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by

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Abstract: We consider rate-independent crystal plasticity with constrained elasticity, and state the variational formulation of the incremental problem. For generic boundary data, even the first time increment does not admit a smooth solution, and fine structures are formed. By using the tools of quasiconvexity, we obtain an explicit relaxation of the first incremental problem for the case of a single slip system. Our construction shows that laminates between two different deformation gradients are formed. Plastic deformation concentrates in one of them, the other is a purely elastic strain. For the concrete case of a simple-shear test we also obtain a completly explicit solution.

1. Introduction

This paper studies rate-independent evolution of elastoplastic bodies. We consider the simplest case where the kinematics is maximally restricted in the sense that only one slip-system is active and the only allowed elastic deformations are rigid body rotations, within the standard framework of crystal plasticity. Approximate solutions are constructed by considering sequences that minimize the sum of elastic energy and dissipated energy in the limit, the only source of dissipation being plastic deformation. The corresponding variational problems are denoted *incremental problems.* Due to the interplay of the directional dependence of the plastic deformation with the rotational invariance of the elastic part the existence of minimizers can not be expected. Minimizing sequences develop fine scale oscillations, which are analogous to microstructures found in models for solid-solid phase-transitions. Regular lamellar structures between phases with a different plastic deformation have been observed at large strains in a wide variety of metals, see e.g. [9, 1] and references therein.

The lack of minimizers for the incremental problems leads to instabilities in numerical algorithms that attempt to follow the time-continuous evolution of the elastoplastic deformation. A standard approach to overcome this difficulty is to consider a relaxed evolution problem where the original incremental problem is replaced by the lower semicontinuous envelope, see e.g. [9, 10, 5, 2].

The main objectives of this paper are (i) to demonstrate rigorously that a simple multi-dimensional model predicts the formation of a single-laminate microstructure; and (ii) to give a partial justification of numerical methods that are based on the computation of the relaxation of the incremental problems.

The first objective is achieved by determining an explicit formula for the quasiconvex envelope of the first incremental problem in the case where only one slip-system is active (in two directions) and the elastic strains are negligible. The latter corresponds to the assumption that the elastic energy is infinite whenever the elastic part of the deformation gradient (in a multiplicative decomposition) is not a rotation. We show that microstructure states can approximate a variety of affine deformations of the type $y(x) = Fx$. In particular, in two dimensions this is possible for F in a relatively open subset of the volume-preserving affine maps $\{F : \det F = 1\}$. We show that the relaxation is achieved by first-order laminates and give an explicit formula for the dependency of the lamination-normal on the boundary condition (Theorem 3.1).

The second objective is achieved by considering the evolution problem that is associated to the relaxed incremental problem. We construct explicit solutions for the relaxed evolution problem that can not be interpreted as simple single-slip motions. These solutions correspond to time-evolving microstructure. In addition we prove that there exist sequences of approximate solutions for the original single-slip model that not only converge weakly to the relaxed solution, but also have the property that the associated plasticity-induced dissipation converges to the dissipation predicted by the relaxed system. The analysis is based on the construction of Lipschitz-maps that form a perfect laminate except on a compact set with arbitrary small measure (Theorem 3.5).

Our analysis has nontrivial implications also for crystals with several slip systems. In particular, one can see that in two dimensions, three slip systems generate an effective response which is identical to Tresca plasticity (i.e. to the response obtained by assuming infinitely many slip systems), and that when finitely many slip systems are active almost every macroscopic deformation will lead to the creation of microstructures. These and further issues will be discussed in a forthcoming publication.

2. Rate-independent finite plasticity for single crystals with one slip system

In this introductory section we briefly review for the case of interest here the formulation of the incremental problem for rate-independent finite plasticity, following Ortiz and Repetto [9] and Miehe, Schotte and Lambrecht [5]. See [2, 6] for a mathematically oriented treatment, and [4] for a treatment including higher gradients.

Let $\Omega \subset \mathbb{R}^d$ be the reference configuration of an elastoplastic body, $y : [0, T] \times \Omega \to \mathbb{R}^d$ be the time dependent total deformation (in the following, d is 2 or 3), and $(\gamma, F_{\mathrm{p}}) : [0, T] \times \Omega \to \mathbb{R}^{N_s + d\times d}$ be a set of internal variables (N_s is the number of active slip systems, see below). Finite plasticity is based on the assumption that the local deformation gradient $F = \nabla y$ can be decomposed multiplicatively into a plastic and an elastic part,

$$F = F_{\rm e} F_{\rm p}. \tag{2.1}$$

As customary we assume that the plastic deformation $F_{\rm p}$ conserves the volume, $\det F_{\rm p}(x) = 1$. The decomposition (2.1) is not determined uniquely by the deformation y, but it depends on the internal variables γ as well, which in turn can be obtained as the solution of an initial-boundary value problem that is based on additional mechanical assumptions.

In crystalline plasticity the set of admissible stresses $\mathbb{Q}$ is determined by the slip-systems of the crystal,

$$\mathbb{Q} = \bigcap_\alpha \{ |s_\alpha \cdot Q m_\alpha| \le \tau_\alpha \} \subset \mathbb{R}^{d\times d},$$

where $(s_\alpha, m_\alpha, \tau_\alpha)_{\alpha=1\dots N}$ is a family of slip systems, $s_\alpha \in \mathbb{R}^d$, $m_\alpha \in \mathbb{R}^d$, $\tau_\alpha \in (0,\infty)$ are slip direction, slip plane normal and critical resolved shear stress corresponding to slip system α. The volume preservation results in the constraint $s_\alpha \cdot m_\alpha = 0$.

For the case of rigid elasticity we consider here, however, the stress is not a well-defined quantity. We resort therefore to the variational formulation. This is best understood, and typically applied, by considering a time-discretization $0 = t_0 \le \ldots \le t_n = T$. Given the state $(y(t_k), \gamma(t_k), F_{\rm p}(t_k))$ at time t_k, the deformation y at time t_{k+1} and the internal variables $\gamma, F_{\rm p}$ in the time interval (t_k, t_{k+1}) minimize

$$I'_k(y, F_{\rm p}, \gamma) := \int_\Omega W_{\rm e}(\nabla y F_{\rm p}^{-1}) + \int_\Omega \int_{t_k}^{t_{k+1}} \phi(\dot\gamma, \dot F_{\rm p}, \gamma, F_{\rm p}), \tag{2.2}$$

where the arguments of the first integral are evaluated at the final time t_{k+1}. The deformation obeys boundary conditions $y = y_{\rm bdry}(t_{k+1})$ on $\partial\Omega$. Here, $W_{\rm e}$ is the elastic energy, $W_{\rm p}$ characterizes the plastic stored energy, and ϕ the dissipation. Further, the solution to the sequence of incremental problems satisfies initial conditions (typically $y = y_0$, $F_{\rm p} = \mathrm{Id}$, $\gamma = 0$ for $t = 0$). The dissipation function ϕ relates the evolution of the plastic deformation to that of the internal variables, which in single-crystal plasticity are related by the classical flow rule [11]

$$\dot F_{\rm p} F_{\rm p}^{-1} = \sum_{\alpha=1}^{N_s} \dot\gamma^\alpha s_\alpha \otimes m_\alpha. \tag{2.3}$$

Indeed, we take

$$\phi(\dot\gamma, \dot F_{\rm p}, \gamma, F_{\rm p}) = \begin{cases} \sum_\alpha |\dot\gamma^\alpha| \tau_\alpha & \text{if (2.3) holds} \\ \infty & \text{else.} \end{cases} \tag{2.4}$$

The existence of a time-continuous limit of (2.2) is a deep and interesting issue, which we do not address here (see e.g. [6] and references therein). We instead focus on a precise analysis of the time-discrete problem, which is the one used in most concrete computations. For the present purposes it is sufficient to observe that F_p and γ can be assumed to be affine functions of time, in each time interval, and at each point in space.

