Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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Preprint no.: 73 2003

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July 30, 2003

Abstract

We prove strong ellipticity of chiral bag boundary conditions on even dimensional manifolds. From a knowledge of the heat kernel in an infinite cylinder, some basic properties of the zeta function are analyzed on cylindrical product manifolds of arbitrary even dimension.

Subject Classification
PACS: 02.40.Vh, 12.39.Ba
MSC: 58J50, 35P05, 35J55

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1 Introduction

The influence that boundary conditions have on different spectral functions is an active field of research. In quantum field theory, spectral functions of particular interest are the zeta function and the heat kernel. Their dependence on the boundary condition is well understood for a large variety of boundary conditions (for a recent review see [1]). However, for some boundary conditions an understanding of elementary properties of spectral functions is still lacking. This is the case for generalized (or chiral) local bag boundary conditions [2, 3].

These boundary conditions involve an angle θ, which is a substitute for introducing small quark masses to drive the breaking of chiral symmetry [3, 4, 5]. The influence the parameter θ has on various correlators was analyzed in detail in [3] for the two dimensional Euclidian ball. In reference [6], the heat kernel and the eta function were analyzed in the two dimensional cylinder. The situation of an arbitrary dimensional ball was considered in [7]. The results found in the above articles suggest that general properties of spectral functions, like the pole structure of the associated zeta function and the form of the asymptotic expansion of the heat kernel, are the properties resulting from strongly elliptic local boundary conditions. For the generalized bag boundary conditions considered here, this property has not been proven. In fact, strong ellipticity is not clear at all because, for $\theta \neq 0$, the boundary conditions are of mixed oblique type [7] and, under certain circumstances, oblique boundary conditions are not strongly elliptic [8, 9]. However, after introducing some basic notation and properties of the boundary conditions, we will prove in Sections 3 and 4 that generalized bag boundary conditions are indeed strongly elliptic boundary conditions. Based on this observation, in future investigations one might envisage a determination of heat kernel coefficients for these boundary conditions. In recent years, a conglomerate of methods has been proven to be very effective in the determination of this asymptotics [1]. Among the methods are special case considerations, which form the basis of the second half of our paper. In particular, we will determine the local heat kernel and the zeta function on cylindrical product manifolds. Results are given in terms of boundary data, much in the way it is possible for spectral boundary conditions [10]. The Conclusions will summarize the main results and describe their possible future applications.

2 Basic properties of chiral bag boundary conditions

In this section, we will establish some notations for the problem at hand, i.e., the Euclidean Dirac operator acting on spinors satisfying local (chiral bag) boundary conditions [3].

Let $m = 2\bar{m}$ be even and let $P = i\gamma_j \nabla_j$ be an operator of Dirac type on a compact oriented Riemannian manifold of dimension m. Let V denote the spinor space; $dim(V) = 2^{\bar{m}}$. An explicit representation of the γ-matrices is provided in Appendix A. They are self-adjoint and satisfy the Clifford anti-commutation relation (A.1). Near the boundary, let e_m be the inward unit normal and let γ_m be the projection of the γ-matrix on the inward unit normal. In addition let $\tilde{\gamma}$ be the generalization of 'γ_5' to arbitrary even dimension, $\tilde{\gamma} = (-i)^{\bar{m}} \gamma_1 ... \gamma_m$.

We set

$$\chi = i\tilde{\gamma} e^{\theta\tilde{\gamma}} \gamma_m$$

and use the relation $\gamma_i \tilde{\gamma} + \tilde{\gamma}\gamma_i = 0$ to compute

$$\chi^2 = -\tilde{\gamma} e^{\theta\tilde{\gamma}} \gamma_m \tilde{\gamma} e^{\theta\tilde{\gamma}} \gamma_m = \tilde{\gamma} e^{\theta\tilde{\gamma}} \tilde{\gamma} e^{-\theta\tilde{\gamma}} \gamma_m \gamma_m = \mathbf{1}.$$

We define

$$\Pi_\pm := \frac{1}{2}(1 \pm \chi)$$

and have

$$\Pi_\pm^2 = \Pi_\pm \text{ and } \Pi_- \Pi_+ = \Pi_+ \Pi_- = 0.$$

Note that these two projectors are not self-adjoint (except for the particular case $\theta = 0$). Rather, calling their respective adjoints $\Pi_-^\star$ and $\Pi_+^\star$, one has $\Pi_\pm^\star := \frac{1}{2}(1 \pm i\tilde{\gamma} e^{-\theta\tilde{\gamma}} \gamma_m)$, and the following equations hold

