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Rank-One Convex Hulls in $\mathbb{R}^{2\times 2}$

by

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Abstract

We study the rank-one convex hull of compact sets $K \subset \mathbb{R}^{2\times 2}$. We show that if K contains no two matrices whose difference has rank one, and if K contains no four matrices forming a T_4 configuration, then the rank-one convex hull K^{rc} is equal to K. Furthermore, we give a simple numerical criterion for testing for T_4 configurations.

1 Introduction

A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be rank-one convex if f is convex along rank-one directions, in other words if $t \mapsto f(A + tB)$ is a convex function whenever $\operatorname{rank} B = 1$. The rank-one convex hull of a compact set $K \subset \mathbb{R}^{m\times n}$ is defined by separation with rank-one convex functions as

$$K^{rc} := \{X \in \mathbb{R}^{m\times n} : f(X) \le \sup_K f \quad \forall f : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ rank-one convex}\}.$$

Rank-one convexity is important in the theory of partial differential equations and in the calculus of variations. In particular the rank-one convex hull is an inner approximation of the *quasiconvex hull*. There are a number of papers dealing with this connection, for example [Mor52],[Šve92],[Mül99a] and the surveys [Bal87] and [Mül99b].

In this paper we concentrate on the following question: under what conditions is $K^{rc} = K$ (i.e. when is the rank-one convex hull trivial)? An immediate necessary condition is that K contains no rank-one connections (that is, $\operatorname{rank}(A - B) > 1$ for any two distinct $A, B \in K$). That this condition is in fact not sufficient for triviality of the hull has been known for some time ([Sch74], [AH86], [CT93], [Tar93], [NM91]), and can be demonstrated on an example consisting of four diagonal matrices (see Example 1 in Section 2).

A natural way of reformulating our question is to look for *nontrivial inclusion-minimal* configurations. Here and in what follows, a set $K \subset \mathbb{R}^{m\times n}$ is nontrivial inclusion-minimal (with respect to rank-one convexity) if $K^{rc} \neq K$ but $\tilde{K}^{rc} = \tilde{K}$ for any proper subset $\tilde{K} \subset K$.

Nontrivial inclusion-minimal sets are well understood in the case of *separate convexity* in $\mathbb{R}^d$. This is a special case of rank-one convexity, arising when we identify the subspace of diagonal matrices in $\mathbb{R}^{d\times d}$ with $\mathbb{R}^d$ (so that the rank-one cone consists of the coordinate directions in $\mathbb{R}^d$). Separate convexity has been treated in [Tar93] by L. Tartar and in [Mat01] and [MP98] by J. Matoušek and P. Plecháč. The main feature is that different directions in the rank-one cone are linearly independent. The consequence is that the structure of separate convex hulls depends only on the ordering of the coordinates of the points in K. This makes the combinatorial aspect very transparent. For the case of separate convexity in $\mathbb{R}^2$ (which corresponds to diagonal matrices in $\mathbb{R}^{2\times 2}$), L. Tartar observed (Remark 10 in [Tar93]) that any nontrivial (finite) set $K \subset \mathbb{R}^2$ with no rank-one connections necessarily contains a T_4 configuration.

A related issue is the following: For usual convexity in $\mathbb{R}^d$, Carathéodory's theorem says that if $K \subset \mathbb{R}^d$ and $x \in K^{co}$ (the usual convex hull), then there exists at most $(d+1)$ points $x_1, \ldots, x_{d+1} \in K$ such that x lies in the convex hull of $\{x_1, \ldots, x_{d+1}\}$. We say that the *Carathéodory number* for usual convexity in $\mathbb{R}^d$ is $(d+1)$. Matoušek and Plecháč proved in [MP98] that the Carathéodory number for separate convexity in $\mathbb{R}^2$ is 5. In [Mat01] Matoušek gave examples (essentially T_N configurations) in $\mathbb{R}^3$ of nontrivial inclusion-minimal sets for separate convexity of arbitrary cardinality. Consequently separate convexity in $\mathbb{R}^d$ for $d \geq 3$ has no finite Carathéodory number. Since separate convexity also arises when restricting rank-one convexity to appropriate subspaces, e.g. to $\begin{pmatrix} x & 0 & z \\ 0 & y & z \end{pmatrix}$, the same assertion holds also for rank-one convexity in $\mathbb{R}^{m\times n}$ if $\max\{m,n\} \geq 3$. Furthermore, J. Kolář showed (see [Kol03]) that there is no finite Carathéodory number for rank-one convexity in $\mathbb{R}^{2\times 2}$. These results can be summarised in the table below:

	Inclusion-minimal configurations	Carathéodory number
Separate convexity in $\mathbb{R}^2$	T_4 [Tar93]	5 [MP98]
Separate convexity in $\mathbb{R}^3$	$T_N, N \geq 4$ [Mat01]	∞ [Mat01]
Rank-one convexity in $\mathbb{R}^{2\times 2}$	?	∞ [Kol03]

In this paper we fill the gap in the table with the following theorem:

THEOREM 1. *Let $K \subset \mathbb{R}^{2\times 2}$ be a compact set with no rank-one connections, and suppose that K is nontrivial, i.e. $K^{rc} \neq K$. Then K contains a T_4 configuration.*

In particular the only nontrivial inclusion-minimal configurations in $\mathbb{R}^{2\times 2}$ are the T_4 configurations. The underlying reason (which also made separate convexity in $\mathbb{R}^2$ special) is that the rank-one cone has codimension 1. The significance of this observation is highlighted in the following result, which is standard in the literature (see for example [KMŠ03]):

LEMMA 1. *Let $K \subset \mathbb{R}^{2\times 2}$ be a compact set, and suppose that $X_0 \notin K$ and $\det(X - X_0) > 0$ for all $X \in K$. Then $(K \cup \{X_0\})^{rc} = K^{rc} \cup \{X_0\}$.*

An immediate consequence of this is that if $K \subset \mathbb{R}^{2\times 2}$ consists of three matrices (and no rank-one connections), then $K^{rc} = K$. Indeed, from the three (nonzero) numbers $d_{ij} = \det(X_i - X_j)$, $1 \leq i < j \leq 3$ at least two have to have the same sign, say $d_{12}, d_{13} > 0$, so we may employ Lemma 1 twice (first with $K = \{X_2, X_3\}$) to end up with the required result (see also [Ped93] and [Mül99b]).

The paper is organised as follows. In Section 2 we will introduce T_N configurations, which serve as the primary examples of finite sets with no rank-one connections and a nontrivial hull. In Section 3 we give a classification of four-point sets in terms of the rank-one convex hulls. The proof is based on the algebraic considerations of Section 2. Then we set out to prove Theorem 1 in three stages: First we restrict to finite sets in Section 4, where we prove that the absence of T_4 configurations implies a certain *sign-separation*. The main separation argument for the rank-one convex hull is in Section 5, and ultimately relies on an elementary geometric analysis of how translated copies of the rank-one cone intersect (Lemma 5 and 6). Finally we deal with general compact sets in Section 6.

2 T_N configurations

Definition 1 (T_N Configuration). *An ordered set of $N \geq 4$ matrices $\{X_i\}_{i=1}^N \subset \mathbb{R}^{m\times n}$ without rank-one connections is said form a T_N configuration if there exist matrices $P, C_i \in \mathbb{R}^{m\times n}$ and real numbers $\kappa_i > 1$ such that*

$$\begin{aligned} X_1 &= P + \kappa_1 C_1 \\ X_2 &= P + C_1 + \kappa_2 C_2 \\ &\vdots \\ X_N &= P + C_1 + \ldots + C_{N-1} + \kappa_N C_N, \end{aligned} \tag{1}$$

and moreover $\operatorname{rank}(C_i) = 1$ *and* $\sum_{i=1}^N C_i = 0$.

The following result, which justifies our interest such configurations, is well known, we include it here purely for completeness:

Lemma 2. *Let $\{X_1, \ldots, X_N\}$ be a T_N configuration, and for $i = 1 \ldots N$ let $P_i = P + C_1 + \cdots + C_{i-1}$ (so that $P_1 = P$). Then the segments $[P_i, X_i]$ are contained in the rank-one convex hull $\{X_1, \ldots, X_N\}^{rc}$.*

It is not obvious from the definition of T_N configurations how one can find the C_i's from given X_i, when such C_i exist, and when $\kappa_i > 1$. In this section we give an algebraic criterion which can easily be used in the 2×2 case for finding P_i for a given ordered set of matrices $\{X_1, \ldots, X_N\}$.

We will use the following notation: for $A \in \mathbb{R}^{N\times N}_{\text{sym}}$ such that $A_{ii} = 0$ for all

i and for $\mu \in \mathbb{R}$ write

$$A^\mu \stackrel{\text{def}}{=} \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,N} \\ \mu a_{1,2} & 0 & a_{2,3} & \dots & a_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu a_{1,N} & \mu a_{2,N} & \mu a_{3,N} & \dots & 0 \end{pmatrix}. \tag{2}$$

PROPOSITION 1. *Let $\{X_i\} \subset \mathbb{R}^{2\times 2}$ and let $A = (\det(X_i - X_j))$. Then $\{X_i\}$ is a T_N configuration if and only if there exist positive numbers $\lambda_i \geq 0$ and $\mu > 1$ such that $A^\mu \lambda = 0$.*

PROOF. Let

$$\xi^{(1)} = c_1 \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_N \end{pmatrix} \quad \xi^{(2)} = c_2 \begin{pmatrix} \mu\lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_N \end{pmatrix} \quad \xi^{(3)} = c_3 \begin{pmatrix} \mu\lambda_1 \\ \mu\lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_N \end{pmatrix} \quad \dots,$$

where c_i are normalising constants so that $\sum_j \xi_j^{(i)} = 1$, e.g. $c_1 = (\sum_i \lambda_i)^{-1}$.

