corollary 3.1 *Let (X, ω) be a symplectic Hilbert space and $(\Lambda(t), \Lambda^{'}(t))$, $a \leq t \leq b$ be a C^1 curve of Fredholm pairs of Lagrange subspaces of X with only regular crossing. Then we have*

$$i(\Lambda, \Lambda^{'}) = m^+(\Gamma(\Lambda, \Lambda^{'}, a)) - m^-(\Gamma(\Lambda, \Lambda^{'}, b)) + \sum_{a<t<b} \mathrm{sign}(\Gamma(\Lambda, \Lambda^{'}, t)). \tag{40}$$

Proof. Pick an invertible skew selfadjoint operator $J \in \mathcal{B}(X)$ such that $J^2 = -I$ and $\omega(x,y) = (Jx, y)$. Let $X_1 = \ker(J - \sqrt{-1}I)$ and $X_2 = \ker(J + \sqrt{-1}I)$. By Lemma 3.1 there are curves of isometric $U(t)$, $U^{'}(t)$ in $\mathrm{U}(X_1, X_2)$ such that $\Lambda(t) = \mathrm{Gr}(U(t))$ and $\Lambda^{'}(t) = \mathrm{Gr}(U^{'}(t))$. Apply Lemma 3.2 for X_1 with $j = -\sqrt{-1}I$, for any $x, y \in \ker(U(t) - U^{'}(t))$ and $t \in [a,b]$ we have

$$\begin{aligned}
\frac{d}{ds}|_{s=t}(-\sqrt{-1}U^{'-1}Ux, y) &= (-jU^{'-1}\dot{U}^{'}U^{'-1}Ux, y) + (jU^{'-1}\dot{U}x, y) \\
&= (-jU^{'-1}\dot{U}^{'}x, y) + (U^{'-1}UjU^{-1}\dot{U}x, U^{'-1}Uy) \\
&= (-jU^{'-1}\dot{U}^{'}x, y) + (jU^{-1}\dot{U}x, y) \\
&= -\Gamma(\Lambda, \Lambda^{'}, t)((x, Ux), (y, Uy)).
\end{aligned}$$

By Proposition 2.1 we obtain (40). Q.E.D.

By Proposition 2.3, Lemma 3.2 and the proof of Corollary 3.1 we have

Corollary 3.2 *Let (X, ω) be a symplectic Hilbert space and $M(s) \in \mathrm{Sp}(X, \omega)$, $a \leq s \leq b$ be a C^1 curve of linear symplectic maps. Assume that $-j\dot{M}(t)M(t)^{-1}$ is semi-positive definite. Let $H = (X \oplus X, (-\omega) \oplus \omega)$ and W be a Lagrange subspace of X. Then we have*

$$i(\mathrm{Gr}(M(t)), W) = \sum_{a<s\leq b} \Big(\dim_{\mathbf{C}}(\mathrm{Gr}(M(s)) \cap W) - \lim_{t \to s^-} \dim_{\mathbf{C}}(\mathrm{Gr}(M(t)) \cap W) \Big) \geq 0. \tag{41}$$

Q.E.D.

Corollary 3.3 *(**Symplectic invariance**) Let (X_l, ω_l), $l = 1, 2$ be two symplectic Hilbert spaces, and $M(t) \in \mathrm{Sp}(X_1, X_2)$, $a \leq t \leq b$ be a curve of linear symplectic maps, and $(\Lambda(t), \Lambda^{'}(t))$, $a \leq t \leq b$ be a curve of Fredholm pairs of Lagrange subspaces of X_1. Then we have*

$$i(M\Lambda, M\Lambda^{'}) = i(\Lambda, \Lambda^{'}). \tag{42}$$

Proof. Firstly assume that the curves M, Λ, Λ' are differentiable and the pairs (Λ, Λ') have only regular crossing. By the definition of the crossing form, for any $t \in [a,b]$ we have

$$\Gamma(M\Lambda, M\Lambda', t)(M(t)u, M(t)v) = \Gamma(\Lambda, \Lambda', t)(u,v) \quad \forall u, v \in \Lambda(t) \cap \Lambda'(t).$$

By Corollary 3.1, equation (42) holds. For the general case, we can make a small perturbation of the curves M, Λ, Λ' with their endpoints fixed such that they satisfy the above condition. Then our result follows from the homotopy invariance rel. endpoints of the Maslov indices. Q.E.D.

3.2 The spectral flow formula and Maslov-type indices

Let A be a closed densely defined symmetric operator on a Hilbert space H with domain D_m. Let D_M be the domain of A^*. Define the inner product on D_M by

$$\langle x, y \rangle = (x,y) + (A^*x, A^*y).$$

Then D_M is a Hilbert space and D_m is a closed subspace of D_M. The orthogonal complement of D_m in D_M is $\ker(A^{*2} + I)$. Define $X = D_M/D_m$ and let $\gamma : D_M \to X$ be the canonical map. The map γ is called the **abstract trace map**. Define $\omega : X \times X \to \mathbf{C}$ by

$$\omega(\gamma(x), \gamma(y)) = (A^*x, y) - (x, A^*y), \quad \forall x, y \in D_M.$$

Then (X, ω) is a symplectic Hilbert space.

The following proposition is Theorem 5.1 in [5].

Proposition 3.1 (Spectral flow formula) *Let A be a closed densely defined symmetric operator on a Hilbert space H with domain D_m and let $C_t, t \in [a,b]$ be bounded. We assume that*

1. A has a selfadjoint extension A_D with compact resolvent, where D is the domain of A.

2. there exists a positive constant a such that $D_m \cap \ker(A^* + C_t + s) = \{0\}$ for any $|s| < a$ and any $t \in [a,b]$.

Then we have

$$\mathrm{sf}\{A_D + C_t\} = -i(\gamma(D), \gamma(\ker(A^* + C_t))). \tag{43}$$

Sketch of the proof. Here we only consider the case that C_t is of class C^1. The condition shows that $(\gamma(D), \{\gamma(\ker(A^* + C_t)\})$ is a C^1 Fredholm pairs of Lagrange subspaces of X. Let $x, y \in \ker(A^* + C_t)$. Pick a Lagrange complement W of $\gamma(\ker(A^* + C_t))$. Then for s with $|s-t|$ small, W is also a Lagrange complement of $\gamma(\ker(A^* + C_s))$ and there is $y_s \in \ker(A^* + C_s)$ such that $y_s \to y$ when $s \to t$. So we have

$$\begin{aligned} \omega(\gamma(x), \gamma(y_s - y)) &= (A^*x, y_s - y) - (x, A^*(y_s - y) \\ &= (-C_t x, y_s - y) - (x, -C_s y_s + C_t y) \\ &= ((C_s - C_t)x, y_s). \end{aligned}$$

Differential it with respect to s at $s = t$, we get

$$Q(\gamma(\ker(A^* + C_s)), t) = Q_t \dot{C}_t Q_t, \tag{44}$$

where Q_t is the orthogonal projection from H onto $\ker(A^* + C_t)$.