$$\begin{aligned}
\Pi_+^\star \Pi_+ &= \cosh(\theta\tilde{\gamma}) \exp(-\theta\tilde{\gamma}) \Pi_+ = \Pi_+^\star \cosh(\theta\tilde{\gamma}) \exp(-\theta\tilde{\gamma}) \\
\Pi_-^\star \Pi_+ &= \sinh(\theta\tilde{\gamma}) \exp(-\theta\tilde{\gamma}) \Pi_+ = \Pi_-^\star \sinh(\theta\tilde{\gamma}) \exp(-\theta\tilde{\gamma}) \\
\Pi_-^\star \Pi_- &= \cosh(\theta\tilde{\gamma}) \exp(-\theta\tilde{\gamma}) \Pi_- = \Pi_-^\star \cosh(\theta\tilde{\gamma}) \exp(-\theta\tilde{\gamma}) \\
\Pi_+^\star \Pi_- &= \sinh(\theta\tilde{\gamma}) \exp(-\theta\tilde{\gamma}) \Pi_- = \Pi_+^\star \sinh(\theta\tilde{\gamma}) \exp(-\theta\tilde{\gamma}) .
\end{aligned}$$

We use Π_- to define boundary conditions for P. Similarly, we shall let $\mathcal{B} := \Pi_- \oplus \Pi_- P$ define the associated boundary condition for P^2.

In the following two sections, we will show that (P, Π_-) and $(P^2, \mathcal{B})$ define strongly elliptic boundary conditions and, as a result, we can assume standard results on the meromorphic structure of eta and zeta-invariants hold [11]. Otherwise stated, we will prove statements (1) and (2) of the following Theorem

Theorem 2.1
(1) (P, Π_-) *is strongly elliptic with respect to* $\mathbf{C} - \mathbf{R}^+ - \mathbf{R}^-$.
(2) $(P^2, \mathcal{B})$ *is strongly elliptic with respect to* $\mathbf{C} - \mathbf{R}^+$.
(3) (P, Π_-) *is self-adjoint.*
(4) $(P^2, \mathcal{B})$ *is self-adjoint.*

Statements (3) and (4) are well known to hold [3] and so we concentrate on statements (1) and (2).

3 Ellipticity of the first order boundary value problem

Proof of (1): We use Lemma 1.11.2 (a) of [11] to prove assertion (1). Note that the special case $\theta = 0$ defines standard mixed boundary conditions and the theorem is known to hold for this case. Let $x = (y_1, ..., y_{m-1}, x_m)$ be coordinates near the boundary where x_m is the geodesic distance to the boundary and where $y = (y_1, ..., y_{m-1})$ are coordinates on ∂M. Let $\xi_a dy^a \in T^*(\partial M)$.

Following [11] we define

$$\tilde{q}(\xi, \lambda) = -i\gamma_m \left(\sum_{a<m} \gamma_a \xi_a - \lambda \right) \text{ for } (\xi, \lambda) \neq (\vec{0}, 0) \text{ and } \lambda \notin \mathbf{R} - \{0\}.$$

We then have $\tilde{q}(\xi, \lambda)^2 = (|\xi|^2 - \lambda^2)\mathbf{1}$. As $(|\xi|^2 - \lambda^2) \notin i\mathbf{R}$, we may let $V_\pm^{\tilde{q}}$ be the span of the eigenvectors of $\tilde{q}(\xi, \lambda)$ with positive/negative real parts. We let

$$W := \mathrm{Kernel}(\Pi_-) = \mathrm{Range}(\Pi_+).$$

Using Lemma 1.11.2 (a) of [11], we prove assertion (1) by verifying that Π_- is an isomorphism from $V_-^{\tilde{q}}(\xi, \lambda)$ to $\mathcal{W} = \mathrm{Range}(\Pi_-)$. This is equivalent to showing

$$V_-^{\tilde{q}}(\xi, \lambda) \cap W = \{0\}. \tag{3.1}$$

We change variables slightly setting $\lambda = -i\mu$ where $\mu \notin i\mathbf{R} - \{0\}$ and $(\xi, \mu) \neq (\vec{0}, 0)$. We then have

$$\tilde{q}(\xi, \mu) = -i\gamma_m \sum_{a<m} \gamma_a \xi_a + \mu\gamma_m \text{ and } \tilde{q}(\xi, \mu)^2 = (|\xi|^2 + \mu^2)\mathbf{1}.$$

The next step in the proof is to reduce the problem to a collection of effective two-dimensional ones. We make use of the properties of the γ-matrices. First note that the elements

$$\tau_1 := i\gamma_2\gamma_3, \quad ..., \tau_{\bar{m}-1} := i\gamma_{m-2}\gamma_{m-1},$$

mutually commute and, in addition, they commute with γ_1 and γ_m. Thus, for $j, k = 1, ..., \bar{m} - 1$, we have

$$\begin{aligned} \tau_j\tau_k &= \tau_k\tau_j, & \tau_j^2 &= 1, \\ \tau_j\gamma_1 &= \gamma_1\tau_j, & \tau_j\gamma_m &= \gamma_m\tau_j. \end{aligned}$$

