Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Equivariant Rational maps and Configurations: spherical equidistribution and SO(N,1) contraction

(revised version: September 2003)

by

Sidney Frankel

Preprint no.: 74 2003

MPI–Math, Leipzig, frankel@mis.mpg.de

EQUIVARIANT RATIONAL MAPS AND CONFIGURATIONS: SPHERICAL EQUIDISTRIBUTION AND SO(N,1) CONTRACTION

SIDNEY FRANKEL

ABSTRACT. We build up a class of O(N,1)-intrinsic spherical rational maps, using only stereographic projections and affine centers of mass, and slightly extend it with antipodal maps. The geometric-analysis of their dynamics lends itself to applications to equidistribution of points on the sphere and to canonical global parametrizations of the rational maps of $\mathbb{C}\mathrm{P}^1$. We construct geometrically natural examples of rational maps of $\mathbb{C}\mathrm{P}^n$, and introduce a new approach, "suspension", to producing iterative algorithms for factoring polynomials, and to finding the k-periodic points of rational maps of $\mathbb{C}\mathrm{P}^1$

Maps, f, are understood in terms of a discrete steepest descent method, involving, as Lyapunov function, the log-chordal energy function associated to the fixed-points of f; ie the spherical Green's function rather than Coulomb energy. A transformation of rational maps of $\mathbb{C}\mathrm{P}^1$ which gives singular flat affine connections on $\mathbb{C}\mathrm{P}^1$ (also known as local systems, a complexification of polyhedra) in a natural way, provides an O(N,1)-intrinsic analogue of the Lyapunov force-fields and suggests higher dimensional versions of Schwartz-Christoffel uniformization of polygonal regions.

Relations to the algebraic geometry of configuration and moduli spaces, discriminants and dual curves are touched on, and we begin a discussion of the relation to geometric plethysm–maps as $sl_2\mathbb{C}$–invariants or covariants. We note as well the connection to moment maps, and begin a study of the relation of these constructions to hyperbolic centers of mass (such as Douady-Earle).

A class of self maps O(N,1)–intrinsic for hyperbolic space is constructed in each dimension as restrictions of the spherical rational maps above with fixed-point parameters in a hemisphere, generalizing the class of holomorphic maps of the 2-dimensional disc, and an associated "Schwarz lemma" confirms that the maps have good geometric and topological properties.

0.0.1. *Note on numbering:* References to theorems, and all but sections and equation numbers, are of the form "theorem 2.1" or for short (2.1). Sections, sub^{n}sections , are referred to as section 2.1, sometimes without the "sub"s, or just 7.2.1, when they have 2 decimal points, while equations are sometimes called ((3.1)) for short. Insofar as the following summary is not comprehensive, the reader might supplement it by browsing section and sub^{n}section headings. A list of some open problems is to be found at the end of the article.

1. Introduction

Given a point x in S^N, the sphere, Stereographic-projection from x provides an affine structure A_x on its complement, $S^N - x$. We consider the **Stereographic center-of-mass** $\mathcal{C}\mathrm{en}(P,x)$ of a configuration $P \subset S^N - x$. We focus on the case $N = 2$ in this article (up to section 8). $\mathcal{C}\mathrm{en}(P,x)$ extends *meromorphically* to

Date: August, 2003.

$P \subset S^2 = CP^1$, it is even meromorphic in $(P, x) \subset (\mathrm{CP}^1)^n$; we will allow weighted interactions, and see that $\phi(x) = \mathcal{C}\mathrm{en}(P, W, x)$ is defined by solving for y in

$$y = \phi(z); \quad \frac{1}{y - z} = \sum_j w_j \frac{1}{p_j - z} \tag{1.1}$$

We exploit the symmetries of this expression in section 3, to construct the geometrically natural rational map $\mathcal{F} : (\mathrm{CP}^1)^n \to (\mathrm{CP}^1)^n$ of *configurations*, as well as its quotients $\mathcal{F} : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$, and even meromorphic $\mathcal{F} : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n}$ on moduli space, in 3.1.1. We then begin a study of the dynamics of $\mathcal{F}$, leading to the first main theorem 3.12, relating rational maps and equidistribution, though this leaves open many problems about the global behavior of $\mathcal{F}$. These were our initial motivations for this work and as such they occupy the central role here.

