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On the regularity of critical points of polyconvex functionals

by

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Abstract

In this paper we are concerned with the question of regularity of critical points for functionals of the type

$$I[u] = \int_\Omega F(Du)\,\mathrm{d}x.$$

We construct a smooth, strongly polyconvex $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$, and Lipschitzian weak solutions $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ to the corresponding Euler-Lagrange system, which are nowhere C^1. Moreover we show that F can be chosen in a way that these irregular weak solutions are weak local minimisers.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be the unit ball. We study critical points of the functional

$$I[u] = \int_\Omega F(Du)\,\mathrm{d}x,$$

where $u : \Omega \to \mathbb{R}^2$ and $F : \mathbb{R}^{2\times 2} \mapsto \mathbb{R}$ is a smooth function with bounded second derivatives. The associated Euler-Lagrange equations can be written as

$$\operatorname{div} DF(Du) = 0. \tag{1}$$

In [MŠ03] S. Müller and V. Šverák constructed an example of a *strongly quasiconvex* F so that the corresponding 2×2 system (1) admits weak solutions that are Lipschitz but not C^1 in any open subset of Ω. Their method is based on a modification of M. Gromov's *convex integration* [Nas54, Kui55, Gro86] combined with ideas originating from L. Tartar's programme of *compensated compactness* [Tar79]. A function $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ is strongly quasiconvex if for some $\gamma > 0$ the inequality

$$\int_{\mathbb{T}^n} F(X + D\eta) - F(X)\,\mathrm{d}x \geq \gamma \int_{\mathbb{T}^n} |D\eta|^2\,\mathrm{d}x$$

holds for all $X \in \mathbb{R}^{m\times n}$ and all periodic Lipschitz mappings $\eta : \mathbb{T}^n \to \mathbb{R}^m$. Due to a well known result of L. C. Evans [Eva86], global minimisers of the functional

$I[u]$, assuming F is strongly quasiconvex, are smooth outside a closed subset of Ω of Lebesgue measure zero. This result was extended by J. Kristensen and A. Taheri [KT01] to the case of strong local minimisers (local with respect to variations in $W^{1,p}$ with $p < \infty$). Kristensen and Taheri also show that the counterexample of Müller and Šverák can be extended to weak local minimisers (where one admits only variations small in $W^{1,\infty}$), so that weak local minimisers of strongly quasiconvex functionals can be nowhere C^1.

In this paper we extend the aforementioned result by proving the analogue for strongly polyconvex integrands:

THEOREM 1. *Let Ω be the unit ball in $\mathbb{R}^2$. There exists a smooth, strongly polyconvex function $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ with bounded second derivatives, such that the corresponding 2×2 elliptic system*

$$\operatorname{div} DF(Du) = 0$$

admits weak solutions $u : \Omega \to \mathbb{R}^2$, which are Lipschitz but not C^1 in any open subset of Ω. Moreover F can be chosen so that these weak solutions are weak local minimisers of the corresponding functional $I[u] = \int_\Omega F(Du)\,dx$.

A function is said to be polyconvex if it is a convex function of the minors. More precisely $F : \mathbb{R}^{2\times 2} \mapsto \mathbb{R}$ is said to be *strongly polyconvex* if there exists a convex function $G : \mathbb{R}^5 \mapsto \mathbb{R}$ and $\epsilon > 0$ so that

$$F(X) = \epsilon |X|^2 + G(X, \det X).$$

Polyconvexity is a commonly used structural assumption in mathematical models of elasticity [Bal77, BJ87, CK88]. It is strictly stronger then quasiconvexity. We also remark that if we strengthen the structural assumption by assuming that F is uniformly convex, weak solutions to the 2×2 system (1) are smooth due to a classical result of Morrey [Mor66].

We follow the strategy of S. Müller and V. Šverák. It should be pointed out that a somewhat similar approach has already been pursued by V. Scheffer in his thesis [Sch74] in 1974, which unfortunately never appeared in a journal. He constructed $W^{1,1}$ solutions to an equation of the type (1), albeit with F only satisfying the strong Legendre-Hadamard condition.

We show in Section 3 that under the hypothesis that there exists a T_N configuration in a certain set of matrices arising from the PDE (see (4)) and assuming a certain non-degeneracy (condition (C)), Lipschitz weak solutions can be constructed to the PDE that are nowhere C^1. The construction (Proposition 2) is the same as that appearing in [MŠ03], we give the proof for completeness.

The main difficulty is then finding a function F which satisfies the structural requirement of polyconvexity and still allows for the construction to be carried out. In Section 4 we show how this difficulty can be overcome by essentially reducing the problem to linear programming. Finally, in Section 5 we show how the necessary non-degeneracy in the construction can be achieved in a general situation.

2 T_N configurations

As pointed out in [MŠ03], whether or not weak solutions to the PDE (1) can be constructed via convex integration (resulting in nowhere C^1 solutions) depends mainly on geometrical-combinatorial properties of the mapping $X \to DF(X)$. In order to explain this in detail, in this section we recall the relevant definitions and results regarding *rank-one convexity.* A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is rank-one convex if f is convex along each rank-one line. The *rank-one convex hull* of a set of matrices is defined by separation with rank-one convex functions, as follows. For a compact set $K \subset \mathbb{R}^{m\times n}$ we define

$$K^{rc} := \{X \in \mathbb{R}^{m\times n} : f(X) \leq \sup_K f \quad \forall f : \mathbb{R}^{m\times n} \mapsto \mathbb{R} \quad \text{rank-one convex}\},$$

and for general sets

$$E^{rc} := \bigcup_{K\subset E \text{ compact}} K^{rc}.$$

The dual objects to rank-one convex functions are a subclass of probability measures supported on $\mathbb{R}^{m\times n}$ called *laminates* (see [Ped93]). That is, a probability measure ν on the space of $m \times n$ matrices is a laminate if

$$\langle \nu, f\rangle \geq f(\bar{\nu}) \quad \text{for all rank-one convex } f : \mathbb{R}^{m\times n} \mapsto \mathbb{R},$$

where $\bar{\nu}$ denotes the barycenter of the measure ν. The set of barycenters of laminates with support in a fixed compact set K is exactly the rank-one convex hull K^{rc}.

It is of fundamental importance, in view of applications to elliptic PDEs, that the rank-one convex hull of a set K can be nontrivial (i.e. strictly larger than K) even if K contains no rank-one connections, that is, even if $\operatorname{rank}(X - Y) > 1$ for any two distinct $X, Y \in K$. This fact has been observed independently by a number of authors in different contexts (e.g. [Sch74, AH86, CT93, Tar93, NM91]), and can be illustrated on an example consisting of four diagonal matrices.

$$X_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_4 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}.$$

In fact this set of matrices played a crucial role in the construction in [MŠ03]. The important property is the following cyclic structure:

Definition 1 (T_N Configuration).
An ordered set of $N \geq 4$ matrices $\{X_i\}_{i=1}^N \subset \mathbb{R}^{m\times n}$ without rank-one connections is said form a T_N configuration if there exist matrices $P, C_i \in \mathbb{R}^{m\times n}$ and real numbers $\kappa_i > 1$ such that

$$\begin{aligned} X_1 &= P + \kappa_1 C_1 \\ X_2 &= P + C_1 + \kappa_2 C_2 \\ &\vdots \\ X_N &= P + C_1 + \ldots + C_{N-1} + \kappa_N C_N, \end{aligned} \tag{2}$$

and moreover $\operatorname{rank}(C_i) = 1$ *and* $\sum_{i=1}^N C_i = 0$.

For example a T_5 configuration can be represented by the diagram below. We emphasise that T_N configurations need not be planar.