We now specialize this general framework to elastically rigid single-slip plasticity. First, we consider a single slip system with two opposite orientations, i.e. $N_s = 2$ and $s^1 \otimes m^1 = -s^2 \otimes m^2 = s \otimes m$, where s and m are two fixed orthogonal unit vectors. The latter condition permits an explicit integration of the plastic flow rule (2.3), leading to

$$F_\mathrm{p}(t) = \mathrm{Id} + \gamma(t) s \otimes m \tag{2.5}$$

where $\gamma(t) = \gamma^1(t) - \gamma^2(t)$. From the variational viewpoint, (2.5) holds whenever the second integral in (2.2) is finite. This local relation between γ and F_p permits to eliminate one of them from the variational problem.

The second simplification is the assumption of infinitely hard elastic response, which corresponds to a decoupling of the elastic and plastic problems, as proposed by Ortiz and Repetto [9]. The elastic part of the deformation gradient F_e is then restricted to be a rigid-body rotation, and

$$W_\mathrm{e}(F) = \begin{cases} 0 & \text{if } F \in SO(d) \\ \infty & \text{else.} \end{cases} \tag{2.6}$$

Due to the rigidity of the elastic energy (2.6) we can minimize out locally F_e, and express both F_p and γ in terms of ∇y.

Lemma 2.1. *If $I_k(y)$ is finite, then $\nabla y \in M^{(d)}$ a.e., where*

$$M^{(d)} = \{F \in \mathbb{R}^{d\times d} \mid F = R(\mathrm{Id} + \gamma s \otimes m), R \in SO(d), \gamma \in \mathbb{R}\}. \tag{2.7}$$

Conversely, let $F \in M^{(d)}$. Then, there exists a unique pair $(R, \gamma) \in SO(d) \times \mathbb{R}$ such that $F = R(\mathrm{Id} + \gamma s \otimes m)$ holds.

Proof. The first part is a direct consequence of (2.5) and (2.6). The second follows from the relation

$$\gamma = (Fm) \cdot (Fs)\,. \tag{2.8}$$

□

From now on γ and F_p are implicitly assumed to be given in terms of ∇y via (2.5) and (2.8). The incremental problem (2.2) then becomes

$$I_k(y) = \int_\Omega W_\mathrm{ep}(\nabla y; \nabla y(t_k))\,, \tag{2.9}$$

where

$$W_\mathrm{ep}(F; F_0) = \begin{cases} |\gamma(F) - \gamma(F_0)| & \text{if } F, F_0 \in M^{(d)} \\ \infty & \text{else} \end{cases} \tag{2.10}$$

and $\gamma(F)$ is given by (2.8). We observe that, provided $\nabla y(t_k)$ is constant on some part of the domain, we can assume it to be the identity, since $W_\mathrm{ep}(F; F_0) = W_\mathrm{ep}(FF_0^{-1}; \mathrm{Id})$ for all admissible F_0. The main aim of this paper is the study of minimizing sequences for (2.9-2.10). Simple counting degrees of freedom shows that the existence of solutions can only be expected for very special boundary conditions. To see that, we consider the homogeneous case where $y_{\rm bdry}(t,x) = F(t)x$. Since $\dim M^{(d)} = \frac{1}{2}d(d-1)+1$ is smaller that d^2-1 for $d \geq 2$, it is clear that the set of paths $t \mapsto F(t) \in \mathbb{R}^{d\times d}$, $\det F(t) = 1$ such that $F(t) \in M^{(d)}$ is ungeneric.

3. The relaxation of the time-incremental problem

We now present the main results of this paper, which concern the behavior of minimizing sequences for the incremental problem (2.9-2.10).

For generic boundary conditions, the minimization problem (2.9) admits no minimum, and minimizing sequences form fine-scale structures. An explicit computation is shown below. Such a behavior is frequent in nonlinear elasticity, especially in the study of shape-memory alloys; the mathematical study of such phenomena has been based on the concept of *quasiconvexity*. A function $W : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$ is said to be quasiconvex if affine deformations are minimizers with respect to their own boundary conditions, i.e. if

$$W(F) \leq \frac{1}{\Omega}\int_\Omega W(\nabla\phi) \tag{3.1}$$

for all $\phi \in W^{1,\infty}(\Omega,\mathbb{R}^d)$ such that $\phi(x) = Fx$ for $x \in \partial\Omega$. It is easy to see that this definition does not depend on the chosen open set Ω. The matrix F is the average of $\nabla\phi$ over Ω: this definition differs from the usual one of convexity by means of Jensen's inequality in that the argument in the right-hand side is required to be a gradient field. The quasiconvex envelope $W^{\rm qc}$ of W is defined as the largest quasiconvex function which is less than or equal to W, and determines the effective, macroscopic behavior. For a more detailed presentation of these and related concepts, see [7].

We now assume that at a given time t_k the deformation gradient ∇y_k takes, in an open set $\omega \subset \Omega$, some value $F_0 \in M^{(d)}$, and compute the relaxation of I_k. For the first incremental problem the initial condition gives $F_0 = \mathrm{Id}$ on all of Ω. We state separately the two- and the three-dimensional results.

Theorem 3.1. *In two dimensions, the quasiconvex envelope of $W_{\rm ep}(\cdot, F_0)$ (defined in (2.10)) is given by*

$$W^{\rm qc}(F,F_0) = \begin{cases} \lambda_{\max}(FF_0^{-1}) - \lambda_{\min}(FF_0^{-1}) & \text{if } F, F_0 \in N^{(2)} \\ \infty, & \text{else} \end{cases}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximal and minimal nonnegative singular values of FF_0, and

$$N^{(2)} = \{F \in \mathbb{R}^{2\times 2} \mid \det F = 1, |Fs| \leq 1\}.$$

The rank-one convex and the polyconvex envelopes, $W^{\rm rc}$ and $W^{\rm pc}$, also agree with $W^{\rm qc}$.

We recall that the rank-one convex envelope $W^{\rm rc}(F)$ is defined as the largest function below $W_{\rm ep}$ which is convex along rank-one lines, i.e. for all $F \in \mathbb{R}^{d\times d}$,

$a,\ b \in \mathbb{R}^d$, the function $W^{\rm rc}(F + ta \otimes b)$ is convex in t. Rank-one convexity corresponds to the linearization of quasiconvexity, i.e., is equivalent to (3.1) up to second order in $\nabla\phi - F$. The polyconvex envelope is the largest function below $W_{\rm ep}$ that can be written as a convex function of F, its determinant, and its minors, i.e., for $d = 2$ such that $W^{\rm pc}(F) = h(F, \det F)$, with $h : \mathbb{R}^5 \to \mathbb{R}$ convex, and for $d = 3$ such that $W^{\rm pc}(F) = h(F, \operatorname{cof} F, \det F)$, with $h : \mathbb{R}^{19} \to \mathbb{R}$ convex. Since the determinant and the cofactors of a gradient field are divergences, their integral depends only on the boundary values and polyconvex functions are quasiconvex.

In three dimensions the situation is more rigid.