In section 8 we undertake a study of $\mathcal{C}\mathrm{en}_{PW}$, with P in a subsphere, $S^N \subset S^{N+1}$, (or in a hemisphere, H^{N+1}), to obtain self-maps of hyperbolic space H^{N+1}, and we prove a Schwarz lemma, (8.20,8.21) which provides a complete description of their dynamics. This illustrates how constructions based on $\mathcal{C}\mathrm{en}$ have nicer properties than one might have otherwise expected.

A priori, one does expect $\mathcal{F} : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ to have a nice closed form expression, but its not clear how to produce one –computer algebra packages did not help here. $\mathcal{F}$ provides an algebraically simple but geometrically rich behaviour that we consider from different viewpoints, and a number of equivalent constructions are provided; ((3.1)), based on $\mathcal{C}\mathrm{en}$, (3.6), using fixed-points and equivariances, (3.9), which uses osculating maps, and finally section 7.1, which provides a closed form expression for $\mathcal{F}$, using the resultant.

Other related results spread throughout the text are less developed, the more exploratory parts being preceeded by the advice that they are not to be used further on. We apologize for this, as well as the many questions and occasional speculation that can slow down the reader, but we include them in the hope that some of these directions will prove to be fruitful.

1.1. **Overview.** We begin section 3 with an analysis of standard elementary properties involving fixed-points and equivariances that leads to a relation to steepest descent methods–this would be of interest if only because holomorphic maps normally cannot be constructed using geometric constructions over $\mathbb{R}$ of this nature–moreover this interaction of real and complex geometry can be exploited to produce attractors for the holomorphic dynamical system.

A variety of relations of $\mathcal{F}$ to discriminants, in (3.6), dual curves, in 7.2.1 etc. arise that are surprising from the elementary viewpoint, but less so from the point of view of invariant or representation theory since $\mathcal{F}$ is also a covariant of $sl(2, \mathbb{C})$ in the classical sense. We make some attempt to understand $\mathcal{F}$ in this latter context, in section 7.3, which should provide more unity to the theory in the longer run, and to clarify what other such constructions might be possible.

The suspension construction, section 6, a slight generalization of $\mathcal{F}$ that admits some parameters, underlies a new approach to iterative algorithms for factoring polynomials, and for finding the k-periodic points of rational maps of $\mathbb{C}\mathrm{P}^1$. We first develop a one-variable theory, section 2, 4, 5.1, by fixing some points–treating them as parameters, at each stage of the development of the multi-variable theory. This is not only useful in developing some intuition, but can often be applied directly to establish key points of the multi-variable theory. In addition the one-variable theory features a number of constructions and results interesting in their own right; we cite for example, (2.1), a general NORMAL FORM theorem for rational maps of $\mathbb{C}\mathrm{P}^1$, (4.18), a natural transformation of rational maps of $\mathbb{C}\mathrm{P}^1$ to singular flat affine connections on $\mathbb{C}\mathrm{P}^1$ (also known as local systems, a complexification of polyhedra). We also discuss the relation of the latter to Schwartz-Christoffel uniformization of polygonal regions and hyperbolic centers-of mass in section 8.

The one-variable (or "one-body") theory turns out to be quite rich for higher dimensional spheres, particularly in relation to hyperbolic geometry–it provides an analogue of holomorphic self maps for hyperbolic spaces, $H^N = \mathbb{R}H^N$ (we specialize to real hyperbolic space here) which have both good algebraic and topological-geometric behavior. The analogy to the Schwarz-Pick lemma is secured by (2.1). (The conjunction of "Douady-Earle" and "Schwarz-lemma" probably brings to mind the work of Courtois et al, but as far as we can see there is no immediate relation to their work, the hypotheses and conclusions here being quite different.)