By Theorem 4.22 in [21], we can choose $\delta \in (0, a)$ sufficiently small such that $\ker(A + C_t + sI) = \{0\}$ for $t = a$ or b and $s \in (0, \delta]$, and $A_D + C_t + \delta I$ has only regular crossing. By Proposition 2.1, Corollary 3.1 and (44) we have

$$\begin{aligned}
\mathrm{sf}\{A_D + C_t\} &= \mathrm{sf}\{A_D + C_t + \delta I\} \\
&= -m^-(P_a \dot{C}_a P_a) + m^+(P_b \dot{C}_b P_b) + \sum_{a<t<b} \mathrm{sign}(P_t \dot{C}_t P_t) \\
&= -m^+(\gamma(D), \Gamma(\gamma(\ker(A^* + C_s + \delta I)), a) \\
&\quad +m^-(\gamma(D), \Gamma(\gamma(\ker(A^* + C_s + \delta I)), b) \\
&\quad - \sum_{a<t<b} \mathrm{sign}(\gamma(D), \Gamma(\gamma(\ker(A^* + C_s + \delta I)), t) \\
&= -i(\gamma(D), \gamma(\ker(A^* + C_s + \delta I))) \\
&= -i(\gamma(D), \gamma(\ker(A^* + C_t))),
\end{aligned}$$

where P_t is the orthogonal projection from H onto $\ker(A_D + C_t)$. Q.E.D.

Now we turn to the Maslov-type indices.

Definition 3.4 *Let (X_l, ω_l) be symplectic Hilbert spaces with $\omega_l(x, y) = (j_l x, y)$, $x, y \in X_l$, $j_l \in \mathcal{B}(X)$ are invertible, and $j_l^* = -j_l$, where $l = 1, 2$. Then we have a symplectic Hilbert space $(H = X_1 \oplus X_2, (-\omega_1) \oplus \omega_2)$. Let $W \in \mathcal{L}(H)$. Let $M(t)$, $a \le t \le b$ be a curve in $\mathrm{Sp}(X_1, X_2)$ such that $\mathrm{Gr}(M(t)) \in \mathcal{FL}(W)$ for all $t \in [a, b]$. The Maslov-type index $i_W(M(t))$ is defined to be $i(\mathrm{Gr}(M(t), W)$. If $a = 0$, $b = T$, $(X_1, \omega_1) = (X_2, \omega_2)$ and $M(0) = I$, we denote by $\nu_{T,W}(M(t)) = \dim_{\mathbf{C}}(\mathrm{Gr}(M(T) \cap W)$.*

The Maslov-type indices have the following property.

Lemma 3.3 *Let (X_l, ω_l) be symplectic Hilbert spaces with $\omega_l(x, y) = (j_l x, y)$, where $x, y \in X_l$, $j_l \in \mathcal{B}(X_l)$ are invertible, and $j_l^* = -j_l$, $l = 1, 2, 3, 4$. Let W be a Lagrange subspace of $(X_1 \oplus X_4, (-\omega_1) \oplus \omega_4)$. Let $\gamma_l \in C([0, 1], \mathrm{Sp}(X_l, X_{l+1}))$, $l = 1, 2, 3$ be syplectic paths such that $\mathrm{Gr}(\gamma_3(s)\gamma_2(t)\gamma_1(s)) \in \mathcal{FL}(W)$ for all $(s, t) \in [0, 1] \times [0, 1]$. Then we have*

$$i_W(\gamma_3\gamma_2\gamma_1) = i_{W'}(\gamma_2) + i_W(\gamma_3\gamma_2(0)\gamma_1), \tag{45}$$

where $W' = \mathrm{diag}(\gamma_1(1), \gamma_3(1)^{-1})W$.

Proof. Let $M = \mathrm{diag}(\gamma_1(1), \gamma_3(1)^{-1})$. By the homotopy invariance rel. endpoints of the Maslov-type indices and Corollary 3.3, we have

$$\begin{aligned}
i_W(\gamma_3\gamma_2\gamma_1) &= i_W(\gamma_3(1)\gamma_2\gamma_1(1)) + i_W(\gamma_3\gamma_2(0)\gamma_1) \\
&= i(M\mathrm{Gr}(\gamma_3(1)\gamma_2\gamma_1(1)), MW) + i_W(\gamma_3\gamma_2(0)\gamma_1) \\
&= i_{W'}(\gamma_2) + i_W(\gamma_3\gamma_2(0)\gamma_1).
\end{aligned}$$

Q.E.D.

The following properties of fundamental solutions for linear ODE will be used later.

Lemma 3.4 *Let $j \in C^1([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be a curve of skew selfadjoint matrices, and $b \in C([0,+\infty), \mathrm{gl}(m, \mathbf{C}))$ be a curve of selfadjoint matrices. Let $\gamma \in C^1([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be the fundamental solution of*

$$-j\dot{x} - \frac{1}{2}\dot{j}x = bx. \tag{46}$$

Then we have $\gamma(t)^* j(t)\gamma(t) = j(0)$ for all t.

Proof. By the definition of the fundamental solution, we have $\gamma(0)^* j(0)\gamma(0) = j(0)$. Since $j^* = -j$ and $b^* = b$, we have

$$\begin{aligned}
\frac{d}{dt}(\gamma(t)^* j(t)\gamma(t)) &= \dot{\gamma}^* j\gamma + \gamma^* \dot{j}\gamma + \gamma^* j\dot{\gamma} \\
&= (-b\gamma - \frac{1}{2}\dot{j})^* j^{*-1} j\gamma + \gamma^* \dot{j}\gamma + \gamma^* j j^{-1}(-b\gamma - \frac{1}{2}\dot{j}) \\
&= \gamma^*(b - \frac{1}{2}\dot{j} + \dot{j} - b - \frac{1}{2}\dot{j})\gamma \\
&= 0.
\end{aligned}$$

So we have $\gamma(t)^* j(t)\gamma(t) = j(0)$. Q.E.D.