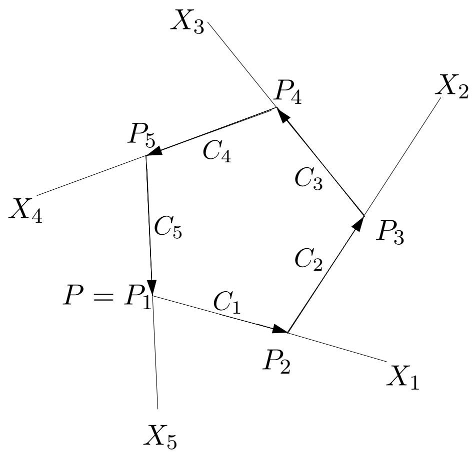

Figure 1: Schematic representation of a T_5

The following result is folklore and is included only for completeness.

LEMMA 1. *Let* $\{X_1, \ldots, X_N\}$ *be a* T_N *configuration, and for* $i = 1 \ldots N$ *let* $P_i = P + C_1 + \cdots + C_{i-1}$ *(so that* $P_1 = P$*). Then*

$$\{P_1, \ldots, P_N\} \subset \{X_1, \ldots, X_N\}^{rc}.$$

In particular for each $k = 1, \ldots, N$ *there exist numbers* $\nu_i^{(k)} \in (0,1)$ *so that the probability measures*

$$\nu^{(k)} = \sum_{i=1}^N \nu_i^{(k)} \delta_{X_i}$$

are laminates with barycenter $\bar{\nu}^{(k)} = P_k$.

It is well known that T_N configurations form locally a manifold in the space of ordered N-tuples of matrices. In the 2×2 case this manifold has the same dimension as the ambient space $(\mathbb{R}^{2\times 2})^N$, in other words T_N configurations are stable with respect to small perturbations. In higher dimensions this is no longer true, but using the implicit function theorem together with an easy dimension counting, one can find the right dimension for manifolds formed by T_N configurations. For the case $N = 4$ this has been done in [MŠ03] (in Section 4.2) for $\mathbb{R}^{4\times 2}$ and in [Kir03] (Proposition 4.26) for $\mathbb{R}^{2\times 2}$. Here we essentially repeat the proof to record the necessary result for general $N \geq 4$ in $\mathbb{R}^{4\times 2}$.

Lemma 2. *(Stability of T_N in $\mathbb{R}^{4\times 2}$) Suppose the ordered set of matrices*

$$(X_1^0, \dots, X_N^0) \in (\mathbb{R}^{4\times 2})^N$$

is a T_N configuration. Then locally around $(X_1^0, \dots, X_N^0)$ there exists a smooth manifold $\mathcal{M}_N \subset (\mathbb{R}^{4\times 2})^N$ of dimension $6N$ such that all N-tuples

$$(X_1, \dots, X_N) \in \mathcal{M}_N$$

are T_N-configurations.

Proof. Suppose (P^0, C_i^0, κ_i^0) is the parametrisation of $\{X_i^0\}$ corresponding to (2), in other words (P^0, C_i^0, κ_i^0) is a solution to the equations (2) with LHS given by $\{X_i^0\}$. We show that the set of (P, C_i, κ_i) nearby satisfying $\operatorname{rank} C_i = 1$ and $\sum_i C_i = 0$ is a manifold of dimension $6N$, using the implicit function theorem.

Write $C_i^0 = a_i^0 \otimes b_i^0$ for $a_i^0 \in \mathbb{R}^4$, $b_i^0 \in \mathbb{R}^2$, and $p = (P, a_i, b_i, \kappa_i)$. Consider the map

$$\Phi : (\mathbb{R}^4 \times \mathbb{R}^2)^N \to \mathbb{R}^{4\times 2}, \quad \Phi\big((a_1, b_1), \dots, (a_N, b_N)\big) = \sum_{i=1}^N a_i \otimes b_i.$$

The derivative at (a_i^0, b_i^0) is given by

$$D\Phi(a^0, b^0)[a, b] = \sum_{i=1}^N (a_i \otimes b_i^0 + a_i^0 \otimes b_i).$$

Since $\operatorname{rank}(C_1^0 - C_2^0) = 2$, we see that $\{b_1^0, b_2^0\}$ is a basis for $\mathbb{R}^2$. Hence $D\Phi(a^0, b^0)$ is surjective and thus full rank 8. So $\Phi^{-1}(0)$ is locally a $(6N-8)$-dimensional manifold in $(\mathbb{R}^4 \times \mathbb{R}^2)^N$, invariant under $(a_i, b_i) \mapsto (\lambda_i a_i, \frac{1}{\lambda_i} b_i)$. Hence the image of $(a_i, b_i) \mapsto (a_i \otimes b_i)$ restricted to $\Phi^{-1}(0)$ is locally a $(5N-8)$ dimensional manifold. This, together with the parameters P and κ_i gives the required $6N$-dimensional local manifold. Q.E.D.

3 Solutions by Convex Integration

Following [Šve95] we can rewrite the 2×2 system

$$\operatorname{div} DF(Du) = 0 \tag{3}$$

as a first order differential inclusion. Namely, we note that in two dimensions the divergence-free field $DF(Du)$ is a rotated curl-free field,

$$\operatorname{curl} DF(Du)J = 0\,, \text{ where } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since the domain Ω is simply connected, there exists a potential $\tilde{u} : \Omega \to \mathbb{R}^2$ such that $DF(Du)J = D\tilde{u}$. Thus with $w = \begin{pmatrix} u \\ \tilde{u} \end{pmatrix}$ we have that (3) is equivalent to the inclusion

$$Dw(x) \in K\,, \qquad \text{with } K = \left\{ \begin{pmatrix} X \\ DF(X)J \end{pmatrix} : X \in \mathbb{R}^{2\times 2} \right\}. \tag{4}$$

Note that K is by the assumptions a smooth 4-dimensional manifold in $\mathbb{R}^{4\times 2}$. To emphasize the dependence of K on the function F we will occasionally also write K_F. If F satisfies the strong Legendre-Hadamard condition, then K_F is elliptic in the sense that the tangent space at any point contains no rank-one lines. Indeed, the tangent space at a point is given by

$$T_{X_0}K = \left\{ \begin{pmatrix} X \\ D^2F(X_0)XJ \end{pmatrix} : X \in \mathbb{R}^{2\times 2} \right\},$$

therefore $T_{X_0}K$ contains rank-one matrices if and only if there exists $a, b, n \in \mathbb{R}^2$ and $X_0 \in \mathbb{R}^{2\times 2}$ such that

$$D^2F(X_0)(a \otimes n)J = b \otimes n.$$

Using that $(a \otimes n)J = a \otimes n^\perp$ we get from the strong Legendre-Hadamard condition

$$0 < \left\langle D^2F(X_0)a \otimes n^\perp, a \otimes n^\perp \right\rangle = \left\langle b \otimes n, a \otimes n^\perp \right\rangle = 0,$$

a contradiction. In fact, by an observation of J. M. Ball in [Bal80] more is true: there are no rank-one connections in K. The building block is the following result from [MŠ03]:

Proposition 1.
Let $U \subset \mathbb{R}^{m\times n}$ be open and bounded, and let $A \in U^{rc}$. Then for any $\delta > 0$ there exists a piecewise affine Lipschitz map $w : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\begin{aligned} Dw(x) &\in U && \text{in } \Omega \text{ a.e.}, \\ w(x) &= Ax && \text{on } \partial\Omega, \\ |w(x) - Ax| &< \delta && \text{in } \Omega. \end{aligned} \tag{5}$$

