Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Dynamical scaling in Smoluchowski's coagulation equations: uniform convergence

by

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Preprint no.: 63 2003

Dynamical scaling in Smoluchowski's coagulation equations: uniform convergence

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June 2003

Abstract

We consider the approach to self-similarity (or dynamical scaling) in Smoluchowski's coagulation equations for the solvable kernels $K(x,y) = 2, x+y$ and xy. We prove the uniform convergence of densities to the self-similar solution with exponential tails under the regularity hypothesis that a suitable moment have an integrable Fourier transform. For the discrete equations we prove uniform convergence under optimal moment hypotheses. Our results are completely analogous to classical local convergence theorems for the normal law in probability theory. The proofs are simple and rely on the Fourier inversion formula and the solution by the method of characteristics for the Laplace transform.

1 Introduction

Smoluchowski's coagulation equation

$$\partial_t n(t,x) = \frac{1}{2}\int_0^x K(x-y,y)n(t,x-y)n(t,y)dy - \int_0^\infty K(x,y)n(t,x)n(t,y)dy, \tag{1.1}$$

is a widely studied mean-field model for cluster growth [4, 8, 17]. We study the evolution of $n(t,x)$, the number of clusters of mass x per unit volume at time t, which coalesce by binary collisions with a symmetric rate kernel $K(x,y)$. Equation (1.1) has been used as a model of cluster growth in a

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surprisingly diverse range of fields such as physical chemistry, astrophysics, and population dynamics (see [4] for a review of applications). In addition, over the past few years a rich mathematical theory has been developed for these equations. (Aldous [1] provides an excellent introduction.)

Most kernels in applications are homogeneous, that is $K(\alpha x, \alpha y) = \alpha^\gamma K(x,y)$, $x, y, \alpha > 0$, for some exponent γ [4]. A mathematical problem of scientific interest is to study self-similar or dynamical scaling behavior for homogeneous kernels. There are no general mathematical results for this problem despite an extensive scientific literature (especially formal asymptotics and numerics [13, 16]). It is known that γ plays a crucial role. On physical grounds, we expect solutions to (1.1) to conserve total mass $\int_0^\infty xn(t,x)dx$. For $\gamma \leq 1$, mass-conserving solutions exist globally in time under suitable moment hypotheses [5]. It is then typical in applications to assert that the solutions approach "scaling form" [16], but there is no rigorous justfication for this in general. For $\gamma > 1$, there is no solution that preserves mass for all time. This breakdown phenomenon is known as gelation. It was first demonstrated by McLeod with an explicit solution to the kernel $K = xy$. The general result using only the growth of the kernel was proved probabilistically by Jeon [10] (see also [6] for a simple analytical proof). It is natural to ask whether the blow-up is self-similar, but there are no general results on this problem yet.

There are a number of results, however, for the solvable kernels $K = 2$, $x + y$ and xy (see [14] and references therein). It is quite remarkable that two central theorems in probability– the Lévy-Khintchine characterization of infinitely divisible distributions, and the characterization of the (weak) domains of attraction of stable laws –have exact analogues for Smoluchowski's coagulation equations [2, 14]. These results suggest that a variety of limit theorems for Smoluchowski's equation can be proved in a simple manner by exploiting the probabilistic analogy. They also suggest that the analytical methods used to prove these classical theorems in probability, actually apply to a wider range of problems involving scaling phenomenon for integral equations of convolution type. In [14] we proved necessary and sufficient conditions for convergence in distribution to self-similar solutions for these kernels. The main tools were the solution for the Laplace transform of the number density n and a fundamental rigidity lemma for scaling limits in terms of functions of regular variation.

Under stronger regularity hypotheses on the initial data, these weak convergence results can be strengthened to uniform convergence of densities using the Fourier inversion formula. Kreer and Penrose [12] follow this approach for the constant kernel; also see [3]. In this article, for $K = 2$

and $x+y$ we present uniform convergence theorems to the self-similar solutions with exponential tails for the continuous and discrete Smoluchowski equations. For $K=2$, this considerably strengthens the result of Kreer and Penrose. For $K=x+y$ the convergence theorem is new. For $K=xy$, we prove uniform convergence of densities to self-similar form as t approaches the gelation time T_{gel}. The task is simplified by a well-known change of variables that reduces the problem to a study of $K=x+y$ [4]. Uniform convergence to the self-similar solutions with "fat" or "heavy" tails is more delicate, and will be considered separately.

Use of the Fourier inversion formula is classical in probability theory. It is used by Feller to prove uniform convergence of densities in the normal law [7, XV.5.2]. Feller's argument is simple and robust, and can be easily extended to the solvable kernels. Our main new idea is to use the method of characteristics in the right half plane to obtain strong decay estimates on the Fourier transform.

In addition to simplicity, the proofs reveal the role of regularity of initial data in the uniform convergence of densities. Equation (1.1) is hyperbolic and it is easy to see that discontinuities in the initial data persist for all finite times. On the other hand, the self-similar solutions are analytic. Thus, one expects some regularity on the initial data is necessary to obtain uniform convergence to a self-similar solution. Loosely speaking, regularity of the initial data $n_0(x)$ translates into a decay hypothesis on its Fourier transform. We need only the weak decay implied by integrability.

The uniform convergence theorem for the continuous Smoluchowski equation with kernels $K(x,y)=2$, $x+y$ and xy, corresponding to $\gamma=0,1,2$ respectively, may be stated in a unified manner as follows. Presuming the γ-th and $(\gamma+1)$-st moments are finite, we may scale so both moments are 1. Let $T_\gamma=\infty$ for $\gamma=0,1$, $T_\gamma=T_{\text{gel}}=1$ for $\gamma=2$. The self-similar solutions with exponential tails are explicitly described by [14]

$$n(t,x)=m_\gamma\lambda_\gamma^{-\gamma-1}n_{*,\gamma}(x\lambda_\gamma^{-1}), \tag{1.2}$$

where

$$m_0(t)=t^{-1}, \quad m_1(t)=1, \quad m_2(t)=(1-t)^{-1}, \tag{1.3}$$

$$\lambda_0(t)=t, \quad \lambda_1(t)=e^{2t}, \quad \lambda_2(t)=(1-t)^{-2}, \tag{1.4}$$

and

$$n_{*,0}(x)=e^{-x}, \quad xn_{*,1}(x)=x^2n_{*,2}(x)=\frac{1}{\sqrt{4\pi}}x^{-1/2}e^{-x/4}. \tag{1.5}$$

Theorem 1.1. *Let* $n_0 \geq 0$, $\int_0^\infty x^\gamma n_0(x)dx = \int_0^\infty x^{1+\gamma} n_0(x)dx = 1$, *and* $x^{1+\gamma} n_0 \in A(\mathbb{R})$. *Then the solution to Smoluchowski's equation with* $K = 2$, $x + y$ *or* xy *(for* $\gamma = 0, 1, 2$ *resp.) with initial data* $n(0, x) = n_0(x)$ *satisfies*

$$\lim_{t \to T_\gamma} \sup_{\hat{x}>0} \hat{x}^{1+\gamma} |m_\gamma^{-1} \lambda_\gamma^{1+\gamma} n(t, \hat{x}\lambda_\gamma) - n_{*,\gamma}(\hat{x})| = 0.$$

Here $A(\mathbb{R})$ is the Wiener algebra of functions with integrable Fourier transform [11]. It is classical that this is the optimal hypothesis for uniform convergence to the normal law [7]. However, we do not know whether this is the optimal hypothesis for the Smoluchowski equation.