This work began as a study of $\mathcal{F}$ as a dynamical system, and the relation to steepest descent dominates our presentation, but the other directions (algebraic, hyperbolic) that have emerged from their study could be of more interest in the long run. In any case the analysis relating to steepest descent is very useful as regards the latter.

1.1.1. *steepest descent, equidistribution, moment map and equivariance.* The relation of the rational map $\mathcal{F}$ to a discrete steepest descent method, for an associated log-chordal energy function, $\mathcal{E}$ (3.11); the spherical Green's function rather than Coulomb energy, is one of the main themes, and it's specialization to the one-variable theory which is already quite rich can be found in section (5.1). This is the energy of interest in problem 7 of Smale's problem list for the 21st century, [37] , but we do not consider computational complexity questions here. We do discuss the hessian of energy at Fekete-Tsuji points, ie the global (or local) minimal energy configurations; in fact it seems that not only is nothing known about uniqueness of the global minimum, Z_0, but even nondegeneracy of the hessian at Z_0, (3.14), is non-trivial, and as far as we know, open.

The fundamental observation regarding equidistributed configurations is that there are very few solutions coming from finite symmetry groups, (though see [26]). The fact that there is an equivariant rational map that finds equidistributed configurations seems to provide a reasonable alternative, though specific applications will require different notions of equidistribution. Diverse energy functionals on the space of sets of k points, as well as the packing problem are among notions considered, [25, 34, 12] , and our canonical rational map provides local minima of one of these. It seems that $\mathcal{F}$ has not been considered elsewhere, and, in particular, not in relation to equidistribution.

The approach to the theory of rational maps presented here should notably be of interest with regards to the global structure of the space of rational maps, we briefly consider a holomorphic surgery construction in this context, section 2.1, though the emphasis here is on the construction of special maps, their geometry and dynamics.

We also emphasize that the theory here is naturally tied to the O(N+1,1) structure of the sphere. This is the natural equivariance of $\mathcal{F}$. A variant of the moment map arises as a natural normalization of configurations; it reduces an O(N+1,1)–orbit of period 2 attractors for the natural rational map $\mathcal{F}$, to an O(N+1)–orbit of local minima of the energy function. There is a strong coincidence in how well this works for precisely the energy of interest here, and we make some attempt to find an underlying reason for this coincidence. This leads us to also consider relations of the moment map to hyperbolic centers of mass using these energy functions. Moment maps are a standard tool for reducing from noncompact-holomorphic to compact-riemannian-isometric group actions, but we do not know of previous cases of its use in reducing properties of equivariant-holomorphic maps (of higher degree) to properties of equivariant-(geo)metric maps. (The maps themselves *do not* commute with this reduction and we observe that no reasonable reduction will commute with the maps, but particular properties of the maps such as fixed-points are well behaved under reduction.) The Schwarz lemma we prove exhibits the Schwarz-Pick contraction phenomenon as an aspect of the O(N+1,1) geometric structure, in contrast to the usual holomorphic viewpoint.

The name "*elliptic center-of-mass*" is likely to win out over alternatives such as *stereographic centrum,* not to mention *stereogravity* or *stereographity* as a name for our central construction, especially in view of its relation to hyperbolic centers of mass, section 8, such as the well known construction of Douady-Earle.