Theorem 3.2. *In three dimensions, the function $W_{\rm ep}(\cdot, F_0)$ is quasiconvex. Its rank-one convex and polyconvex envelopes are given by*

$$W^{\rm pc}(F, F_0) = W^{\rm rc}(F, F_0) = \begin{cases} \lambda_{\max}(FF_0^{-1}) - \lambda_{\min}(FF_0^{-1}) & \text{if } F, F_0 \in N^{(3)} \\ \infty, & \text{else} \end{cases}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximal and minimal nonnegative singular values of FF_0, and

$$N^{(3)} = \{F \in \mathbb{R}^{3\times3} \mid \det F = |F(s \wedge m)| = |\operatorname{cof} F(s \wedge m)| = 1, |Fs| \le 1\}.$$

The proofs of Theorems 3.1 and 3.2 are based on the construction of matching upper and lower bounds. Lower bounds on the quasiconvex envelope can be derived by constructing suitable polyconvex functions. Upper bounds are obtained by explicitly constructing test functions. The simplest construction is a simple laminate, i.e., a test function v whose gradient ∇v takes essentially only two values $F_\pm$ (except for a negligible small region around the boundary). Continuity of v across interfaces enforces $F_+ - F_-$ to be rank one. A fine-scale mixture of $F_\pm$ can approximate the affine deformation Fx, and hence be made to satisfy the boundary condition by a small correction around the boundary, if $\mu F_+ + (1-\mu)F_- = F$, where $\mu \in [0, 1]$ is the volume fraction in which $\nabla v = F_+$. The only subtle point here is that the small correction around $\partial\Omega$ has to be chosen so as to remain in the set $M^{(d)}$ where the energy is finite. In two dimensions this can be done, using the convex integration results by Müller and Šverák, see Theorems 3.4 and 3.5 below. In three dimensions instead we show that any Lipschitz function whose gradient is almost everywhere in $M^{(3)}$ is affine, hence no such boundary layer can be constructed, and the quasiconvex envelope differs from the lamination-convex one. We remark that this results depends crucially on the assumption of rigid elasticity. If one would replace ∞ with a large constant in (2.6), or define $W_{\rm e}$ as a large multiple of the squared distance from $SO(d)$, then the rigidity result would fail, and the quasiconvex envelope would be less than the rank-one convex one.

Proof of Theorem 3.1. The result clearly depends only on FF_0^{-1}. We write $W_{\rm ep}(F) = W_{\rm ep}(F, \mathrm{Id})$, and study $W_{\rm ep}(F)$.

Firstly, we note that the conditions defining $N^{(2)}$ are polyconvex, in the sense that the function

$$h(F,g)=\begin{cases}0 & \text{if } g=1 \text{ and } |Fs|\le 1\\ \infty & \text{else}\end{cases}$$

is convex on $\mathbb{R}^5$, and $N^{(2)}=\{F: h(F,\det F)=0\}$. The proof of the lower bound is at this point immediate, since

$$|\lambda_1(F)-\lambda_2(F)|=\sqrt{|F|^2-2\det F}=\sqrt{(F_{11}-F_{22})^2+(F_{12}+F_{21})^2}$$

is a convex function on $\mathbb{R}^{2\times 2}$, it equals W on $M^{(2)}$ (indeed, $|\gamma(F)|^2=|F|^2-2=|F|^2-2\det F$), and it equals $W^{\rm qc}$ on $N^{(2)}$. This constitutes automatically also a lower bound for $W^{\rm rc}$ and $W^{\rm pc}$.

To prove the upper bound, we give an explicit construction of a laminate. We take $F\in N^{(2)}\setminus M^{(2)}$ and seek unit vectors a, b such that

$$F_\mu=F+\mu a\otimes b$$

is in $M^{(2)}$ for two values of μ, with different sign. The constraint $\det F_\mu=1$ corresponds to $a^\perp F b^\perp=0$ (here and below, $(x,y)^\perp=(-y,x)$). To determine a and b, we use the fact that this laminate is optimal iff $W^{\rm qc}$ is linear along it. Hence we impose that

$$|F_\mu|^2-2=|F|^2-2+2\mu(aFb)+\mu^2 \tag{3.2}$$

is the square of a binomial in μ, i.e. $(aFb)^2=|F|^2-2$. In turn, and using the determinant constraint, this gives

$$(a^\perp F b)^2+(aFb^\perp)^2=2$$

which corresponds to $|Fb^\perp|=1$, which has two solutions for b (apart from an irrelevant global sign). From the determinant constraint we get then $a=Fb^\perp$ (again, with an irrelevant sign freedom).

Now observe that

$$q(\mu)=|F_\mu s|^2=(b\cdot s)^2\mu^2+2\mu(b\cdot s)(a\cdot Fs)+|Fs|^2$$

is quadratic in μ, and its leading coefficient is strictly positive for all $F\in N^{(2)}\setminus M^{(2)}$ (if $b\cdot s=0$, then $b^\perp=\pm s$, which gives $|Fs|=|Fb^\perp|=1$, i.e. $F\in M^{(2)}$). Since $q(0)<1$, it follows that there are two values $\mu_\pm$ with different sign such that $q(\mu_\pm)=1$, i.e. $F_{\mu_\pm}\in M^{(2)}$. We had already checked that the expression given in the statement equals $W_{\rm ep}$ on $M^{(2)}$, and since $|\lambda_1-\lambda_2|$ is linear along the segment $[F_{\mu_-},F_{\mu_+}]$ we obtain for all F in that segment a simple laminate between F_{μ_-} and F_{μ_+} with energy $\lambda_{\max}(F)-\lambda_{\min}(F)$ (see Section 4 for a more explicit characterization of the constructed laminate in a specific example). This concludes the construction of the laminate, and hence the upper bound on $W^{\rm rc}$ and $W^{\rm pc}$.

To conclude the proof of the upper bound for the quasiconvex envelope $W^{\rm qc}$, we still need to show that for any $\varepsilon>0$ we can construct a test function $v:\Omega\to\mathbb{R}^2$ with boundary values Fx so that $\int W_{\rm ep}(\nabla v)$ is less than $|\Omega|W^{\rm qc}(F)+\varepsilon$.

The construction strongly relies on the convex integration results by Müller and Šverák. We proceed in two steps. First, by Theorem 3.5 for any small δ we can obtain a piecewise affine function u_1 whose gradient is everywhere in $N^{(2)}$, and which on a large subset Ω_δ coincides with the laminate between A and B. Indeed, we choose $v = s$, $F_{\mu_\pm}$ as A and B. The vectors As and Bs have unit length, but are different since their weighted average Fs does not have unit length. Then, from Theorem 3.5 we obtain $|\nabla u_1 s| \le 1$, which gives $\nabla u_1 \in N^{(2)}$. The closeness of ∇u_1 to $[A, B]$ further shows that $\nabla u_1 \in U^{(k)}$ a.e., where

$$U^{(k)} = \left\{F \in \mathbb{R}^{2\times 2} : \det F = 1, |Fs| \le 1,\ |Fs^\perp|^2 < k^2\right\} . \tag{3.3}$$

and $k = 1+\max(|A|, |B|)$. We then apply Lemma 3.3 to each affine piece of ∇u_1 where $\nabla u_1 \ne M^{(2)}$. We obtain u_2 such that its gradient is everywhere in $M^{(2)}$, uniformly bounded, and on Ω_δ still coincides with the laminate above. Then, it is clear that

$$\int_\Omega W_{\rm ep}(\nabla u_2) \le |\Omega_\delta| \left(\lambda W_{\rm ep}(A) + (1-\lambda)W_{\rm ep}(B)\right) + \delta k.$$

Since δ can be made arbitrarily small, this concludes the proof. ☐

Lemma 3.3. *For any $\Omega \subset \mathbb{R}^2$ open, $k > 0$ and F in the set $U^{(k)}$, defined in (3.3) above, there is $v \in W^{1,\infty}(\Omega, \mathbb{R}^2)$ such that $v = Fx$ on $\partial\Omega$ and $\nabla v \in U^{(k)} \cap M^{(2)}$ a.e.*

Proof. This follows from Theorem 3.4 below by Müller and Šverák, if we can show that the sequence

$$U_j^{(k)} = \left\{F \in \mathbb{R}^{2\times 2} : \det F = 1, 1-2^{-j} < |Fs| < 1,\ |Fs^\perp|^2 < k^2\right\} .$$

constitutes (as $j \to \infty$) an in-approximation of $U^{(k)} \cap M^{(2)}$. Indeed, if $F_j \in U_j^{(k)}$ and $F_j \to F$, then $|Fs| = 1$, hence $F \in U^{(k)} \cap M^{(2)}$. We now show that $F \in U_j$ can be obtained as a simple laminate supported in U_{j+1}. To do so, we define

$$F_\mu = F + \mu(Fs^\perp) \otimes s$$

and observe that $|F_\mu s^\perp|$ and $\det F_\mu$ do not depend on μ, whereas

$$|F_\mu s|^2 = |Fs|^2 + 2\mu(Fs) \cdot (Fs^\perp) + \mu^2|Fs^\perp|^2$$

By assumption $|Fs^\perp|^2 > 1$ and $|Fs| < 1$, hence we find two values of μ with opposite sign such that F_μ lies in U_{j+1}, and the proof is concluded. ☐

We now come to the proof of Theorem 3.2.