So we can choose a set of simultaneous eigenvectors of τ_j with eigenvalues $\rho_j = \pm 1$. We denote by $\vec{\rho} = (\rho_1, ..., \rho_{\bar{m}-1})$ the collection of simultaneous eigenvalues of τ_j and we define the associated simultaneous eigenspaces by

$$V_{\vec{\rho}} = \{v \in V : \tau_i v = \rho_i v\}.$$

The vector space $V_{\vec{\rho}}$ is preserved by γ_1, γ_m and $\tilde{\gamma}$. We use this fact to decompose $V_{\vec{\rho}}$ into its chiral parts,

$$V_{\vec{\rho}} = V_{\vec{\rho}}^+ \oplus V_{\vec{\rho}}^- \text{ where } V_{\vec{\rho}}^\pm = \{v \in V_{\vec{\rho}} : \tilde{\gamma}v = \pm v\}.$$

Let $V_{\vec{\rho}}^+ = \text{span}\{v_{\vec{\rho}}\}$. Then since $\gamma_m\tilde{\gamma} = -\tilde{\gamma}\gamma_m$ we have

$$V_{\vec{\rho}} = \text{span}\{v_{\vec{\rho}}, \gamma_m v_{\vec{\rho}}\}.$$

The vector spaces $V_{\vec{\rho}}$ provide the decomposition

$$V = \bigoplus_{\vec{\rho}} V_{\vec{\rho}}, \quad \text{dim} V_{\vec{\rho}} = 2,$$

and the problem completely decouples into two-dimensional spaces.

On $V_{\vec{\rho}}$ one easily computes, using the definition $\epsilon(\vec{\rho}) = \rho_1 \times ... \times \rho_{\bar{m}-1}$,

$$\gamma_1 = \epsilon(\vec{\rho}) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma_m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

thus reproducing the two-dimensional Pauli-matrices up to the standard sign ambiguity.

We note that the kernel of Π_- is determined by the eigenvectors of χ. Thus to establish (3.1) we shall need explicit representations of χ and $\tilde{q}(\xi, \mu)$

acting on $V_{\vec{\rho}}$. To calculate $\tilde{q}(\xi,\mu)$ it is possible to choose coordinates such that $\xi_2 = ... = \xi_{m-1} = 0$. It is then easy to see that

$$\chi = i\begin{pmatrix} 0 & e^{\theta} \\ -e^{-\theta} & 0 \end{pmatrix}, \quad \tilde{q}(\xi,\mu) = \begin{pmatrix} \epsilon(\rho)\xi_1 & \mu \\ \mu & -\epsilon(\rho)\xi_1 \end{pmatrix}.$$

The eigenvectors of χ follow,

$$\chi\begin{pmatrix} \varrho \\ -ie^{-\theta} \end{pmatrix} = \varrho\begin{pmatrix} \varrho \\ -ie^{-\theta} \end{pmatrix},$$

and we compute

$$\tilde{q}(\xi,\mu)\begin{pmatrix} \varrho \\ -ie^{-\theta} \end{pmatrix} = \begin{pmatrix} \epsilon(\vec{\rho})\varrho\xi_1 - i\mu e^{-\theta} \\ \mu\varrho + i\epsilon(\vec{\rho})\xi_1 e^{-\theta} \end{pmatrix}$$

where $\varrho = \pm 1$. Assertion (1) holds if these are not multiples of each other, that is if

$$\det\begin{pmatrix} \epsilon(\vec{\rho})\varrho\xi_1 - i\mu e^{-\theta} & \varrho \\ \mu\varrho + i\epsilon(\vec{\rho})\xi_1 e^{-\theta} & -ie^{-\theta} \end{pmatrix} = -2i\epsilon(\vec{\rho})\varrho\xi_1 e^{-\theta} - \mu\left(1 + e^{-2\theta}\right) \neq 0.$$

Since the first term is purely imaginary and $\mu \notin i\mathbf{R} - \{0\}$ this proves assertion (1).

In fact, to prove what we announced (i.e., that Π_- is an isomorphism from $V_-^{\tilde{q}}(\xi,\lambda)$ to $\mathcal{W} = \text{Range}(\Pi_-)$), it is enough to consider only $\varrho = +1$. But since $V_-^{\tilde{q}}(-\xi,-\lambda) = V_+^{\tilde{q}}(\xi,\lambda)$, considering $\rho = \pm 1$ gives the same condition.

Although assertion (2) follows from assertion (1) and Lemma 1.11.2 (b) in [11], we prefer to give a second proof showing that the boundary operator $\mathcal{B}$ involves tangential derivatives. This makes apparent that the chiral boundary conditions are non-standard boundary conditions.