We digress briefly into some issues of analysis that we do not consider in the rest of the paper. It is known that functions in $A(\mathbb{R})$ possess some delicate regularity properties. For example, a function in $A(\mathbb{R})$ has a logarithmic modulus of continuity in a neighborhood where it is monotonic. It is definitely not obvious whether this regularity is truly necessary to obtain uniform convergence. If $v_0(ik) = \int_0^\infty e^{-ikx} x^{1+\gamma} n_0(x)dx$ it also follows that $v_0 \in H^1(\mathbb{R}) \cap A(\mathbb{R})$, since v_0 is the boundary limit of an analytic function (the Laplace transform of $x^{1+\gamma} n_0$). Here H^1 denotes the classical Hardy space. This is turn means that v_0 has some hidden regularity and integrability properties. It is worth remarking that the precise characterization of $A(\mathbb{R})$ remains an outstanding open problem in harmonic analysis, though several sufficient conditions are known (see [11]).

It has been traditional to treat the discrete Smoluchowski equations separately from the continuous equations. Yet, within the framework of measure valued solutions [14, 15], the discrete Smoluchowski equations simply correspond to the special case of a lattice distribution, a measure valued solution supported on the lattice $h\mathbb{N}$ and taking the form $\nu_t = \sum_{l=1}^\infty n_l(t)\delta_{hl}(x)$, where $\delta_{hl}(x)$ is a Dirac delta at hl. If h is maximal we call ν_t a lattice measure with *span* h. The coefficients n_l satisfy the discrete Smoluchowski equations

$$\partial_t n_l(t) = \frac{1}{2} \sum_{j=1}^{l-1} \kappa_{l-j,j} n_{l-j}(t) n_j(t) - \sum_{j=1}^{\infty} \kappa_{l,j} n_l(t) n_j(t), \tag{1.6}$$

where $\kappa_{l,j} = K(lh, jh)$. Physically, this case is of importance, since some mass aggregation processes (e.g., polymerization) have a fundamental unit of mass (e.g., a monomer). The uniform convergence theorem for the continuous Smoluchowski equations has a natural extension to this case. Again, our analysis mimics Feller's [7] treatment of convergence of lattice distributions to the normal law, and the proof follows easily after the uniform convergence theorem. The hypotheses are simpler — we only need the optimal moment hypotheses.

Theorem 1.2. *Let $\nu_0 \geq 0$ be a lattice measure with span h such that $\int_0^\infty x^\gamma \nu_0(dx) = \int_0^\infty x^{1+\gamma}\nu_0(dx) = 1$. Then with $\hat{l} = lh\lambda_\gamma^{-1}$ and $\hat{n}_l(t) = h^{-1}m_\gamma^{-1}\lambda_\gamma^{1+\gamma}n_l(t)$ we have*

$$\lim_{t\to T_\gamma} \sup_{l\in\mathbb{N}} \hat{l}^{1+\gamma}\left|\hat{n}_l(t) - n_{*,\gamma}(\hat{l})\right| = 0.$$

2 Uniform convergence of densities for the constant kernel $K = 2$

2.1 Evolution equations and scaling

Let $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$. For $s \in \mathbb{C}_+$ we let

$$\tilde{u}(t,s) = \int_0^\infty e^{-sx} n(t,x)\, dx$$

denote the Laplace transform of the number density n. We take the Laplace transform of (1.1) with $K = 2$, and its limit as $s \to 0$ to see that $\tilde{u}(t,s)$ solves

$$\partial_t \tilde{u} = \tilde{u}^2 - 2\tilde{u}(t,0)\tilde{u}, \quad \partial_t \tilde{u}(t,0) = -\tilde{u}(t,0)^2. \tag{2.1}$$

There is no loss of generality in supposing that the initial time $t = 1$, and we assume that the initial data $n_0(x) = n(1,x)$ satisfy $\int_0^\infty n_0(x)dx = \int_0^\infty x n_0(x)dx = 1$, whence $\tilde{u}(t,0) = m_0(t) = t^{-1}$.

Under these assumptions, the weak convergence result of [1, 14] implies that

$$\lim_{t\to\infty} t\tilde{u}(t, st^{-1}) = u_*(s) := \frac{1}{1+s}, \quad s \in \mathbb{C}_+. \tag{2.2}$$

(We use the classical equivalence between pointwise convergence of the Fourier transform and weak convergence of probability measures [7].) Here $u_*(s)$ is the Laplace transform of $n_*(x) = e^{-x}$. In the present situation, (2.2) is easy to verify from the explicit solution formula

$$\tilde{u}(t,s) = \frac{1}{t}\frac{\tilde{u}_0(s)}{t(1-\tilde{u}_0(s)) + \tilde{u}_0(s)}, \tag{2.3}$$

since $\tilde{u}_0(0) = -\partial_s \tilde{u}_0(0) = 1$.

Notice that u_* is not absolutely integrable on the imaginary axis, since $|u_*(ik)| \sim |k|^{-1}$ as $|k| \to \infty$. The weak decay of the Fourier transform is caused by the jump discontinuity at $x = 0$, since $n_*(x) = 0$ for $x < 0$. In order to obtain a uniform convergence result, we must smooth this discontinuity. Thus, we consider the mass density xn which has Laplace transform $\tilde{v}(s) = -\partial_s \tilde{u}$. In particular, the self-similar solution satisfies

$$xn_*(x) = xe^{-x}, \quad v_*(s) = \frac{1}{(1+s)^2}, \quad |v_*(ik)| = \frac{1}{1+k^2}, k \in \mathbb{R}. \tag{2.4}$$

Henceforth, we switch to self-similar variables. Let

$$\tau = \log t, \quad u(\tau, s) = e^\tau \tilde{u}(e^\tau, se^{-\tau}) = t\tilde{u}(t, s/t). \tag{2.5}$$