Proof of Theorem 3.2. As above, we denote $W_{\rm ep}(F) = W_{\rm ep}(F, \mathrm{Id})$, and change variables so that $s = e_1$, $m = e_2$.

We first show that $W_{\rm ep}(F)$ is quasiconvex. To do this, we need to show that

$$\int_\Omega W_{\rm ep}(\nabla u) \ge W_{\rm ep}(F)$$

for all Lipschitz vector fields $u : \Omega = [0, 1]^3 \to \mathbb{R}^3$ such that $u(x) = Fx$ on $\partial\Omega$ (since quasiconvexity does not depend on the domain we can focus on the unit cube). If the integral is infinite, there is nothing to prove. We can therefore assume that $W_{\rm ep}(\nabla u)$ is finite almost everywhere. We now show that in this case u is affine, hence equality holds. Indeed, if $\nabla u \in M^{(3)}$ a.e. we have

$$|Fe_3| = \left|\int_\Omega \partial_3 u\right| \le \int_\Omega |\partial_3 u| = 1\,,$$

$$|\operatorname{cof} Fe_3| = \left|\int_\Omega \partial_1 u \wedge \partial_2 u\right| \le \int_\Omega |\partial_1 u \wedge \partial_2 u| = 1\,,$$

and

$$\det F = \int_\Omega \det \nabla u = 1\,.$$

Since $\det F = Fe_3 \cdot \operatorname{cof} Fe_3 \le |Fe_3|\,|\operatorname{cof} Fe_3|$, equality holds throughout, and in particular we get $\partial_3 u = Fe_3$ a.e., which implies $u(x_1,x_2,x_3) = u(x_1,x_2,0) + Fe_3x_3$. The boundary condition on $x_3 = 0$ then gives $u(x) = Fx$ on Ω, and the proof is concluded.

We now come to the second part of the statement. The inequality $W^{\rm rc} \ge W^{\rm pc}$ follows from general arguments, hence it is sufficient to show that $W^{\rm rc}$ is less or equal, and $W^{\rm pc}$ larger or equal, than the function given in the statement. Firstly, we note that $N^{(3)}$ is a polyconvex set (i.e. it is the intersection of sublevel sets of polyconvex functions). Indeed,

$$\tilde N = \{F \in \mathbb{R}^{3\times 3} \;:\; \det F = 1, |Fe_3| \le 1, |\operatorname{cof} Fe_3| \le 1, |Fe_1| \le 1\}.$$

is clearly polyconvex. But $N^{(3)} = \tilde N$, since $\det F = Fe_3 \cdot \operatorname{cof} Fe_3 = 1$. The polyconvexity of $N^{(3)}$, together with the fact that W is infinite outside $M^{(3)} \subset N^{(3)}$, implies that $W^{\rm pc} = W^{\rm rc} = +\infty$ outside $N^{(3)}$. Hence we only need to consider matrices inside $N^{(3)}$, which is a two-dimensional problem. Indeed, if $F \in N^{(3)}$, then

$$F = \begin{pmatrix} \tilde F & 0 \\ 0 & 1 \end{pmatrix}$$

where $\tilde F \in N^{(2)}$. Replacing F with $\tilde F$, the construction of the laminate done in two dimensions applies also here, and gives the desired upper bound on $W^{\rm rc}$ and $W^{\rm pc}$. □

Before concluding this Section, we state the results regarding the constrained construction which has been used in the construction above.

Theorem 3.4 ([8], Theorem 1.3). *Let $\Sigma = \{F \in \mathbb{R}^{d\times d} : \det F = 1\}$, and let K be a subset of Σ. Suppose that U_i is an in–approximation of K, i.e. the U_i are open in Σ, uniformly bounded, U_i is contained in the rank–one convex hull of U_{i+1}, and U_i converges to K in the following sense: if $F_i \in U_i$ and $F_i \to F$, then $F \in K$. Then, for any $F \in U_1$ and any open domain $\Omega \subset \mathbb{R}^d$ there exists a Lipschitz solution of the partial differential inclusion*

$$\begin{aligned} Du &\in K && \text{a.e. in } \Omega \\ u(x) &= Fx && \text{on } \partial\Omega\,. \end{aligned}$$

The following result is obtained [8] (Theorem 6.1 and Remark 2 thereafter) without the convex constraint on $|(\nabla u)v|$. We give a full proof in the appendix.

Theorem 3.5. *Let* $A, B \in \mathbb{R}^{2\times 2}$, *with* $\det A = \det B = 1$ *and* $\operatorname{rank}(A - B) = 1$, $v \in \mathbb{R}^2$ *be such that* $|Av| = |Bv|$ *and* $Av \neq Bv$, *and* Ω *be an open domain in* $\mathbb{R}^2$. *For any* $\lambda \in (0, 1)$, *and any* $\delta > 0$, *there are* $h_0 > 0$ *and* $\Omega_\delta \subset \Omega$, *with* $|\Omega \setminus \Omega_\delta| \leq \delta$, *such that the restriction to* Ω_δ *of any simple laminate between the gradients* A *and* B *with weights* λ *and* $1 - \lambda$ *and period* $h < h_0$ *can be extended to a finitely piecewise affine* $u : \Omega \to \mathbb{R}^2$ *so that* $u = (\lambda A + (1 - \lambda)B)x$ *on* $\partial\Omega$ *and* $\det \nabla u = 1$, $|(\nabla u)v| \leq |Av| = |Bv|$, *and* $\operatorname{dist}(\nabla u, [A, B]) \leq \delta$ *on* Ω.

By finitely piecewise affine we mean that the domain can be decomposed in finitely many pieces such that the function is affine on each of them. A simple laminate of period h is a function of the form

$$y(x) = (\lambda A + (1 - \lambda)B)x + ah\chi_\lambda \left(\frac{n \cdot x + c}{h}\right)$$

where $A - B = a \otimes n$, and $\chi_\lambda(t)$ is a continuous, one-periodic real function of t such that $\chi' = 1 - \lambda$ for $t \in (0, \lambda)$, $\chi' = -\lambda$ for $t \in (\lambda, 1)$.

4. Sequence of incremental problems and explicit solution for simple shear

Theorems 3.1 and 3.2 show that existence of minimizers cannot be expected even for a single incremental problem, and give an explicit relaxation formula. We consider now the full sequence of incremental problems, and define a simple relaxation scheme for the sequence. For the concrete case of simple shear, this leads to an explicit solution. Concrete computations are done here only in two dimensions.

Given an initial condition y_0, and a small positive number ε, an approximate solution of the sequence of incremental problems is a sequence $\{y_k\}_{k=1,\ldots,K}$ such that y_{k+1} is an approximate minimizer of

$$I_k(y) = \int_\Omega W_{\text{ep}}(\nabla y; \nabla y_k), \tag{4.1}$$

in the sense that

$$I_k(y_{k+1}) \leq \inf I_k(y) + \varepsilon\,.$$

We observe that in the single-slip case, one has

$$\sum_{k=0}^{K-1} W_{\text{ep}}(\nabla y_{k+1}, \nabla y_k) \geq W_{\text{ep}}(\nabla y_K, \text{Id}) \tag{4.2}$$

If equality holds, replacing $W_{\text{ep}}(\nabla y, \nabla y_k)$ with the simpler $W_{\text{ep}}(\nabla y, \text{Id})$ in each minimization problem (4.1) corresponds to an irrelevant shift by a constant. In turn, equality in (4.2) is guaranteed if γ is locally monotone in t, and ∇y is always in the allowed set $M^{(d)}$. This is a condition that can be directly checked on an explicitly known candidate minimizing sequence y_k.