Lemma 3.5 *Let $B \in C([0,+\infty), \mathrm{gl}(m, \mathbf{C}))$ and $P \in C^1([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be two curves of matrices. Let $\gamma \in C^1([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be the fundamental solution of*

$$\dot{x} = Bx, \tag{47}$$

and $\gamma' \in C^1([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be the fundamental solution of

$$\dot{y} = (PBP^{-1} + \dot{P}P^{-1})y. \tag{48}$$

Then we have

$$\gamma' = P\gamma P(0)^{-1}. \tag{49}$$

Proof. Direct calculation shows

$$\frac{d}{dt}(P\gamma P(0)^{-1}) = (PBP^{-1} + \dot{P}P^{-1})P\gamma P(0)^{-1}$$

and $P(0)\gamma P(0)^{-1} = I$. By definition, $P\gamma P(0)^{-1}$ is the fundamental solution of (48). Q.E.D.

Corollary 3.4 *Let $j_1, j_2 \in C^1([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be two curves of skew selfadjoint matrices. Let $P \in C^1([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be a curve of matrices such that $P^* j_2 P = j_1$, and $b \in C([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be a curve of selfadjoint matrices. Let $\gamma \in C^1([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be the fundamental solution of*

$$-j_1\dot{x} - \frac{1}{2}\dot{j}_1 x = bx, \tag{50}$$

and $\gamma' \in C^1([0,+\infty), \mathrm{Gl}(m, \mathbf{C}))$ be the fundamental solution of

$$j_2\dot{y} - \frac{1}{2}\dot{j}_2 y = (P^{*-1}bP^{-1} + Q)y, \tag{51}$$

where $Q = \frac{1}{2}(P^{*-1}\dot{P}^* j_2 - j_2\dot{P}P^{-1})$. *Then we have*

$$\gamma' = P\gamma P(0)^{-1}. \tag{52}$$

In particular, when j_1 *and* j_2 *are constant matrices, we have*

$$Q = P^{*-1}\dot{P}^* j_2 = -j_2\dot{P}P^{-1}.$$

Proof. Take $B = -j_1^{-1}(b + \frac{1}{2}\dot{j}_1)$ in Lemma 3.5, we have

$$\begin{aligned}
-j_2(PBP^{-1} + \dot{P}P^{-1}) - \frac{1}{2}\dot{j}_2 &= -j_2(P(-j_1)^{-1}(b + \frac{1}{2}\dot{j}_1)P^{-1} + \dot{P}P^{-1}) - \frac{1}{2}\dot{j}_2 \\
&= P^{*-1}(b + \frac{1}{2}\dot{j}_1)P^{-1} - j_2\dot{P}P^{-1} - \frac{1}{2}\dot{j}_2 \\
&= P^{*-1}bP^{-1} - j_2\dot{P}P^{-1} + \frac{1}{2}(P^{*-1}\dot{j}_1P^{-1} - \dot{j}_2) \\
&= P^{*-1}bP^{-1} - j_2\dot{P}P^{-1} + \frac{1}{2}(P^{*-1}\frac{d}{dt}(P^* j_2 P)P^{-1} - \dot{j}_2) \\
&= P^{*-1}bP^{-1} + Q.
\end{aligned}$$

By Lemma 3.5, our results holds. Q.E.D.

The following is a special case of the spectral flow formula.

Let $j \in C^1([0,T], \mathrm{Gl}(m, \mathbf{C}))$ be a curve of skew selfadjoint matrices. Then we have symplectic Hilbert spaces $(\mathbf{C}^m, \omega(t))$ with standard Hermitian inner product and $\omega(t)(x,y) = (j(t)x, y)$, for all $x, y \in \mathbf{C}^m$ and $t \in [0,T]$. Then we have a symplectic Hilbert space $(V = \mathbf{C}^m \oplus \mathbf{C}^m, (-\omega(0)) \oplus \omega(T))$. Let $W \in \mathcal{L}(V)$. Let $b_s(t) \in \mathcal{B}(\mathbf{C}^m)$, $0 \le s \le 1$, $0 \le t \le T$ be a continuous family of selfadjoint matrices such that $b_0(t) = 0$. By Lemma 3.4, there are continuous family of matrices $M_s(t) \in \mathrm{Gl}(m, \mathbf{C})$ such that $M_s(0) = I$ and

$$-j\frac{d}{dt}M_s(t) - \frac{1}{2}(\frac{d}{dt}j)M_s(t) = b_s(t)M_s(t).$$

Set

$$\begin{aligned}
X &= L^2([0,T], \mathbf{C}^m), \quad D_m = H_0^1([0,T], \mathbf{C}^m), \\
D_M &= H^1([0,T], \mathbf{C}^m), \quad D_W = \{x \in D_M; (x(0), x(t)) \in W\}.
\end{aligned}$$

Let $A_M \in \mathcal{C}(X)$ with domain D_M be defined by

$$A_M x = -j\frac{d}{dt}x - \frac{1}{2}(\frac{d}{dt}j)x.$$

Set $x \in D_M$, $A = A_M|_{D_m}$, $A_W = A_M|_{D_W}$. Let $C_s \in \mathcal{B}(X)$ be defined by $(C_s x)(t) = b_s(t)x(t)$, $x \in X$, $t \in [0,T]$.

Proposition 3.2 *Set* $W' = \mathrm{diag}(I, M_0(T)^{-1})W$. *Then we have*

$$I(A_W, A_W - C_1) = i_{W'}(M_0^{-1}M_1). \tag{53}$$

Proof. The Sobolev embedding theorem shows that $D_M \subset C([0,T], \mathbf{C}^m)$. For any $x \in D_M$, define $\gamma(x) = (x(0), x(T))$. Direct calculation shows that $D_M/D_m = \mathbf{C}^m \oplus \mathbf{C}^m$ with symplectic structure $(\mathrm{diag}(j(0), -j(T))\gamma(x), \gamma(y))$, $x, y \in D_M$, and γ is the abstract trace map. Moreover, $A^* = A_M$, $\gamma(A^* - C_s) = \mathrm{Gr}(M_s(T))$, and $\gamma(D_W) = W$. By Proposition 3.1 and Lemma 3.3, we have

$$
\begin{aligned}
I(A_W, A_W - C_1) &= -\mathrm{sf}\{A_W - C_s\} \\
&= i(\{M_s(T); 0 \le s \le 1\}, W) \\
&= i_W(M_0(T)(M_0(T)^{-1}M_s(T))I; 0 \le s \le 1) \\
&= i_{W'}(M_0(T)^{-1}M_s(T); 0 \le s \le 1) \\
&= -i_{W'}(M_0(t)^{-1}M_0(t); 0 \le t \le T) + i_{W'}(M_0(0)^{-1}M_s(0); 0 \le s \le 1) \\
&\quad + i_{W'}(M_0(t)^{-1}M_1(t); 0 \le t \le T) \\
&= i_{W'}(M_0^{-1}M_1).
\end{aligned}
$$

Q.E.D.