Suppose $A^\mu \lambda = 0$. Then we claim

$$(A\xi^{(i)})_i = 0 \text{ and } \xi^{(i)} \cdot A\xi^{(i)} = 0 \quad \text{for } i = 1, \dots, N. \tag{3}$$

Indeed, the first set of equalities follows from the definition of the $\xi^{(i)}$'s, and for the second set we have (using that A is symmetric, zero on the diagonal and taking $c_i = 1$ without loss of generality):

$$\begin{aligned}
\xi^{(1)} \cdot A\xi^{(1)} &= 2\sum_{i>j} \lambda_i \lambda_j a_{ij} = \frac{2}{1+\mu} \lambda \cdot A^\mu \lambda = 0, \\
\xi^{(2)} \cdot A\xi^{(2)} &= (\xi^{(1)} + \lambda_1(\mu-1)e_1) \cdot A(\xi^{(1)} + \lambda_1(\mu-1)e_1) \\
&= \xi^{(1)} \cdot A\xi^{(1)} + 2\lambda_1(\mu-1)e_1 \cdot A\xi^{(1)} + \lambda_1^2(\mu-1)^2 e_1 \cdot Ae_1 \\
&= 0 \\
&\vdots \\
\xi^{(N)} \cdot A\xi^{(N)} &= (\xi^{(N-1)} + \lambda_{N-1}(\mu-1)e_{N-1}) \cdot A(\xi^{(N-1)} + \lambda_{N-1}(\mu-1)e_{N-1}) \\
&= \xi^{(N-1)} \cdot A\xi^{(N-1)} \\
&= 0.
\end{aligned}$$

Now it is easy to check that if $P_i = \sum_j \xi_j^{(i)} X_j$, then (3) is equivalent to

$$P_i = \sum_{j=1}^n \xi_j^{(i)} \det X_j \quad \text{and} \quad \det(X_i - P_i) = 0 \quad \text{for } i = 1, \dots, N. \tag{4}$$

Indeed, writing $P = \sum_i \xi_i X_i$, we have

$$\begin{aligned}\det P &= \frac{1}{2}\sum_{i,j} \xi_i\xi_j \langle X_i, \operatorname{cof} X_j\rangle \\ &= \frac{1}{2}\sum_{i,j} \xi_i\xi_j(\det X_i + \det X_j - \det(X_i - X_j)) \\ &= \sum_{i=1}^{n} \xi_i \det X_i - \frac{1}{2}\xi\cdot A\xi,\end{aligned} \tag{5}$$

and in the same way, since $P - X_k = \sum_i^n \xi_i(X_i - X_k)$,

$$\begin{aligned}\det(P - X_k) &= \frac{1}{2}\sum_{i,j} \xi_i\xi_j \langle (X_i - X_k), \operatorname{cof}(X_j - X_k)\rangle \\ &= \frac{1}{2}\sum_{i,j} \xi_i\xi_j(\det(X_i - X_k) + \det(X_j - X_k) - \det(X_i - X_j)) \\ &= (A\xi)_k - \frac{1}{2}\xi\cdot A\xi.\end{aligned} \tag{6}$$

Moreover, P_i and P_{i+1} lie on the same rank-one line connecting them to X_i:

$$\begin{aligned}P_{i+1} = \nu_i P_i + (1-\nu_i)X_i &= X_i + \nu_i(P_i - X_i) \\ \text{where } \nu_i &= \frac{c_{i+1}}{c_i}\end{aligned}$$

with the convention that $\xi^{(N+1)} = \xi^{(1)}$ (and hence $\mu c_{N+1} = c_1$).

Now if $\mu > 1$ then $0 < \nu_i < 1$ and so P_{i+1} lies between P_i and X_i, so the rank-one N-gon given by P_i is the required one. Moreover $\kappa_i = \frac{1}{1-\nu_i}$ in the definition, so that

$$\kappa_i = \frac{\mu\lambda_1 + \cdots + \mu\lambda_i + \lambda_{i+1} + \cdots + \lambda_N}{(\mu - 1)\lambda_i}.$$

Conversely, if $K = \{X_i\}$ are in T_N, then labelling the corners of the n-gon P_i it is clear that these corners lie in the convex hull of K and have a convex (barycentric) representation $\xi^{(i)}$ of the form as above. Q.E.D.

Remark 1. *It follows from the proof of the Proposition that the probability measures*

$$\mu^{(k)} = \sum_{i=1}^{N} \xi_i^{(k)} \delta_{X_i} \tag{7}$$

are laminates, with barycenter $\bar{\mu}^{(k)} = P_i$. In fact, we see from the equivalence of (3) and (4) that for N probability measures of the special form (7), they are laminates if and only if they commute with the determinant.

Example 1. *The first example is standard in the literature for demonstrating that a set can have a nontrivial rank-one convex hull even if there are no rank-one connections. Let* $K = \{X_1, \ldots, X_4\}$*, where*

$$X_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_4 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}.$$

These matrices can be represented in the plane, as in Figure 1. The shaded area together with the four segments shows the rank-one convex hull of K*.*

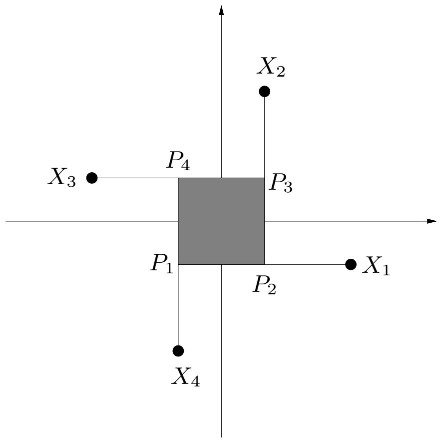


Figure 1: T_4 configuration in the diagonal plane

Example 2. *The second example shows that four-point sets can produce six* T_4*'s at the same time, one corresponding to each ordering. This example is taken from B. Kirchheim [Kir03]. In the plots we represent* 2×2 *symmetric matrices in* $\mathbb{R}^3$ *with the identification* $(x, y, z) \cong \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix}$ *and the hyperboloid is the set* $\{\det = -1\}$*. Let*

$$X_1 = \begin{pmatrix} \sqrt{3} & -2 \\ -2 & \sqrt{3} \end{pmatrix}, X_2 = \begin{pmatrix} \sqrt{3} & 2 \\ 2 & \sqrt{3} \end{pmatrix},$$

$$X_3 = \begin{pmatrix} -\sqrt{3}+2 & 0 \\ 0 & -\sqrt{3}-2 \end{pmatrix}, X_4 = \begin{pmatrix} -\sqrt{3}-2 & 0 \\ 0 & -\sqrt{3}+2 \end{pmatrix}.$$

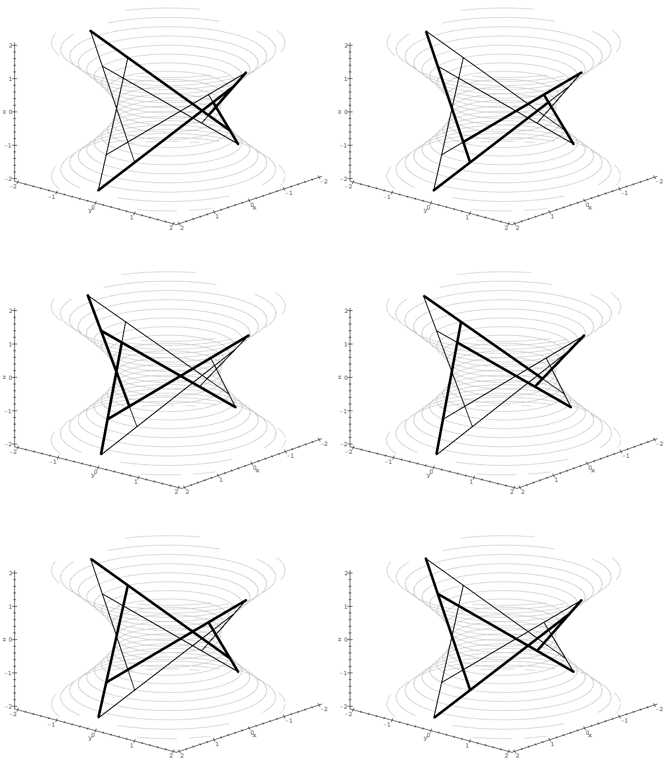


Figure 2: Maple plot showing six different T_4's

Example 3. *The limiting case $\mu \to 1^+$ in Proposition 1 corresponds to a degenerate configuration with nontrivial rank-one convex hull, that appeared in the work of B. Kirchheim whilst studying rank-one extreme points ([Kir03] Example 4.18), and in [NM91] in a slightly different context. In the original definition this limit corresponds to fixing the rank-one matrices $\tilde{C}_i := \kappa_i C_i$, defining $\kappa_i^\epsilon = \epsilon^{-1}\kappa_i$, $C_i^\epsilon = \epsilon C_i$ and letting $\epsilon \to 0$. This scaling fixes the length of the segments $[P_i^\epsilon, X_i^\epsilon]$ whilst shrinking the N-gon down to the point P. Then*

$$P_i^\epsilon \to P \ \text{ and } \ X_i^\epsilon \to X_i^0 := P + \tilde{C}_i \quad \text{ for all } i.$$

In particular, since $\sum_{i=1}^N \kappa_i^{-1}\tilde{C}_i = 0$, by writing $\xi_i^{(0)} := \kappa_i^{-1}$ we see that

$$\mu_\epsilon^{(k)} \overset{*}{\rightharpoonup} \mu^{(0)} := \sum_{i=1}^N \xi_i^{(0)} \delta_{X_i(0)}$$

and so by definition the measure $\mu^{(0)}$ is a laminate. Notice that the only condition on the support of $\mu^{(0)}$ is that $\operatorname{rank}(\tilde{C}_i) = 0$ *and* $0 \in \{\tilde{C}_1, \ldots, \tilde{C}_N\}^{co}$.

3 Four-point sets

In contrast to the diagonal case, in the full space $\mathbb{R}^{2\times 2}$ not all T_4 configurations (in the sense of definition 1) are similar copies of each other. In fact there are two distinct types of T_4 (and a degenerate case, see Example 3), as we shall see. In order to prove that these are the only inclusion-minimal configurations, we need to obtain simple criteria for when a four-point set is a T_4. Thus in this section we consider four-point sets $K = \{X_1, X_2, X_3, X_4\}$ with no rank-one connections.