Assume now that the macroscopic deformation is homogeneous, and that there is a minimizing sequence which at each time step is a laminate in most of the domain. To be precise, given a macroscopic deformation $y_{\mathrm{bdry}}(x,t) = F(t)x$, we seek $y_k^{(\varepsilon)}$ such that $\nabla y_k^{(\varepsilon)}$ takes only two rank-one connected values, which we call $F_\pm^{\varepsilon,k}$, with volume fractions $\mu^{\varepsilon,k}$ and $1-\mu^{\varepsilon,k}$. The average gradient coincides with the one imposed on the boundary provided that

$$\mu^{\varepsilon,k} F_+^{\varepsilon,k} + (1-\mu^{\varepsilon,k}) F_-^{\varepsilon,k} = F(t_k)\,. \tag{4.3}$$

We show below that for approximate minimizers this simpler condition can replace the Dirichlet boundary conditions.

The displacement field can be explicitly written as (dropping the index (ε,k))

$$y(x) = (\mu F_+ + (1-\mu)F_-)x + a h_\varepsilon \chi_\mu \left(\frac{n\cdot x + c}{h_\varepsilon}\right)$$

where $F_+ - F_- = a\otimes n$, with $|n| = 1$, $c = c^{\varepsilon,k} \in \mathbb{R}$, h_ε sets the scale of the laminate, and $\chi_\mu(t)$ is a continuous, one-periodic real function of t such that $\chi' = 1-\mu$ for $t \in (0,\mu)$, $\chi' = -\mu$ for $t \in (\mu,1)$. The numbers $c^{\varepsilon,k}$ represent the phase relation between successive laminates. This assumption of a laminate structure has been used in both in analytical [9] and numerical computations [10, 5, 1], and it also constitutes the most popular method of construction approximate solutions to the incremental variational problem and its time-continuous counterpart, cf. [3, 2]. In [10] multiple-order laminates are used. We notice, however, that there is no general reason why laminates (and, even more so, simple laminates) should be sufficient to relax W_{ep}, and also the fact that the first incremental problem can be relaxed with simple laminates does not imply that the same holds for the whole sequence. We shall now show that this is actually the case if additional geometrical assumptions are satisfied.

As discussed above, we have equality in (4.2) if γ is locally monotone and $\nabla y \in M^{(d)}$ everywhere. In turn, this holds if the following conditions *on the simplified problem* are satisfied: (i) all lamination directions are the same; (ii) $\gamma(F_\pm)$ are monotone in t, (iii) if $\gamma(F_i)$ is not constant, then the volume fraction of F_i is nondecreasing in t, where $i \in \{+,-\}$. Indeed, in such a case it is immediate to construct a laminate such that pointwise γ is a monotone function.

Before going into the explicit construction of the laminate, we show that (4.3) can replace the Dirichlet boundary condition $y(x) = F(t)x$ on $\partial\Omega$. Let y^{lam} be a laminate solution defined on Ω. As the discussion at the end of the proof of Theorem 3.1 shows, for any $\delta > 0$ there is a lamination period h_δ, such that if y^{lam} has period less then h_δ we can find $y^\delta(x)$ which coincides with y^{lam} up to a set of measure δ, and which satisfies the Dirichlet boundary condition on $\partial\Omega$. Further, $|I_k(y^{\mathrm{lam}}) - I_k(y^\delta)|$ is controlled by a constant times δ, since in the construction only bounded gradients are used (the constant depends on the Lipschitz norm of y^{lam}). Therefore by choosing δ sufficiently small the energy corrections coming from the boundary layer can be made arbitrarily small. Hence we can focus on the bulk contributions alone. Note that in this argument it is important that the laminate can be made arbitrarily fine without affecting the bulk term $I_k(y^{\text{lam}})$.

We now focus on the case of a simple shear experiment, in which the applied deformation is

$$y_{\text{bdry}}(x,t) = F(t)x\,, \qquad F(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Let the slip system be characterized by $s = (s_1, s_2)$ and $m = s^\perp = (-s_2, s_1)$, where we can assume without loss of generality that $s_2 \geq 0$. We construct y_k based on the lamination obtained in the computation of the quasiconvex envelope of $W_{\text{ep}}(F(t_k), \text{Id})$. First observe that $F(t) \in N^{(2)}$ for

$$0 \leq t \leq T = -2\frac{s_1}{s_2}\,,$$

hence we obtain solutions only if $s_1 \leq 0$. In the degenerate case $s_2 = 0$ we get $F(t) \in M^{(2)}$, and no microstructure is generated. For $s_2 > 0$, the construction above gives two solutions for b,

$$b_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad b_2 = \frac{1}{\sqrt{1+(t/2)^2}}\begin{pmatrix} 1 \\ t/2 \end{pmatrix}.$$

In the first case, the lamination direction does not depend on t. We explicitly evaluate the result in this case. This gives $a_1 = Fb_1^\perp = (-1, 0)$, and $\mu_\pm$ are the roots of

$$q(\mu) - 1 = |F_\mu s|^2 - 1 = s_2^2\mu^2 - 2\mu s_2(s_1 + s_2 t) + s_2 t(2s_1 + s_2 t)$$

i.e.

$$\mu_+ = t \qquad \mu_- = t + 2\frac{s_1}{s_2}$$

and correspondingly

$$\gamma(F_+) = 0, \qquad \gamma(F_-) = 2\frac{s_1}{s_2}$$

Hence $\gamma(F_\pm)$ are constant, and the volume fraction of F_-, $|\mu_+|/(\mu_+ - \mu_-) = s_2 t/2|s_1|$, is increasing in t. We conclude that this laminate satisfies the three conditions mentioned above for equality in (4.2), and is therefore an approximate solution. The construction of the boundary layer can also be performed straightforwardly: the laminates are uniformly Lipschitz, hence for each ε, there is a δ (independent on k) such that the contribution of the boundary layer to the energy is less than $\varepsilon/2$; in turn, this δ gives an h_ε which sets the scale of the laminates needed for having an ε-approximate solution. As $\varepsilon \to 0$, also δ and $h_\varepsilon \to 0$.

The total dissipation up to time T is

$$\int_\Omega W_{\text{ep}}(\nabla y(T,x), \text{Id}) = |\Omega| W^{\text{qc}}(F(T), \text{Id}) = |\Omega||T|.$$

These results demonstrate that we have indeed constructed a relaxed evolution system that is solved by weak limits of approximate solutions of the original single-slip system.

The averaged, macroscopic evolution of the system can be completely described by the quasiconvex envelope $W^{\rm qc}$. It is interesting to observe that the latter has, in its domain, exactly the same form as the plastic dissipation in Tresca plasticity, which is based on the assumption that any pair of orthonormal vectors is a possible slip system and the resolved shear stresses are all equal. Indeed, take a matrix F with unit determinant. Then, there is a unit vector a such that $|Fa| = 1$. Hence we can write $F = Q(\mathrm{Id} + \gamma a \otimes a^\perp)$, where Q is the rotation that brings F to upper triangular form in the basis $(a, a^\perp)$. Then, the Tresca dissipation corresponding to F is simply given by $W_{\rm Tr}(F) = |\gamma|$. But it is a simple check (see beginning of the proof of Theorem 3.1) that $W_{\rm Tr}(F) = (\lambda_2 - \lambda_1)(F)$. We conclude that the effective behavior of single-slip plasticity can, in some part of the domain, be reproduced by isotropic Tresca plasticity. If three slip systems are present, with a generic orientation, it can be shown that this result holds for all F with unit determinant in an open neighborhood of the identity.