Moreover if A is the barycenter of a laminate ν supported on a finite subset of U, with $\nu = \sum \nu_i \delta_{Z_i}$, then for each $\epsilon > 0$ we can choose w in addition so that

$$|\{x \in \Omega : \ \mathrm{dist}(Dw(x), Z_i) < \epsilon\}| = \nu_i |\Omega|. \tag{6}$$

The basic philosophy is to find enough T_N configurations in K so that they "generate" open sets and use Proposition 1 iteratively. To do this, it suffices in general to find just one T_N configuration and combine it with a transversality argument, which yields a submanifold of T_N configurations in K. To explain this, recall Lemma 2 which says that locally around an ordered N-tuple

$$(X_1^0, \ldots, X_N^0) \in \left(\mathbb{R}^{4\times 2}\right)^N$$

which is a T_N-configuration, there exists a smooth manifold of (ordered) N-tuples, $\mathcal{M}_N$, consisting of T_N configurations. Moreover $\dim \mathcal{M}_N = 6N$. Let

$$\mathcal{K}_F = K_F \times \cdots \times K_F$$

be the N-fold Cartesian product of the manifold K_F, so that $\mathcal{K}_F$ is a $4N$-dimensional smooth manifold in $\left(\mathbb{R}^{4\times 2}\right)^N$.
Define the maps π_k, $\phi_k : \mathcal{M}_N \to \mathbb{R}^{4\times 2}$ as

$$\pi_k(Z_1, \ldots, Z_N) = P_k \quad \text{and} \quad \phi_k(Z_1, \ldots, Z_N) = Z_k \tag{7}$$

for $k = 1, \ldots, N$, where P_k is as in Lemma 1.

Let us recall the basic facts about transversality (a possible reference is [GP74]): suppose two smooth manifolds $\mathcal{M}$ and $\mathcal{K}$ embedded in $\mathbb{R}^d$ intersect at a point z. The intersection is said to be *transversal* if the tangent spaces at the point z satisfy

$$T_z\mathcal{M} + T_z\mathcal{K} = \mathbb{R}^d. \tag{8}$$

A direct consequence of the implicit function theorem is that if $\mathcal{M}$ and $\mathcal{K}$ intersect transversely at z, then locally the intersection $\mathcal{M} \cap \mathcal{K}$ is a smooth manifold. Furthermore $\dim \mathcal{M} \cap \mathcal{K} = \dim \mathcal{M} + \dim \mathcal{K} - d$.

Therefore in our case if $\mathcal{M}_N$ and $\mathcal{K}_F$ intersect transversely then the intersection is a manifold of dimension $2N$. As $N \geq 4$, we can expect that generically the map π_k restricted to $\mathcal{M}_N \cap \mathcal{K}_F$ is a submersion. That is, the image under π_k of the intersection (which by Lemma 1 is contained in the rank-one convex hull of K_F) is an open set. Now we formally define the necessary condition for this genericity:

Definition 2 (Condition (C)).

Suppose $F \in C^2(\mathbb{R}^{2\times 2})$ is such that K_F contains a T_N configuration $\{Z_i\}$ and $\mathcal{M}_N$ is the manifold of T_N configurations given by Lemma 2. If $\mathcal{M}_N$ and $\mathcal{K}_F$ intersect transversely, and if for each $k = 1, \ldots, N$ the map

$$\pi_k : (Z_1, \ldots, Z_N) \mapsto P_k$$

is a local submersion on $\mathcal{M}_N \cap \mathcal{K}_F$ then F is said to satisfy condition (C) at $\{Z_i\}$.

After these preliminary considerations, we are ready for the main construction which appears in [MŠ03]:

Proposition 2.

Suppose $F \in C^2(\mathbb{R}^{2\times 2})$ is such that the associated manifold K given by (4) contains a T_N configuration $\{Z_1^0, \ldots, Z_N^0\}$ and suppose F satisfies condition (C) at $\{Z_i^0\}$. Let $P_0 \in \{Z_1^0, \ldots, Z_N^0\}^{rc}$. Then for any $\delta > 0$ there exists a Lipschitz map $w : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^4$ with the following properties:

1. *$Dw(x) \in K \cap \left(\bigcup_{k=1}^N B_\delta(Z_k^0)\right)$ a.e. in Ω,*

2. *In particular $u = (w_1, w_2)$ is a weak solution to (3),*

3. *$w(x) = P_0 x$ on $\partial\Omega$, and $|w(x) - P_0 x| < \delta$ in Ω,*

4. *Dw has essential oscillation of order 1 in any subdomain of Ω, so that w is nowhere C^1.*

Proof. We will denote the manifold of T_N configurations given by Lemma 2 near $z_0 = (Z_1^0, \dots, Z_N^0)$ by $\mathcal{M}$ (i.e. dropping the subscript N).

The aim is to define a sequence of *approximate solutions* $w^{(i)}$ using Proposition 1. To this end we need to define a sequence of open sets $U_i \subset \mathbb{R}^{4\times 2}$ such that $U_i \subset U_{i+1}^{\mathrm{rc}}$ and $U_i \to K_F$ in the sense that if $Z_i \in U_i$ with $Z_i \to Z$ then $Z \in K_F$. In Gromov's original terminology such a sequence of sets is called an *in-approximation*. To define these open sets we use the maps π_k and ϕ_k (see (7)), since condition (C) guarantees that the image of π_k is an open set. To move from these open sets towards K we take a convex combination of π_k and ϕ_k.

By our assumptions $D\pi_k$ restricted to the tangent space $T_{z_0}(\mathcal{M}\cap\mathcal{K})$ has full rank, and so for all but finitely many values of λ the linear map

$$(1-\lambda)D\pi_k + \lambda D\phi_k \tag{9}$$

has full rank. Let $\lambda_i \in (0,1)$ be an increasing sequence with $\lambda_i \to 1$ so that the maps in (9) have full rank for all i and k. Let

$$\Phi_i^k \stackrel{\text{def}}{=} (1-\lambda_i)\pi_k + \lambda_i \phi_k.$$

Then $\Phi_i^k : \mathcal{M}\cap\mathcal{K} \to \mathbb{R}^{4\times 2}$ are local submersions. In order to ensure that in addition $U_i \subset U_{i+1}^{\mathrm{rc}}$, we choose an increasing sequence of relatively open sets

$$\mathcal{O}_{i-1} \subset \mathcal{O}_i \subset \mathcal{M}\cap\mathcal{K}\cap\left(B_\delta(Z_1^0)\times\cdots\times B_\delta(Z_N^0)\right)$$

and let $U_{i,k} = \Phi_i^k(\mathcal{O}_i)$, $U_i = \bigcup_{k=1}^N U_{i,k}$. By adjusting the sequence λ_i if necessary, we may assume that $P_0 \in U_1^{rc}$.

In order to apply Proposition 1, we pick laminates in the following way. Let $Z \in U_i$, say $Z \in U_{i,1}$. By our construction, there exists $(Z_1, \dots, Z_N) \in \mathcal{O}_i$ forming a T_N such that Z is contained in the segment $[P_1, Z_1]$.