In view of Lemma 1 and the observation following it, a necessary condition for $K^{\mathrm{rc}} \neq K$ is that $\det(X_i - X_j)$ changes sign for any fixed j as i varies. Thus, by possibly renumbering the matrices and multiplying by a matrix of determinant -1 we have one of the following sign-configurations: (dashed lines denote negative determinant and solid lines positive determinant)

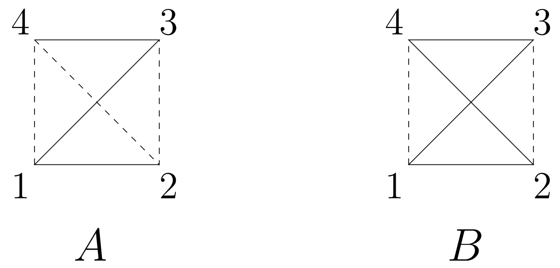


Figure 3: Possible sign-configurations

Theorem 2. *Suppose $K = \{X_i : i = 1, \dots, 4\} \subset \mathbb{R}^{2\times 2}$ contains no rank-one connections, with signs as in (A) or (B).*

1. *If the signs are as in (A), then exactly one ordering is in T_4.*
2. *If the signs are as in (B), then exactly one of the following three holds.*
   - *(i) There exists $P \in K^{co}$ with $\det(X_i - P) > 0$ and then K^{rc} is trivial.*
   - *(ii) There exists $P \in K^{co}$ with $\det(X_i - P) = 0$ and then*

$$K^{rc} = \{Y : \det(Y - P) = 0\} \cap K^{co}.$$

   - *(iii) There exists $P \in K^{co}$ with $\det(X_i - P) < 0$ and then each ordering is in T_4.*

Remark 2. *The theorem shows that there are exactly two combinatorially different types of T_4 configurations. The classical T_4 in Example 1 is type (A), whereas Example 2 is type (B). Example 3 shows how case 2. (ii) arises. The formula for the rank-one convex hull in case 2.(ii) is taken from [Kir03] (p.*

84) and is included only for completeness (to show that in this case the hull is nontrivial).

The triviality of the hull if $\det(X_i - P) > 0$ *will follow in a more general setting from Theorem 4, here we will just prove that in case some ordering is not a* T_4*, then there exists* $P \in K^{co}$ *with* $\det(X_i - P) > 0$ *(c.f. Lemma 6).*

PROOF. We split the proof into two parts according to whether the signs are as in (A) or (B).

Case (A)

Consider the matrix $A = (d_{ij})$ where $d_{ij} = \det(X_i - X_j)$. From Proposition 1 we know that K (for the ordering (X_1, X_2, X_3, X_4)) is a T_4 if and only if there exists $\mu > 1$ and $\lambda_i > 0$ with $A^\mu \lambda = 0$. Recall that A^μ denotes the matrix obtained by multiplying the entries in A below the diagonal by μ, as in (2).

So for the existence of a T_4 we first require the existence of $\mu > 1$ satisfying $\det A^\mu = 0$. Now $\det A^\mu$ is a cubic polynomial with a trivial root $\mu = 0$. Furthermore, note that $\mu^{-1}(A^\mu)^T = A^{(\mu^{-1})}$, so nonzero roots come in pairs $\mu_1\mu_2 = 1$. Let $p(\mu) = \mu^{-1} \det A^\mu$. Then $p(0) = -ab$, and

$$p(1) = \det A = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc, \tag{8}$$

where $a = d_{12}d_{34}, b = d_{14}d_{23}, c = d_{13}d_{24}$.

Because the signs of d_{ij} are as in (A), $a, b > 0$, $c < 0$, $p(0) = -ab < 0$, and so $p(1) = \det A = (a-b)^2 + c^2 - 2ac - 2bc > 0$. Therefore a root $\mu > 1$ of p exists. Now consider permutations of $(X_1, \dots, X_4)$: each corresponds to a permutation of (a, b, c) and since $p(1) > 0$ for each by symmetry, the only permutations admitting a root $\mu_\sigma > 1$ are the ones leaving c invariant (otherwise $p(0) > 0$). Hence only the orderings $(1, 2, 3, 4)$ and $(1, 4, 3, 2)$ can be in T_4.

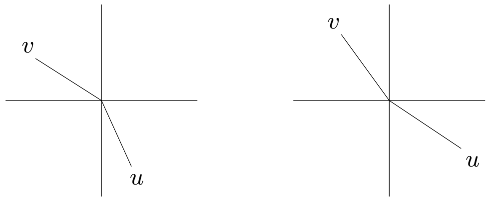
Suppose now that $A^\mu \lambda = 0$ for some $\mu > 1$ and $\lambda \in \mathbb{R}^4$. We need to analyse the sign of λ_i. Firstly, $\lambda_i \neq 0$ for all i, because the principal 3×3 minors are all nonzero: e.g.

$$\begin{vmatrix} 0 & d_{12} & d_{13} \\ \mu d_{12} & 0 & d_{23} \\ \mu d_{13} & \mu d_{23} & 0 \end{vmatrix} = \mu(\mu + 1) d_{12} d_{23} d_{13}. \tag{9}$$

As a first tool we note that for a 2×2 matrix M with signs $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$,

$$\begin{aligned} \{x : Mx > 0\} \cap \{x : x > 0\} \neq \emptyset \text{ if and only if } \det M > 0, \\ \{x : Mx > 0\} \cap \{x : x < 0\} \neq \emptyset \text{ if and only if } \det M < 0. \end{aligned} \tag{10}$$

This is elementary and best illustrated by the following diagram, where we write $u = Me_1$ and $v = Me_2$:



$\det M < 0$ $\qquad$ $\det M > 0$

By assumption A^μ has signs

$$\begin{pmatrix} 0 & + & + & - \\ + & 0 & - & - \\ + & - & 0 & + \\ - & - & + & 0 \end{pmatrix}.$$

Suppose without loss of generality that $\lambda_1 > 0$. Then we need to eliminate the possibility of $\lambda_i < 0$ for some $i > 1$. By considering an appropriate row of the matrix, we see that the only possibilities for the signs of λ_i are $(+,-,+,-)$ or $(+,+,+,+)$ (for example $(+,+,+,-)$ is ruled out by the first row).

Suppose the signs alternate as in the first possibility. Now λ in particular satisfies the equations

$$\begin{aligned} \begin{pmatrix} \lambda_3 \\ \mu\lambda_1 \end{pmatrix} &= \frac{-1}{d_{13}} \begin{pmatrix} d_{12} & d_{14} \\ \mu d_{23} & d_{34} \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_4 \end{pmatrix}, \\ \begin{pmatrix} \lambda_4 \\ \mu\lambda_2 \end{pmatrix} &= \frac{-1}{d_{24}} \begin{pmatrix} d_{12} & d_{23} \\ d_{14} & \mu d_{34} \end{pmatrix} \begin{pmatrix} \mu\lambda_1 \\ \lambda_3 \end{pmatrix}. \end{aligned} \tag{11}$$

Then (10) yields $d_{12}d_{34} - \mu d_{14}d_{23} > 0$ and $\mu d_{12}d_{34} - d_{14}d_{23} < 0$, i.e.

$$1 < \mu < \frac{a}{b}, \frac{b}{a},$$

which is a contradiction. Hence all entries of λ must be positive, and so (X_1, X_2, X_3, X_4) is a T_4. In a similar fashion, if instead we had that $d_{13} < 0, d_{24} > 0$ (corresponding to $(1,4,3,2)$ together with a sign-change), we would get the same contradiction when assuming all λ_i are positive. To summarize, in case (A) exactly one ordering of the matrices $\{X_1, \ldots, X_4\}$ is a T_4 configuration.

Case (B)

Now assume the signs of $\det(X_i - X_j)$ are as in (B). To arrive at the classification in part 2. of the theorem (see also Remark 2), we need to do three things:

a) show that there exists $P \in K^{co}$ for which $\det(X_i - P)$ has the same sign for all i (so that *at least* one of (i), (ii) or (iii) occurs),

b) show that *at most* one of the cases (i), (ii), (iii) can occur,

c) show that if $\{X_1, X_2, X_3, X_4\}$ do not form a T_4 for some ordering, then there exists $P \in K^{\text{co}}$ with $\det(X_i - P) \geq 0$ for all i.

For a) consider the convex hull of $\{X_1, X_4, Y\}$ where Y is such that

$$\det(X_1 - Y) > 0, \, \det(X_4 - Y) > 0$$

(and remember that $\det(X_1 - X_4) < 0$). The convex hull is a (non-degenerate) triangle as shown in Figure 4 a) below.

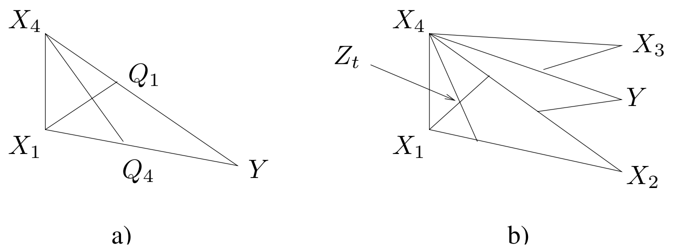


Figure 4: The convex hull of $\{X_1, X_2, X_3, X_4\}$

Since $\det(X_4 - X_1) < 0$ and $\det(Y - X_1) > 0$, there exists a unique Q_1 on the segment $[X_4, Y]$ with $\det(Q_1 - X_1) = 0$. Uniqueness follows because $\det(Q - X_1)$ restricted to the line going through X_4 and Y is a quadratic polynomial which is positive at X_4 and negative at Y. Similarly there exists a unique Q_4 on the segment $[X_1, Y]$ with $\det(Q_4 - X_4) = 0$. The (unique) intersection of the segments $[X_1, Q_1]$ and $[X_4, Q_4]$, call it Z, lies in the interior of the triangle, and in any neighbourhood of Z there exists Z_1 and Z_2 such that

$$\det(Z_1 - X_i) < 0 \quad \text{and} \quad \det(Z_2 - X_i) > 0 \quad \text{for } i = 1, 4.$$

Furthermore this unique point Z depends continuously on Y. In particular, taking $Y = Y_t = tX_2 + (1 - t)X_3$ we obtain a continuous, compact curve

$$\Gamma = \{Z_t : t \in [0, 1]\} \subset K^{\text{co}}$$

such that $\det(X_1 - Z) = \det(X_4 - Z) = 0$ on Γ (see Figure 4 b) above). Of course if K^{co} is planar, the curve degenerates to a point. Consider the sets

$$C_i = \{Z \in \Gamma : \det(X_i - Z) > 0\}$$

for $i = 2, 3$. Suppose $\Gamma = C_2 \cup C_3$. Since C_i is open and Γ is connected, necessarily $C_2 \cap C_3$ is nonempty. But then in a neighbourhood of $C_2 \cap C_3$ there exists $P \in K^{co}$ with $\det(X_i - P) > 0$ for all i. Otherwise, if $\Gamma \setminus (C_2 \cup C_3)$ is not empty, then it has either nonempty interior (relative to Γ), or $\partial C_2 \cap \partial C_3$ is nonempty. In the former case there exists $P \in K^{co}$ with $\det(X_i - P) < 0$ for all

i (by a similar argument to before), and in the latter case there exists $P \in K^{co}$ with $\det(X_i - P) = 0$ for all i.