In real space, this corresponds to the mass-preserving rescaling

$$\hat{x} = xt^{-1} = xe^{-\tau}, \quad \hat{n}(\tau, \hat{x}) = e^{2\tau} n(e^\tau, e^\tau \hat{x}) = t^2 n(t, x). \tag{2.6}$$

In these variables, the transforms become

$$u(\tau, s) = \int_0^\infty e^{-s\hat{x}} \hat{n}(\tau, \hat{x})\, d\hat{x}, \quad v(\tau, s) = \int_0^\infty e^{-s\hat{x}} \hat{x}\hat{n}(\tau, \hat{x})\, d\hat{x}.$$

We will use the Fourier inversion formula for integrable $v(t, ik)$,

$$\hat{x}n(\tau, \hat{x}) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik\hat{x}} v(\tau, ik)\, dk. \tag{2.7}$$

In self-similar variables the equation of evolution for u is

$$\partial_\tau u + s\partial_s u = -u(1-u). \tag{2.8}$$

Equation (2.8) may be solved by the method of characteristics. A characteristic originating at s_0 is denoted $s(\tau; s_0)$ and solves

$$\frac{ds}{d\tau} = s, \quad s(0; s_0) = s_0, \quad \text{hence } s(\tau; s_0) = e^\tau s_0. \tag{2.9}$$

The geometry of characteristics is particularly simple: they are rays emanating from the origin. In particular, the imaginary axis is invariant under the flow of (2.9). Along characteristics we have

$$\frac{du}{d\tau} = -u(1-u), \quad \text{whence } u(\tau, s) = \frac{u_0(s_0)e^{-\tau}}{(1 - u_0(s_0)(1 - e^{-\tau}))}. \tag{2.10}$$

We are interested in the growth of the derivative $v = -\partial_s u$. Differentiating equation (2.8), we see that on characteristics the derivative solves

$$\frac{dv}{d\tau} = -2(1-u)v, \quad \text{thus } v(\tau, s) = \frac{v_0(s_0)e^{-2\tau}}{(1 - u_0(s_0)(1 - e^{-\tau}))^2}. \tag{2.11}$$

It is easy to see that $|u|$ and $|v|$ decay along characteristics. Indeed, we have from (2.10) and (2.11) that

$$|u(\tau,s)| \le \frac{|u_0(s_0)|e^{-\tau}}{1-|u_0(s_0)|(1-e^{-\tau})}, \tag{2.12}$$

and

$$|v(\tau,s)| \le \frac{|v_0(s_0)|e^{-2\tau}}{\left(1-|u_0(s_0)|(1-e^{-\tau})\right)^2} \le \frac{|v_0(s_0)|e^{-2\tau}}{\left(1-|u_0(s_0)|\right)^2}. \tag{2.13}$$

2.2 The main theorem

Theorem 2.1. *Suppose* $n_0(x) \ge 0$, $\int n_0(x)dx = \int_0^\infty x n_0(x)dx = 1$, *and* $xn_0 \in A(\mathbb{R})$. *Then in terms of the rescaling in (2.6) we have*

$$\lim_{\tau\to\infty} \sup_{\hat{x}>0} \hat{x}|\hat{n}(\tau,\hat{x}) - n_*(\hat{x})| = 0. \tag{2.14}$$

Proof. The assumption $xn_0 \in A(\mathbb{R})$ guarantees that $\int_{\mathbb{R}} |v_0(ik)|dk < \infty$. By the Fourier inversion formula (2.7) it is sufficient to show that

$$\lim_{\tau\to\infty} \int_{\mathbb{R}} |v(\tau,ik) - v_*(ik)|dk = 0. \tag{2.15}$$

The main goal is to control the integrals over the tails, since for fixed $R > 0$, (2.2) together with the dominated convergence theorem implies

$$\lim_{\tau\to\infty} \int_{-R}^{R} |v(\tau,ik) - v_*(ik)|dk = 0.$$

We will control the tails of $v(\tau,ik)$ and v_* separately. It is sufficient to consider only $k \ge 0$, since $|v(\tau,ik)| = |v(\tau,-ik)|$. Note

$$\int_R^\infty |v(\tau,ik) - v_*(ik)|dk \le \int_R^\infty |v(\tau,ik)|dk + \int_R^\infty |v_*(ik)|dk.$$

But $|v_*(ik)| = (1+|k|^2)^{-1}$ by (2.4), so that $\int_R^\infty |v_*(ik)|dk \le R^{-1}$. Moreover, since u and v converge to u_* and v_* pointwise, we can choose T so large depending upon R that

$$\sup_{\tau\ge T} |u(\tau,iR)| \le \frac{2}{R}, \quad \sup_{\tau\ge T} |v(\tau,iR)| \le \frac{2}{R^2}. \tag{2.16}$$

The control obtained from (2.16) propagates as the characteristics flow outwards. Precisely, for any point ik with $R \le k \le e^{\tau-T}R$ choose a starting time $\tau_0(k)$ so that the preimage at time $\tau_0(k)$ of ik is iR. Explicitly,

$e^{-(\tau_0-T)}k = R$. Then using the decay estimate (2.13) on the time interval $[\tau_0(k),\tau]$ and the boundary control (2.16) we have

$$|v(\tau,ik)| \le \frac{|v(\tau_0,iR)|e^{-2(\tau_0-T)}}{(1-|u(\tau_0,iR)|)^2} \le CR^{-2}\left(\frac{R}{k}\right)^2 = Ck^{-2}.$$

Integrating this estimate over the transition region we have

$$\int_R^{Re^{\tau-T}} |v(\tau,ik)|dk \le C\int_R^\infty k^{-2}dk = CR^{-1}.$$

We now consider the tail region $k \ge Re^{(\tau-T)} = \tilde{R}e^\tau$. We use (2.13) again on the time interval $[0,\tau]$ to obtain

$$\int_{\tilde{R}e^\tau}^\infty |v(\tau,ik)|dk \le e^{-2\tau}\int_{\tilde{R}e^\tau}^\infty \frac{|v_0(ike^{-\tau})|}{(1-|u_0(ike^{-\tau})|)^2}dk$$
$$= e^{-\tau}\int_{\tilde{R}}^\infty \frac{|v_0(ik)|}{(1-|u_0(ik)|)^2}dk \le \left(\sup_{|k|\ge\tilde{R}} \frac{1}{(1-|u_0(ik)|)^2}\right)e^{-\tau}\|v_0\|_{L^1}.$$

Since $|u_0(ik)| < 1$ for $k \neq 0$, and $\lim_{|k|\to\infty} u_0(ik) = 0$ by the Riemann-Lebesgue lemma, $\sup_{|k|\ge\tilde{R}}(1-|u_0(ik)|)^{-1} < \infty$. ☐

Remark 2.2. The proof of the theorem implies the stronger assertion that if $v(\tau,ik) \in L^1$ for any $\tau > 0$, then we have uniform convergence of the mass density.