In Figure 2, solid lines show the original T_N contained in K, and dashed lines the perturbed T_N with $Z \in [P_1, Z_1]$. As $(Z_1, \dots, Z_N) \in \mathcal{O}_{i+1}$ also, there exist new points $\tilde{Z}_k \in U_{i+1,k}$ on the segments $[P_k, Z_k]$. But then, since $\tilde{Z}_k$ themselves form a T_N with $\pi_k(\tilde{Z}_1, \dots, \tilde{Z}_N) = P_k$, there exist coefficients $\nu_i \in (0,1)$ such that the probability measure

$$\nu = \sum_{k=1}^N \nu_k \delta_{\tilde{Z}_k}$$

is a laminate with barycenter P_1. Consequently

$$\mu \overset{\text{def}}{=} \frac{\lambda_i}{\lambda_{i+1}}\delta_{\tilde{Z}_1} + \big(1 - \frac{\lambda_i}{\lambda_{i+1}}\big)\nu$$

is a laminate supported in U_{i+1} with barycenter Z. Moreover

$$\mu(U_{i+1,1}) > \frac{\lambda_i}{\lambda_{i+1}}. \tag{10}$$

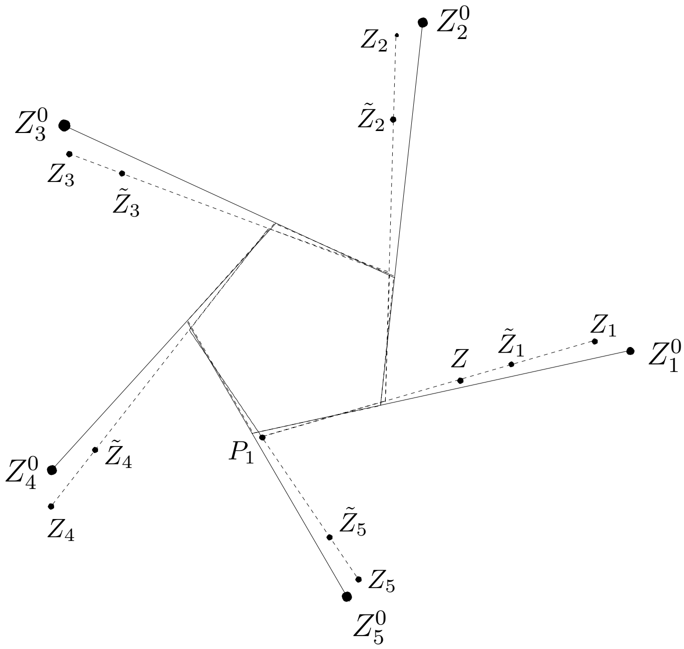

Figure 2: Original and perturbed T_5's

For any subdomain $\tilde{\Omega} \subset \Omega$ Proposition 1 now gives a function $w : \tilde{\Omega} \to \mathbb{R}^4$ with the following properties:

(i) $w(x) = Zx$ on $\partial\tilde{\Omega}$, and $Dw(x) \in U_{i+1}$ in $\tilde{\Omega}$,

(ii) $|w(x) - Zx| < 2^{-(i+1)}\delta$ in $\tilde{\Omega}$,

(iii) $|\{x \in \tilde{\Omega} :\ Dw(x) \in U_{i+1,1}\}| > \frac{\lambda_i}{\lambda_{i+1}}|\tilde{\Omega}|$,

(iv) $\int_{\tilde{\Omega}} |Dw - Z|\,\mathrm{d}x \le C(\lambda_{i+1} - \lambda_i)|\tilde{\Omega}|$.

Indeed, (i) and (ii) follow directly from Proposition 1 and since U_i are open sets, and (iii) follows from the estimate (10) together with (6). To prove (iv) note that by (iii) the gradient Dw takes values near $\tilde{Z}_1$ in a large portion of the domain $\tilde{\Omega}$, and $|Z - \tilde{Z}_1| = (\lambda_{i+1} - \lambda_i)|P_1 - Z_1|$. Hence

$$\begin{aligned}\int_{\tilde{\Omega}} |Dw - Z|\,\mathrm{d}x &= \int_{\{Dw \in U_{i+1,1}\}} |Dw - Z|\,\mathrm{d}x + \int_{\{Dw \notin U_{i+1,1}\}} |Dw - Z|\,\mathrm{d}x \\ &\le C|\tilde{\Omega}|(\lambda_{i+1} - \lambda_i) + C|\tilde{\Omega}|(1 - \frac{\lambda_i}{\lambda_{i+1}}) \\ &\le C(1 + \frac{1}{\lambda_1})|\tilde{\Omega}|(\lambda_{i+1} - \lambda_i).\end{aligned}$$

We are now ready to define a sequence of functions on Ω inductively in the following way. Let $w^{(0)}(x) \equiv P_0 x$. To obtain $w^{(i+1)}$ from $w^{(i)}$, decompose Ω into a union of pairwise disjoint open sets of diameter no more than $\frac{1}{i}$,

$$|\Omega \setminus \bigcup_\alpha \Omega^i_\alpha| = 0,$$

so that $w^{(i)}$ is affine in each open set. In each Ω^i_α we can apply the above construction and obtain $w^{(i+1)}$ by replacing the affine function with the newly constructed one.

That our sequence $w^{(i)}$ converges uniformly and in $W^{1,1}$ to some limit w follows from (ii) and (iv). Moreover, w is Lipschitz with $w(x) = P_0 x$ on $\partial\Omega$, $|w(x) - P_0 x| < \delta$ in Ω and

$$Dw(x) \in K \cap \big(\bigcup_{k=1}^N B_\delta(Z^0_k)\big) \qquad \text{a.e. in } \Omega.$$

To show that Dw has essential oscillation of order 1 in any open set, take an open subset $\tilde{\Omega} \subset \Omega$. For large enough i_0 there exists α such that $\Omega^{i_0}_\alpha \subset \tilde{\Omega}$. Now the way we obtain $w^{(i_0+1)}$ from $w^{(i_0)}$ means that there exist $\epsilon_k > 0$ (depending on i_0 as well) so that for each $k = 1, \dots, N$

$$|\{x \in \Omega^{i_0}_\alpha : Dw^{(i_0+1)}(x) \in U_{i_0+1,k}\}| > \epsilon_k |\Omega^{i_0}_\alpha|.$$

But then, from (iii) follows that for each $i > i_0$ and each k

$$|\{x \in \Omega^{i_0}_\alpha : Dw^{(i)}(x) \in U_{i,k}\}| > \frac{\lambda_{i-1}}{\lambda_i}\frac{\lambda_{i-2}}{\lambda_{i-1}} \dots \frac{\lambda_{i_0}}{\lambda_{i_0+1}}\epsilon_k|\Omega^{i_0}_\alpha| = \frac{\lambda_{i_0}}{\lambda_i}\epsilon_k|\Omega^{i_0}_\alpha|,$$

and passing to the limit gives

$$|\{x \in \Omega^{i_0}_\alpha : Dw(x) \in B_\delta(Z^0_k)\}| \ge \lambda_{i_0}\epsilon_k|\Omega^{i_0}_\alpha|$$

for all $k = 1, \dots, N$. This proves that

$$|\{x \in \tilde{\Omega} : Dw(x) \in B_\delta(Z^0_k)\}| > 0$$

for any open $\tilde{\Omega} \subset \Omega$ and thus Dw has non-vanishing essential oscillation in any open set. Therefore it is nowhere C^1. Q.E.D.

Following a suggestion of J. Kristensen, we immediately obtain the corollary below:

Corollary 1. *Assume, as in Proposition 2, that* $F \in C^2(\mathbb{R}^{2\times 2})$*, that* K_F *contains a* T_N *configuration* $\{Z_1^0, \ldots, Z_N^0\}$ *with* $Z_k^0 = \begin{pmatrix} X_k^0 \\ Y_k^0 \end{pmatrix}$*,* F *satisfies condition (C) at* $\{Z_i^0\}$*, and in addition that* $D^2F(X_k^0)$ *is positive definite for each* k*.*

Then for sufficiently small $\delta > 0$ *the map* w *constructed in Proposition 2 is such that* $u = (w_1, w_2)$ *is a weak local minimiser of*

$$\int_\Omega F(Du(x))\, dx.$$

In the paper [MŠ03] Müller and Šverák constructed a strongly quasiconvex function F for which K_F contains a T_4 configuration. Then they explicitly calculated the tangent space to $\mathcal{M}_4$ at the point of intersection with $\mathcal{K}_{F_0}$ to prove that a suitable perturbation can move into the non-degenerate situation (C). In Section 5 we will show that (C) can be achieved in a general situation, for any T_N configuration. In view of this, and the fact that small enough perturbations of strongly polyconvex functions remain strongly polyconvex, it is sufficient to exhibit one T_N configuration for one specific strongly polyconvex function to prove Theorem 1. We will do this in the next section, with $N = 5$.