Let $P \in K^{\text{co}}$, so that $P = \sum_{i=1}^4 x_i X_i$ for some $x \geq 0$, $\sum_i x_i = 1$. From the proof of Proposition 1 we get

$$\det(X_i - P) = (Ax)_i - \frac{1}{2} x \cdot Ax.$$

Suppose $\det(X_i - P) > 0$ for all i. Then summing over i gives $\frac{1}{2} x \cdot Ax > 0$, and hence $Ax > 0$. Similarly if $\det(X_i - P) < 0$ for all i then $Ax < 0$, and if $\det(X_i - P) = 0$ then $Ax = 0$. But since A is symmetric, at most one of these three cases can occur. Indeed, if $x, y \geq 0$ with $Ax > 0$ and $Ay \leq 0$, then $0 \leq y \cdot Ax = x \cdot Ay \leq 0$ and since $(Ax)_i > 0$ for all i, necessarily $y = 0$.
Summarizing the above: for any $\diamond \in \{<, >, =\}$

$$\begin{gathered} \text{there exists } P \in K^{\text{co}} \text{ with } \det(X_i - P) \diamond 0 \text{ for all } i \\ \text{if and only if} \\ \text{there exists } x \in \mathbb{R}^4 \text{ with } x_i > 0, (Ax)_i \diamond 0 \text{ for all } i. \end{gathered} \tag{12}$$

Let $a = d_{12}d_{34}\,, b = d_{14}d_{23}\,, c = d_{13}d_{24}$ as before, and let us assume that (X_1, X_2, X_3, X_4) do not form a T_4 configuration (the argument for all other orderings is the same). By the assumption on the signs of d_{ij} we have $a, b, c > 0$. From Proposition 1 we deduce that either there exists no $\mu > 1$ with $\det A^\mu = 0$, or there exists such a $\mu > 1$ and then the corresponding $\lambda \in \ker A^\mu$ has coordinates with mixed signs.

Recall from (8) that if $\mu > 1$ with $\det A^\mu = 0$ does not exist, then p does not vanish in $(0,1)$. As $p(0) = -ab < 0$, we deduce that $p(1) = \det A \leq 0$. Suppose $b > a + c$ and observe that

$$x_1 = \frac{b - a - c}{2d_{12}d_{13}},\ x_2 = \frac{b + a - c}{-2d_{12}d_{23}},\ x_3 = \frac{b - a + c}{-2d_{13}d_{23}},\ x_4 = 1$$

gives

$$(Ax)_1 = (Ax)_2 = (Ax)_3 = 0,\ (Ax)_4 = \frac{-\det A}{-2d_{12}d_{13}d_{23}}.$$

(remember that $d_{23} < 0$ and d_{12}, $d_{13} > 0$). By symmetry we can get similar x's where $(Ax)_i > 0$ for $i = 1, 2, 3$ respectively. Summing up gives $x \in \mathbb{R}^4$ with $x_i > 0$ and $(Ax)_i \geq 0$ for each i.
On the other hand, if $b \leq a + c$ then

$$x_1 = -d_{23},\ x_2 = d_{13},\ x_3 = d_{12},\ x_4 = 0$$

yields

$$(Ax)_1 = 2d_{12}d_{13},\ (Ax)_2 = 0,\ (Ax)_3 = 0,\ (Ax)_4 = a + c - b,$$

and again by symmetry we can obtain $x's$ with $(Ax)_i > 0$ for $i = 2, 3, 4$ respectively, so that again by summing we obtain $x \in \mathbb{R}^4$ with $x_i > 0$ and $(Ax)_i > 0$.

We conclude using (12) and using that at most one of the cases in (12) can occur, that if $\det A \leq 0$ (or if $b \leq a+c$) then there exists $P \in K^{\text{co}}$ with $\det(X_i - P) > 0$ for all i (or $\det(X_i - P) = 0$ for all i).

Finally suppose that there exists $\mu > 1$ and $\lambda \in \mathbb{R}^4$ with $A^\mu \lambda = 0$, and suppose that λ has mixed signs. As in (9) we see that $\lambda_i \neq 0$ for each i. Furthermore we may assume that $\lambda_1 > 0$. Observe that A^μ has signs

$$\begin{pmatrix} 0 & + & + & - \\ + & 0 & - & + \\ + & - & 0 & + \\ - & + & + & 0 \end{pmatrix}.$$

As before in case (A), we can eliminate possibilities for the signs of λ_i by considering the appropriate row of the matrix. The only remaining are

$$(+,-,+,-) \text{ or } (+,+,-,-).$$

In the first case the first identity in (11) together with (10) implies that $a > \mu b$. In particular $a > b$. In the second case similarly to (11) we have

$$\begin{pmatrix} \lambda_2 \\ \mu\lambda_1 \end{pmatrix} = \frac{-1}{d_{12}} \begin{pmatrix} d_{13} & d_{14} \\ d_{23} & d_{24} \end{pmatrix} \begin{pmatrix} \lambda_3 \\ \lambda_4 \end{pmatrix},$$

and then (10) implies $c > b$. Therefore in both cases we get $b < a + c$ using which the solution above gives $x \in \mathbb{R}^4$ with $x_i > 0$ and $(Ax)_i > 0$. In view of (12) this finishes the proof of c), and hence the proof of the theorem.

Q.E.D.

4 Finite sets

THEOREM 3. *Let $K = \{X_i\}$ be a finite set of 2×2 matrices with no rank-one connections. If $K^{\text{rc}} \neq K$, then K contains four matrices which form a (possibly degenerate) T_4.*

Instead of giving the proof directly, we split it up into a graph-theoretical part in this section and a separation argument in the next section. Arguing by contradiction we assume that K is a finite set with no rank-one connections and a nontrivial rank-one convex hull but doesn't contain a T_4 configuration. Then we may assume without loss of generality that K is inclusion-minimal (otherwise we can remove points until the remaining set is nontrivial inclusion-minimal).

We use part 1. of Theorem 2 to show (in Lemma 4 below) that if K contains no T_4 of type (A) (recall Figure 3), then it must have a decomposition $K = K_1 \cup K_2$ where $\det(X - Y) > 0$ for all $X \in K_1$, $Y \in K_2$. Then in Section 5 we use part 2. of Theorem 2 to show that if K has such a decomposition and it doesn't contain a T_4 of type (B) then the rc-hull separates: $K^{\text{rc}} = K_1^{\text{rc}} \cup K_2^{\text{rc}}$, and this will contradict the inclusion-minimality.

A set K of N matrices gives rise to an N-point complete graph G where all edges are labelled either $\oplus$ or $\ominus$ depending whether $\det(X_i - X_j)$ for the corresponding matrices is positive or negative. The assumption on inclusion minimality implies that for each X_i there exists X_{j_1} and X_{j_2} such that $\det(X_i - X_{j_1}) < 0$ and $\det(X_i - X_{j_2}) > 0$ (see Lemma 1). In the corresponding graph this means at each vertex there are both $\oplus$ and $\ominus$ edges.

LEMMA 3. *If G is an N-point graph with each edge $\oplus$ or $\ominus$, and at each vertex there are both $\oplus$ and $\ominus$ edges, then there exist 4 points P, Q, R, S in G where the edges alternate, i.e. PQ and RS are $\ominus$ and QR, SP are $\oplus$.*

PROOF. Assume there exists a point P such that there is only one $\ominus$ edge at P, all others are $\oplus$ (or other way round). Suppose the $\ominus$ edge is PQ. Now at Q there must be at least one $\oplus$ edge, say QR. Going on, at R there must be a $\ominus$ edge, say RS. Now $S \neq P$ since $R \neq Q$ (by assumption the only $\ominus$ edge at P is PQ), and SP must be $\oplus$ by the same reason (since $S \neq Q$). Hence we are done (see Figure 5 a) below).

If there doesn't exist a point P with only one $\ominus$ edge (or only one $\oplus$ edge), then at all points there is at least two $\oplus$ and two $\ominus$ edges. So $G' = G \setminus \{P\}$ (for any P) satisfies the assumptions of the claim. Hence we are done by induction. Q.E.D.

A 4-tuple P, Q, R, S with alternating signs as in Lemma 3 looks (up to swapping $\oplus$ and $\ominus$) like either (A) or (B) in Figure 3. We know from Theorem 2 that (A) is necessarily a T_4. Now we show that if G does not contain (A) then it "separates" as a graph.

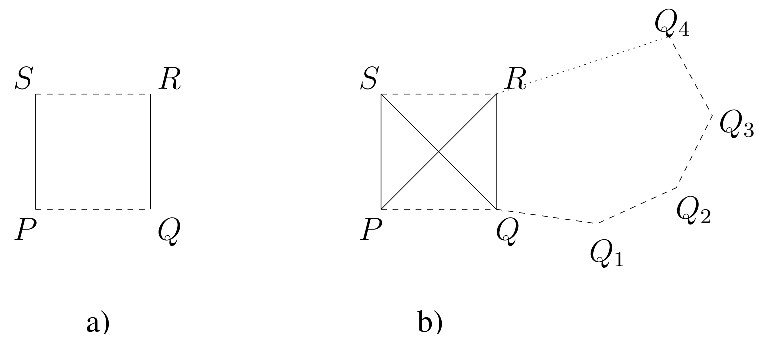


Figure 5: Alternating signs and a $\ominus$-path from Q to R

LEMMA 4. *Suppose in addition that G does not contain (A). Then $G = G_1 \cup G_2$ where G_i are nonempty and whenever $P_1 \in G_1$ and $P_2 \in G_2$, then the edge P_1P_2 is $\oplus$ (up to swapping $\oplus$ and $\ominus$).*

PROOF. As in the previous claim, take away points from G until there is a point P with only one $\ominus$ edge, PQ. Call the new graph G'. As before, there exist edges S, R so that SP, SQ, RP, RQ are all $\oplus$ (upto swapping signs in the whole graph). Now suppose there exists a $\ominus$ path from Q to R and take the shortest such: $Q, Q_1, Q_2, \ldots, Q_k, R$ (shortest in the sense that $k \leq k'$ for any other path $Q, Q'_1, \ldots, Q'_{k'}, R$). See Figure 5 b) above. Then in particular QQ_2 is $\oplus$, otherwise our path could be shortened. By assumptions on P also PQ_1 and PQ_2 are $\oplus$. Now $k \geq 1$ and so (regardless of whether $Q_2 = R$ or not) we can consider the square P, Q_1, Q_2, Q, which looks like (A). This is a contradiction, so there is no $\ominus$-path from Q to R. Then $G' = \bigcup_i G'_i$ where G'_1 consists of the points reachable from Q with a $\ominus$-path, and G'_2 consists of the points reachable from R with a $\ominus$-path and $G'_3, \ldots$ are possible other "$\ominus$-connected" components.