1.1.2. *natural correspondences.* $\mathcal{E}$ is privileged with a multitude of seemingly unrelated connections to conformal maps and symplectic geometry, which arose here. The two nice properties of $\mathcal{E}$ in (4.1) that make it useful in studying rational maps are explained and unified by showing how $\mathcal{E}$ is related to homogeneous polynomials. But there is a direct and natural relation of homogeneous polynomials to rational maps in one variable, in [9]. This motivates us to look further at the underlying unifying geometric structures, and a large part of section 4 involves this somewhat exploratory material. It seems that the payoff in applications might come from the connection to hyperbolic geometry in section 8, especially the relation of smooth maps to curvature measures which is somewhat akin to quasiconformal distortion measures. Various sections contain extended discussions of how these pieces fit together, in particular we systematically develop some *natural correspondences,* section 4.2.1, providing a geometric explanation and tying together rational maps, local systems, energy functions, etc. For example, (4.18) gives an easy correspondence of Gauss-Bonnet for polyhedra to the holomorphic Lefschetz fixed point theorem. We provide some of these natural correspondences, with a few applications, and the hope that more will follow. One such correspondence is closely related to the work of Doyle and McMullen,[9] , on icosahedra and quintics, (as we near the 120th anniversary of Klein's 1884 book on the subject). There are also tempting links to work described in [28] , on moduli spaces of linkages, configurations, etc. The local minima of the energy functions have been much studied numerically, in the context of equidistribution, but it is difficult to give conceptual proofs of any of their properties. Transforming to the context of holomorphic functions and spherical harmonics, or rational maps may help, (5.13). While most of this work applies directly to the main goal; relating rational maps and equidistribution, the relation to hyperbolic geometry which grew partly out of the "coincidences" involving $\mathcal{E}$ and the moment map provides an equally good motivation.

We have tried to strike a balance between comprehensibility of the proofs and use of machinery (e.g. Kahler geometry); more elementary approaches to some points might rely only on trigonometry, and other constructions might benefit from more powerful algebrao-geometric tools.

A summary of some open problems is provided at the end. Computer experiments will probably be useful in guiding the next stages of this subject.

We owe much thanks to Chris Connell for help with computer experiments and discussions leading to the initial conjectures, especially a version of conjecture 5.12 bearing some elements of theorem 3.12. The confirmation of the latter constitutes the core of this article. Most of this work was done in the summer of 1999, and part of it was presented, November of that year, in a seminar at Stanford. We also thank our host, MPI for the possibility to pursue this and other matters farther, and for their gracious hospitality. In the remainder of this section we introduce the main actors of the paper, with some clues as to their hidden characters.

1.2. **Geometric structure of maps, GRas .** The main ingredients used in constructing the *geometric subclass* of rational maps are described in 1.2.1–1.2.4 below, as well as compositions of maps, and we use the term GRas to refer quite generally to the class of all such possible constructions. This includes many things not studied here, such as $\mathcal{C}\text{en}(\{f_1(x), f_2(x), f_3(x)\}, x)$ where $f_1(x) = \mathcal{C}\text{en}_{PW}(x), f_2(x) = A\mathcal{C}\text{en}_{P'W'}(x), f_3(x) = g\mathcal{C}\text{en}_{P'W'}(x)$, $g \in Aut$, combining holomorphic and anti-holomorphic objects, so we will specify some "tamer" classes below, (see 1.2.5).

We construct rational maps of the sphere $S^N = SO(N+1)/SO(N) \subset \mathbb{R}^{N+1}$, and since an important component of what is used will be *conformally intrinsic* for S^N, ie intrinsic for the structure associated to $\partial H^{N+1} = S^N$, where $H^{N+1} =$ real hyperbolic space, with its *isometry* group of Automorphisms, including reflections that reverse orientation, [11] , we denote $Aut(S^N) = SO(N+1,1) = Aut(H^{N+1})$. (Note: we will use the *geometric* properties of the 2 groups, (of oriented or all isometries) in this article but not the algebraic properties of the matrix groups just alluded to. Subtleties regarding the exact identifications of these matrix groups to *isometry* groups, such as the use of $SO^+(2,1)$ as (notation for) oriented isometries are not important in this article, so we do not strictly insist that the matrix groups act faithfully and our use of $SO(N+1,1)$ as convenient notation might be off by a Z_2 kernel as above. Most often the notation is simply a way to indicate the dimension, N, in use, as well as the associated Lie *algebra* that identifies the relevant geometric structure.)