4 Polyconvex examples

Instead of fixing a specific strongly polyconvex function and looking for T_5's in the corresponding set K_F, we look for a specific T_5 which lies in K_F for some strongly polyconvex function F. The difference is computational: for the former one has to solve 15 nonlinear equations in 25 variables, whereas the latter can be reduced to linear programming.

Lemma 3. *There exists a smooth, strongly polyconvex function* $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ *and a* T_5*-configuration* $\{Z_i\} \subset \mathbb{R}^{4\times 2}$ *such that* $\{Z_i\} \subset K_F$*. Moreover* $D^2F(X_i)$ *is positive definite for each* i*, where* $Z_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$*.*

Proof. By the definition of the set K_F, a T_N configuration $\{Z_i\} = \left\{ \begin{pmatrix} X_i \\ Y_i \end{pmatrix} \right\}$ is contained in K_F exactly if

$$DF(X_i)J = Y_i. \tag{11}$$

Recall that $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is strongly polyconvex if there exists a convex function $G : \mathbb{R}^5 \to \mathbb{R}$ and $\epsilon > 0$ such that $F(X) = \frac{\epsilon}{2}|X|^2 + G(X, \det X)$.

Therefore there exists a strongly polyconvex function F for which $Z_i \in K_F$ for all $i = 1, \ldots, N$ if and only if there exists $\epsilon > 0$ and a convex function G satisfying

$$\partial_X G(\tilde{X}_i) + \partial_d G(\tilde{X}_i) \operatorname{cof} X_i = -Y_i J - \epsilon X_i \quad \text{for } i = 1, \ldots, N. \tag{12}$$

Here ∂_d means derivative with respect to the determinant term, and for $X \in \mathbb{R}^{2\times 2}$ we write $\tilde{X} = (X, \det X) \in \mathbb{R}^5$. Suppose we are given real numbers c_i and vectors B_i, $\tilde{X}_i \in \mathbb{R}^5$ for $i = 1 \dots n$. It is well known that there exists a (smooth) convex function G with the property that $G(\tilde{X}_i) = c_i$ and $DG(\tilde{X}_i) = B_i$ if the data satisfies the system of $n(n-1)$ inequalities

$$c_j > c_i + \left\langle B_i, \tilde{X}_j - \tilde{X}_i \right\rangle_{\mathbb{R}^5} \qquad \text{for all } i \neq j. \tag{13}$$

Indeed, let $G_0(\tilde{X}) = \max_i \Big(c_i + \left\langle B_i, \tilde{X} - \tilde{X}_i \right\rangle \Big)$. Take a smooth mollifier ϕ on $\mathbb{R}^5$ supported in a small ball around the origin and satisfying $\int \phi(\tilde{Y})\,\mathrm{d}\tilde{Y} = 1$ and $\int \tilde{Y}\phi(\tilde{Y})\,\mathrm{d}\tilde{Y} = 0$. Since the inequalities (13) are strict, taking the support of ϕ sufficiently small we ensure that in a neighbourhood of each $\tilde{X}_i$

$$\begin{aligned} \phi * G_0(\tilde{X}) &= \int \big(c_i + \left\langle B_i, (\tilde{X} - \tilde{Y}) - \tilde{X}_i \right\rangle \big) \phi(\tilde{Y}) \mathrm{d}\tilde{Y} \\ &= c_i + \left\langle B_i, \tilde{X} - \tilde{X}_i \right\rangle = G_0(\tilde{X}). \end{aligned}$$

Therefore $G = \phi * G_0$ gives the required smooth and convex function.

Substituting (12) into (13) gives

$$\begin{aligned} c_j &> c_i + \left\langle B_i, \tilde{X}_j - \tilde{X}_i \right\rangle_{\mathbb{R}^5} \\ &= c_i + \left\langle \partial_X G(\tilde{X}_i), X_j - X_i \right\rangle + \partial_d G(\tilde{X}_i)(\det X_j - \det X_i) \\ &= c_i - \left\langle Y_i J + \epsilon X_i + \partial_d G(\tilde{X}_i) \mathrm{cof}\, X_i, X_j - X_i \right\rangle + \partial_d G(\tilde{X}_i)(\det X_j - \det X_i). \end{aligned}$$

Writing $d_i = \partial_d G(\tilde{X}_i)$ we see that a convex function G satisfying (12) exists if there exists real numbers c_i, d_i satisfying the system

$$c_i - c_j + d_i \det(X_i - X_j) + \langle X_i - X_j, Y_i J \rangle < -\epsilon \langle X_i, X_i - X_j \rangle. \tag{14}$$

In particular if

$$c_i - c_j + d_i \det(X_i - X_j) + \langle X_i - X_j, Y_i J \rangle < 0 \tag{15}$$

for all $i \neq j$, then we can choose $\epsilon > 0$ so that (14) is also satisfied. We conclude the proof by presenting an explicit example at the end of this section of a T_5 configuration, for which the system (15) is feasible.

In order to achieve that $D^2 F(X_i)$ is positive definite, we modify G_0 slightly. Namely, let

$$\Psi(X) = \begin{cases} \gamma |X|^2 & \text{if } |X| < \delta \\ \gamma\delta |X| & \text{if } |X| \geq \delta, \end{cases} \tag{16}$$

where $\gamma, \delta > 0$ are to be determined later. Let

$$G_0(\tilde{X}) = \max_i \Big(c_i + \left\langle B_i, \tilde{X} - \tilde{X}_i \right\rangle + \Psi(X - X_i) \Big)$$

and $G = \phi * G_0$ as before. Since G_0 is a pointwise maximum of convex functions (in $\tilde{X}$), it is convex, and so G is also convex. Since $\Psi(X) \leq \delta\gamma|X|$, for any given $\gamma > 0$ there exists $\delta > 0$ so that

$$c_j > c_i + \left\langle B_i, \tilde{X}_j - \tilde{X}_i \right\rangle_{\mathbb{R}^5} + \Psi(X_j - X_i)$$

for all i, j. Therefore in a neighbourhood of $\tilde{X}_i$

$$G(\tilde{X}) = c_i + \left\langle B_i, \tilde{X} - \tilde{X}_i \right\rangle + \phi * \Psi(X - X_i),$$

hence in a neighbourhood of X_i

$$F(X) = \frac{\epsilon}{2}|X|^2 + c_i - \langle Y_i J + \epsilon X_i, X - X_i \rangle + d_i \det(X - X_i) + \phi * \Psi(X - X_i).$$

Thus

$$D^2 F(X_i)[Z, Z] = -d_i \det Z + (\gamma + \epsilon)|Z|^2$$

so that $D^2 F(X_i)$ is positive definite if $\gamma > \max d_i$. This finishes the proof of the lemma. Q.E.D.