4 Proof of the main results

In this section we will use the notations in §1 and §3. Let R be a subspace of $\mathbf{C}^{2n}$. Set $H = L^2([0,T], \mathbf{C}^n)$, where $T > 0$. Let K_R be a closed operator on H. Its domain is H_R defined by (11), and $H_R x = \dot{x}$ for all $x \in H_R$. Set

$$X = L^2([0,T], \mathbf{C}^{2n}), \quad D_{W(R)} = \{x \in H^1([0,T], \mathbf{C}^{2n}); (x(0), x(t)) \in W(R)\}.$$

For any $x, y \in H^1([0,T], \mathbf{C}^{2n})$, define

$$(x, y)_1 = (x, y) + (\dot{x}, \dot{y}).$$

Let $A_{W(R)} \in \mathcal{C}(X)$ with domain $D_{W(R)}$ be defined by $A_{W(R)}x = -J\frac{d}{dt}x$, $x \in D_{W(R)}$. Let $b_s(t)$ de defined by (13) and $C_s \in \mathcal{B}(X)$ be defined by $(C_s x)(t) = b_s(t)x(t)$, $x \in X$, $t \in [0,T]$. Then we have $K_R^* = -K_{R^b}$. Consider the standard orthogonal decomposition

$$\mathbf{C}^{2n} = (\mathbf{C}^n \times \{0\}) \oplus (\{0\} \times \mathbf{C}^n).$$

It induces orthogonal decompositions $X = H \oplus H$ and $D_{W(R)} = H_{R^b} \oplus H_R$. Under such orthogonal decompositions, $A_{W(R)}$ is in block form $A_{W(R)} = \begin{pmatrix} 0 & K_R \\ K_R^* & 0 \end{pmatrix}$. Let $b_s(t), C_s$ be in block form

$$
\begin{aligned}
b_s(t) &= \begin{pmatrix} b_{11}(s,t) & b_{12}(s,t) \\ b_{21}(s,t) & b_{22}(s,t) \end{pmatrix} \\
C_s &= \begin{pmatrix} C_{11}(s) & C_{12}(s) \\ C_{21}(s) & C_{22}(s) \end{pmatrix}.
\end{aligned}
$$

Define $P_s, Q_s, R_s \in \mathcal{B}(H_R)$ by

$$
\begin{aligned}
(P_s x, y)_1 &= (C_{11}(s)^{-1}K_R x, K_R y), \\
(Q_s x, y)_1 &= -(C_{11}(s)^{-1}C_{12}(s)x, K_R y), \\
(R_s x, y)_1 &= ((C_{21}(s)C_{11}(s)^{-1}C_{12}(s) - C_{22}(s))x, y)
\end{aligned}
$$

for all $x, y \in \mathrm{H}_R$. Then the index forms $\mathcal{I}_s$ defined by (12) satisfy

$$\mathcal{I}_s(x,y) = ((P_s + Q_s + Q_s^* + R_s)x, y)_1.$$

Lemma 4.1 *The operator $P_s + Q_s + Q_s^* + R_s \in \mathcal{B}(H_R)$ is a curve of Fredholm selfadjoint operators.*

Proof. Since $\mathcal{I}_s$ are bounded symmetric quadratic forms on H_R, by Riesz representation theorem, $P_s + Q_s + Q_s^* + R_s \in \mathcal{B}(H_R)$. Similarly we can see that they form a continuous curve.

Let $Q_s^{'} = -C_{11}(s)^{-1}C_{12}(s)$ and $R_s^{'} = C_{21}(s)C_{11}(s)^{-1}C_{12}(s) - C_{22}(s)$ be two bounded operators on H. Then $Q_s^* = (K_R^*K_R + I)^{-1}Q_s^{'*}K_R$ and $R_s = (K_R^*K_R + I)^{-1}R_s^{'}$. So Q_s^* and R_s maps bounded subset of H_R into the bounded subset in the domain $\mathbf{D}(K_R^*K_R)$ of $K_R^*K_R$. Since $\mathbf{D}(K_R^*K_R)$ is a closed subspace of $H^2([0,T], \mathbf{C}^n)$, by Sobolev embedding theorem, the embedding of $\mathbf{D}(K_R^*K_R)$ into H_R is compact. So Q_s^*, Q_s, and R_s are compact.

Now we prove that P_s is Fredholm and then our lemma is proved. If $p_s(t)$ is positive definite, we can choose $q_s(t) = 0$ and a positive definite curve $r_s(t)$ such that $P_s + R_s$ is positive definite. So $P_s + R_s$ is invertible. Since R_s is compact, P_s is Fredholm. Here it is only required that $p_s(t)$ continuous. In the general case, we have to assume that $p_s(t)$ is C^1 in t. Note that $H_{\mathbf{C}^{2n}} = H^1([0,T], \mathbf{C}^n)$. Consider the operator $p_s : H_{\mathbf{C}^{2n}} \to H_{\mathbf{C}^{2n}}$. Let $k : H_R \to H_{\mathbf{C}^{2n}}$ be the injection. Then p_s is invertible and $p_s k$ is Fredholm. For any $x \in H_R$ and $y = H_{\mathbf{C}^{2n}}$, the inner product $((P_s - p_s)x, y)_1$ consists only the lower-order terms (i.e., no second-order differential involved) and some boundary terms. Similar to the above proof, we can conclude that the lower-order terms correspond to compact operators. The boundary terms correspond to finite rank operators. So $kP_s - p_s k$ is compact. Since $p_s k$ and k are Fredholm, kP_s and P_s are Fredholm. Q.E.D.

The following lemma is the key to the proof of Theorem 1.1.

Lemma 4.2 *Let $u_{b_s}(x) = (p_s K_R x + q_s x, x)$ for all $x \in H_R$ and $0 \le s \le 1$. Then we have*

$$\ker(A_{W(R)} - C_s) = \{u_{b_s}(x); x \in \ker\ \mathcal{I}_s\}. \tag{54}$$

Moreover, for any $x, y \in H_R$, we have

$$-\left(\left(\frac{d}{ds}C_s\right)u_{b_s}(x), u_{b_s}(y)\right) = \left(\left(\frac{d}{ds}\mathcal{I}_s\right)x, y\right). \tag{55}$$

Q.E.D.