C*onformal* structure usually refers to structures local in nature, (but we use this term for lack of better alternatives, the possible use of C*enformal* is suggested by the relation to center-of-mass) and it is more appropriate to think of $Aut(S^N)$ as preserving a *global* structure, for example, that provided by round circles, (see section 4.0.4), or cross ratios of round subspheres, ie S^2s.

(1) Complete totally geodesic subspaces of H^{N+1} limit to round spheres in $\partial H^{N+1} = S^N$, as follows by symmetry considerations such as those in the claims in section 4.0.4; any limiting sphere S^k is invariant by an $O(k+1) \subset O(N+1,1)$. But $Aut(S^N)$ acts transitively on the space of all such S^k (for each k).
(2) On each such S^2 , (with an orientation, thus a $\mathbb{C}\mathrm{P}^1$), there is a $\mathbb{C}$-valued cross ratio, XR, and these are $Aut(S^N)$-invariant.
(3) The space RC of round circles determines the C*onformal* structure of S^N uniquely, by a passage to the limit giving round circles in T_xS^N.

Proposition 1.1. *$Aut(S^N)$ is the largest subgroup of twice differentiable diffeomorphisms of S^N preserving either XR or RC.*

This is included for completeness, but isn't essential in what follows. The structures XR or RC are typically studied in relation to geometrization of negatively curved spaces, and some related rigidity conjectures, [21]. These last remarks are relevant to section 8, if anything.

Rational *maps* of S^N are often denoted $Rat_d S^N$, $d = degree$, they are maps whose components are quotients of polynomials, using co-ordinates (by Stereographic–projection from p) of $\mathbb{R}^N = S^N - p$. This is independent of p, since Aut is itself rational (Moebius) in these co-ordinates. A natural alternative is to define Rat using homogeneous coordinates. Certainly with this definition the antipodal map A is in Rat, however in the $S^2 = \mathbb{C}\mathrm{P}^1$ case, which is our main concern here, we prefer to reserve Rat to denote the construction involving complex coordinates and holomorphic maps, so A falls outside. We are generally considering *maps* of $\mathbb{C}\mathrm{P}^1$, or $(\mathrm{CP}^1)^n$ to itself, rather than rational *functions* (which are of interest in control theory or ODEs, via the laplace transform). Rat is used somewhat ambiguously in the literature both for self-maps and functions. Note that the two are quite different insofar as the natural group-actions transform only the domain, for functions, or both domain and range, for self-maps. This is a common source of confusion when first looking at the $\mathcal{C}$en construction (people tend to think of it as a function rather than a self map).

1.2.1. *Center-of-mass:*

Proposition 1.2. *Given x in S^N there is a canonical (conformally intrinsic) affine structure A_x on its complement, $S^N - x$; it is the affine structure produced by Stereographic-projection from x, and it is invariant by the (Borel) isotropy group of x, in $SO(N+1,1)$.*

This leads us to consider configurations, $P \subset S^N - x$, of cardinality $|P| = n-1$, and their **Stereographic center-of-mass** $\mathcal{C}\mathrm{en}(P,x)$ with respect to A_x. $\mathcal{C}\mathrm{en}(P,x)$ extends *meromorphically* to $P \subset S^2 = CP^1$, it is even meromorphic in $(P,x) \subset (\mathrm{CP}^1)^n$, see equation 2.1, corollary 2.12.

Uniqueness of the affine structure determined by x can be seen in a multitude of ways. We provide a few here quickly to get started, but others arise farther on that may be more interesting and intrinsic to the theory developed here, and might be preferable in the long run. To keep the discussion clear we will number the approaches to uniqueness by Aff(i). Each might be useful for different generalizations of GRas , for example they will not all work over other fields, (a generalization not taken up in this article). The first couple, using the invariance of XR or RC introduced above, might be regarded as asides, (mentioned here in passing) whereas Aff(3,4) are more standard:

Aff(1) –fixing p, XR determines a well defined midpoint $m(x,y)$ of any $x, y \in S^N - p$ and the midpoints satisfy the necessary relation $m(m(x,y), m(x,z)) = m(x, m(y,z))$ in any $S^2 \ni p$ to get a bonafide affine structure, [14] .