2.3 The discrete Smoluchowski equations

We consider measure solutions of the form $\nu_t = \sum_{l=1}^\infty n_l(t)\delta_{hl}(x)$, where $\delta_{hl}(x)$ denotes a Dirac mass at hl. To avoid redundancy, we always assume that h is the *span* of the lattice, that is the maximal $h > 0$ so that all initial clusters, and thus clusters at any time $t > 0$, are concentrated on $h\mathbb{N}$. We will call ν_t a lattice measure with span h. Notice that if the initial number of clusters and mass are finite, by rescaling n_l and h we may assume that $\int_0^\infty \nu_0(dx) = \int_0^\infty x\nu_0(dx) = 1$. Under these conditions, the weak convergence theorem of [14] asserts that $\lim_{t\to\infty} t\tilde{u}(t,s/t) = u_*(s)$. We show that this theorem may be strengthened by use of Fourier series. The Fourier transform of ν_t is the Fourier series

$$\tilde{u}(t,ik) = \sum_{l\in\mathbb{N}} n_l(t)e^{-ilhk},$$

which has minimal period $2\pi/h$. Thus, $n_l(t) = (h/2\pi)\int_{-\pi/h}^{\pi/h} e^{ilhk}\tilde{u}(t,ik)\,dk$, or

$$t^2 n_l(t) = \frac{h}{2\pi}\int_{-\pi e^\tau/h}^{\pi e^\tau/h} \exp(ilhke^{-\tau})u(\tau,ik)\,dk, \tag{2.17}$$

in self-similar variables from (2.5). We integrate by parts and let

$$\hat{l} = lhe^{-\tau} = lht^{-1}, \quad \hat{n}_l(t) = h^{-1}t^2 n_l(t) \tag{2.18}$$

to obtain

$$\hat{l}\hat{n}_l(t) = tln_l(t) = \frac{1}{2\pi}\int_{-\pi e^\tau/h}^{\pi e^\tau/h} e^{i\hat{l}k}v(\tau,ik)\,dk. \tag{2.19}$$

As in Theorem 2.1 we expect the right hand side to converge to $\hat{l}n_*(\hat{l})$.

Theorem 2.3. *Let $\nu_0 \geq 0$ be a lattice measure with span h such that $\int_0^\infty \nu_0(dx) = \int_0^\infty x\nu_0(dx) = 1$. Then with the scaling (2.18) we have*

$$\lim_{t\to\infty}\sup_{l\in\mathbb{N}} \hat{l}\left|\hat{n}_l(t) - n_*(\hat{l})\right| = 0. \tag{2.20}$$

Proof. By (2.19) and the continuous Fourier inversion formulas it suffices to show that

$$\lim_{\tau\to\infty}\sup_{\hat{l}\geq 0}\left|\int_{-\pi e^\tau/h}^{\pi e^\tau/h} e^{i\hat{l}k}v(\tau,ik)dk - \int_{\mathbb{R}} e^{i\hat{l}k}v_*(ik)dk\right| = 0.$$

As earlier it suffices to consider $k > 0$. The integrals

$$\int_{-R}^{R}|v(\tau,ik) - v_*(ik)|dk, \qquad \int_R^{\tilde{R}e^\tau}|v(\tau,ik)|dk, \qquad \int_R^\infty |v_*(ik)|dk$$

are controlled exactly as in the proof of Theorem 2.1. It only remains to estimate the integral of $|v - v_*|$ over the tail region $\tilde{R}e^\tau < k < \pi e^\tau/h$. We assume that $\pi/h > \tilde{R}$, for otherwise there is nothing to prove. But then by the exact solution (2.9) and the uniform decay estimate (2.13) we have

$$\int_{\tilde{R}e^\tau}^{\pi e^\tau/h}|v(\tau,ik)|dk \leq e^{-\tau}\int_{\tilde{R}}^{\pi/h}\frac{|v_0(ik)|}{|1 - u_0(ik)(1-e^{-\tau})|^2}dk \leq C(\tilde{R},u_0,v_0)e^{-\tau}.$$

This estimate is true for the following reason. Since the domain of integration is finite, it suffices to show that the integrand is uniformly bounded. It is only necessary to control the denominator. Since $u_0(ik) = \sum_{l\in\mathbb{N}} n_l(0)e^{-ilkh}$ with $n_l(0) \geq 0$ we have $|u_0(ik)| \leq 1$, and

$$u_0(ik) = 1 \quad \text{if and only if} \quad k = \frac{2\pi m}{h}, m \in \mathbb{Z}.$$

In particular, we have the strict inequality

$$\min_{k\in[\tilde{R},\frac{\pi}{h}]} |1 - u_0(ik)| \geq \delta > 0.$$

Therefore,

$$\left|1 - u_0(ik)(1 - e^{-\tau})\right| \geq |1 - u_0(ik)| - |u_0(ik)|e^{-\tau} \geq \delta - e^{-\tau} \geq \frac{\delta}{2}$$

for sufficiently large τ. □

3 Uniform convergence of densities for the additive kernel

3.1 Self-similar solution and rescaling

In this section we prove the analogue of Theorem 2.1 for the additive kernel. The proof is similar to the previous section. However, the characteristics are nonlinear in this case, and this changes the analysis. The self-similar solution with exponential tail found by Golovin [9] is

$$n(t, x) = e^{-4t} n_*(xe^{-2t}) \quad \text{where } n_*(\hat{x}) = \frac{1}{\sqrt{2\pi\hat{x}^3}} e^{-\hat{x}/2}. \tag{3.1}$$

The total number of clusters, $\int_0^\infty n_*(x)dx$, is infinite. This situation may be resolved by working with the variables

$$\tilde{u}(t, \tilde{s}) = \int_0^\infty e^{-\tilde{s}x} xn(t, x)\, dx \quad \text{and} \quad \tilde{\varphi}(t, \tilde{s}) = \int_0^\infty (1 - e^{-\tilde{s}x})n(t, x)dx,$$

which are the Laplace transform of the mass density $xn(t, x)$ and a renormalised number density in unscaled variables. It is easy to check that $\tilde{\varphi}$ solves the simple equation

$$\partial_t\tilde{\varphi} - \tilde{\varphi}\partial_{\tilde{s}}\tilde{\varphi} = -\tilde{\varphi}. \tag{3.2}$$

We consider the mass-preserving rescaling

$$u(t, s) = \tilde{u}(t, e^{-2t}s), \quad \varphi(t, s) = e^{2t}\tilde{\varphi}(t, e^{-2t}s). \tag{3.3}$$