4 Ellipticity of the second order boundary problem

When considering spectral properties of the square of the operator of Dirac type, $P^2 = (i\gamma_j\nabla_j)^2$, the boundary condition imposed through $\mathcal{B}$ is

$$\Pi_-\psi\,|_{\partial M} = 0, \tag{4.1}$$
$$\Pi_-\gamma_j\nabla_j\psi\,|_{\partial M} = 0. \tag{4.2}$$

The second boundary condition (4.2) can be rewritten as an oblique boundary condition involving tangential derivatives. To this end note

$$\begin{aligned} \gamma_m \Pi_\mp &= \Pi^\star_\pm \gamma_m , \\ \gamma_a \Pi_\mp &= \Pi^\star_\mp \gamma_a , \end{aligned}$$

with the tangential γ-matrices γ_a, where $a = 1, ..., m-1$. This allows the boundary condition (4.2) to be written as

$$\begin{aligned} 0 &= \Pi_-(-i\gamma_j \nabla_j)\psi \,|_{\partial M} = i\gamma_m \Pi_-(-i\gamma_j \nabla_j)\psi \,|_{\partial M} \\ &= \gamma_m \Pi_-(\gamma_m \nabla_m + \gamma_a \nabla_a)\psi \,|_{\partial M} = (\Pi^\star_+ \nabla_m + \gamma_m \gamma_a \Pi^\star_- \nabla_a)\psi \,|_{\partial M} \\ &= (\Pi^\star_+ \nabla_m + \gamma_m \gamma_a \Pi^\star_- \nabla_a)(\Pi_- + \Pi_+)\psi \,|_{\partial M} \ , \end{aligned} \tag{4.3}$$

where ∇_m is the interior normal derivative. The boundary condition contains tangential derivatives and the conditions imposed through $\mathcal{B}$ could thus be termed of mixed oblique type.

Proof of (2): To study the ellipticity of the boundary value problem, we introduce the "partial" leading symbol of P^2

$$\sigma_L(y, x_m, \omega, -i\partial_m, \lambda) = -\partial_m^2 + \omega^2 - \lambda$$

and the graded symbol, σ_g, of $\mathcal{B}$

$$\sigma_g = \begin{pmatrix} \Pi_- & 0 \\ -\gamma_a \omega_a \Pi^\star_- & i\gamma_m \Pi^\star_+ \end{pmatrix} .$$

Strong ellipticity requires that the problem

$$\sigma_L(y, x_m, \omega, -i\partial_m, \lambda)\Psi(y, x_m, \omega, \lambda) = 0 \tag{4.4}$$

with

$$\Psi \to_{r\to\infty} 0 \tag{4.5}$$

and

$$\sigma_g \begin{pmatrix} \Psi \\ \partial_m \Psi \end{pmatrix} \Bigg|_{r=0} = \begin{pmatrix} \Pi_- \alpha \\ i\gamma_m \Pi^\star_+ \partial_m \alpha \end{pmatrix} \Bigg|_{r=0} \tag{4.6}$$

has an unique solution.

Now, the solutions to (4.4) and (4.5) are

$$\Psi(x_m, \omega, \lambda) = \Psi_0 \exp(-\Lambda x_m) ,$$

where $\Lambda = +\sqrt{\omega^2 - \lambda}$. Note that $\Re(\Lambda) > 0$ for $\lambda \in \mathbf{C} - \mathbf{R}_+$.

The condition (4.6), when applied to them, reads

$$\begin{pmatrix} \Pi_- & 0 \\ -\gamma_a\omega_a\Pi_-^\star & i\gamma_m\Pi_+^\star \end{pmatrix} \begin{pmatrix} \Psi_0 \\ -\Lambda\Psi_0 \end{pmatrix} = \begin{pmatrix} \Pi_-\alpha \\ -i\gamma_m\Pi_+^\star\Lambda\alpha \end{pmatrix} .$$

This gives a system of two equations. After multiplying the second one by $i\gamma_m$, and using $\Psi_0 = \Pi_-\Psi_0 + \Pi_+\Psi_0$, one obtains

$$\Pi_-\Psi_0 = \Pi_-\alpha \,, \tag{4.7}$$

and

$$\left(-i\gamma_m(\gamma_a\omega_a)\Pi_-^\star\Pi_+ + \Lambda\Pi_+^\star\Pi_+\right)\Psi_0 =$$

$$\Lambda\Pi_+^\star\alpha - \left(-i\gamma_m(\gamma_a\omega_a)\Pi_-^\star\Pi_- + \Lambda\Pi_+^\star\Pi_-\right)\Psi_0 \,. \tag{4.8}$$