Proof. Since $\mathcal{I}_s(x,y) = (p_s K_R x + q_s x, K_R y) + (q_s^* K_R x + r_s x, y)$ for all $x, y \in H_R$, by the definition of K_R^*, we have $x \in \ker\ \mathcal{I}_s$ if and only if $x \in \mathbf{D}(K_R^*)$, and $K_R^*(p_s K_R x + q_s x) + (q_s^* K_R x + r_s x) = 0$, if and only if $u_{b_s}(x) \in \ker(A_{W(R)} - C_s)$. So equation (54) is proved.

Now we turn to equation (55). Set $Z_s = \begin{pmatrix} p_s & q_s \\ 0 & 1 \end{pmatrix}$. By the definition of b_s and direct computation we have

$$-Z_s^* b_s Z_s = \begin{pmatrix} -p_s & 0 \\ 0 & r_s \end{pmatrix}, \qquad -Z_s^* b_s = \begin{pmatrix} -I & q_s \\ 0 & r_s \end{pmatrix}, \qquad -b_s Z_s = \begin{pmatrix} -I & 0 \\ q_s^* & r_s \end{pmatrix}. \tag{56}$$

So we have

$$\begin{aligned}
-Z_s^*\left(\frac{d}{ds}b_s\right)Z_s &= -\frac{d}{ds}\left(Z_s^*b_sZ_s\right)+Z_s^*b_s\frac{d}{ds}Z_s+Z_s^*\frac{d}{ds}(b_sZ_s)\\
&= \begin{pmatrix} -\frac{d}{ds}p_s & 0\\ 0 & \frac{d}{ds}r_s\end{pmatrix} - \begin{pmatrix} -I & q_s\\ 0 & r_s\end{pmatrix}\begin{pmatrix} \frac{d}{ds}p_s & \frac{d}{ds}q_s\\ 0 & 0\end{pmatrix} - \begin{pmatrix} \frac{d}{ds}p_s & 0\\ \frac{d}{ds}q_s^* & 0\end{pmatrix}\begin{pmatrix} -I & 0\\ q_s^* & r_s\end{pmatrix}\\
&= \begin{pmatrix} -\frac{d}{ds}p_s & 0\\ 0 & \frac{d}{ds}r_s\end{pmatrix} - \begin{pmatrix} -\frac{d}{ds}p_s & -\frac{d}{ds}q_s\\ 0 & 0\end{pmatrix} - \begin{pmatrix} -\frac{d}{ds}p_s & 0\\ -\frac{d}{ds}q_s^* & 0\end{pmatrix}\\
&= \frac{d}{ds}\begin{pmatrix} p_s & q_s\\ q_s^* & r_s\end{pmatrix}.
\end{aligned}$$

Hence for all $x, y \in H_R$, we have

$$\begin{aligned}
-\left(\left(\frac{d}{ds}C_s\right)u_{b_s}(x), u_{b_s}(y)\right) &= -\int_0^T\left(\left(\frac{d}{ds}b_s\right)Z_s\begin{pmatrix}\dot{x}\\ x\end{pmatrix}, Z_s\begin{pmatrix}\dot{x}\\ x\end{pmatrix}\right)dt\\
&= \int_0^T\left(\frac{d}{ds}\begin{pmatrix} p_s & q_s\\ q_s^* & r_s\end{pmatrix}\begin{pmatrix}\dot{x}\\ x\end{pmatrix}, \begin{pmatrix}\dot{x}\\ x\end{pmatrix}\right)dt\\
&= \left(\left(\frac{d}{ds}\mathcal{I}_s\right)x, y\right).
\end{aligned}$$

Q.E.D.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 4.1, $\mathrm{sf}\{\mathcal{I}_s\}$ is well-defined. Since the spectral flow is invariant under homotopy with endpoints fixed, by small perturbation which fixes the endpoint, we can assume that C_s is a C^1 curve. By Theorem 4.22 in [21], we can take a sufficient small $\delta > 0$ such that $C_{11}(s) - \delta I$ is invertible, $A_{W(R)} - C_s + \delta I$, $0 \le s \le 1$ has only regular crossing, and $A_{W(R)} - C_s + aI$ is invertible for $s = 0, 1$ and $a \in (0, \delta]$. Let $\mathcal{I}_{s,a}$ be the correspondent index form of $C_s - aI$. By Proposition 2.1, Proposition 3.2 and Lemma 4.2 we have

$$\begin{aligned}
I(\mathcal{I}_0, \mathcal{I}_1) &= -\mathrm{sf}\{\mathcal{I}_s, 0 \le s \le 1\}\\
&= -\mathrm{sf}\{\mathcal{I}_{s,\delta}, 0 \le s \le 1\}\\
&= -\mathrm{sf}\{A_{W(R)} - C_s + \delta I, 0 \le s \le 1\}\\
&= -\mathrm{sf}\{A_{W(R)} - C_s, 0 \le s \le 1\}\\
&= i_{W(R)}(\gamma_1) - i_{W(R)}(\gamma_0).
\end{aligned}$$

Q.E.D.

To prove Theorem 1.2, we need some preparations.

Lemma 4.3 *Theorem 1.2 holds in the following two cases:*

(i) $R = \mathbf{C}^{2n}$,

(ii) $p \in C([0, T], \mathrm{gl}(n, \mathbf{C}))$ *is a path of positive definite matrices, and* $P(t) = \int_0^t p(s)ds$ *for all* $t \in [0, T]$.

Proof. By the definition of $W(R)$ we have

$$\mathrm{Gr}(\gamma(t))\cap W(R)=\{(x,y,x,y+P(t)x);(x,x)\in R^b,(y,y+P(t)x)\in R\}.$$

In the case (i), we have $R^b=\{0\}$ and $\mathrm{Gr}(\gamma(t))\cap W(R)=\{(0,y,0,y);y\in\mathbf{C}^n\}$ for all $t\in[0,T]$. So we have $i_{W(R)}(\gamma)=0$ and Theorem 1.2 holds.

Now we consider the case (ii). Since $P(t)$ is positive definite for all $t\in(0,T]$, for all $(x,y,x,y+P(t)x)\in\mathrm{Gr}(\gamma(t))\cap W(R)$, we have $-(x,P(t)x)=((x,-x),(y,y+P(t)x))=0$ and hence $x=0$. So we have

$$\mathrm{Gr}(\gamma(t))\cap W(R)=\begin{cases}\mathrm{Gr}(I)\cap W(R), & \text{if } t=0,\\ \{(0,y,0,y);(y,y)\in R\}, & \text{if } t\in(0,T].\end{cases}$$

By Corollary 3.2 we have

$$\begin{aligned}i_{W(R)}(\gamma) &= \dim_{\mathbf{C}}(\mathrm{Gr}(\gamma(0))\cap W(R))-\dim_{\mathbf{C}}(\mathrm{Gr}(\gamma(T))\cap W(R))\\ &= \dim_{\mathbf{C}}(\mathrm{Gr}(I)\cap W(R))-\dim_{\mathbf{C}}(\mathrm{Gr}(I)\cap R)\\ &= \dim_{\mathbf{C}}(\mathrm{Gr}(I)\cap R^b)\\ &= \dim_{\mathbf{C}} S.\end{aligned}$$

Q.E.D.