To finish we need to add the points back that we removed at the start. Adding back in the same order we see that at each step the new point X has both $\oplus$ and $\ominus$ edges to the existing graph. We claim that after each step there are at least two $\ominus$-connected components. If not, then at some step the point X that we add will be $\ominus$-connected to all components G'_i. Of course X needs to be $\oplus$-connected to at least one G'_i, say to G'_1. Then $G'_1 = H_1 \cup H_2$ where XY is $\ominus$ for all $Y \in H_1$ and $\oplus$ for all $Y \in H_2$. By assumption H_1, H_2 are nonempty. Moreover, since G'_1 is $\ominus$-connected, there exists $P_1 \in H_1$, $P_2 \in H_2$ such that P_1P_2 is $\ominus$. Take $Q \in G'_2$ such that XQ is $\ominus$ and consider Q, P_1, P_2, X. It is easy to see that this has signs as in (A), contradicting our assumption.

Q.E.D.

Let us say that two compact sets K_1 and K_2 are **sign-separated** if

$$\det(X - Y) > 0 \text{ whenever } X \in K_1,\ Y \in K_2.$$

Lemma 4 shows that if K is a finite set with no rank-one connections and no T_4's of type (A), then K can be decomposed into $K_1 \cup K_2$ so that K_1 and K_2 are sign-separated.

In the next section we show that such sets have separate rank-one convex hulls, i.e. $(K_1 \cup K_2)^{\mathrm{rc}} = K_1^{\mathrm{rc}} \cup K_2^{\mathrm{rc}}$ unless $K_1 \cup K_2$ contains a T_4 "connecting" the hulls. That will complete the proof of Theorem 3.

5 Separation

Let us introduce conformal-anticonformal coordinates on $\mathbb{R}^{2\times 2}$ in the following way: For each $X \in \mathbb{R}^{2\times 2}$ there exists a unique $z, w \in \mathbb{R}^2$ such that

$$X = \begin{pmatrix} z_1 + w_1 & w_2 - z_2 \\ w_2 + z_2 & z_1 - w_1 \end{pmatrix}$$

so that with considerable abuse of notation we write $\mathbb{R}^{2\times 2} = \mathbb{C} \times \overline{\mathbb{C}}$. Here $\mathbb{C}$ denotes conformal matrices, and $\overline{\mathbb{C}}$ denotes anticonformal matrices and both are identified with $\mathbb{R}^2$. The norm $|\cdot|$ is the Euclidean norm on $\mathbb{R}^2$. Then for each matrix $X = (x^+, x^-)$, $\det X = |x^+|^2 - |x^-|^2$, so that

$$\det X > 0 \quad \text{if and only if } |x^+| > |x^-|. \tag{13}$$

We will also use the Euclidean inner-product on 2×2 matrices, defined as

$$\langle X, Y \rangle \stackrel{\text{def}}{=} \text{trace } (X^T Y). \tag{14}$$

THEOREM 4. *Suppose $K \subset \mathbb{R}^{2\times 2}$ such that $K = K_1 \cup K_2$ where K_1 and K_2 are disjoint compact sets that are sign-separated in the sense that*

$$\det(X - Y) > 0 \text{ whenever } X \in K_1,\, Y \in K_2.$$

If for any $X_1, X_2 \in K_1$ and $Y_1, Y_2 \in K_2$ the four-point set $\{X_1, X_2, Y_1, Y_2\}$ is not a T_4, then there exists a continuous curve $\Gamma : \mathcal{S}^1 \mapsto \mathbb{R}^{2\times 2}$ with the following properties

(i) $\det(X - \Gamma(t)) > 0$ *for all $t \in \mathcal{S}^1$ and all $X \in K$.*

(ii) The projection γ of Γ onto the conformal plane is a Jordan curve.

(iii) If $\tilde{K}_i$ is the projection of K_i onto the conformal plane, then $\tilde{K}_1$ and $\tilde{K}_2$ lie in different components.

In particular $K^{rc} = K_1^{rc} \cup K_2^{rc}$.

The main ingredient in the proof is Helly's theorem on compact convex sets in $\mathbb{R}^d$ (see for example [DGK63]):
HELLY'S THEOREM. *Let $\{C_\alpha\}$ be a collection of compact, convex sets in $\mathbb{R}^d$ and suppose that for any $\alpha_1, \ldots, \alpha_{d+1}$ the intersection*

$$C_{\alpha_1} \cap \cdots \cap C_{\alpha_{d+1}}$$

is nonempty. Then the whole intersection $\bigcap_\alpha C_\alpha$ is nonempty.

In conformal-anticonformal coordinates, since K_1 and K_2 are sign-separated,

$$|x^+ - y^+| > |x^- - y^-| \text{ for all } X \in K_1,\, Y \in K_2. \tag{15}$$

With $\text{proj}_{\mathbb{C}}$ denoting the projection onto the conformal plane $\mathbb{C}$, let $\tilde{K}_1 = \text{proj}_{\mathbb{C}} K_1$ and $\tilde{K}_2 = \text{proj}_{\mathbb{C}} K_2$. In particular from (15) we have $\tilde{K}_1 \cap \tilde{K}_2 = \emptyset$. Let

$$S := \{Z : \det(X - Z) > 0 \text{ for all } X \in K\}. \tag{16}$$

Suppose we fix $z \in \mathbb{C}$ and look for $w \in \overline{\mathbb{C}}$ such that $Z := (z, w)$ satisfies $Z \in S$. By definition $Z = (z, w)$ is in S if $|w - x^-| < |z - x^+|$ for all $X \in K$, in other words

$$w \in B_{|x^+ - z|}(x^-) \text{ for any } X \in K.$$

Hence

$$\text{proj}_{\mathbb{C}} S = \{z \in \mathbb{C} : \bigcap_{X \in K} B_{|x^+ - z|}(x^-) \neq \emptyset\}. \tag{17}$$

Note that for any $X \in K_1$, $Y \in K_2$ and $z \notin \{x^+, y^+\}$ we have

$$B_{|x^+-z|}(x^-) \cap B_{|y^+-z|}(y^-) \neq \emptyset, \tag{18}$$

since $|x^- - y^-| < |x^+ - y^+| \leq |x^+ - z| + |y^+ - z|$.

In view of Helly's theorem we study the intersection of any three balls. Note that Helly's theorem is not directly applicable to an infinite family of *open* balls, but we will deal with this later. For any $X_1, X_2, X_3 \in K$ we define

$$E(X_1, X_2, X_3) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \bigcap_{i=1}^{3} B_{|x_i^+ - z|}(x_i^-) = \emptyset\}. \tag{19}$$

Let us say that $\mathcal{E} \subset \mathbb{R}^2$ is a **solid ellipse** if $\mathcal{E}$ is a (possibly empty) closed convex set whose boundary is an ellipse.

LEMMA 5. *Under the assumptions of Theorem 4 the sets $E(X_1, X_2, X_3)$ as in (19) satisfy*

(1) $E(X_1, X_2, X_3)$ is closed and bounded for any $X_i \in K$

(2) $x_1^+, x_2^+, x_3^+ \in E(X_1, X_2, X_3)$ for any $X_i \in K$

(3) For any $X_1, X_2, X_3 \in K_1$ and $Y \in K_2$ we have $y^+ \notin E(X_1, X_2, X_3)$

(4) Suppose $X_1^{(i)}, X_2^{(i)}, X_3^{(i)} \in K$ such that $X_j^{(i)} \to X_j$ for $j = 1, 2, 3$ as $i \to \infty$. Then for all $z \in \mathbb{C}$ for which there exists $z_i \in E(X_1^{(i)}, X_2^{(i)}, X_3^{(i)})$ with $z_i \to z$ we have $z \in E(X_1, X_2, X_3)$.

(5) For any $X_1, X_2, X_3 \in K$ there exists a solid ellipse $\mathcal{E} \subset \mathbb{C}$ such that

$$E(X_1, X_2, X_3) = \mathcal{E} \cup \{x_1^+, x_2^+, x_3^+\}.$$

(6) If $X_1, X_2 \in K_1$ and $X_3 \in K_2$, then there exists $\delta > 0$ such that

$$z \notin E(X_1, X_2, X_3) \text{ whenever } 0 < |z - x_3^+| < \delta,$$

moreover

(i) if $\det(X_1 - X_2) < 0$, then $x_1^+, x_2^+ \in \mathcal{E}$,

(ii) if $\det(X_1 - X_2) > 0$, then $E(X_1, X_2, X_3) = \{x_1^+, x_2^+, x_3^+\}$.

PROOF. The first and second statements follow directly from the definition. Part (3) follows from (15). For part (4) suppose that $z \notin E(X_1, X_2, X_3)$. By definition this means

$$\bigcap_{j=1}^{3} B_{|z-x_j^+|}(x_j^-) \neq \emptyset.$$

But then the intersection remains nonempty for small perturbations of the three balls, and in particular $z_i \notin E(X_1^{(i)}, X_2^{(i)}, X_3^{(i)})$ for sufficiently large i.