We use (2.1) and substitute (4.7) into (4.8) to see

$$\exp\left(-\theta\tilde{\gamma}\right)\left[-i\gamma_m(\gamma_a\omega_a)\sinh\left(\theta\tilde{\gamma}\right) + \Lambda\cosh\left(\theta\tilde{\gamma}\right)\right]\Pi_+\Psi_0 =$$

$$\Lambda\Pi_+^\star\alpha - \left[-i\gamma_m(\gamma_a\omega_a)\cosh\left(\theta\tilde{\gamma}\right) + \Lambda\sinh\left(\theta\tilde{\gamma}\right)\right]\exp\left(-\theta\tilde{\gamma}\right)\Pi_-\alpha \,.$$

So, the problem has an unique solution if the matrix

$$M = -i\gamma_m(\gamma_a\omega_a)\sinh\left(\theta\tilde{\gamma}\right) + \Lambda\cosh\left(\theta\tilde{\gamma}\right) = A\sinh\left(\theta\right) + \Lambda\cosh\left(\theta\tilde{\gamma}\right)$$

is nonsingular, where we introduced $A = -i\tilde{\gamma}\gamma_m\gamma_a\omega_a$.

But $A^\star = A$; so, it is diagonalizable. Moreover, $A^2 = \left(\sum_a \omega_a^2\right)\mathbf{1}$. As a consequence, A has eigenvalues $\pm\sqrt{\sum_a \omega_a^2}$, except in two dimensions, where A is proportional to the identity. Then, the determinant of M can be evaluated in the basis of eigenvectors of A, and in all cases, $detM$ can be seen to vanish if $\lambda = \frac{\sum_a \omega_a^2}{\cosh^2\theta}$. For $\lambda = 0$, the determinant can only vanish if $\omega_a = 0$. Otherwise, it can only happen for $\lambda \in \mathbf{R}_+$, which proves strong ellipticity in $\mathbf{C} - \mathbf{R}_+$, and any even dimension. This completes the proof of Theorem 2.1. □

5 Heat kernel in an infinite cylinder

In what follows, we present the heat kernel for $(P^2, \mathcal{B})$ in an infinite cylinder $\mathcal{M} = \mathbf{R}_+ \times \mathcal{N}$ of any even dimension. By cylinder we mean that the metric is of the type $ds^2 = dx_m^2 + ds_{\mathcal{N}}^2$, where $ds_{\mathcal{N}}^2$ is the metric of the closed boundary $\mathcal{N}$.

In order to determine the heat kernel, it is useful to note that the chiral bag boundary conditions in equations (4.1) and (4.2) are equivalent, for each eigenvalue of the tangential part B of the operator P, to Dirichlet boundary conditions on part of the fibre, and Robin (modified Neumann) on the rest.

In fact, let's first notice that the operators $\mathcal{P}_+ = \frac{\Pi_+\Pi_+^\star}{\cosh^2\theta}$ and $\mathcal{P}_- = \frac{\Pi_-^\star\Pi_-}{\cosh^2\theta}$ are self-adjoint projectors, and they satisfy $\mathcal{P}_+ + \mathcal{P}_- = \mathbf{1}$ splitting V into two complementary subspaces.

Let $\xi = x_m - x_m'$, and $\eta = x_m + x_m'$ and as before, let $y = (y_1, y_2, ..., y_{m-1})$ be the coordinates on the boundary and $x = (y, x_m)$. If we call $\phi_\omega(y)$ the eigenspinors of the operator $B = \tilde\gamma\gamma_m\gamma_a\partial_a$ (with $a = 1, 2, ..., m-1$) corresponding to the eigenvalue ω, normalized such that

$$\sum_\omega \phi_\omega^\star(y)\phi_\omega(y') = \delta^{m-1}(y - y')$$

with δ^{m-1} the Dirac delta function, and

$$\int_{\partial M} dy\ \phi_\omega^\star(y)\phi_\omega(y) = 1\,,$$

we can expand $\psi(y, x_m) = \sum_\omega f_\omega(x_m)\phi_\omega(y)$. If $\psi = \mathcal{P}_+\psi$, then the condition (4.1) is identically satisfied, and only (4.3) must be imposed at the boundary which, for each ω, reduces to

$$\cosh\theta e^{-\theta\tilde\gamma}\left(\partial_m + \omega\tanh\theta\right) f_\omega = 0\,.$$

Since the factor to the left of the parenthesis is invertible, this is nothing but a Robin boundary condition.

In the subspace $\psi = \mathcal{P}_-\psi$, the boundary condition (4.1) reduces to

$$\cosh\theta e^{\theta\tilde\gamma} f_\omega = 0\,,$$

while (4.3) requires

$$\omega f_\omega = 0\,.$$

Thus, in this subspace, both boundary conditions are nothing but homogeneous Dirichlet ones.