For $N = 4$ the system (15) does not admit solutions for any T_4, as shown in [KMŠ03] (Proposition 9). For $N = 5$ however we can find solutions by essentially fixing the "base" configuration $\{X_i\}$ and treating the Y_i's as variables. The simple observation is that in this way the Y_i appear in (15) linearly. From the numerical point of view the easiest is to consider the parametrisation (2) with the rank-one pentagon given by $C_i = \begin{pmatrix} a_i \otimes n_i \\ b_i \otimes n_i \end{pmatrix}$. If we fix a_i, n_i, κ_i, then the b_i are an additional 10 variables in (15) subject to the constraint $\sum_i b_i \otimes n_i = 0$. In this way we obtain a system of 20 linear inequalities in 16 variables. So the corresponding adjoint system should have a reasonably small kernel, meaning that the set of obstructions to (15) is small. To check whether a linear system of inequalities has solutions we used the simplex algorithm in Maple V. After a few tries for the parameters (κ_i, a_i, n_i) one can obtain a soluble linear system and a solution.

Example 1.

$$Z_1 = \begin{pmatrix} 2 & 2 \\ -2 & -2 \\ 20 & 20 \\ 14 & 14 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 3 & 5 \\ -5 & -9 \\ 0 & -10 \\ 3 & -1 \end{pmatrix}, \quad Z_3 = \begin{pmatrix} 4 & 3 \\ -9 & -5 \\ -41 & 0 \\ -21 & 3 \end{pmatrix},$$

$$Z_4 = \begin{pmatrix} -3 & -3 \\ 8 & 9 \\ 54 & 72 \\ 30 & 41 \end{pmatrix}, \quad Z_5 = \begin{pmatrix} 0 & 0 \\ -1 & -2 \\ -18 & -36 \\ -11 & -22 \end{pmatrix}.$$

The corresponding rank-one pentagon is

$$C_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 10 & 10 \\ 7 & 7 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 2 \\ -2 & -4 \\ -5 & 10 \\ -2 & -4 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & 0 \\ -3 & 0 \\ -23 & 0 \\ -13 & 0 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -3 & -3 \\ 7 & 7 \\ 36 & 36 \\ 19 & 19 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 0 & 0 \\ -1 & -2 \\ -18 & -36 \\ -11 & -22 \end{pmatrix},$$

and $P = 0$, $\kappa_1 = \cdots = \kappa_5 = 2$.

One can check that plugging this T_5 *into (15) gives a feasible linear system of inequalities (with RHS=*10^{-2}*).*

5 Stable embedding of T_N

The purpose of this section is to prove that if for a function F_0 there is a T_N configuration contained in K_{F_0}, then for certain small perturbations F of F_0 the same T_N configuration is contained in K_F in a stable way (i.e. condition (C) holds). The requirement that K_F contains the *same* T_N means that we are not dealing with any generic perturbation of F_0. Thus we need to carefully analyse the structure of the tangent space $T\mathcal{M}_N$. On the other hand, once F is such that $\mathcal{K}_F$ and $\mathcal{M}_N$ intersect transversely, **any** small perturbation of F leads to F' with $K_{F'}$ still containing some (possibly different) T_N configuration.

Theorem 2. *Suppose* $F_0 \in C^2(\mathbb{R}^{2\times 2})$ *such that* K_{F_0} *contains a* T_N *configuration. Then for any* $\delta > 0$ *there exists* $F \in C^2(\mathbb{R}^{2\times 2})$ *with* $\sup |D^2F - D^2F_0| < \delta$ *and such that* K_F *contains the same* T_N *configuration and moreover* F *satisfies the non-degeneracy condition (C).*

Proof. Let the T_N configuration be $\left\{ \begin{pmatrix} X_i \\ Y_i \end{pmatrix} : i = 1, \ldots, N \right\}$. Following [MŠ03] we will prove that

$$F(X) = F_0(X) + \delta \sum_{k=1}^{N} H_k(X - X_k) \tag{17}$$

gives the required perturbation for suitable $H_k \in C^2(\mathbb{R}^{2\times 2})$ compactly supported in a neighbourhood of the origin with $DH_k(0) = 0$ and $D^2H_k(0) = A_k$.

Condition (C) requires that the tangent space to $\mathcal{K}_F = (K_F)^{\times N}$ at the "point" $z = (Z_1, \ldots, Z_N) \in (\mathbb{R}^{4\times 2})^{\times N}$ satisfies

$$\begin{aligned} T_z\mathcal{K}_F + T_z\mathcal{M}_N &= (\mathbb{R}^{4\times 2})^{\times N}, \\ \dim(\mathrm{im} D\pi_k\big|_{T_z\mathcal{K}_F}) &= 8 \ \text{ for } k = 1, \ldots, N. \end{aligned} \tag{18}$$

As $\dim \mathcal{K}_F = 4N$ and $\dim \mathcal{M}_N = 6N$, we have that $\dim(T_z\mathcal{M}_N \cap T_z\mathcal{K}_F) \geq 2N$ and hence $\dim(\ker D\pi_k \cap T_z\mathcal{K}_F) \geq 2N - 8$, with equality corresponding to transversal intersection. Thus (18) is equivalent to

$$\dim(\ker D\pi_k \cap T_z\mathcal{K}_F) = 2N - 8,$$

that is,

$$T_z\mathcal{K}_F + \ker D\pi_k = (\mathbb{R}^{4\times 2})^{\times N}. \tag{19}$$

Given any symmetric linear map $A_i : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $i = 1, \ldots, N$, we can choose H_i in (17) so that the tangent space to $\mathcal{K}_F$ is given by

$$T_z\mathcal{K}_F = V_1 \times \cdots \times V_N,$$

where

$$V_i = \left\{ \begin{pmatrix} Y \\ A_i[Y] \end{pmatrix} : Y \in \mathbb{R}^{2\times 2} \right\}. \tag{20}$$

Let us say that a property $\mathbf{P}$ holds for *generic* $(V_1, \ldots, V_N)$ if whenever $A_i^0 : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ are symmetric linear maps, then there exist symmetric linear maps A_i in any neighbourhood of the A_i^0 so that the N-tuple formed from the corresponding subspaces V_i satisfies the property $\mathbf{P}$.

If there is a choice of $(V_1, \ldots, V_N)$ for which (19) holds for *some* k, then the set of such choices is generic. Hence, in order to prove that there is a choice of $(V_1, \ldots, V_N)$ for which (19) holds for *all* k, it suffices to prove this for $k = 1$.

Suppressing the subscript we write

$$\pi(Z_1, \ldots, Z_N) = P_1. \tag{21}$$

We will show that $\ker D\pi$ contains a $4N$-dimensional subspace L such that for generic V_i as above we have

$$L \cap (V_1 \times \cdots \times V_N) = \{0\}. \tag{22}$$

Since $\dim(V_1 \times \cdots \times V_N) = 4N$, this shows that

$$L + (V_1 \times \cdots \times V_N) = (\mathbb{R}^{4\times 2})^{\times N},$$

and this will finish the proof of Theorem 2. To construct L we derive a necessary and sufficient condition for (22) in Lemma 4 below. Finally, in Lemma 5 we prove that $\ker D\pi$ contains a subspace L satisfying the conditions in Lemma 4.

Q.E.D.