This corresponds to the mass-preserving rescaling

$$\hat{x} = xe^{-2t}, \quad \hat{n}(t,\hat{x}) = e^{4t}n(t,\hat{x}e^{2t}) = e^{4t}n(t,x). \tag{3.4}$$

It follows that u and φ are related through $u = \partial_s \varphi$, with

$$u(t,s) = \int_0^\infty e^{-s\hat{x}}\hat{x}\hat{n}(t,\hat{x})d\hat{x}, \ \varphi(t,s) = \int_0^\infty (1-e^{-s\hat{x}})\hat{n}(t,\hat{x})d\hat{x}. \tag{3.5}$$

We restrict ourselves to initial data with finite mass and second moment normalized so that $\int_0^\infty xn_0(x)dx = 1$ and $\int_0^\infty x^2 n_0(x)dx = 1$. The scaling solution with these moments is a trivial rescaling of Golovin's solution and its profile $n_{*,1}$ from (1.5) has the Laplace transform

$$u_*(s) = \frac{1}{\sqrt{1+2s}}. \tag{3.6}$$

The weak convergence theorem of [14] ensures that

$$\lim_{t\to\infty} u(t,s) = u_*(s), \quad s \in \mathbb{C}_+, \tag{3.7}$$

and the convergence is uniform on compact sets by Montel's theorem. Note that the convergence also holds on the imaginary axis, by the equivalence between weak convergence of probability measures and pointwise convergence of their Fourier transforms. On the imaginary axis, $|u_*(ik)| \sim |k|^{-1/2}$ as $|k| \to \infty$. Hence, it is not in L^1. As earlier, we resolve the situation by considering $v = -\partial_s u$.

3.2 Evolution equations and characteristics

The equations of evolution for φ and u are

$$\partial_t \varphi + (2s-\varphi)\partial_s \varphi = \varphi, \tag{3.8}$$

$$\partial_t u + (2s-\varphi)\partial_s u = -u(1-u). \tag{3.9}$$

These equations may be solved by the method of characteristics. A characteristic originating at s_0 is denoted $s(t;s_0)$ and solves

$$\frac{ds}{dt} = 2s - \varphi, \quad s(0;s_0) = s_0. \tag{3.10}$$

Along characteristics we have

$$\frac{d\varphi}{dt} = \varphi, \quad \text{and} \quad \frac{du}{dt} = -u(1-u). \tag{3.11}$$

Integrating (3.11) we have

$$\varphi(t,s) = e^t \varphi_0(s_0), \quad u(t,s) = \frac{u_0(s_0)e^{-t}}{1 - u_0(s_0)(1-e^{-t})}. \tag{3.12}$$

We are interested in the growth of the derivative $v = -\partial_s u$. Differentiating equation (3.9), and using (3.5), we obtain along characteristics

$$\frac{dv}{dt} = -3(1-u)v. \tag{3.13}$$

We substitute for u from (3.12) and integrate to obtain

$$v(t,s) = \frac{v_0(s_0)e^{-3t}}{\left(1 - u_0(s_0)(1-e^{-t})\right)^3}. \tag{3.14}$$

$|u|$ decays along characteristics as in (2.12). By (3.14), $|v|$ decays on characteristics according to

$$|v(t,s)| \leq \frac{|v_0(s_0)|e^{-3t}}{|1 - u_0(s_0)(1-e^{-t})|^3} \leq \frac{|v_0(s_0)|e^{-3t}}{(1 - |u_0(s_0)|)^3}. \tag{3.15}$$

3.3 Geometry of characteristics

Our analysis relies on the key observation that the domain of analyticity of $\varphi(t,s)$ (and hence $u(t,s)$ and $v(t,s)$) grows as t does. This may be explained using the explicit solution for characteristics.

We substitute for $\varphi(t,s)$ from (3.11) in (3.10) and integrate, to obtain

$$e^{-2t}s(t;s_0) = s_0 - \varphi_0(s_0)(1-e^{-t}). \tag{3.16}$$

We therefore define the flow map $\Phi_t : \mathbb{C}_+ \to \mathbb{C}$ by

$$\Phi_t(s_0) = s(t;s_0) = e^{2t}\left[s_0 - \varphi_0(s_0)(1-e^{-t})\right]. \tag{3.17}$$

Φ_t is analytic for $\operatorname{Re} s_0 > 0$, as we have from (3.17)

$$\frac{d\Phi_t(s_0)}{ds_0} = e^{2t}\left(1 - u_0(s_0)(1-e^{-t})\right). \tag{3.18}$$

Let D_t denote the image $\Phi_t(\mathbb{C}_+)$. We claim that D_t is strictly larger than $\mathbb{C}_+$. This follows by considering the image of the imaginary axis under Φ_t and a continuity argument. For any $t \geq 0$, let Γ_t denote the image of the imaginary axis, and let Γ_{-t} denote its preimage. By the explicit formula (3.16) we have

$$\begin{aligned} \Gamma_t &= \{z \mid z = ik - \varphi_0(ik)(1-e^{-t}), k \in \mathbb{R}\}, \\ \Gamma_{-t} &= \{z \mid ik = z - \varphi_0(z)(1-e^{-t}), k \in \mathbb{R}\}. \end{aligned}$$

Clearly, $\Phi_t(0) = 0$ so that Γ_t always includes the origin. But, if $0 \neq z \in \Gamma_t$ then $\operatorname{Re} z < 0$. Indeed, $\operatorname{Re} z = -(1-e^{-t}) \operatorname{Re} \varphi_0(ik)$ and

$$\operatorname{Re} \varphi_0(ik) = \int_0^\infty (1 - \cos kx) n_0(x) dx > 0, \quad k \neq 0,$$

since n_0 is continuous. The inverse of Φ_t is denoted by $\Phi_t^{-1} : D_t \to \mathbb{C}_+$. It is easy to see that Φ_t^{-1} is analytic in the interior of D_t, and thus from the explicit solution formulas (3.12) we see that $\varphi(t,s)$ (and hence $u(t,s)$ and $v(t,s)$) are analytic in D_t. The analyticity of $v(t,s)$ is used in the proof.