Proposition 4.1 *Let $\gamma\in C([0,T],\mathrm{Sp}(2n,\mathbf{C}))$ be such that $\gamma(0)=I$. Let $R_1\subset R_2$ be two linear subspaces of $\mathbf{C}^{2n}$. Define*

$$\mathcal{N}=\{(x,y,z,u)\in\mathrm{Gr}(\gamma(T));(x,z)\in R_1^b,(y,u)\in R_2\},$$

and

$$Q((x_1,y_1,z_1,u_1),(x_2,y_2,z_2,u_2))=(z_1,u_2)-(x_1,y_2)$$

for all $(x_1,y_1,z_1,u_1),(x_2,y_2,z_2,u_2)\in\mathcal{N}$. Then Q is a quadratic form on $\mathcal{N}$, and we have

$$i_{W(R_2)}(\gamma)-i_{W(R_1)}(\gamma)=C(\gamma(T);R_1,R_2)+\dim_{\mathbf{C}}(\mathrm{Gr}(I)\cap R_2^b)-\dim_{\mathbf{C}}(\mathrm{Gr}(I)\cap R_1^b), \qquad (57)$$

where

$$C(\gamma(T);R_1,R_2)=m^-(Q)+\dim_{\mathbf{C}}\ker Q-\dim_{\mathbf{C}}(\mathrm{Gr}(\gamma(T))\cap W(R_2)).$$

We call $C(\gamma(T);R_1,R_2)$ the **Morse concavity** *of $\gamma(T)$ with respect to R_1,R_2.*

Proof. By [24], there exist paths of matrices $p,q,r\in\mathrm{gl}(n,\mathbf{C})$ such that $p(t)$ are positive definite, $r(t)=r(t)^*$ for all $t\in[0,T]$, and $\gamma_1(T)=\gamma(T)$ if $p_s=p$, $q_s=sq$, $r_s=sr$ for all $s\in[0,1]$, and γ_s are the fundamental solution of (14). Let $\mathcal{I}_s$ be defined by (12). By Theorem 1.1 and Lemma 4.3, we have

$$m^-(\mathcal{I}_1|_{H_{R_k}})=i_{W(R_k)}(\gamma_1)-\dim_{\mathbf{C}}(\mathrm{Gr}(I)\cap R_k^b),\quad k=1,2. \qquad (58)$$

Let N be the $\mathcal{I}_1$-complement of H_{R_1} in H_{R_2}. Direct computation shows that

$$N=\left\{\begin{array}{c} x\in H_{R_2};(u_{b_1}(x))(t)=\gamma_1(t)(u_{b_1}(x))(0) \text{ for all } t\in[0,T]\\ \text{and } (p(0)\dot{x}(0)+q(0)x(0),p(T)\dot{x}(T)+q(T)x(T))\in R_1^b\end{array}\right\}.$$

Define $\varphi : N \to \mathcal{N}$ by

$$\varphi(x) = ((u_{b_1}(x))(0), (u_{b_1}(x))(T)).$$

Then φ is a linear isomorphism. Direct computation shows that

$$\mathcal{I}_1(x, y) = Q(\varphi(x), \varphi(y))$$

for all $x, y \in N$. By Proposition 2.4 and Lemma 4.2 we have

$$\begin{aligned}
m^-(\mathcal{I}_1|_{H_{R_2}}) - m^-(\mathcal{I}_1|_{H_{R_1}}) &= m^-(\mathcal{I}_1|_N) + \dim_{\mathbf{C}} \ker(\mathcal{I}_1|_N) - \dim_{\mathbf{C}} \ker(\mathcal{I}_1|_{H_{R_2}}) \\
&= m^-(Q) + \dim_{\mathbf{C}} \ker Q - \dim_{\mathbf{C}}(\mathrm{Gr}(\gamma_1(T)) \cap W(R_2)) \\
&= m^-(Q) + \dim_{\mathbf{C}} \ker Q - \dim_{\mathbf{C}}(\mathrm{Gr}(\gamma(T)) \cap W(R_2)) \\
&= C(\gamma(T); R_1, R_2). \qquad (59)
\end{aligned}$$

By the fact that γ and γ_1 has the same end points, we have

$$i_{W(R_1)}(\gamma) - i_{W(R_1)}(\gamma_1) = i_{W(R_2)}(\gamma) - i_{W(R_2)}(\gamma_1).$$

By (58) and (59), we have

$$\begin{aligned}
i_{W(R_2)}(\gamma) - i_{W(R_1)}(\gamma) &= i_{W(R_2)}(\gamma_1) - i_{W(R_1)}(\gamma_1) \\
&= (m^-(\mathcal{I}_1|_{H_{R_2}}) + \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R_2^b)) \\
&\quad -(m^-(\mathcal{I}_1|_{H_{R_1}}) + \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R_1^b)) \\
&= C(\gamma(T); R_1, R_2) + \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R_2^b) - \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R_1^b).
\end{aligned}$$

Q.E.D.

Proof of Theorem 1.2. Firstly we assume that $P(0) = 0$. Set $R_1 = R$ and $R_2 = \mathbf{C}^{2n}$. Let $Q, \mathcal{N}$ be defined by Proposition 4.1. By the definition of $\mathcal{N}$ we have

$$\mathcal{N} = \{(x, y, x, P(T)x + y); (x, x) \in R^b\}.$$

Define $\varphi : S \times \mathbf{C}^n \to \mathcal{N}$ by

$$\varphi(x, y) = (x, y, x, P(T)x + y).$$

Then φ is a linear isomorphism. So we have $\dim_{\mathbf{C}} \mathcal{N} = \dim_{\mathbf{C}} S + n$. By the definition of Q we have

$$\begin{aligned}
Q(\varphi(x_1, y_1), \varphi(x_2, y_2)) &= (x_1, P(T)x_2 + y_2) - (x_1, y_2) \\
&= (P(T)x_1, x_2).
\end{aligned}$$