Aff(2) –Likewise, RC provides a class of straight lines $S^1 \ni p$ in any plane $S^2 \ni p$ satisfying the parallel postulate etc. we will not elaborate here.

Aff(3) –In fact it can also be seen for $\mathbb{R}^N, N > 2$ by Darboux's theorem, [20] , (a weak form–for global maps), it is standard textbook material for $N = 2$, a corollary of the uniformization theorem, the main point being the relation of degree to growth for holomorphic functions .

Aff(4) –Uniqueness can also be seen from a Lie-theoretic approach, noting that the Borel (isotropy) groups, in $S^N = SO(N+1,1)/B_p$, are affine. In terms of Fractional-Linear-Transformations, fixing the point at infinity eliminates the denominator.

Aff(5) –If we choose a Moebius transform, M of $\bar{\mathbb{C}}$ such that $x \mapsto \infty$, A_x is the pullback from $\mathbb{C}$, if N=2, (or $\mathbb{R}^N$, $N > 2$) of the usual affine structure and this is independent of the Moebius M chosen. This is how we use uniqueness in practice.

1.2.2. *Automorphisms: $Aut(S^N)$*, as given above, is defined with respect to the invariant conformal structure. As just indicated, (Aff(5)) automorphisms are used to study $\mathcal{C}$en explicitly. On the other hand, many oriented automorphisms can be constructed directly using the center-of-mass, see theorem 2.1, but not so for the ANTIPODAL map, denoted A, which plays a major role here. We use equivariance properties systematically to simpify proofs. We note that *antipodal maps, A_x* are parametrized by $x \in H^{N+1}$ (we almost *never* use the subscript for antipodal maps*)* and *affine structures, A_x or $A(x)$* by $x \in \partial H^{N+1} = S^N$. Note the obvious unifying property, that they are *each centralized* by the isotropy group in SO(N+1,1) of x.

In fact we introduce a natural correspondence of affine connections to maps in (4.18), and one can apply isotropy equivariance to see that the Levi-Civita connection of the round metric on S^N associated to $x \in H^{N+1}$ corresponds precisely to the antipodal maps*, A_x,* see also section 8.2.

1.2.3. *clamped and variable points:* We distinguish *clamped* from *variable* points, in the construction of functions; clamped points are denoted p, q, and *configurations* of clamped points are denoted P, Q. They are *constants* in S^N that we use to parametrize self-maps of the configuration of *variable* points X in S^N. Thus $\mathcal{C}\mathrm{en}(X, P, W, i)$ is the (W-weighted) center-of-mass of $X - x_i, P$ with respect to the affine structure of x_i, it is a function of $X = (x_1, \ldots, x_k)$. We will see that $p_j \in P$ are *fixed-points* of this map (viewed as a function of x_i). Given a configuration $X \subset S^N$, one should ideally formalize it not as a set or a tuple, but rather, as a *labelled* (or marked) *set* $X = \{\ldots, x_j, \ldots\}$, thus j is not so much an index as a label or marker, though in the generic local case this doesn't matter much. For degenerate configurations (double points) and global topology it is essential. (In section 5, we use Z for a configuration of *variable* points, but in the beginning we hold all but one fixed, calling it x to study the one variable case, so Z are clamped. Other such minor variations in notation are clarified locally in the relevant section.) Unless stated otherwise one should *always* suppose that the points in P are DISTINCT. We use the prime, P' to denote the complementary space, $\mathbb{C}\mathrm{P}^1 - P$, while in configuration space we often refer to the complement $\mathcal{D}'$ of the diagonal $\mathcal{D}$ of degenerate configurations, see (*3.3*). $|X|$ denotes the cardinality of a configuration, (and never a vector norm).