3.4 The main theorem

The proof of the main theorem is similar to the proof of Theorem 2.1, however it is more delicate to control the tails of the integrals uniformly. We will need the following uniform Riemann-Lebesgue lemma. Let us denote the positive semicircle of radius R by $C_R = \{s \in \mathbb{C}_+ \mid |s| = R\}$

Lemma 3.1. *Let $x^2 n_0(x) \in L^1$ and $v_0(s) = \int_0^\infty e^{-sx} x^2 n_0(x) dx$. Then*

$$\lim_{R\to\infty} \sup_{s \in C_R} |v_0(s)| = 0. \tag{3.19}$$

Proof. Let $\varepsilon > 0$, $s \in \mathbb{C}_+$. We choose a step function $g_\varepsilon = \sum_{k=1}^{N_\varepsilon} c_k \mathbf{1}_{[a_k,b_k]}$ so that $\|x^2 n_0 - g_\varepsilon\|_{L^1} < \varepsilon$. But then, $\|e^{-sx}(x^2 n_0 - g_\varepsilon)\|_{L^1} < \varepsilon$. Therefore,

$$|v_0(s)| \leq \varepsilon + \left| \int_0^\infty e^{-sx} g_\varepsilon(x) dx \right| = \varepsilon + \left| \sum_{k=1}^{N_\varepsilon} c_k \int_{a_k}^{b_k} e^{-sx} dx \right| \leq \varepsilon + \frac{C_\varepsilon}{|s|}.$$

□

Theorem 3.2. *Suppose $n_0(x) \geq 0$, $\int_0^\infty x n_0(x) dx = \int_0^\infty x^2 n_0(x) dx = 1$, and $x^2 n_0 \in A(\mathbb{R})$. Then in terms of the rescaling (3.4) we have*

$$\lim_{t\to\infty} \sup_{\hat{x}>0} \hat{x}^2 |\hat{n}(t,\hat{x}) - n_{*,1}(\hat{x})| = 0. \tag{3.20}$$

Proof. The assumption $x^2 n_0 \in A(\mathbb{R})$ implies that $\int_{\mathbb{R}} |v_0(ik)|\, dk < \infty$. It is sufficient to prove that

$$\lim_{t\to\infty} \sup_{x>0} \left| \int_{\mathbb{R}} e^{ikx} \left[v(t,ik) - v_*(ik) \right] \, dk \right| = 0. \tag{3.21}$$

Firstly, by the weak convergence result (3.7) and the dominated convergence theorem, for fixed $R > 0$,

$$\lim_{t\to\infty} \int_{-R}^{R} |v(t,ik) - v_*(ik)|\, dk = 0.$$

It remains to control the tails $|k| \geq R$. It suffices to consider $k > 0$ because $|v(t,ik)| = |v(t,-ik)|$. We control the tails of v and v_* separately. It follows from the explicit formula (3.6) that

$$\int_R^\infty |v_*(ik)|\, dk \leq C \int_R^\infty k^{-3/2}\, dk \leq CR^{-1/2}.$$

It again follows from formula (3.6) and the uniform convergence of v on compact subsets, that for sufficiently large T

$$\sup_{t\geq T} \max_{s\in C_R} |u(t,s)| \leq CR^{-1/2} \quad \text{and} \quad \sup_{t\geq T} \max_{s\in C_R} |v(t,s)| \leq CR^{-3/2}. \tag{3.22}$$

Let us consider the "transition zone" $R \leq k \leq Re^{2(t-T)}$. For any point ik, there exists a unique time $\tau(k) \geq T$ so that ik is the image of some point $s_0 \in C_R$ after time $\tau(k) - T$. Thus, from the explicit solution formula (3.16)

$$e^{-2(\tau(k)-T)} ik = s_0 - \varphi(T,s_0)(1 - e^{-(\tau(k)-T)}).$$

But $|\varphi(T,s_0)| \leq |s_0|$ so that

$$e^{-2(\tau(k)-T)} \leq 2\frac{|s_0|}{k} = 2\frac{R}{k}.$$

It then follows from the decay estimate (3.15) and (3.22) that

$$|v(t,ik)| \leq \frac{|v(\tau,s_0)| e^{-3(\tau-T)}}{(1-|u(\tau,s_0)|)^3} \leq Ck^{-3/2}.$$

Integrating this estimate we have

$$\int_R^{Re^{2(t-T)}} |v(t,ik)|\, dk \leq C \int_R^\infty k^{-3/2}\, dk = CR^{-1/2}.$$

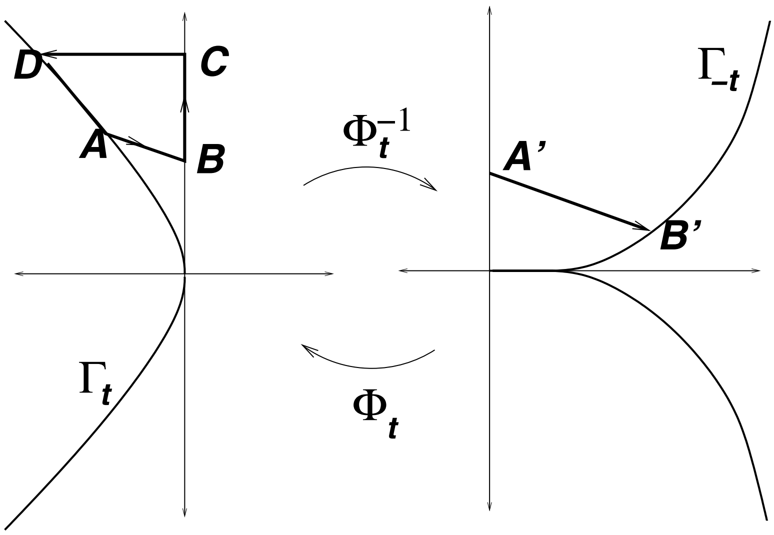

Figure 3.1: $A = \Phi_t(i\tilde{R})$, $B = i\tilde{R}e^{2t}$, $C = iR_2$, $\mathrm{Im}(D) = R_2$, $A' = i\tilde{R}$, $B' = \Phi_t^{-1}(i\tilde{R}e^{2t})$

Finally, consider the tail region $k > Re^{2(t-T)} = \tilde{R}e^{2t}$. To complete the proof of Theorem 3.2 we will use the analyticity of $v(t,s)$ in D_t and contour deformation. For large finite $R_2 < \infty$ consider the domain $ABCD$ shown in Figure 3.4. The path AB is chosen so that $A'B' = \Phi_t^{-1}(AB)$ is a straight line. CD is parallel to the real axis. Then by Cauchy's theorem

$$\begin{aligned}\left|\int_{\tilde{R}e^{2t}}^{R_2} e^{ikx} v(t,ik)\,dk\right| &= \left|\int_{BC} e^{ikx} v(t,ik)\,dk\right| \\ &= \left|\int_{DA} e^{sx} v(t,s)\,ds + \int_{AB} e^{sx} v(t,s)\,ds + \int_{CD} e^{sx} v(t,s)\,ds\right|.\end{aligned}$$

Let σ denote $\mathrm{Re}\, s$. Since $\sigma < 0$ in D_t for $s \in CD$ we see that the last integral is estimated by

$$\left|\int_{CD} e^{sx} v(t,s)\,ds\right| \le \sup_{s\in CD} |v(t,s)| \int_{-\infty}^{0} e^{\sigma x} d\sigma = \frac{\sup_{s\in CD} |v(t,s)|}{x}.$$