So we have $m^+(Q) = m^+(P(T)|_S)$. By the definition of $C(\gamma(T); R, \mathbf{C}^{2n})$ we have

$$\begin{aligned}
C(\gamma(T); R, \mathbf{C}^{2n}) &= m^-(Q) + \dim_{\mathbf{C}} \ker Q - \dim_{\mathbf{C}}(\mathrm{Gr}(\gamma(T)) \cap W(R)) \\
&= \dim_{\mathbf{C}} \mathcal{N} - m^+(Q) - n \\
&= \dim_{\mathbf{C}} S - m^+(P(T)|_S).
\end{aligned}$$

By Proposition 4.1 we have

$$\begin{aligned}
i_{W(R)}(\gamma) &= i_{W(\mathbf{C}^{2n})}(\gamma) - C(\gamma(T); R, \mathbf{C}^{2n}) - \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap (\mathbf{C}^{2n})^b) + \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R^b) \\
&= 0 - (\dim_{\mathbf{C}} S - m^+(P(T)|_S)) - 0 + \dim_{\mathbf{C}} S \\
&= m^+(P(T)|_S).
\end{aligned}$$

Now we consider the general case. Define $M_t(s) = \begin{pmatrix} I & 0 \\ sP(t) & 0 \end{pmatrix}$ for all $s \in [0,1]$ and $t \in [0,T]$. Then $M_t(s) \in \mathrm{Sp}(2n, \mathbf{C})$, and we have

$$\begin{aligned} i_{W(R)}(\gamma) &= i_{W(R)}(M_T) - i_{W(R)}(M_0) \\ &= m^+(P(T)|_S) - m^+(P(0)|_S). \end{aligned}$$

Q.E.D.

Now we prove Theorem 1.3. Let a, $p_1^{'}$, $q_1^{'}$, $r_1^{'}$, $b_1^{'}$, $R^{'}$ be as in §1. Let $C_1^{'}$ be the corresponding bounded operator of C_1. The following lemma follows from direct calculation.

Lemma 4.4 *We have*

$$\begin{aligned} \begin{pmatrix} p_1^{'} & q_1^{'} \\ q_1^{'*} & r_1^{'} \end{pmatrix} &= \begin{pmatrix} a^* & 0 \\ \dot{a}^* & a^* \end{pmatrix} \begin{pmatrix} p_1 & q_1 \\ q_1^* & r_1 \end{pmatrix} \begin{pmatrix} a & \dot{a} \\ 0 & a \end{pmatrix}, & (60) \\ b_1^{'} &= \mathrm{diag}(a^{-1}, a^*) b_1 \mathrm{diag}(a^{*-1}, a) + \begin{pmatrix} 0 & -a^{-1}\dot{a} \\ -\dot{a}^* a^{*-1} & 0 \end{pmatrix}, & (61) \\ A_{W(R^{'})} - C_1^{'} &= \mathrm{diag}(a^{-1}, a^*)(A_{W(R)} - C_1)\mathrm{diag}(a^{*-1}, a), & (62) \\ a^{-1} K_R a &= K_{R^{'}} + a^{-1}\dot{a}. & (63) \end{aligned}$$

Q.E.D.

By Corollary 3.4 we have

Corollary 4.1 *We have*

$$\gamma_1^{'} = \mathrm{diag}(a^*, a^{-1})\gamma_1 \mathrm{diag}(a(0)^{*-1}, a(0)). \quad (64)$$

Lemma 4.5 *Let $a \in C([0,T], \mathrm{Gl}(n, \mathbf{C}))$. Set $\gamma = \mathrm{diag}(a^*, a^{-1})$. Then $\gamma \in C([0,T], \mathrm{Sp}(2n, \mathbf{C}))$, and we have*

$$\begin{aligned} i_{W(R)}(\gamma) &= \dim_{\mathbf{C}}(\mathrm{Gr}(a(0)^{-1}) \cap R) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^{-1}) \cap R) \\ &= \dim_{\mathbf{C}}(\mathrm{Gr}(a(0)^*) \cap R^b) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^*) \cap R^b). \quad (65) \end{aligned}$$

Proof. Clearly we have $\gamma \in C([0,T], \mathrm{Sp}(2n, \mathbf{C}))$. We divide the proof into three steps.
Step 1. $a(0) = I$ and $a \in C^1([0,T], \mathrm{Gl}(n, \mathbf{C}))$.
In this case, set

$$c = \begin{pmatrix} 0 & -a^{-1}\dot{a} \\ -\dot{a}^* a^{*-1} & 0 \end{pmatrix}.$$

Define $C \in \mathcal{B}(X)$ by $(Cu)(t) = c(t)u(t)$. Then γ, a^{-1} and a^* are the fundamental solutions of $\dot{u} = Jcu$, $\dot{x} = -a^{-1}\dot{a}x$ and $\dot{x} = \dot{a}^* a^{*-1} x$ respectively. Since $K_R^* = -K_{R^b}$, we have

$$\begin{aligned} \dim_{\mathbf{C}} \ker(K_R + a^{-1}\dot{a}) &= \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^{-1}) \cap R), \\ \dim_{\mathbf{C}} \ker(K_R^* + \dot{a}^* a^{*-1}) &= \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^*) \cap R^b). \end{aligned}$$

By Proposition 3.2 and Lemma 2.7, we have

$$\begin{aligned}
i_{W(R)}(\gamma) &= I(A_{W(R)}, A_{W(R)} - C) \\
&= -\mathrm{sf}\{A_{W(R)} - sC; 0 \leq s \leq 1\} \\
&= -\mathrm{sf}\{\begin{pmatrix} 0 & -K_R - sa^{-1}\dot{a} \\ -K_R^* - s\dot{a}^*a^{*-1} & 0 \end{pmatrix}; 0 \leq s \leq 1\} \\
&= \dim_{\mathbf{C}} \ker(K_R) - \dim_{\mathbf{C}} \ker(K_R + a^{-1}\dot{a}) \\
&= \dim_{\mathbf{C}} \ker(K_R^*) - \dim_{\mathbf{C}} \ker(K_R^* + \dot{a}^*a^{*-1}) \\
&= \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^{-1}) \cap R) \\
&= \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R^b) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^*) \cap R^b).
\end{aligned}$$

Step 2. $a(0) = I$.

Since $\mathrm{Gl}(n, \mathbf{C})$ is a connected Lie group, there exists $H_s(t) \in \mathrm{Gl}(n, \mathbf{C})$ such that $H_0(t) = a(t)$, H_1 is smooth, $H_s(0) = I$, and $H_s(T) = a(T)$ for all $s \in [0, 1]$, $t \in [0, T]$. By Step 1 we have

$$\begin{aligned}
i_{W(R)}(\gamma) &= i_{W(R)}(\mathrm{diag}(H_1^*, H_1^{-1})) \\
&= \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^{-1}) \cap R) \\
&= \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R^b) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^*) \cap R^b).
\end{aligned}$$

Step 3. General case.