As a consequence, the complete heat kernel can be written as a Dirichlet heat kernel on $\mathcal{P}_-V$ and a Robin heat kernel on $\mathcal{P}_+V$. For the convenience of the reader we make the single ingredients explicit [12] and write

$$\begin{aligned}
K(t;x,x') &= K(t;x,x')(\mathcal{P}_- + \mathcal{P}_+) \\
&= \frac{1}{\sqrt{4\pi t}} \sum_\omega \phi^\star_\omega(y')\phi_\omega(y) e^{-\omega^2 t}\left(e^{\frac{-\xi^2}{4t}} - e^{\frac{-\eta^2}{4t}}\right)\mathcal{P}_- \\
&+ \frac{1}{\sqrt{4\pi t}} \sum_\omega \phi^\star_\omega(y')\phi_\omega(y) e^{-\omega^2 t}\left\{\left(e^{\frac{-\xi^2}{4t}} + e^{\frac{-\eta^2}{4t}}\right)\right. \\
&\quad \left. +2\sqrt{\pi t}\ \omega \tanh\theta e^{\omega^2 t \tanh^2\theta - \omega\eta\tanh\theta} erfc[u_\omega(\eta,t)]\right\}\mathcal{P}_+ \\
&= \frac{1}{\sqrt{4\pi t}} \sum_\omega \phi^\star_\omega(y')\phi_\omega(y) e^{-\omega^2 t}\left\{\left(e^{\frac{-\xi^2}{4t}} - e^{\frac{-\eta^2}{4t}}\right)\mathbf{1}\right. \\
&\quad \left. +\frac{2\Pi_+\Pi_+^\star}{\cosh^2(\theta)}\left[1+\sqrt{(\pi t)}\omega\tanh\theta e^{u_\omega(\eta,t)^2} erfc[u_\omega(\eta,t)]\right] e^{\frac{-\eta^2}{4t}}\right\}
\end{aligned} \tag{5.1}$$

where $u_\omega(\eta,t) = \frac{\eta}{\sqrt{4t}} - \sqrt{t}\omega\tanh(\theta)$, and

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty d\xi e^{-\xi^2}$$

is the complementary error function. Note that (5.1) is a direct generalization of the heat kernel given in [6] for the two-dimensional case, which, in turn, coincides with the Fourier transform of equation (101) in [4] for an antiperiodic boundary fiber.

6 Meromorphic properties of the zeta function

Let us now analyze the boundary contributions to the global zeta function related to (5.1). We first note that global quantities are necessarily divergent due to the non-compact nature of our manifold $\mathcal{M} = \mathbf{R}_+ \times \mathcal{N}$. This is not a severe problem because the result (5.1) allows us to identify easily the bulk term leading to a divergent contribution when integrated. In particular, it is the first term in (5.1) that represents the heat kernel on the manifold $\mathbf{R} \times \mathcal{N}$. In the following, without changing the notation, we will first ignore this term and this will allow us to determine the boundary contributions to the global zeta function. Alternatively, as we will show afterwards, one can introduce a localizing function of compact support in (5.1) such that the trace gives a finite result.

Let us consider the trace of (5.1) ignoring the first term. It is convenient to perform the Dirac trace (tr) first. Since

$$tr\left(\frac{2\Pi_+\Pi_+^\star}{\cosh^2(\theta)}\right) = 2^{\tilde{m}}\,,$$

the trace of the 'boundary' heat kernel reduces to

$$\begin{aligned} TrK(t;x,x) &= \frac{2^{\tilde{m}}}{2}\sum_\omega \omega\tanh\theta e^{-\omega^2 t} \\ &\times \int_0^\infty dx_m\, erfc[u_\omega(2x_m,t)]e^{\frac{-x_m^2}{t}+u_\omega^2(2x_m,\,t)}\,, \end{aligned}$$

where the second and third term in (5.1) have cancelled each other. Now, using that

$$-\frac{1}{2}\frac{\partial}{\partial x_m}\left[e^{-x_m^2/t+u_\omega^2(2x_m,t)}erfc\left[u_\omega(2x_m,t)\right]\right] =$$

$$e^{-x_m^2/t}\left[\frac{1}{\sqrt{\pi t}}+\omega\tanh\theta\, e^{u_\omega^2(2x_m,t)}erfc\left[u_\omega(2x_m,t)\right]\right]\,,$$

we get

$$TrK(t;x,x) = \frac{2^{\tilde{m}}}{4}\sum_\omega e^{-\omega^2 t}\left[e^{u_\omega^2(0,t)}erfc\left[u_\omega(0,t)\right]-1\right] =$$

$$\frac{2^{\tilde{m}}}{4}\sum_\omega\left[e^{\frac{-\omega^2 t}{\cosh^2\theta}}\left[1+erf(\omega\sqrt{t}\tanh\theta)\right]-e^{-\omega^2 t}\right]\,.$$

Here we used $erf(x) = -erf(-x) = 1 - erfc(x)$.