To prove (5) let us assume that $E(X_1, X_2, X_3) \neq \{x_1^+, x_2^+, x_3^+\}$. For $z \notin E(X_1, X_2, X_3)$ the intersection $\bigcap_i B_{|x_i^+ - z|}(x_i^-)$ is nonempty, and

$$z \mapsto \operatorname{diam} \bigcap_i B_{|x_i^+ - z|}(x_i^-)$$

is continuous. Hence if z is on the boundary of $E(X_1, X_2, X_3)$ (and z is different from x_1^+, x_2^+, x_3^+), then the corresponding three circles

$$\mathcal{C}_i = \{w \in \overline{\mathbb{C}} : |w - x_i^-| = |z - x_i^+|\}$$

need to intersect in a single point w, which lies in the convex hull of $\{x_1^-, x_2^-, x_3^-\}$. We prove that the set of points z with this property, i.e. the set

$$\big\{z \in \mathbb{C} : \text{there exists } w \in \overline{\mathbb{C}} \text{ with } |w - x_i^-| = |z - x_i^+| \text{ for } i = 1, 2, 3\big\} \tag{20}$$

is an ellipse. Notice that this set is exactly the projection onto $\mathbb{C}$ of

$$\{P \in \mathbb{R}^{2\times 2} : \det(X_i - P) = 0 \text{ for } i = 1, 2, 3\}.$$

Consider the equations

$$\begin{aligned} |w - x_1^-|^2 - |z - x_1^+|^2 &= 0, \\ |w - x_2^-|^2 - |z - x_2^+|^2 &= 0, \\ |w - x_3^-|^2 - |z - x_3^+|^2 &= 0. \end{aligned} \tag{21}$$

Subtracting the ith from the jth equation gives

$$2w \cdot (x_j^- - x_i^-) = 2z \cdot (x_j^+ - x_i^+) + |x_i^+|^2 - |x_j^+|^2 - |x_i^-|^2 + |x_j^-|^2.$$

In this way we obtain three linear equations for w in terms of z:

$$\nu_{ij} \cdot w = a_{ij} \cdot z + b_{ij} \text{ for } 1 \le i < j \le 3. \tag{22}$$

Suppose that ν_{12} and ν_{13} are parallel (or one of them is zero). Then $\{x_1^-, x_2^-, x_3^-\}$ is contained in a line. Suppose for definiteness that the ordering of the points on the line is such that $x_2^- \in [x_1^-, x_3^-]$. Then the three balls $B_{|z-x_i^+|}(x_i^-)$ have an empty intersection if and only if $B_{|z-x_1^+|}(x_1^-) \cap B_{|z-x_3^+|}(x_3^-)$ is empty. The necessary and sufficient condition for this is that the sum of the radii is less then the distance of the centers, i.e.

$$|z - x_1^+| + |z - x_3^+| \le |x_1^- - x_3^-|. \tag{23}$$

But equality in (23) gives the equation of an ellipse with focal points x_1^+ and x_3^+.

Now suppose ν_{12} and ν_{13} are not parallel. Then we can solve the first two equations in (22) for w, as an affine function of z, say, $w = l(z)$. Substituting back into the first equation in (21) gives a quadratic equation for z:

$$|l(z) - x_3^-|^2 - |z - x_3^+|^2 = 0. \tag{24}$$

The way we obtained (22) implies that if z satisfies (24), then $w = l(z)$ also satisfies the other two equations in (21). This proves that (24) is the equation defining the set (20). Since (24) is quadratic and since $E(X_1, X_2, X_3)$ (and hence (20)) is bounded, (20) is an ellipse.

It remains to prove (6). Firstly note that from (15) there exists $\delta > 0$ such that

$$|x_3^- - x_j^-| + \delta < |x_3^+ - x_j^+| \text{ for } j = 1, 2$$

and hence

$$x_3^- \in B_{|z-x_1^+|}(x_1^-) \cap B_{|z-x_2^+|}(x_2^-) \cap B_{|z-x_3^+|}(x_3^-) \tag{25}$$

whenever $0 < |z - x_3^+| < \delta$.

Suppose $\det(X_1 - X_2) < 0$. As in (23),

$$\{z \in \mathbb{C} : |z - x_1^+| + |z - x_2^+| \le |x_1^- - x_2^-|\} \subset \mathcal{E}$$

But since $|x_1^+ - x_2^+| < |x_1^- - x_2^-|$, the set of such z is a nonempty solid ellipse, so $x_1^+, x_2^+ \in \mathcal{E}$.

Finally assume that $\det(X_1 - X_2) > 0$. Suppose the solid ellipse $\mathcal{E}$ given by part (5) is nonempty. Consider the triangle $T = \{x_1^+, x_2^+, x_3^+\}^{co}$. If $z \in \mathcal{E}$ and z is outside the triangle T, then we can move z towards T in a direction perpendicular to a line separating T from z whilst remaining in $E(X_1, X_2, X_3)$, since along such a direction all three distances $|z - x_i^+|$ decrease. So we may assume that $\mathcal{E} \cap T \neq \emptyset$. But from (25) we also know that $T \backslash \mathcal{E}$ is nonempty. Hence there exists $z \in T \cap \partial\mathcal{E}$. But for each $z \in \partial\mathcal{E}$ there exists $w \in \{x_1^-, x_2^-, x_3^-\}^{co}$ such that $|z - x_i^+| = |w - x_i^-|$ for $i = 1, 2, 3$. Since $|x_i^+ - x_j^+| > |x_i^- - x_j^-|$, the angle between $(z - x_i^+)$ and $(z - x_j^+)$ needs to be greater than the angle between $(w - x_i^-)$ and $(w - x_j^-)$. This gives a contradiction, since z and w both lie inside the triangles and so the sum of the three angles equals in both cases 2π. Therefore $\mathcal{E}$ is empty.

Q.E.D.

Now we will make use of the assumption that for any $X_1, X_2 \in K_1$ and $Y_1, Y_2 \in K_2$ the set $\{X_1, X_2, Y_1, Y_2\}$ is not a T_4 configuration.

Lemma 6. *Suppose $X_1, X_2, Y_1, Y_2 \in \mathbb{R}^{2\times 2}$ such that $\det(X_i - Y_j) > 0$, and suppose that there exists $P \in \{X_1, X_2, Y_1, Y_2\}^{co}$ such that*

$$\det(X_i - P) > 0 \textit{ and } \det(Y_j - P) > 0 \textit{ for all } i, j.$$

Then $E(X_1, X_2, Y_1) \cap E(X_1, Y_1, Y_2) \subset \{x_1^+, x_2^+, y_1^+, y_2^+\}$.

Proof. If $\det(X_1 - X_2) > 0$, then Lemma 5 part (6) implies that

$$E(X_1, X_2, Y_1) = \{x_1^+, x_2^+, y_1^+\}.$$

So let us assume that $\det(X_1 - X_2) < 0$ and $\det(Y_1 - Y_2) < 0$. In this case we know from Lemma 5 that

$$E(X_1, X_2, Y_1) = \mathcal{E}_1 \cup \{y_1^+\} \text{ and } E(X_1, Y_1, Y_2) = \mathcal{E}_2 \cup \{x_1^+\},$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are two nonempty solid ellipses containing x_1^+, x_2^+ and y_1^+, y_2^+ respectively. If they intersect, then $E(X_1, X_2, Y_1) \cup E(X_1, Y_1, Y_2)$ is a connected set. We claim that this is not possible.

Consider the subspace L spanned by

$$\{X_2 - X_1, Y_1 - X_1, Y_2 - X_1\}.$$

If there exists nonzero $R \in L^\perp$ with $\det R \geq 0$, then let $Q = \operatorname{cof} R$. Since then

$$\langle \operatorname{cof} Q, X_i - P \rangle = \langle R, X_i - P \rangle = 0,$$

we have that

$$\det(X_i - (P + tQ)) = \det(X_i - P) + t^2 \det Q \geq \det(X_i - P) > 0$$

and similarly with Y_i. Thus the line $P + tQ$ is contained in the set

$$\{Z \in \mathbb{R}^{2\times 2} : \det(X_i - Z) > 0, \det(Y_i - Z) > 0 \text{ for } i = 1, 2\}.$$

Since $\det Q \geq 0$, the projection onto $\mathbb{C}$ is a (non-degenerate) line l that is contained in $E(X_1, X_2, Y_1)^c \cup E(X_1, Y_1, Y_2)^c$ (the union of the complements). Since $P \in \{X_1, X_2, Y_1, Y_2\}^{co}$, the points x_1^+, x_2^+ and y_1^+, y_2^+ cannot all lie on the same side of l. But then $E(X_1, X_2, Y_1) \cup E(X_1, Y_1, Y_2)$ cannot be connected.

Now suppose $R \in L^\perp$ with $\det R < 0$, i.e. $\langle Z, R \rangle = 0$ for all $Z \in L$. Let $\tilde{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\langle \tilde{J} R^T Z, \tilde{J} \rangle = -\langle Z, R \rangle = 0 \quad \text{ for all } Z \in L.$$

That is, $\tilde{X}_i = \tilde{J} R^T X_i$ and $\tilde{Y}_i = \tilde{J} R^T Y_i$ lie in an affine space $\tilde{L}$ orthogonal to $\tilde{J}$. In particular the projections $\tilde{x}_i^-$ and $\tilde{y}_i^-$ lie in a line $l \subset \overline{\mathbb{C}}$. Consider the set

$$\tilde{\mathcal{C}}_\delta = \{\tilde{P} \in \tilde{L} : \det(\tilde{X}_i - \tilde{P}) = \delta \text{ for } i = 1, 2\},$$

where $\delta > 0$ is such that $0 < \delta < \det(X_i - Y_j)$ for all i, j. In coordinates $\tilde{P} = (\tilde{z}, \tilde{w})$, and $\tilde{P} \in \tilde{\mathcal{C}}_\delta$ if and only if $\tilde{w} \in l$ and $|\tilde{z} - \tilde{x}_i^+| = |\tilde{w} - \tilde{x}_i^-| + \delta$ for $i = 1, 2$. This implies that $\tilde{w} \in [\tilde{x}_1^-, \tilde{x}_2^-]$, and hence $\tilde{z}$ satisfies

$$|\tilde{z} - \tilde{x}_1^+| + |\tilde{z} - \tilde{x}_2^+| = |\tilde{x}_1^- - \tilde{x}_2^-| + 2\delta.$$

Thus $\operatorname{proj}_{\mathbb{C}} \tilde{\mathcal{C}}_\delta$ and hence $\tilde{\mathcal{C}}_\delta$ is an ellipse, with $\tilde{x}_1^+$ and $\tilde{x}_2^+$ contained in the interior of $\operatorname{proj}_{\mathbb{C}} \tilde{\mathcal{C}}_\delta$.

Suppose that there exists $\tilde{P} \in \tilde{\mathcal{C}}_\delta$ so that $\det(\tilde{Y}_1 - \tilde{P}) \leq 0$. Since $\tilde{y}_1^- \in l$, we may assume that $\tilde{w} \in [\tilde{y}_1^-, \tilde{x}_1^-]$. But then

$$\begin{aligned} |\tilde{y}_1^+ - \tilde{x}_1^+| &\leq |\tilde{y}_1^+ - \tilde{z}| + |\tilde{z} - \tilde{x}_1^+| \\ &\leq |\tilde{y}_1^- - \tilde{w}| + |\tilde{w} - \tilde{x}_1^-| + \delta \\ &= |\tilde{y}_1^- - \tilde{x}_1^-| + \delta \\ &< |\tilde{y}_1^+ - \tilde{x}_1^+|, \end{aligned}$$

which is a contradiction. We deduce therefore that $\det(\tilde{Y}_i - \tilde{P}) > 0$ for all $\tilde{P} \in \tilde{\mathcal{C}}_\delta$ and $i = 1, 2$ and thus $\tilde{y}_1^+$ and $\tilde{y}_2^+$ lie outside the ellipse $\mathrm{proj}_{\mathbb{C}}\tilde{\mathcal{C}}_\delta$.