1.2.4. *weights:* It is useful, (remarks after proposition 3.6) to generalize to the **weighted center-of-mass** using weights, W_{ij};

$$\forall i, \sum_j W_{ij} = 1, \tag{1.2}$$

with i indexing X, and j indexing X and P. If X is a single point we denote the associated center-of-mass, $\mathcal{C}\mathrm{en}(P, W, x)$, and W is a vector. In the very special case N=2, weights can be arbitrary *complex* numbers, and $\mathcal{C}$en is fully meromorphic in

X, P, W. Note that complex weights are justified because the intrinsic affine structures produced in proposition 1.2 have an additional complex structure preserved by $O(2,1)$, and that commutativity of $\mathbb{C}^*$ is essential here, so that generalization to the quaternionic case is not (obviously) possible.

We will see that w_{ij} essentially reperesents the *multiplier i.e.* the linearization at the fixed-point p_j of $\mathcal{C}\text{en}(X, P, W, i)$. One naturally conceives X, P as the nodes of a directed graph $G(X, P, W)$ and W as weights on it's directed edges. **We'll assume throughout that** every $p \in P$ has at least one nonzero weight–(otherwise it can be left out), and that no x_i has a nonzero self-weight w_{ii} (otherwise x_i is constant under $\mathcal{C}\text{en}$). When discussing ENERGY functions, $\mathcal{E}$ the edge weights are always assumed REFLEXIVE, ie $w_{ij} = w_{ji}$. As a general principle, weights of energy functions are dictated by the condition that the associated force vectors are determined by a matrix of weights, which must agree with the matrix used by $\mathcal{C}\text{en}$ to construct the meromorphic map $\mathcal{F}$. Thus one should also appreciate that there is a slight difference in handling the subsets in P and X when constructing $\mathcal{E}$, (3.11).

1.2.5. *compositions, Ras classes.* There is not much to say about compositions, but formally one only can substitute a map for a variable, compositions come up notably in the construction of *suspensions,* section 6, as well as in the iteration of maps. In fact, for the most part we consider maps that are constructed using only the weighted center-of-mass, and we propose to use the shorthand term Ras (pronounced as the 1st syllable of *rational* in your preferred language) to denote the geometric subclass of real rational maps consisting of self-maps of the form $\mathcal{C}\text{en}_{PW}$ or $A\mathcal{C}\text{en}_{PW}$, notably P with distinct points,. In dimension 2 we will allow complex weights here and by (2.1) this just extends Rat by A. But on $S^N, N > 2$, one only has real weights and this class will be much smaller than real–Rat, see (8.4). Furthermore ARas is the subclass of Ras of maps, $A\mathcal{C}\text{en}(P, W, x)$, which reverse orientation.

Rass allows for compositions of maps in Ras , (see 8.1.1). *GRas* includes the multivariable constructions, such as (6.1), viewed as a subclass of Rat. Finally we will use reflections in hyperspheres (codim 1) in section 8, to study a class of maps of hyperbolic space which we denote *HRas* . We often use Rat without explicitly indicating the space (it is usually $\mathbb{C}\mathrm{P}^1$) or even whether maps are defined over $\mathbb{R}$ or $\mathbb{C}$, where this is clear from context.

It might seem that such simple ingredients wouldn't produce anything of interest. While we leave this for the reader to judge, our first goal is thus to show that this *is* a non-trivial class, already in the case $N = 2, S^2 = CP^1$, which is our focus in most of this article. Note that for $N = 2$ we get a subclass of meromorphic maps, if one restricts to orientation preserving automorphisms, but the antipodal map turns out to play an important role even in this case (tying maps to steepest descent methods). In section 8 we begin to develop $GRas(S^N), N > 2$, but not beyond the one-variable case, and recover some of the the nice properties of holomorphic maps, such as a Schwarz lemma, as well as a related "tautness" property, 8.6.1.