To show that the situation is not always so simple, we now consider a different shear experiment,

$$\widetilde{y}_{\rm bdry}(x,t) = \tilde F(t)\,, \qquad \tilde F(t) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}\,.$$

for $0 \le t \le T = s_1/s_2$ (we assumed here that $s_1 > s_2 > 0$). Here the rank-one connection is given by

$$b_{1,2} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 \\ \pm t \end{pmatrix}\,, \qquad a_{1,2} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} t \\ \pm 1 \end{pmatrix}\,.$$

hence the lamination direction depends on t in all cases. An approximate solution of a sequence of incremental problems cannot any more be found with the simple scheme used above. Inequality (4.2) gives

$$\sum_{k=0}^{K-1} W_{\rm ep}(\nabla y_{k+1}, \nabla y_k) \ge W_{\rm ep}(\nabla y_K, \mathrm{Id}) \ge W^{\rm qc}(\tilde F(t_K), \mathrm{Id})\,,$$

and gives the lower bound $W^{\rm qc}(\tilde F(t_K), \mathrm{Id}) = t_K - 1/t_K$ on the total dissipation.

In closing, we remark that for the case of perfect single-slip plasticity the fundamental obstacle that prevents the construction of more accurate approximations of the evolution is given by the fact that $W^{\rm qc}(F, F_0)$ is only known for $F_0 \in M^{(d)}$, whereas for time steps after the first one $F_0 \in N^{(d)}$. The approach above corresponds to replacing F_0 with a gradient Young measure, which is then assumed to be a laminate. This permits a straightforward solution in the simple-shear case, but is not applicable to more general problems.

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Appendix A. Proof of Theorem 3.5

Theorem 3.5 can be proven by a explicit construction. The basic construction step, given here in Lemma A.2, is a slight variant of the one used by Müller and Šverák to obtain Theorem A.1. This result was, however, stated without proof in [8]. The main point in our construction is that our maps form perfect laminates in sets with small compact complement and have gradients in a set $K \cap GL(2)$ where $K \subset \mathbb{R}^{2\times 2}$ is convex.

Theorem A.1 ([8], Theorem 6.1 and Remark 2 thereafter)**.** *Let* $A, B \in \mathbb{R}^{2\times 2}$*, with* $\det A = \det B = 1$ *and* $\operatorname{rank}(A - B) = 1$*, and* Ω *be an open set in* $\mathbb{R}^2$*. For any* $\lambda \in (0, 1)$*, and any* $\delta > 0$*, there is a piecewise linear map* $u : \Omega \to \mathbb{R}^2$ *such that* $\det \nabla u = 1$*,* $\operatorname{dist}(\nabla u, \{A, B\}) \le \delta$ *a.e., and* $u = (\lambda A + (1 - \lambda)B)x$ *on* $\partial\Omega$*.*

Proof. A suitable change of variables, which is discussed in more detail at the beginning of the proof of Theorem 3.5 below, shows that it is sufficient to prove the statement for the matrices of the form given in Eq. (A.1), whose (weighted) average is the identity. Lemma A.2 with $\varepsilon = \delta$ gives then a proof of Theorem A.1 for a special domain ω. The open set Ω can be covered by countably many disjoint scaled copies of ω, plus a null set. Since the construction can be scaled down to each scaled copy of ω, Theorem A.1 follows. □

We now give the basic construction, with the additional quantitative estimates needed for the proof of Theorem 3.5.

Lemma A.2. *For any* $\varepsilon > 0$*,* $\lambda \in (0, 1)$ *and* $t > 0$*, such that* $t^{1/2}\varepsilon$ *is small enough, there is* $\xi > 0$ *such that for all* $\Omega = [-L, L] \times [-H, H]$ *with* $H < L/\xi$ *one can construct a finitely piecewise affine* $u : \Omega \to \mathbb{R}^2$ *such that* $\det \nabla u = 1$ *a.e.,* $u(x) = x$ *on the boundary,* u *coincides with a laminate with period* H *(as in (A.2)) between the matrices*

$$A = \begin{pmatrix} 1 & (1-\lambda)t \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -\lambda t \\ 0 & 1 \end{pmatrix} \tag{A.1}$$

on $[-L + H\xi, L - H\xi] \times [-H, H]$*, and* $\operatorname{dist}(\nabla u, [A, B]) \le ct^{1/2}\varepsilon(1 + t)$*. The parameter* ξ *can be chosen as* $1/\varepsilon$*.*

Further, on the open subset where $\nabla u \ne \mathrm{Id}$ *(i.e.* $\omega = \Omega \cap \{|x| + |y|\xi \le L\}$*), the stronger bound* $\operatorname{dist}(\nabla u, \{A, B\}) \le ct^{1/2}\varepsilon(1 + t)$ *holds.*

By finitely piecewise affine we mean that the domain can be decomposed in finitely many pieces such that the function is affine on each of them. In the following proofs we just call them piecewise affine for simplicity.

Proof. Consider the simple laminate on the set $[-L, L] \times [-H, H]$ defined by $u_L(0, 0) = (0, 0)$ and

$$\nabla u_L(x, y) = \begin{cases} A & \text{for } |y| < H\lambda \\ B & \text{for } H\lambda < |y| < H \end{cases} \tag{A.2}$$

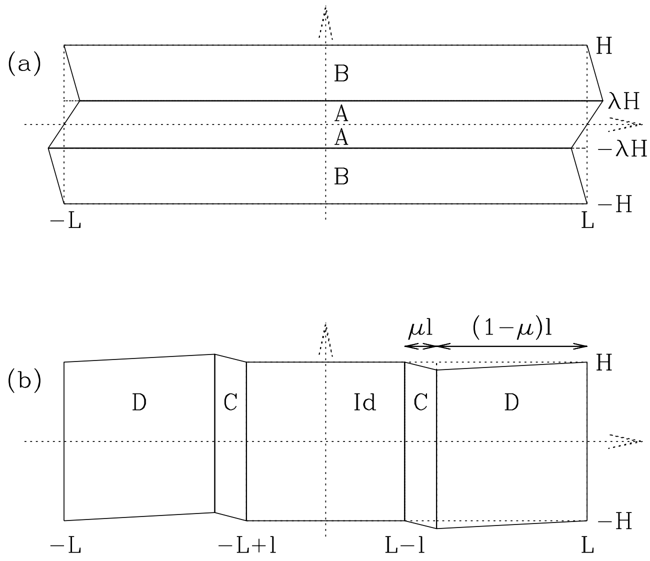

FIGURE A.1. The two laminates whose composition is used in the construction of Lemma A.2. Dotted curves: reference configuration. Full curves: deformed configuration. (a): u_L, defined in (A.2), is a horizontal shear, which satisfies the top and bottom boundary conditions, its gradient takes values A and B. (b): v, defined in (A.3), is a vertical shear, which is the identity in the central part, and satisfies the left and right boundary conditions. Its gradient takes values Id, C and D which are all close to the identity.

(see Fig. A.1a), which satisfies the boundary condition on the top and bottom sides, but not on the left and right ones. The construction is based on a modification of u_L in the region $L - \xi H < |x| < L$ in order to enforce the boundary conditions on the latter two sides. This is done by first composing u_L with another piecewise affine function, and then modifying further a boundary layer.