Transforming back, let $\mathcal{C}_\delta = R^{-T}\tilde{J}^{-1}\tilde{\mathcal{C}}_\delta$. Then

$$\det(X_i - P) > 0 \text{ and } \det(Y_i - P) > 0$$

for all $P \in \mathcal{C}_\delta$ and $i = 1, 2$. In particular the projection $\mathrm{proj}_{\mathbb{C}}\mathcal{C}_\delta$ cannot intersect $E(X_1, X_2, Y_1) \cup E(X_1, Y_1, Y_2)$.To see that $\mathrm{proj}_{\mathbb{C}}\mathcal{C}_\delta$ is also an ellipse with x_1^+ and x_2^+ lying inside, connect the identity matrix and $R^{-T}\tilde{J}^{-1}$ with a continuous path lying in the set $\{Q \in \mathbb{R}^{2\times 2} : \det Q > 0\}$. If, say, x_1^+ is not contained in the interior of the convex hull of $\mathrm{proj}_{\mathbb{C}}\mathcal{C}_\delta$, then there exists a matrix Q with $\det Q > 0$ such that $(QX)_1^+ \in \mathrm{proj}_{\mathbb{C}}Q\mathcal{C}_\delta$. But that means that there exists $P \in \mathcal{C}_\delta$ so that $QP - QX_1$ is anticonformal. This however cannot be, since $\det(QP - QX_1) = \det Q \det(P - X_1) > 0$.

Q.E.D.

Lemma 6 motivates the following definition: Suppose $\{a_i\}$, $\{b_j\}$ are two families of open balls in the plane with the property that whenever $(a_i \cap a_j) \cap b_k = \emptyset$, then $a_i \cap (b_k \cap b_l)$ and $a_j \cap (b_k \cap b_l)$ are nonempty (and same with a and b swapped). Let us then say that these two families satisfy the **T_4-property**.

Then lemma 6 implies that if $K_1 \cup K_2$ contain no T_4, then for any $z \in \mathbb{C}$ the corresponding balls $a_i = B_{|x_i^+ - z|}(x_i^-)$, $b_i = B_{|y_i^+ - z|}(y_i^-)$ for $X_i \in K_1$ and $Y_i \in K_2$ satisfy the T_4-property.

Lemma 7. *Suppose $\{a_i\}$, $\{b_j\}$ are two families of open balls in the plane with the T_4-property. Then for any $c_1, c_2 \in \{a_i\} \cup \{b_j\}$ the sets $a_1 \cap a_2 \cap c_1$ and $b_1 \cap b_2 \cap c_2$ cannot be both empty.*

Proof. We split the proof into cases depending on which family of balls c_1 and c_2 belong to.

(1) $a_1 \cap a_2 \cap b_3$ **and** $b_1 \cap b_2 \cap a_1$

Suppose that both sets are empty. Applying the T_4-property, the following sets are nonempty: $(a_1 \cap b_3) \cap b_2$, $(a_1 \cap b_3) \cap b_1$, $(a_2 \cap b_3) \cap b_2$, $(a_2 \cap b_3) \cap b_1$, $(a_1 \cap a_2) \cap b_2$, $(a_1 \cap a_2) \cap b_1$. In particular the picture is as shown in Figure 6, with c being the bounded component of $\mathbb{R}^2 \setminus (a_1 \cup a_2 \cup b_3)$.

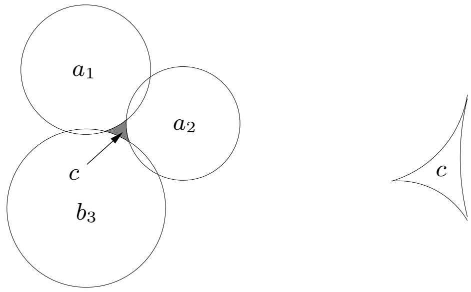


Figure 6: Intersection of three balls

Now since b_2 intersects all of $(a_1 \cap a_2)$, $(a_1 \cap b_3)$, $(a_2 \cap b_3)$, it contains c. Similarly $c \subset b_1$. But then $(b_1 \cap b_2)$ contains the convex hull of c, hence intersects a_1.

(2) $a_1 \cap a_2 \cap a_3$ **and** $b_1 \cap b_2 \cap b_3$

Suppose again both sets are empty. Suppose in addition $b_1 \cap b_2 \cap a_1 = \emptyset$. By part (1) b_1 and b_2 both have a nonempty intersection with $a_i \cap a_j$ (for all i, j), and if c is the bounded component of $\mathbb{R}^2 \setminus (a_1 \cup a_2 \cup a_3)$, we see that $c \subset b_1 \cap b_2$. But then $(b_1 \cap b_2) \cap a_1$ cannot be empty. So in fact $b_i \cap b_j \cap a_k$ is nonempty (for all i, j, k). Let $\tilde{c}$ be the bounded component of $\mathbb{R}^2 \setminus (b_1 \cup b_2 \cup b_3)$. Then $\tilde{c} \subset a_1 \cap a_2 \cap a_3$, a contradiction.

(3) $a_1 \cap a_2 \cap b_3$ **and** $b_1 \cap b_2 \cap a_3$

If both sets are empty, then by part (1) a_1 and a_2 intersect all $(b_i \cap b_j)$, and b_1 and b_2 intersect all $(a_i \cap a_j)$. In particular if c is as in the picture above, then $c \subset b_1$ and $c \subset b_2$. Moreover by part (2) we may assume $a_1 \cap a_2 \cap a_3 \neq \emptyset$. Suppose now that $a_3 \cap (a_1 \cap b_3) = \emptyset$. Then $a_3 \cap (b_i \cap b_j) \neq \emptyset$ for all i, j, which contradicts our assumptions. Hence $a_3 \cap (a_1 \cap b_3) \neq \emptyset$, and similarly $a_3 \cap (a_2 \cap b_3) \neq \emptyset$. But then $c \subset a_3$ and in particular $a_3 \cap b_1 \cap b_2 \neq \emptyset$, a contradiction.

Q.E.D.

Proof of Theorem 4.
Suppose for a moment that $z \in \mathbb{C}$ such that for any $X_1, X_2, X_3 \in K$ the intersection $\bigcap_{j=1}^{3} B_{|x_j^+ - z|}(x_j^-)$ is nonempty. We claim that then the whole family of balls

$$\mathcal{B} \stackrel{\text{def}}{=} \{B_{|x^+ - z|}(x^-) : X \in K\}$$

has a nonempty intersection. This would be a direct consequence of Helly's theorem once we can pass from open to closed balls. For this we employ compactness of K. Firstly, $z \notin \tilde{K}_1 \cup \tilde{K}_2$ otherwise one of the balls would be empty, so there exists $r_0 > 0$ so that $r \geq r_0$ for all $B_r(x) \in \mathcal{B}$. Furthermore, for each triple $B_{r_j}(x_j) \in \mathcal{B}$, $j = 1, 2, 3$, there exists $\epsilon > 0$ such that

$$\bigcap_{j=1}^{3} B_{r_j - \epsilon}(x_j) \neq \emptyset. \tag{26}$$

Suppose that there is no lower bound for $\epsilon > 0$ as the triple varies. Then there exists $B_1^{(i)}, B_2^{(i)}, B_3^{(i)} \in \mathcal{B}$ with the property that

$$\operatorname{diam} \bigcap_{j=1}^{3} B_j^{(i)} < \frac{1}{i}.$$

But then for appropriate subsequences $B_j^{(i)} \to B_j \in \mathcal{B}$ (in the sense that the radii and the centers converge), and in the limit $\bigcap_{j=1}^{3} B_j = \emptyset$. This contradicts our initial assumption. Hence there exists $\epsilon > 0$ so that (26) holds for all triples $B_{r_1}(x_1), B_{r_2}(x_2), B_{r_3}(x_3) \in \mathcal{B}$. Then we apply Helly's theorem to the family of closed balls

$$\{\overline{B}_{|x^+ - z| - \epsilon}(x^-) : X \in K\},$$

and thus finish the proof of the claim that $\bigcap_{B \in \mathcal{B}} B$ is nonempty.

Our assumption that $X_1, X_2 \in K_1$ and $Y_1, Y_2 \in K_2$ do not form a T_4 implies (by Theorem 2) that there exists $P \in \{X_1, X_2, Y_1, Y_2\}^{\mathrm{co}}$ with

$$\det(X_i - P) > 0 \text{ and } \det(Y_i - P) > 0 \text{ for } i = 1, 2.$$

Thus, Lemma 6 together with Lemma 7 implies that

$$E(X_1, X_2, Z_1) \cap E(Y_1, Y_2, Z_2) \subset \{x_1^+, x_2^+, y_1^+, y_2^+, z_1^+, z_2^+\} \tag{27}$$

for any $X_i \in K_1$, $Y_i \in K_2$ and $Z_i \in K_1 \cup K_2$. Let

$$\mathcal{E}_1 = \Bigg(\bigcup_{X_1, X_2 \in K_1,\, Z \in K} E(X_1, X_2, Z) \Bigg) \setminus \tilde{K}_2$$

$$\mathcal{E}_2 = \Bigg(\bigcup_{Y_1, Y_2 \in K_2,\, Z \in K} E(Y_1, Y_2, Z) \Bigg) \setminus \tilde{K}_1.$$

(Recall that $\tilde{K}_i = \mathrm{proj}_{\mathbb{C}} K_i$). By (27) above $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$. From Lemma 5 we deduce that $\mathcal{E}_1$ and $\mathcal{E}_2$ are compact sets and (17) combined with the argument above concerning the use of Helly's theorem implies that

$$\mathrm{proj}_{\mathbb{C}} S = \mathbb{C} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2), \tag{28}$$

where, as in (16),

$$S := \{Z : \det(X - Z) > 0 \text{ for all } X \in K\}.$$

But then we can find a smooth curve $\gamma \in \mathbb{C}$ separating $\tilde{K}_1$ and $\tilde{K}_2$ and lying in $\mathrm{proj}_{\mathbb{C}} S$. For each point $z \in \gamma$ there exists $w \in \overline{\mathbb{C}}$ such that $Z = (z, w) \in S$. In addition we can choose $w = w(z)$ so that it varies continuously with z. But then $\Gamma(t) := (\gamma(t), w(\gamma(t)))$ satisfies the required conditions.