In the following we make the identification $\mathbb{R}^{4\times 2} \cong \mathbb{R}^8$. To construct the subspace L, let us introduce the following notation: For any $k \geq 1$ and integers $1 \leq i_1 < \cdots < i_k \leq N$, let $p_{i_1 \ldots i_k} : \mathbb{R}^{8N} \to \mathbb{R}^{8N}$ be the orthogonal projection onto the subspace which is the Cartesian product of k $\{0\}$'s and $N - k$ $\mathbb{R}^8$'s with the $\{0\}$'s at the $i_1, \ldots, i_k$'s places. So for example

$$\operatorname{im} p_1 = \{0\} \times \mathbb{R}^8 \times \cdots \times \mathbb{R}^8 \text{ and } \operatorname{im} p_{12} = \{0\} \times \{0\} \times \mathbb{R}^8 \times \cdots \times \mathbb{R}^8.$$

The main issue is the following: If for example L is a $4N$ (or less) dimensional subspace of $\mathbb{R}^{8N}$, such that its intersection with $\mathbb{R}^8 \times \{0\} \times \cdots \times \{0\}$ is at least 5-dimensional, then it will nontrivially intersect any subspace of the form $V_1 \times \cdots \times V_N$ (with $\dim V_i = 4$). Similarly if L has at least 9-dimensional intersection with $\mathbb{R}^8 \times \mathbb{R}^8 \times \{0\} \times \cdots \times \{0\}$, and so on. On the other hand suppose that for any $1 \leq i_1 < \cdots < i_k \leq N$, the intersection $L \cap \operatorname{im} p_{i_1 \ldots i_k}$ is at most $4(N-k)$-dimensional. We claim that in this case there exist symmetric $A_i : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ for which $L \cap (V_1 \times \cdots \times V_N) = \{0\}$. More precisely the following holds (here we identify $\mathcal{L}_{\mathrm{sym}}(\mathbb{R}^{2\times 2}, \mathbb{R}^{2\times 2}) \cong \mathbb{R}^{4\times 4}_{\mathrm{sym}}$):

Lemma 4. *Suppose $L \subset \mathbb{R}^{8N}$ is a subspace such that $\dim L \leq 4N$ and with the property that for any $1 \leq k \leq N$ and $1 \leq i_1 < \cdots < i_k \leq N$ we have*

$$\dim(L \cap \operatorname{im} p_{i_1 \ldots i_k}) \leq 4(N-k).$$

Then the set of $(V_1, \ldots, V_N)$ for which

$$L \cap (V_1 \times \cdots \times V_N) = \{0\} \tag{23}$$

is generic, i.e. whenever $A_i^0 \in \mathbb{R}^{4\times 4}_{sym}$, there exist $A_i \in \mathbb{R}^{4\times 4}_{sym}$ in any neighbourhood of A_i^0 so that the corresponding subspaces V_i (as in (20)) satisfy (23).

We will now show that $\ker D\pi$ contains a subspace of the above type.

Lemma 5. *Let $(Z_1^0, \ldots, Z_N^0) \in (\mathbb{R}^{4\times 2})^N$ be a non-degenerate T_N configuration with no rank-one connections, let $\mathcal{M}_N$ be the local manifold given in Lemma 2, and let $\pi(Z_1, \ldots, Z_N) = P_1$ as in (21).*

Then $\ker D\pi$ contains a $4N$-dimensional subspace L with the property that $\dim(L \cap \operatorname{im} p_{i_1 \ldots i_k}) \leq 4(N-k)$ for any $k \geq 1$ and $1 \leq i_1 < \cdots < i_k \leq N$.

Proof of Lemma 5. From (the proof of) Lemma 2 we know that $\ker D\pi$ is a $(6N-8)$-dimensional vector space given by N-tuples $(Z_1, \ldots Z_N)$ of the form

$$Z_i = \sum_{j=1}^{i-1} (a_j^0 \otimes b_j + a_j \otimes b_j^0) + \nu_i^0 a_i^0 \otimes b_i + \nu_i^0 a_i \otimes b_i^0 + \nu_i a_i^0 \otimes b_i^0,$$

where the parameters (a_i, b_i, ν_i) satisfy $\sum_{i=1}^N (a_i \otimes b_i^0 + a_i^0 \otimes b_i) = 0$ (and by convention $\sum_{j=1}^0 := 0$).

It suffices to prove that

$$\dim(\ker D\pi \cap \operatorname{im} p_{i_1 \ldots i_k}) \leq (6N-8) - 4k, \tag{24}$$

since generic $4N$-dimensional subspaces L of $\ker D\pi$ intersect $\operatorname{im} p_{i_1 \ldots i_k}$ transversely, in which case

$$\begin{aligned}
\dim(L \cap \operatorname{im} p_{i_1 \ldots i_k}) &= \dim L + \dim(\ker D\pi \cap \operatorname{im} p_{i_1 \ldots i_k}) - \dim \ker D\pi \\
&\leq 4N + ((6N-8) - 4k) - (6N-8) \\
&= 4(N-k).
\end{aligned}$$

Then we can choose a $4N$-dimensional subspace which intersects $\operatorname{im} p_{i_1\ldots i_k}$ transversely for all $k \geq 1$ and $1 \leq i_1 < \cdots < i_k \leq N$.

Let $R_i = \{(a_i^0 \otimes b_i + a_i \otimes b_i^0) : a_i \in \mathbb{R}^4,\ b_i \in \mathbb{R}^2\}$. Since the T_N configuration is assumed to contain no rank-one connections, $\{b_i^0, b_{i+1}^0\}$ and $\{a_i^0, a_{i+1}^0\}$ are linearly independent and so $R_i + R_{i+1} = \mathbb{R}^{4\times 2}$. Moreover $\dim R_i = 5$ for all i. Let us write $\ker D\pi = R \oplus C$, where

$$C = \big(\langle a_1^0 \otimes b_1^0\rangle\big) \times \cdots \times \big(\langle a_N^0 \otimes b_N^0\rangle\big)$$
$$R = \{(Y_1, \ldots, Y_N) : Y_j = \sum_{i=1}^{j-1} X_i + \nu_j^0 X_j \text{ where } X_i \in R_i,\ \sum_{i=1}^{N} X_i = 0\}.$$

Now $R \cap \operatorname{im} p_{i_1\ldots i_k}$ is precisely the solution space of the following system of $k+1$ (matrix) equations in the unknowns $(X_1, \ldots, X_N) \in R_1 \times \cdots \times R_N$:

$$\begin{aligned}
X_1 + \cdots + X_{i_1-1} + \nu_{i_1}^0 X_{i_1} &= 0\\
(1-\nu_{i_1}^0)X_{i_1} + X_{i_1+1} + \cdots + X_{i_2-1} + \nu_{i_2}^0 X_{i_2} &= 0\\
(1-\nu_{i_2}^0)X_{i_2} + X_{i_2+1} + \cdots + X_{i_3-1} + \nu_{i_3}^0 X_{i_3} &= 0\\
&\ \vdots\\
(1-\nu_{i_{k-1}}^0)X_{i_{k-1}} + X_{i_{k-1}+1} + \cdots + X_{i_k-1} + \nu_{i_k}^0 X_{i_k} &= 0\\
(1-\nu_{i_k}^0)X_{i_k} + X_{i_k+1} + \cdots + X_N &= 0.
\end{aligned}$$

Note that $\nu_i^0 > 1$. If $k \leq N-2$, then at least one equation contains two consecutive X_i's which are not in the previous equations. Then this equation has rank 8 (since $\dim(R_i + R_{i+1}) = 8$), and all the others at least rank 5 independently (since $\dim R_i = 5$). Thus the total rank is at least $5k+8$, and so we deduce

$$\dim(R \cap \operatorname{im} p_{i_1\ldots i_k}) \leq 5N - (5k+8),$$

which in turn implies (24).

If $k = N-1$, then the above system has rank $5N$, hence

$$R \cap \operatorname{im} p_{i_1\ldots i_{N-1}} = \{0\}.$$

But then $\ker D\pi \cap \operatorname{im} p_{i_1\ldots i_{N-1}}$ is at most N-dimensional, and since $N \geq 4$,

$$\dim(\ker D\pi \cap \operatorname{im} p_{i_1\ldots i_{N-1}}) \leq N \leq 2N-4 = (6N-8) - 4(N-1).$$

This finishes the proof of (24) and hence the proof of Lemma 5.