More precisely, let $l = H\xi$, and consider the function defined by $v(0,0) = (0,0)$ and

$$\nabla v(x,y) = \begin{cases} \mathrm{Id} & \text{for } |x| < L - l \\ C & \text{for } L - l < |x| < L - (1-\mu)l \\ D & \text{for } L - (1-\mu)l < |x| < L \end{cases} \tag{A.3}$$

(see Fig. A.1b) where $C = \mathrm{Id} - q(1-\mu)e_2 \otimes e_1$, and $D = \mathrm{Id} + q\mu e_2 \otimes e_1$. The small parameters q and μ will be chosen later. Note that ∇v has unit determinant, is everywhere close to the identity, and that v is the identical map for $|x| < L - l$

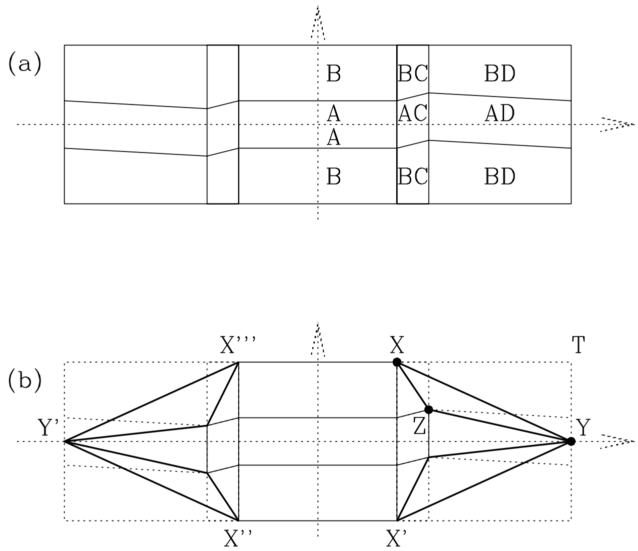

FIGURE A.2. Construction used in proving Lemma A.2. (a): the composition of u_L with v generates a piecewise affine function, which on the considered domain takes six different gradients. (b): the four triangles on which the function is replaced by the affine interpolation.

and for $x = \pm L$. Consider now the composition $u_L \circ v$. This is a piecewise affine map, whose gradient has unit determinant and is close to A or B everywhere. Quantitatively, we get

$$|AC - A| = |A||C - \mathrm{Id}| \leq (2+t)q \tag{A.4}$$

and the same estimate for the other products.

The pieces on which $u_L \circ v$ is affine are shown in Figure A.2a. Consider the four triangles shown in Fig. A.2b, where for definiteness we focus on the one in the first quadrant, XYZ. We now set $u = u_L \circ v$ in the central region, and equal to the affine interpolation between the values of $u_L \circ v$ at the three corners in each of the four triangles (XYZ and the analogous ones in the other quadrants). Since in X and Y both u_L and v are the identical map, this function satisfies the boundary condition $u(x, y) = (x, y)$ on the boundary of the domain $\omega = XYX'X''Y'X'''$. It can therefore be extended to the full domain by using the identical map in the remaining triangles (XYT and its copies in the other quadrants). This results in a continuous, piecewise affine map. It only remains to check that the two new gradients used in the four boundary triangles have unit determinant and are close to B. Consider XYZ, for definiteness. The map is the identity on the side XY. The determinant of the affine interpolation is unity if the area is conserved, namely, if the vertex Z moves parallel to XY. This corresponds to the condition that $u(Z)-Z=(t\lambda(1-\lambda)H,-q\mu(1-\mu)l)$ is parallel to $(l,-H)$, namely, that $ql^2\mu(1-\mu)=t\lambda(1-\lambda)H^2$. This permits to determine q as a function of μ and $\xi=l/H$. If this relation is satisfied, the gradient in the triangle is an area-preserving shear along XY. More precisely, we get $u(Z)-Z=\lambda(1-\lambda)Ht(1,-\xi^{-1})$, and

$$\nabla u|_{XYZ}=\mathrm{Id}-t\lambda p(1,-\xi^{-1})\otimes(\xi^{-1},1)$$

for some p. For large ξ, we see that p approaches unity and this gradient approaches B. To determine quantitatively the distance from B we compute p from the relation

$$u(Z)-Z=(\nabla u|_{XYZ}-\mathrm{Id})\,(Z-X)$$

which follows from $u(X)=X$ and the fact that u is affine in this triangle. A straightforward computation leads to

$$\frac{1}{p}=1-\frac{\mu}{1-\lambda}-\frac{\lambda t}{\xi}\,. \tag{A.5}$$

We remark that the fact that the right-hand side of this relation is positive shows that Z lies below the line XY, as was drawn in Figure A.2b. Then, notice that

$$\Big|\nabla u|_{XYZ}-B\Big|\le t|p-1|+2\frac{tp}{\xi} \tag{A.6}$$

Combining (A.4), (A.5) and (A.6), we get

$$\mathrm{dist}(\nabla u,\{A,B\})\le c\frac{\mu}{1-\lambda}+c\frac{t}{\xi}+(2+t)q$$

Finally, choose $\mu=\lambda(1-\lambda)t^{1/2}\varepsilon$ and $\xi\ge 1/\varepsilon$. We then get $q<2t^{1/2}/\xi^2\varepsilon\le 2t^{1/2}\varepsilon$, and we can check that all conditions are satisfied.

In summary, we have obtained a construction on ω which is composed by 15 affine pieces, uses 7 different gradients, all with unit determinant and close either to A or to B, matches continuously with the identity on the boundary, and is a laminate in the central part of the domain. By taking $L=l$ we can eliminate the central region, and reduce to 12 affine pieces and 5 gradients. □

Proof of Theorem 3.5. We first reduce to a canonical form by a change of variables. Let $F=\lambda A+(1-\lambda)B$. We set $\tilde u(x)=Qu(F^{-1}Q^Tx)$, where $\tilde u:\tilde\Omega=FQ\Omega\to\mathbb{R}^2$, and $Q\in SO(2)$ is such that $\tilde A=QAF^{-1}Q^T$ is an upper triangular matrix as in (A.1). The latter exists since AF^{-1} is a rank-one perturbation of the identity. By the rank-one condition, the same holds for $\tilde B=QBF^{-1}Q^T$. The vector v is in turn replaced by $\tilde v=QFv$. The boundary condition becomes $\tilde u(x)=x$ on $\partial\tilde\Omega$.

From now on we assume that $A=\mathrm{Id}+t(1-\lambda)e_x\otimes e_y$, and $B=\mathrm{Id}-t\lambda e_x\otimes e_y$. We can further assume $t>0$, since $t=0$ is trivial, and if $t<0$ we can swap A and B. We cannot, however, set $t=1$, since that would require a non-isometric change of variables also in the target, which would change the norm in $|\nabla uv|$.

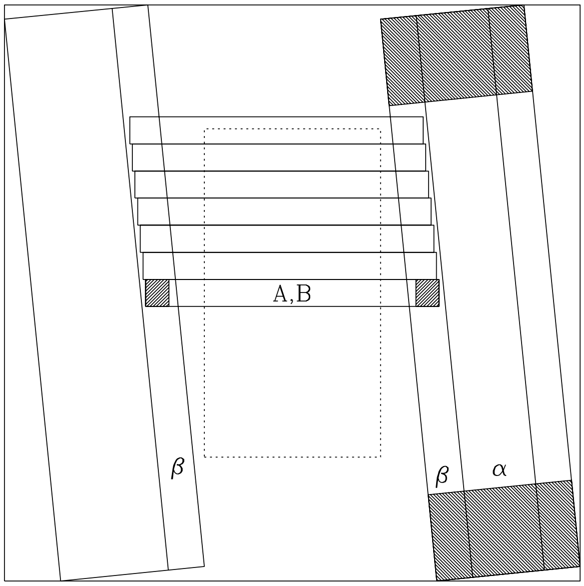

FIGURE A.3. Construction used in the proof of Theorem 3.5. The two oblique rectangles are R_1 and R_2, outside which u_1 is the identity map. In the part toward the interior of the square Ω, $\nabla u_1 = A_1$. The dashed parts are where the u_1 differs from a laminate. The thin rectangle in the center is S, which is one of the blocks used in the construction of u_2. The construction differs from a laminate only in the two dashed end parts, which are contained in the region where $\nabla u_1 = A_1$. The dashed rectangle is Ω_δ, where the final construction is a laminate between A and B.