To see that $K^{\mathrm{rc}} = K_1^{\mathrm{rc}} \cup K_2^{\mathrm{rc}}$ we may apply the so-called Structure Theorem ([Ped93],[MP98],[Kir03]), since $\gamma \times \overline{\mathbb{C}}$ defines a hypersurface disconnecting K^{rc}.

Alternatively we can consider the function $f : \mathbb{R}^{2\times 2} \mapsto \mathbb{R}$ defined by

$$f(X) = \begin{cases} \sup_{Y\in\Gamma}(-\det(X-Y)) & \text{if } X \in U_1 \cup U_3 \\ \sup_{Y\in\Gamma}(-\det^-(X-Y)) & \text{if } X \in U_2 \cup U_3. \end{cases}$$

Here U_1 and U_2 are the two components of $\{X : \det(X-Y) > 0 \text{ for all } Y \in \Gamma\}$, and $U_3 = \mathbb{R}^{2\times 2} \setminus (U_1 \cup U_2)$. Since all rank-one lines in $\mathbb{R}^{2\times 2}$ lie entirely in $U_1 \cup U_3$ or $U_2 \cup U_3$ and since f is rank-one convex (locally polyconvex) in both these regions, f is globally rank-one convex. Moreover $f < 0$ in U_1 and $f = 0$ in U_2.

Suppose now that $\nu \in \mathcal{M}^{rc}(K)$, and without loss of generality assume that the barycenter $\bar{\nu} \in U_2$ (it is clear that $K^{rc} \subset U_1 \cup U_2$). Then $f(\bar{\nu}) = 0$, and by the definition of laminates

$$0 = f(\bar{\nu}) \leq \langle \nu, f \rangle .$$

This implies that supp $\nu \subset U_2$.

6 Compact sets

Proposition 2. *Suppose $K \subset \mathbb{R}^{2\times 2}$ is compact with no rank-one connections. If K contains no T_4 configuration of type (A), then (upto changing signs) either*

$$\det(X-Y) > 0 \textit{ for all } X, Y \in K \textit{ with } X \neq Y,$$

or K admits a decomposition of the following type:

$$K = K_1 \cup K_2,$$

where K_1 and K_2 are both nonempty, disjoint compact sets, and $\det(X-Y) > 0$ *for all $X \in K_1$, $Y \in K_2$.*

Proof. We study the $\ominus$-, and $\oplus$-connectedness of K. We call the set $\ominus$-connected if for any $X, Y \in K$ there exists $X_1, X_2, \ldots, X_N \in K$ such that

$$\det(X - X_1) < 0, \det(X_1 - X_2) < 0, \ldots, \det(X_N - Y) < 0.$$

In fact we can always assume that if such a path exists, then it has at most length 2 (that is, $X_2 = Y$). Indeed, let us assume a $\ominus$-path between X and Y exists, and take the shortest such path. If the shortest path has length 3 at least, then we have the following sign assertions:

1. $\det(X - X_1) < 0$, $\det(X_1 - X_2) < 0$, $\det(X_2 - X_3) < 0$,
2. $\det(X - X_2) > 0$, $\det(X - X_3) > 0$, $\det(X_1 - X_3) > 0$.

But this is exactly the sign-configuration (A) which cannot exist by assumption. This proves our first claim.

Secondly, for any $X \in K$, the set

$$CC_\ominus(X) \stackrel{\text{def}}{=} \{Y \in K : \text{there exists a } \ominus\text{-path from } X \text{ to } Y\}$$

is compact. This is clear since if $Y_i \in K$ are $\ominus$-connected to X, then there exist $X_i \in K$ with $\det(X_i - X) \le 0$ and $\det(Y_i - X_i) \le 0$ (with equality if and only if the matrices are equal), and for appropriate subsequences $Y_i \to Y$ and $X_i \to P$ with $Y, P \in K$ satisfying

$$\det(X - P) \le 0 \text{ and } \det(P - Y) \le 0.$$

Thus Y is also $\ominus$-connected to X.

On the other hand, if $X_0 \in K$ such that there exists $Y_0 \in K$ with

$$\det(X_0 - Y_0) < 0,$$

then $CC_\ominus(X_0)$ is also open (relative to K): for if $Y \in CC_\ominus(X_0) \setminus \{X_0\}$, then either $\det(X_0 - Y) < 0$, or there exists $P \in K$ such that $\det(X_0 - P) < 0$ and $\det(P - Y) < 0$. Then there exists an $\epsilon > 0$ such that for any $\tilde{Y} \in B_\epsilon(Y)$ we have $\det(\tilde{Y} - X_0) < 0$ in the first case, or $\det(\tilde{Y} - P) < 0$ in the second case. This means that $B_\epsilon(Y) \cap K \subset CC_\ominus(X_0)$. Furthermore, as $\det(Y_0 - X_0) < 0$, there is a neighbourhood $B_\epsilon(X_0)$ of X_0 such that $\det(X - Y_0) < 0$ for all $X \in B_\epsilon(X_0)$. Hence $B_\epsilon(X_0) \cap K \subset CC_\ominus(X_0)$.

Assume now, for the moment, that K contains a matrix X_0 with the property that $\det(X - X_0) > 0$ for all $X \in K \setminus \{X_0\}$. If $\det(X - Y) > 0$ for all $X, Y \in K$ with $X \ne Y$, then we are done. Otherwise fix $Y_0 \in K$ for which there exists $Y_1 \in K$ with $\det(Y_0 - Y_1) < 0$. By the above, $CC_\ominus(Y_0)$ is both closed and open in K, and $X_0 \notin CC_\ominus(Y_0)$. But then

$$K_1 = CC_\ominus(Y_0) \text{ and } K_2 = K \setminus CC_\ominus(Y_0)$$

give the required nontrivial decomposition.

Finally consider the general case. For any n take an $\frac{1}{n}$-net $X_1, \ldots, X_N$, with $N = N_n$. In other words for any $Y \in K$ there exists $i \le N$ such that

$$|Y - X_i| \le \frac{1}{n}.$$

We can apply the considerations of Section 4 to get a decomposition

$$\{X_1, \ldots, X_N\} = K_1^n \cup K_2^n$$

where K_1^n and K_2^n are nonempty, and there exists $c_n > 0$ such that

$$\det(X_i - X_j) \ge c_n \text{ for } X_i \in K_1^n,\, X_j \in K_2^n. \tag{29}$$

Now suppose that there is no lower bound for $c_n > 0$ as we let $n \to \infty$ (in a way that $\{X_1, \ldots, X_{N_n}\} \subset \{X_1, \ldots, X_{N_{n+1}}\}$). Then there exist

$$X_n \in K_1^n,\, Y_n \in K_2^n \text{ with } \det(X_n - Y_n) = c_n \to 0.$$

In particular, since K contains no rank-one connections, X_n, $Y_n \to P \in K$. We claim that $\det(P - X) > 0$ for all $X \in K \setminus \{P\}$. If there exists $Q \in K$ with $\det(P - Q) < 0$, then for some $\delta > 0$ we have

$$\det(P_1 - Q_1) < 0 \text{ whenever } |P - P_1|, |Q - Q_1| < \delta. \tag{30}$$

Take n sufficiently large so that $n > \frac{1}{\delta}$ and $|X_n - P|, |Y_n - P| < \delta$. Then there exists a matrix X_i in the $\frac{1}{n}$-net for which $|X_i - Q| < \delta$. Furthermore either $\det(X_i - X_n) > 0$ or $\det(X_i - Y_n) > 0$ (depending on whether X_i is in K_1^n or K_2^n, see (29)). But that gives a contradiction with (30) and thus proves that if $c_n \to 0$, then there exists $P \in K$ with $\det(P - X) > 0$ for all $X \in K \setminus \{P\}$. In this case the previous claim yields a sign-decomposition. In the case where $c_n \geq c > 0$, we automatically get the decomposition K_1 and K_2, obtained as the limits of K_1^n and K_2^n. This concludes the proof. Q.E.D.

We recall the following result from Šverák [Šve93]:

LEMMA 8. *Let K be a bounded Borel measurable subset of $\mathbb{R}^{2\times 2}$ with no rank-one connections. If $\det(X - Y) > 0$ for any distinct $X, Y \in K$, then $\mathcal{M}^{pc}(K)$ is trivial, i.e. contains Dirac masses only. In particular K^{pc} and hence K^{rc} is trivial.*

Now we are ready to prove the main result of this chapter:

Proof of Theorem 1. Suppose $X \in K^{rc} \setminus K$, and consider all compact subsets $\tilde{K}$ of K such that $X \in \tilde{K}^{rc}$. If

$$K \supset K_1 \supset K_2 \supset \dots$$

is a decreasing sequence of compact sets such that $X \in K_i^{rc}$ for all i, then $K_\infty = \bigcap_i K_i$ is a nonempty compact subset of K. Suppose that $X \notin K_\infty^{rc}$. Then there exists $f : \mathbb{R}^{2\times 2} \mapsto \mathbb{R}$ rank-one convex such that $f \equiv 0$ on K_∞ and $f(X) = 1$. But since f is in particular continuous (in fact Lipschitz), there exists i_0 such that $f < \frac{1}{2}$ on K_i for $i \geq i_0$ (otherwise $K_i \cap \{f \geq \frac{1}{2}\}$ is a decreasing chain of nonempty compact sets, and so $K_\infty \cap \{f \geq \frac{1}{2}\}$ cannot be empty).

But then $g = \max\{0, f - \frac{1}{2}\}$ is a rank-one convex function such that $g \equiv 0$ on K_i (for $i \geq i_0$) and $g(X) > 0$, and this contradicts the assumption that $X \in K_i^{rc}$. So $X \in K_\infty^{rc}$.

But then Zorn's Lemma can be applied to give a minimal set $K_0 \subset K$, i.e. K_0 satisfies

1. $X \in K_0^{rc} \setminus K_0$,
2. if $K_1 \subset K_0$ is compact with $X \in K_1^{rc}$, then $K_1 = K_0$.

If K_0 does not contain a T_4 configuration, then Proposition 2 implies that either $\det(X - Y) > 0$ for all distinct $X, Y \in K_0$, or $K_0 = K_1 \cup K_2$ is a nontrivial sign-separation as described in Proposition 2. In the former case Lemma 8 gives a contradiction, and in the latter case we use Theorem 4 to get $K_0^{rc} = K_1^{rc} \cup K_2^{rc}$. Then either $X \in K_1^{rc}$ or $X \in K_2^{rc}$. In both cases we contradict the minimality of K_0.

Q.E.D.

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