Proof of Lemma 4.

The proof is by induction on N. Suppose first of all $N = 1$. Let $L \subset \mathbb{R}^8$ be a subspace with $\dim L \leq 4$. We need to prove that for a generic set of $A_1 \in \mathbb{R}^{4\times 4}_{\text{sym}}$ the corresponding subspace V_1 satisfies $L \cap V_1 = \{0\}$. For this we may assume that $\dim L = 4$. Consider the matrix representation of a basis of $V_1 + L$ in block form:

$$\begin{pmatrix} I & B \\ A_1 & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ A_1 & I \end{pmatrix} \begin{pmatrix} I & B \\ 0 & C - A_1 B \end{pmatrix},$$

where $B, C \in \mathbb{R}^{4\times 4}$, and $\begin{pmatrix} B \\ C \end{pmatrix}$ (corresponding to L) has rank 4.

We show that $C - A_1 B$ is nonsingular for generic $A_1 \in \mathbb{R}^{4\times 4}_{\mathrm{sym}}$. We know that $\ker B \cap \ker C = \{0\}$, since otherwise $\dim L < 4$. Choose a subspace W of $\mathbb{R}^4$ such that

$$W \cap \operatorname{im} C = \{0\} \text{ and } W^\perp \cap B(\ker C) = \{0\}.$$

This is possible since $\dim B(\ker C) = \dim \ker C = 4 - \dim \operatorname{im} C$, hence both requirements are satisfied for a generic subspace W with $\dim W = 4 - \dim \operatorname{im} C$. Then let A_1 be the orthogonal projection onto W (which is symmetric). Now suppose $v \in \ker A_1 B \cap \ker C$. Then $Bv \in B(\ker C) \cap W^\perp$ and hence $Bv = 0$, $v \in \ker B \cap \ker C$, so finally $v = 0$. Thus $\ker C \cap \ker A_1 B = 0$, and moreover $\operatorname{im} C \cap \operatorname{im} A_1 B \subset \operatorname{im} C \cap W = 0$. This implies that $C - A_1 B$ is nonsingular. But this means that

$$A_1 \mapsto \det(C - A_1 B),$$

which is a (fourth-order) polynomial in $A_1 \in \mathbb{R}^{4\times 4}_{\mathrm{sym}}$, is not identically zero. Hence the set of A_1 for which $C - A_1 B$ is nonsingular is generic. We are finished with the proof for $N = 1$.

In order to prove the induction step, let us first consider the case $N = 2$ for clarity. Let $L \subset \mathbb{R}^8 \times \mathbb{R}^8$ such that $\dim L \leq 8$,

$$\dim L \cap (\mathbb{R}^8 \times \{0\}) \leq 4, \text{ and } \dim L \cap (\{0\} \times \mathbb{R}^8) \leq 4. \tag{25}$$

Let $p_1 : L \to \{0\} \times \mathbb{R}^8$ and $p_2 : L \to \mathbb{R}^8 \times \{0\}$ be the orthogonal projections restricted to L.

We first claim that the set of A_1 for which $V_1 \times \{0\} \subset \mathbb{R}^8 \times \mathbb{R}^8$ is transversal to $\operatorname{im} p_2$ and $\ker p_1$ is generic. Indeed, if $\dim \operatorname{im} p_2 \leq 4$ then we may apply the case $N = 1$ directly (with $\tilde{L} = \operatorname{im} p_2$ now considered as a subspace of $\mathbb{R}^8$), and if $\dim \operatorname{im} p_2 > 4$, then we may take any 4-dimensional subspace $\tilde{L} \subset \operatorname{im} p_2$ and apply the step $N = 1$ with this $\tilde{L}$. Thus we deduce that the set of A_1 for which $V_1 \times \{0\}$ is transversal to $\operatorname{im} p_2$ is generic. Applying the same argument to $\ker p_1$ and noting that the intersection of two generic sets is generic, we deduce our claim.

Secondly, we claim that for V_1 as above we have

(i) $\dim(L \cap (V_1 \times \mathbb{R}^8)) \leq 4$

(ii) $L \cap (V_1 \times \{0\}) = \{0\}$.

The second assertion follows directly, since from the assumption (25) we have $\dim \ker p_1 \leq 4$ and thus transversality of the intersection $\ker p_1 \cap (V_1 \times \{0\})$ implies

$$\ker L \cap (V_1 \times \{0\}) = \ker p_1 \cap (V_1 \times \{0\}) = \{0\}.$$

For the first assertion note that

$$\begin{aligned}\dim(L\cap(V_1\times\mathbb{R}^8)) &= \dim p_2(L\cap(V_1\times\mathbb{R}^8)) + \dim(\ker p_2\cap(V_1\times\mathbb{R}^8))\\ &\le \dim((V_1\times\{0\})\cap \operatorname{im} p_2) + \dim\ker p_2.\end{aligned}$$

If $\dim\operatorname{im} p_2 \ge 4$, then transversality implies $(V_1\times\{0\})\cap\operatorname{im} p_2 = \{0\}$ and the assertion follows from $\dim\ker p_2 \le 4$.

If $\dim\operatorname{im} p_2 > 4$, then transversality implies $(V_1\times\{0\}) + \operatorname{im} p_2 = \mathbb{R}^8\times\{0\}$ and hence

$$\begin{aligned}\dim(V_1\cap\operatorname{im} p_2) + \dim\ker p_2 &= \dim V_1 + \dim\operatorname{im} p_2 - 8 + \dim\ker p_2\\ &\le \dim L - 4 \le 4.\end{aligned}$$

Finally, observe that from (i) we have

$$\dim(p_1(L\cap(V_1\times\mathbb{R}^8))) \le \dim(L\cap(V_1\times\mathbb{R}^8)) \le 4$$

and so again, by using the case $N=1$, generic $A_2\in\mathbb{R}^{4\times 4}_{\mathrm{sym}}$ gives V_2 satisfying

$$p_1(L\cap(V_1\times\mathbb{R}^8))\cap(\{0\}\times V_2) = \{0\}. \tag{26}$$

Let $v=(v_1,v_2)\in L\cap(V_1\times V_2)$. Then in particular

$$(0,v_2)\in p_1(L\cap(V_1\times\mathbb{R}^8))\cap(\{0\}\times V_2),$$

so $v_2=0$, and hence $v_1=0$ by (ii). Thus $L\cap(V_1\times V_2)=\{0\}$, and this proves our statement for $N=2$.

For general N the argument is the same. For any $k\ge 0$ let W be a ($8k$-dimensional) subspace of $\mathbb{R}^{8(N-1)}$ which is the product of k $\mathbb{R}^8$'s and $(N-k-1)$ $\{0\}$'s. Then by an analogous argument to above, we see that for generic choice of A_1 we have

$$\dim L\cap(V_1\times W)\le 4k.$$

Then we can choose A_1 so that it satisfies this for all W of this form (since there is a finite number of conditions on A_1, and each is satisfied by "most" A_1's). In this way V_1 satisfies the analogue of (i) and (ii).

Let L' be the orthogonal projection of $L\cap(V_1\times\mathbb{R}^{8(N-1)})$ onto $\{0\}\times\mathbb{R}^{8(N-1)}$. Then L' satisfies the conditions for $N-1$, and so for generic $A_2,\dots A_N$ we have the analogue of (26):

$$L'\cap(\{0\}\times V_2\times\cdots\times V_N)=\{0\},$$

and thus $L\cap(V_1\times\cdots\times V_N)=\{0\}$ as above.

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