The vector v is uniquely determined by A and B, up to a factor. Indeed, let $v = (v_x, v_y)$. Then, $|Av|^2 = |Bv|^2$ gives $2t(1-\lambda)v_xv_y + t^2(1-\lambda)^2v_y^2 = -2t\lambda v_xv_y + t^2\lambda^2v_y^2$. This has two solutions, which up to a scaling are $v_1 = (t(\lambda - \frac{1}{2}), 1)$ and $v_2 = (1, 0)$. The assumption $Av \neq Bv$ eliminates the second one, hence from now on we can assume $v = (t(\lambda - \frac{1}{2}), 1)$.

It is sufficient to prove the statement for the reference square $(-1, 1)^2$. Indeed, by Vitali's theorem we can find N and k such that Ω can be covered by N disjoint copies of $(-2^{-k}, 2^{-k})^2$, plus a remainder Ω' with measure less than $\delta/2$. The construction for Ω is then done using the result $\tilde{u}$ for the reference square, with $\tilde{\delta} = \min(\delta, 2^{2k-1}\delta/N)$, scaled and translated onto each of the N small squares, and $u(x) = x$ in Ω'.

For the case $\Omega = (-1, 1)^2$ we construct u explicitly, as the composition of two deformations. The first deformation is a laminate between A and B on a large part of the domain (containing Ω_δ), satisfies the determinant constraint, and has $\mathrm{dist}(\nabla u, [A, B])$ small. In the central part $|(\nabla u)v| = |Av|$, in the boundary layer the the inequality $|(\nabla u)v| \le |Av|$ is violated by a small amount (controlled by $\mathrm{dist}(\nabla u, [A, B])$). The second deformation is the identity in the central part (covering Ω_δ), and corrects the inequality in the region around the boundary.

We start with the latter, which will be a single step of a laminate in each of two rectangles, close to the left and right boundaries of the domain Ω, as illustrated in Figure A.3. For small θ and q, consider the matrices $\alpha = \mathrm{Id} + qn \otimes n^\perp$ and $\beta = \mathrm{Id} - qn \otimes n^\perp$, where $n = (-\sin\theta, \cos\theta)$. We shall choose them so that

$$|A\beta v| < |Av| \quad \text{and} \quad |B\beta v| < |Bv| \,. \tag{A.7}$$

To show that this is possible, consider for $C \in \{A, B\}$ the expansion

$$|C\beta v|^2 - |Cv|^2 = -2q(n \cdot C^T Cv)(n^\perp \cdot v) + O(q^2) \,.$$

We first choose $\theta < \delta/20$ such that $n^\perp \cdot v \ne 0$ and $n \cdot A^T Av$, $n \cdot B^T Bv > 1/2$. This is possible since for $\theta = 0$ they are $1 + \frac{t^2}{2}(1 - \lambda)$ and $1 + \frac{t^2}{2}\lambda$ and therefore bigger than 1. Then it is clear that for q small enough we find $\eta > 0$ such that

$$|C\beta v| \le |Av| - \eta \tag{A.8}$$

for all $C \in [A, B]$, by convexity. The function u_1 will be constructed by Lemma A.2 using the pair (α, β) (after suitable rotation, the precise domain is specified below) with $\lambda_1 = 1/2$ and $\varepsilon_1 = 1$. For $q < 1$, the resulting ξ_1 can be taken to be a global constant. For q small enough, we have $|\nabla u_1 v| \le |Av|$, since Lemma A.2 gives

$$|\nabla u_1 v| \le (1 + |\nabla u_1 - \mathrm{Id}|)\, |v| \le (1 + cq^{1/2})|v| \,,$$

and $|v| = |\lambda Av + (1 - \lambda)Bv| < |Av|$ by hypothesis (this is where we need that $Av \ne Bv$).

We now define the domains where this construction is used, which have to be rectangles with sides parallel to n and $n^\perp$. Let R_1 be such a rectangle, with height (along $n^\perp$) $\delta/2$, inscribed into $[1 - \delta, 1] \times [-1, 1]$. R_2 is the same on the other side (see Figure A.3). Since the other dimension of the rectangles is larger than 1, the hypothesis of Lemma A.2 are satisfied for $\delta < \xi_1$ (recall that ξ_1 was fixed). We define u_1 by application of Lemma A.2 to R_1 and R_2, and extend it by $u_1(x) = x$ outside. At this point, u_1 satisfies the following: (i) $|(\nabla u_1)v| \le |Av|$, $\det \nabla u_1 = 1$, $|\nabla u_1 - \mathrm{Id}| \le cq^{1/2}$ everywhere; (ii) $\nabla u_1 = \mathrm{Id}$ for $|x| \le 1 - \delta$, (iii) $\nabla u_1 = \beta$ in the region at distance less than $\delta/8$ from the left side of R_1, and $|y| < 1 - \delta\xi_1$.

To construct the other function, which is a laminate between A and B in the central part of Ω, consider for $h > 0$ small and $|y_0| \le 1 - \delta\xi_1 - h$ a strip of the form $S_{y_0} = [x_0, x_1] \times [y_0 - h, y_0 + h]$. It is clear that we can choose x_0 in $[-1, -1 + \delta]$ and x_1 in $[1 - \delta, 1]$ such that ∇u_1 takes only the values Id and β on S_{y_0}, and that on the two regions $S^1_{y_0} = [x_0, x_0 + \delta/16] \times [y_0 - h, y_0 + h]$ and $S^2_{y_0} = [x_1 - \delta/16, x_1] \times [y_0 - h, y_0 + h]$ it takes value β. We now want to apply

Lemma A.2 to the matrices A and B in S_{y_0}, so that the result is affine outside of $S^1_{y_0}$ and $S^2_{y_0}$, and ε small enough that $c(1+t)t^{1/2}\varepsilon \le \delta$,

$$|A\beta v| + c(1+t)t^{1/2}\varepsilon|\beta v| \le |Av|\,, \tag{A.9}$$

and the same with A replaced by B. The latter is possible by (A.7). Let ε_2 be the largest ε that satisfies these conditions. Then, the corresponding ξ_2 (as in the statement of Lemma A.2) gives the bound h_0 on the lamination period.

For $h < h_0$, we cover $(-1+\delta, 1-\delta) \times (-1+\delta\xi_1, 1+\delta\xi_1)$ with stripes of the form S_{y_0+nh}, and construct u_2 in each of them by application of Lemma A.2 to the matrices A and B, with $\varepsilon = \varepsilon_2$. Outside the stripes, we define $u_2(x) = x$.

The final function will be $u = u_2 \circ u_1$. Now, the gradient

$$\nabla u = (\nabla u_2 \circ u_1)\nabla u_1$$

automatically satisfies the determinant constraint, and the closeness to $[A, B]$ since ∇u_2 is close to it and ∇u_1 is close to Id. Now we check the inequality. In the outer part of the domain, $\nabla u_2 = \mathrm{Id}$ and we had seen that $|(\nabla u_1)v| \le |Av|$. In the intermediate part, ∇u_2 is close to $[A, B]$ and $\nabla u_1 = \beta$, and we get the desired inequality by (A.9). In the inner part, finally, ∇u_2 takes values A and B, and $\nabla u_1 = \mathrm{Id}$, hence equality holds. This concludes the proof of the theorem. □

Remark A.3. The same result, with the minor changes in the proof, holds also with the condition $|(\nabla u)v| \le |Av|$ replaced by $f(\nabla u) \le 0$, for any smooth f which obeys (i) $f(A) = f(B) = 0$; (ii) $f(\lambda A + (1-\lambda)B) < 0$ for $0 < \lambda < 1$; and (iii) there is $\theta \ne \pi/2$ such that $df(A(\mathrm{Id}+tn\otimes n^\perp))/dt$ and $df(B(\mathrm{Id}+tn\otimes n^\perp))/dt$ evaluated in 0 are both nonzero and have the same sign.

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