By the decay estimate (3.15) we have

$$\sup_{s\in CD} |v(t,s)| \le \sup_{s\in CD} \frac{|v_0(\Phi_t^{-1}(s))| e^{-3t}}{\left(1 - |u_0(\Phi_t^{-1}(s))|\right)^3}.$$

It follows from the uniform Riemann-Lebesgue lemma 3.1 that as $R_2 \to \infty$, $\sup_{s\in CD} |v_0(\Phi_t^{-1}(s)))| \to 0$. We thus let $R_2 \to \infty$ to conclude that

$$\left| \int_{\tilde{R}e^{2t}}^{\infty} e^{ikx} v(t,ik)\, dk \right| \leq \left| \int_{\Gamma_{t,A}} e^{sx} v(t,s)\, ds \right| + \left| \int_{AB} e^{sx} v(t,s)\, ds \right|. \qquad (3.23)$$

where $\Gamma_{t,A}$ denotes the path from A to ∞ on Γ_t. Notice that (3.23) holds independent of x. The virtue of deforming the contour is that the integrals are now easy to estimate. We first use the exact solution (3.14) and then a change of variables with (3.18) to obtain

$$\begin{aligned} \int_{\Gamma_{t,A}} e^{sx} v(t,s)\, ds &= \int_{\Gamma_{t,A}} e^{sx} \frac{e^{-3t} v_0(\Phi_t^{-1}(s))}{\left(1 - u_0(\Phi_t^{-1}(s)(1-e^{-t}))\right)^3}\, ds \\ &= e^{-t} \int_{\tilde{R}}^{\infty} e^{\Phi_t(ik)x} \frac{v_0(ik)}{(1-u_0(ik)(1-e^{-t}))^2}\, dk. \end{aligned}$$

Since $\operatorname{Re} \Phi_t(ik)x \leq 0$, this yields the estimate

$$\left| \int_{\Gamma_{t,A}} e^{sx} v(t,s)\, ds \right| \leq e^{-t} \|v_0\|_{L^1} \sup_{|k| \geq \tilde{R}} \left| 1 - u_0(ik)(1-e^{-t}) \right|^{-2}.$$

Similarly, we have by (3.14) and (3.18)

$$\begin{aligned} \left| \int_{AB} e^{sx} v(t,s)\, ds \right| &= e^{-t} \left| \int_{A'B'} e^{\Phi_t(s_0)x} \frac{v_0(s_0)}{(1-u_0(s_0)(1-e^{-t}))^2}\, ds_0 \right| \\ &\leq e^{-t} |A'B'| \sup_{s_0 \in A'B'} \left| 1 - u_0(s_0)(1-e^{-t}) \right|^{-2}. \end{aligned}$$

Now, the point $A' = i\tilde{R}$ is independent of t. It also follows from (3.17) that $B' = \Phi_t^{-1}(i\tilde{R}e^{2t})$ converges to the point s_0 that solves $i\tilde{R} = s_0 - \varphi_0(s_0)$. Thus, we have the exponential decay estimate $|\int_{AB} e^{sx} v(t,s)\, ds| \leq Ce^{-t}$ independent of x. □

3.5 The discrete Smoluchowski equations

With the proof of Theorem 3.2 in hand, it is easy to obtain a uniform convergence theorem for the discrete Smoluchowski equations with additive kernel. Moreover, it is easy to obtain uniform control over the tail region, without the contour deformation argument.

Let $\nu_t = \sum_{l=1}^{\infty} n_l(t)\delta_{hl}(x)$ denote a measure-valued solution to (1.1). We first adapt the inversion formula to self-similar variables. In unscaled variables we have

$$\begin{aligned}
\tilde{v}(t, i\tilde{k}) := -\partial_{\tilde{s}}\tilde{u}(t, i\tilde{k}) &= \int_0^{\infty} e^{-i\tilde{k}x}x^2\nu_t(dx) = \sum_{l=1}^{\infty} h^2 l^2 n_l(t)e^{-i\tilde{k}hl}, \\
h^2 l^2 n_l(t) &= \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ilh\tilde{k}}\tilde{v}(t, i\tilde{k})d\tilde{k},
\end{aligned}$$

Let

$$\hat{l} = lhe^{-2t}, \quad \hat{n}_l(t) = h^{-1}e^{4t}n_l(t). \tag{3.24}$$

Then we can rewrite the above inversion formula using self-similar variables $k = \tilde{k}e^{2t}$ and $v(t, ik) = e^{-2t}\tilde{v}(t, i\tilde{k})$ as

$$\hat{l}^2\hat{n}_l(t) = h^{-1}e^{4t}\hat{l}^2 n_l(t) = \frac{1}{2\pi}\int_{-\pi e^{2t}/h}^{\pi e^{2t}/h} e^{i\hat{l}k}v(t, ik)\, dk. \tag{3.25}$$

Theorem 3.3. *Let $\nu_0 \geq 0$ be a lattice measure with span h such that $\int_0^{\infty} x\nu_0(dx) = \int_0^{\infty} x^2\nu_0(dx) = 1$. Then with the scaling (3.24) we have*

$$\lim_{t\to\infty}\sup_{l\in\mathbb{N}} \hat{l}^2\left|\hat{n}_l(t) - n_{*,1}(\hat{l})\right| = 0.$$

Proof. By (3.25) and the continuous Fourier inversion formulas it suffices to show that

$$\lim_{t\to\infty}\sup_{\hat{l}\geq 0}\left|\int_{-\pi e^{2t}/h}^{\pi e^{2t}/h} e^{i\hat{l}hk}v(t, ik)\, dk - \int_{\mathbb{R}} e^{i\hat{l}hk}v_*(ik)\, dk\right| = 0.$$

The integral over $[-R, R]$ and the transition zone $R < |k| < \tilde{R}e^{2t}$ is controlled as in the proof of Theorem 3.2, and it only remains to control the integral of $|v(t, ik)|$ over the tail region $\tilde{R}e^{2t} < k < \pi e^{2t}/h$. This is considerably simpler than in the previous proof. We use the exact solution (3.14) and change variables using (3.18) to obtain

$$\begin{aligned}
\int_{\tilde{R}e^{2t}}^{\pi e^{2t}/h} e^{ikx}v(t, ik)\, dk &= e^{-3t}\int_{\tilde{R}e^{2t}}^{\pi e^{2t}/h} \frac{e^{ikx}v_0(\Phi_t^{-1}(ik))}{\left(1 - u_0(\Phi_t^{-1}(ik))(1 - e^{-t})\right)^3}\, dk \\
&= e^{-t}\int_{\Gamma_{-t}(\tilde{R},\pi/h)} \frac{e^{x\Phi_t(s_0)}v_0(s_0)}{\left(1 - u_0(s_0)(1 - e^{-t})\right)^2}\, ds_0.
\end{aligned}$$