Now, we can Mellin transform this trace, to obtain the 'boundary' zeta function of the square of the Dirac operator in the infinite cylinder

$$\begin{aligned} \zeta(s,P^2) &= \frac{2^{\tilde{m}}}{4\Gamma(s)}\sum_\omega\int_0^\infty dt\; t^{s-1}\left[e^{\frac{-\omega^2 t}{\cosh^2\theta}}-e^{-\omega^2 t}\right] \\ &\quad+\frac{2^{\tilde{m}}}{4\Gamma(s)}\sum_\omega\int_0^\infty dt\; t^{s-1}e^{\frac{-\omega^2 t}{\cosh^2\theta}}erf(\omega\sqrt{t}\tanh\theta) \\ &= \zeta_1(s,P^2)+\zeta_2(s,P^2)\,. \end{aligned} \tag{6.1}$$

The first contribution can be readily seen to be

$$\zeta_1(s,P^2) = \frac{1}{4}\left(\cosh^{2s}\theta-1\right)\zeta(s,B^2)\,, \tag{6.2}$$

where B is the operator defined in Section 5.

As for the second contribution to (6.1), it is given by

$$\zeta_2(s,P^2)=\frac{2^{\tilde{m}}}{4\Gamma(s)}\sum_\omega\int_0^\infty dt\ t^{s-1}e^{\frac{-\omega^2 t}{\cosh^2\theta}}\frac{2}{\sqrt{\pi}}\int_0^{(\omega\sqrt{t}\tanh\theta)}d\xi e^{-\xi^2}\,.$$

After changing variables according to $y=\frac{\xi\cosh\theta}{\sqrt{t}\omega}$, and interchanging integrals, one finally gets

$$\begin{aligned}
\zeta_2(s,P^2) &= \frac{2^{\tilde{m}}\Gamma\left(s+\frac{1}{2}\right)}{4\Gamma(s)}\cosh^{2s}\theta\sum_\omega sign(\omega)\left(\omega^2\right)^{-s}\\
&\qquad\times\frac{2}{\sqrt{\pi}}\int_0^{\sinh\theta}dy\left(1+y^2\right)^{-s-\frac{1}{2}}\\
&= \frac{\Gamma\left(s+\frac{1}{2}\right)}{4\Gamma(s)}\cosh^{2s}\theta\eta(2s,B)\frac{2}{\sqrt{\pi}}\int_0^{\sinh\theta}dy\left(1+y^2\right)^{-s-\frac{1}{2}}\\
&= \frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s)}\sinh\theta\cosh^{2s}\theta\eta(2s,B)\\
&\qquad\times{}_2F_1\left(\frac{1}{2},\frac{1}{2}+s,\frac{3}{2};-\sinh^2\theta\right)\,.
\end{aligned}\tag{6.3}$$

The structure of the zeta function is similar to the structure found for spectral boundary conditions, see e.g. [10]. In particular, the analysis of the zeta function on $\mathcal{M}$ has been reduced to the analysis of the zeta and eta function on $\mathcal{N}$.

As already commented, from (6.2) and (6.3) one can determine the positions of the poles and corresponding residues for the zeta function in any cylindrical product manifold, in terms of the meromorphic structure of the zeta and eta functions of the boundary operator. For the rightmost poles, explicit results can be given in terms of the geometry of the boundary $\mathcal{N}$. For example, for $s=(m-1)/2$ we see that

$$\begin{aligned}
\text{Res}\ \zeta_1\left(\frac{m-1}{2},P^2\right) &= \frac{1}{4}\left(\cosh^{m-1}\theta-1\right)\text{Res}\ \zeta\left(\frac{m-1}{2},B^2\right)\\
&= \frac{1}{4}\left(\cosh^{m-1}\theta-1\right)\frac{(4\pi)^{-(m-1)/2}}{\Gamma\left(\frac{m-1}{2}\right)}2^m\ \text{Vol}(\mathcal{N}).
\end{aligned}$$

Because ζ_2 does not contribute, given $\eta(2s,B)$ is regular at $s=(m-1)/2$ [11], this equals Res $\zeta(s,P)$ and is the result expected from the calculation on the ball [13]. For $\theta=0$ the residue disappears as is known to happen for the standard local bag boundary conditions [11]. Further results can be obtained by using Theorem 4.4.1 of [11]. Given we considered the case without 'potential', it is immediate that

$$\mathrm{Res}\ \zeta\left(\frac{m-2}{2},P^2\right)=0.$$

Also, for the particular case of $s=0$, the fact that $\zeta(s,B^2)$ and $\eta(2s,B)$ are regular at $s=0$ shows that $\zeta(0,P^2)=\zeta(0,P)=0$.