Since $\mathrm{Gl}(n, \mathbf{C})$ is a connected Lie group, there exists $\alpha \in C([0, T], \mathrm{Gl}(n, \mathbf{C}))$ such that $\alpha(0) = I$, $\alpha(T) = a(0)$. By Step 2 we have

$$\begin{aligned}
i_{W(R)}(\gamma) &= (i_{W(R)}(\mathrm{diag}(\alpha^*, \alpha^{-1})) + i_{W(R)}(\gamma)) - i_{W(R)}(\mathrm{diag}(\alpha^*, \alpha^{-1})) \\
&= (\dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^{-1}) \cap R)) \\
&\quad -(\dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(0)^{-1}) \cap R)) \\
&= \dim_{\mathbf{C}}(\mathrm{Gr}(a(0)^{-1}) \cap R) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^{-1}) \cap R) \\
&= \dim_{\mathbf{C}}(\mathrm{Gr}(a(0)) \cap R^b) - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^*) \cap R^b).
\end{aligned}$$

Q.E.D.

Proof of Theorem 1.3. By Corollary 4.1, Lemma 3.3 and Lemma 4.5, we have

$$\begin{aligned}
i_{W(R')}(\gamma_1') &= i_{W(R')}(\mathrm{diag}(a^*, a^{-1})\gamma_1 \mathrm{diag}(a(0)^{*-1}, a(0))) \\
&= i_{W(R)}(\gamma_1) + i_{W(R')}(\mathrm{diag}(a^*, a^{-1})\mathrm{diag}(a(0)^{*-1}, a(0))^{-1}) \\
&= i_{W(R)}(\gamma_1) + \dim_{\mathbf{C}}(\mathrm{Gr}(a(0)^{-1}a(0)) \cap R') - \dim_{\mathbf{C}}(\mathrm{Gr}(a(T)^{-1}a(0)) \cap R') \\
&= i_{W(R)}(\gamma_1) + \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R') - \dim_{\mathbf{C}}(\mathrm{Gr}(I) \cap R).
\end{aligned}$$

Q.E.D.

Acknowledgements This work was done when the author visited Professor Tian Gang at MIT. The author is most grateful to Professor Liu Chun-gen, Professor Tian Gang and Professor Zhang Weiping for simulating discussions and comments, and MIT for the hospitality and nice research air there.

References

[1] A. A. Agrachev, A. V. Sarychev, Abnormal sub-Riemannian geodesics: Morse index and rigidity. *Ann. Inst. Henri Poincaré, Analyse non lineairé.* 13 (1996). 635-690.

[2] W. Ambrose, The index theorem in Riemannian geometry. *Ann. of Math.* 73 (1961). 49-86.

[3] V.I. Arnol'd, Characteristic class entering quantization conditions. *Funkts. Anal. Priloch.* 1 (1967). 1-14 (Russian). Funct. Anal. Appl. 1 (1967). 1-13 (English transl.).

[4] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Spectral asymmetry and Riemannian geometry. III. *Proc. Camb. Phic. Soc.* 79 (1976). 71-99.

[5] B. Booss and K. Furutani, The Maslov index – a functional analytical definition and the spectral flow formula, *Tokyo J. Math.* 21 (1998). 1–34.

[6] B. Booss and C. Zhu, The general spectral flow formula. In preparation.

[7] S. E. Cappell, R. Lee, and E. Y. Miller, On the Maslov index. *Comm. Pure Appl. Math.* 47. (1994). 121-186.

[8] X. Dai and W. Zhang, Splitting of the familly index. *Comm. Math. Phys.* 182 (1996). 303-317.

[9] X. Dai and W. Zhang, Higher spectral flow. *J. Funct. Analysis.* 157 (1998). 432-469.

[10] J. J. Duistermaat, *On the Morse index in variational calculus.* Adv. Math. 21. (1976). 173-195.

[11] L. Hörmander, Fourier integral Operators I. *Acta Math.* 127(1971). 79-183.

[12] T. Kato, Pertubation Theory for Linear Operators. Springer-Verlag. Berlin. 1980.

[13] Y. Long, Bott formula of the Maslov-type index theory. *Pacific J. Math.* 187 (1999). 113-149.

[14] R. B. Melrose and P. Piazza, Families of Dirac operators, boundaries and the b-calculus. *J. Diff. Geom.* 46 (1997). 99-180.

[15] Mercuri, Francesco; Piccione, Paolo; Tausk, Daniel V. Stability of the conjugate index, degenerate conjugate points and the Maslov index in semi-Riemannian geometry. Pacific J. Math. 206 (2002), no. 2, 375–400.

[16] Y. Long and C. Zhu, Maslov-type index theory for symplectic paths and spectral flow (II). *Chinese Ann. of Math.* 21B:1 (2000). 89-108.

[17] M. Morse, The Calculus of Variations in the Large. A.M.S. Coll. Publ., Vol.18, Amer. Math. Soc., New York, 1934.

[18] P. Piccione and D. V. Tausk, The Maslov index and a generalized Morse index theorem for non-positive definite metrics. *C. R. Acad. Sci. Paris Sér. I Math.* 331(2000). 385-389.

[19] P. Piccione and D. V. Tausk, The Morse index theorem in semi-Riemannian geometry. Topology 41 (2002), no. 6, 1123–1159.

[20] Piccione, Paolo; Tausk, Daniel V. An index theory for paths that are solutions of a class of strongly indefinite variational problems. Calc. Var. Partial Differential Equations 15 (2002), no. 4, 529–551.

[21] J. Robbin and D. Salamon, The spectral flow and the Maslov index. *Bull. London Math. Soc.* 27 (1995). 1–33.

[22] S. Smale, On the Morse index theorem. *J. Math. Mech.* 14(1965). 1049-1056.

[23] K. Uhlenbeck, The Morse index theorem in Hilbert space. *J. Diff.Geom.* 8 (1973). 555-564.

[24] B. Wilking, Index parity of closed geodesics and rigidty of Hopf fibrations. *Invent. math.* 2001. DOI 10.1007/s002220100123.

[25] C. Zhu, Maslov-type index theory and closed characteristic on compact convex hypersurfaces in $\mathbf{R}^{2n}$. Ph. D. Thesis. Nankai Institue of Mathematics.

[26] C. Zhu and Y. Long, Maslov-type index theory for symplectic paths and spectral flow (I). *Chinese Ann. of Math.* 20B:4 (1999). 413-424.