$\Gamma_{-t}(\tilde{R}, \pi/h)$ denotes the segment along Γ_{-t} from $\Phi_t^{-1}(\tilde{R}e^{2t})$ to $\Phi_t^{-1}(\pi e^{2t}/h)$. The exact solution (3.16) shows that $\Gamma_{-t}(\tilde{R}, \pi/h)$ converges to a smooth compact curve defined implicitly by $ik = s_0 - \varphi_0(s_0), \tilde{R} \le k \le \pi/h$. Thus,

$$e^{-t} \left| \int_{\Gamma_{-t}(\tilde{R},\pi/h)} \frac{e^{x\Phi_t(s_0)} v_0(s_0)}{(1 - u_0(s_0)(1 - e^{-t}))^2} \, ds_0 \right| \le C(\tilde{R}, u_0, v_0) e^{-t}.$$

□

4 Self-similar gelation for the multiplicative kernel

For $K = xy$, McLeod solved the coagulation equation explicitly for monodisperse initial data, and showed that a mass-conserving solution failed to exist for $t > 1$. The second moment satisfies $m_2(t) = (1-t)^{-1}$. The divergence of the second moment implies further that the (formal) conservation of mass breaks down at this critical time. A rescaled limit of McLeod's solution is the following self-similar solution for $K = xy$ [1]:

$$n(t, x) = \frac{1}{\sqrt{2\pi}} x^{-5/2} e^{-(1-t)^2 x/2}, \quad x > 0, \quad t < 1. \tag{4.1}$$

The problem of solving Smoluchowski's equation with multiplicative kernel can be reduced to that for the additive kernel by a change of variables [4]. Let us briefly review this. In unscaled variables we define

$$\tilde{\psi}(t, \tilde{s}) = \int_0^\infty (1 - e^{-\tilde{s}x}) x n(t, x) dx. \tag{4.2}$$

Then $\tilde{\psi}$ solves the inviscid Burgers equation:

$$\partial_t \tilde{\psi} - \tilde{\psi} \partial_s \tilde{\psi} = 0. \tag{4.3}$$

The gelation time for initial data with finite second moment is $T_{\mathrm{gel}} = (\int_0^\infty x^2 \nu_0(dx))^{-1}$ and this is exactly the time for the first intersection of characteristics [14]. We presume the solution is scaled so the second and third moments are 1; then $T_{\mathrm{gel}} = 1$. The connection between the additive and multiplicative kernels is that $\tilde{\psi}$ solves (4.3) with initial data $\tilde{\psi}_0$, if and only if $\tilde{\varphi}(\tau, \tilde{s})$ is a solution to (3.2) with the same initial data, where

$$\tilde{\psi}(t, \tilde{s}) = e^\tau \tilde{\varphi}(\tau, \tilde{s}), \quad \text{with } \tau = \log(1-t)^{-1}. \tag{4.4}$$

For solutions $n(t,x)$ and $\tilde{n}(t,x)$ to Smoluchowski's equation with multiplicative and additive kernels respectively, this means that

$$xn(t,x) = (1-t)^{-1}\tilde{n}(\tau,x) \tag{4.5}$$

for all $t \in (0,1)$, if and only if the same holds at $t = 0$.

We thus obtain a scaling limit as $t \to T_{\text{gel}}$ directly from Theorem 3.2. The self-similar variables in real space are

$$\hat{x} = (1-t)^2 x, \quad \hat{n}(t,\hat{x}) = \frac{n(t,\hat{x}(1-t)^{-2})}{(1-t)^5} = \frac{n(t,x)}{(1-t)^5}, \tag{4.6}$$

and the self-similar profile is

$$n_{*,2}(\hat{x}) = \frac{1}{\sqrt{4\pi x^5}} e^{-\hat{x}/4}. \tag{4.7}$$

Notice that (4.6) is *not* a mass-preserving rescaling; indeed, the rescaled mass diverges:

$$\int_0^\infty \hat{x}\hat{n}(t,\hat{x})d\hat{x} = \frac{1}{1-t}\int_0^\infty xn(t,x)dx = \frac{1}{1-t} \to \infty,$$

Instead, (4.6) preserves the second moment:

$$\int_0^\infty \hat{x}^2\hat{n}(t,\hat{x})d\hat{x} = (1-t)\int_0^\infty x^2 n(t,x)dx = 1, \quad t \in [0,1).$$

The explanation is that the scaling in (4.6) is designed to capture the behavior of the distribution of large clusters as t approaches T_{gel} — the average cluster size is $(1-t)^{-1}$. Correspondingly, the mass of the self-similar solution is infinite.

Theorem 4.1. *Suppose* $n_0(x) \geq 0$, $\int_0^\infty x^2 n_0(x)dx = \int_0^\infty x^3 n_0(x)dx = 1$, *and* $x^3 n_0 \in A(\mathbb{R})$. *Then*

$$\lim_{t\to 1} \sup_{\hat{x}>0} \hat{x}^3 |\hat{n}(t,\hat{x}) - n_{*,2}(\hat{x})| = 0. \tag{4.8}$$

Theorem 3.3 may be similarly adapted to $K = xy$. In the discrete case, the correspondence (4.5) between solutions of Smoluchowski's equations with multiplicative and additive kernels becomes

$$hln_l(t) = (1-t)^{-1}\tilde{n}_l(\tau). \tag{4.9}$$

We introduce self-similar variables via

$$\hat{l} = lh(1-t)^2, \quad \hat{n}_l(t) = h^{-1}(1-t)^{-5}n_l(t). \tag{4.10}$$

Then directly from Theorem 3.3 we obtain the following.

Theorem 4.2. *Let $\nu_0 \geq 0$ be a lattice measure with span h such that $\int_0^\infty x^2 \nu_0(dx) = \int_0^\infty x^3 \nu_0(dx) = 1$. Then with the rescaling (4.10) we have*

$$\lim_{t \to 1} \sup_{l \in \mathbb{N}} \hat{l}^3 \left| \hat{n}_l(t) - n_{*,2}(\hat{l}) \right| = 0. \tag{4.11}$$

Acknowledgements

The authors thank the Max Planck Institute for Mathematics in the Sciences, Leipzig for hospitality during part of this work. G.M. thanks Timo Seppäläinen for his help during early stages of this work. This material is based upon work supported by the National Science Foundation under grant nos. DMS 00-72609 and DMS 03-05985.

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