Given the local heat kernel (5.1), a local version of the results of this section is easily obtained. To this end, we use a localizing function with compact support near the boundary, such that its normal derivatives at the boundary vanish,

$$\left.\frac{\partial^n}{\partial x_m^n}f(y,x_m)\right|_{x_m=0}=0,\quad n\in\mathbb{N}.$$

Furthermore, we let $\tilde{P}^2$ denote the operator P^2 on the double $\mathbf{R}\times\mathcal{N}$ of $\mathbf{R}_+\times\mathcal{N}$, and we extend the localizing function as an even function to the double. We use the notation $f=f(y,x_m)$, $f_{\mathcal{N}}=f(y,x_m=0)$ and $\tilde{f}$ for f on the double. Introducing the local versions $\zeta(\tilde{f},s,\tilde{P}^2)$, $\zeta(f,s,P^2)$, $\zeta(f_{\mathcal{N}},s,B^2)$ and $\eta(f_{\mathcal{N}},2s,B)$ of the zeta functions and the eta function, (5.1) and previous calculations show that the following theorem holds:

Theorem 6.1

$$\begin{aligned}\Gamma(s)\zeta(f,s,P^2)&=\Gamma(s)\left\{\frac{1}{2}\zeta(\tilde{f},s,\tilde{P}^2)\right.\\&+\frac{1}{4}\left(\cosh^{2s}\theta-1\right)\zeta(f_{\mathcal{N}},s,B^2)\\&\left.+\frac{1}{2\sqrt{\pi}}\frac{\Gamma(s+1/2)}{\Gamma(s)}\sinh\theta\cosh^{2s}\theta\ {}_2F_1\left(\frac{1}{2},\frac{1}{2}+s,\frac{3}{2},-\sinh^2\theta\right)\eta(f_{\mathcal{N}},2s,B)\right\}\\&+h(s)\end{aligned}\tag{6.4}$$

where $h(s)$ is an entire function.

This result parallels Theorem 2.1 in [10] for spectral boundary conditions.

7 Conclusions

In this article we have considered the influence of generalized bag boundary conditions on the heat kernel and the zeta function. In order to guarantee certain structural properties we have first shown the strong ellipticity of the boundary conditions. Work by Seeley [14, 15] then shows the standard heat kernel expansion holds and so the zeta function can have only simple poles at $s = m/2, (m-1)/2, ..., 1/2$, and $s = -(2l+1)/2$, $l \in \mathbb{N}$. This is the main result of this paper.

Based on the strong ellipticity one might envisage the determination of the leading heat kernel coefficients for generalized bag boundary conditions as they are needed for the calculation of effective actions in gauge theories in Euclidean bags [3]. Special case calculations can serve to restrict the general form that coefficients can have, cylindrical manifolds providing a valuable example. Here, for $P = i\gamma_j \nabla_j$, we have expressed the heat kernel and the zeta function of the associated second order operator on $\mathcal{M} = \mathbf{R}_+ \times \mathcal{N}$ in terms of the boundary data on $\mathcal{N}$. In fact, this result, under certain restrictions, can be straightforwardly generalized to $P = i\gamma_j\nabla_j - \phi$. In order that a separation of variables as presented succeeds we need $\partial_{x_m}\phi = 0$ and $\{\gamma_m, \phi\} = \{\tilde{\gamma}, \phi\} = 0$. If this is satisfied, equations (6.2) and (6.3) remain valid, once the operator B incorporates the potential, $B = \tilde{\gamma}\gamma_m(\gamma_a\nabla_a - i\phi)$. We have thus a particular case involving a potential and Riemannian curvature and various restrictions on heat kernel coefficients will follow.

A Appendix: γ-matrices

Let $m = 2\bar{m}$ be the dimension of a Riemannian manifold. We denote by $\gamma_j^{(m)}$, $j = 1, ..., m$, the self-adjoint γ-matrices projected along some m-bein system. These are defined inductively by

$$\gamma_j^{(m)} = \begin{pmatrix} 0 & i\gamma_j^{(m-2)} \\ -i\gamma_j^{(m-2)} & 0 \end{pmatrix}, \qquad j = 1, ..., m-1,$$

$$\gamma_m^{(m)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_m^{(m+1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

starting from the Pauli matrices

$$\gamma_1^{(2)} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma_2^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_3^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The γ-matrices satisfy the Clifford anti-commutation formula

$$\gamma_j^{(m)}\gamma_k^{(m)} + \gamma_k^{(m)}\gamma_l^{(m)} = 2\delta_{kl}. \tag{A.1}$$

In the main body of the paper we will simplify the notation and we will not indicate the dimension explicitly. In addition, we set

$$\gamma_{m+1}^{(m)} = \tilde{\gamma} = (-i)^{\bar{m}} \gamma_1 ... \gamma_m,$$

which is the generalization of 'γ_5' to arbitrary even dimension.

Acknowledgements: Research of CGB and EMS was partially supported by CONICET(PIP 0459/98) and UNLP(11/X298). Research of PG was partially supported by the MPI (Leipzig, Germany). KK acknowledges support by the Baylor University Summer Sabbatical Program and by the MPI (Leipzig, Germany).

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