1.3. **Divisors.** Many of the constructions in this article are parametrized by weighted configurations, and the set of pairs $\{(p_i, w_i), 0 < i < n\}$ are best formalized as *divisors.* For a divisor in $\mathbb{C}\mathrm{P}^1$, we allow coefficients in $\mathbb{C}$. Since it is also useful to formalize P,W, as n-tuples of points or weights, divisors are also denoted in terms of these vectors by PW or WP, the pairing of $\{(p_i, w_i), 0 < i < n\}$ being the essential structure; PW is an n-tuple of pairs quotiented by permutations.

Definition 1.3. A divisor in $\mathbb{C}\mathrm{P}^1$ is a formal sum, denoted $WP \in S\mathbf{Div} = \{\sum w_i p_i : p_i \neq p_j \in \mathbb{C}\mathrm{P}^1, w_i \in \mathbb{C}\}$. The S prefix emphasizes the distinctness of the p_i. $SDiv_1$ is the subspace with $\sum w_i = 1$. These are clearly complex manifolds.

Since we always assume $\sum w_i = 1$ in this article, and distinctness in P is our default mode, we generally drop the modifiers and simply write Div rather than $SDiv_1$. If P alone is used *as a divisor*, then weights are assumed symmetric, ie permutation invariant. *Compactification* of $SDiv$ is nontrivial; there are different compactifications, at the set of double points, of $S\mathbf{Div}$, corresponding to rational maps and local systems for example. We will consider their relation below in section 2.1.

2. The One variable case

We study maps of the form $r_{PW}(x) = \mathcal{C}\mathrm{en}(P, W, x)$ here, with $N = 2, S^2 = CP^1$. Following remarks in 1.2.1 and choosing the Moebius transform, $m_z(u) = \frac{1}{u-z}$, of $\bar{\mathbb{C}}$, such that $z \mapsto \infty$ to construct $\mathcal{C}\mathrm{en}$, $\phi(x) = \phi_{PW}(x) = \mathcal{C}\mathrm{en}(P, W, x)$ is defined by solving for y in

$$y = \phi(z); \quad \frac{1}{y - z} = \sum_j w_j \frac{1}{p_j - z}. \tag{2.1}$$

This also shows that y is meromorphic in (z, P, W), (and that Ras is a subset of the extension by A of Rat).

We emphasize that this definition of ϕ is independent of the choice of Moebius transform, m_z, subject to $z \mapsto \infty$, (Aff(5) above; by direct calculation). ϕ depends only on PW.

Theorem 2.1. *Every (holomorphic) Rational map of $\mathbb{C}P^1$ with simple fixed points $\{p_i, 0 \leq i \leq d\} = P$ is of the form $r_{PW}(x) = \mathcal{C}\mathrm{en}(P, W, x)$, where $w_i = (1 - m_i)^{-1}$, $m_i = r'(p_i)$. In particular $r, (= r_{PW})$, is uniquely determined by its fixed-points and multipliers, ie by the divisor $PW = \{(w_i, p_i)\}$.*

Note that oriented automorphisms are constructed by using $|P| = 2$, making the ingredient list above somewhat redundant. The $\mathcal{C}$en representation of r in this case is in effect a diagonalization (normal form) theorem. In low degrees one can simplify the form of $r \in$Ras alot, using automorphisms, by chosing special fixed-points , Milnor, [30] and Thurston, [39] (preprint), discuss degrees up to 4. Despite all efforts, we cannot find evidence that the general Normal form theorem implicit in theorem 2.1 was already known. Although the proof we give here is technically simplest, there is a proof based on (4.18) which might be considered a better *explanation* of this phenomenon. The latter approach is taken farther in section 8.2, where we extend the theorem here to smooth maps, using a symplectic construction to recover the measure (generalizing weights) which is also realized as the curvature of a natural connection. The holomorphic maps studied here are thus those whose graphs are Lagrangian with respect to a complexified symplectic form.

Lemma 2.2. *The fixed points of $\phi = r_{PW}$ are precisely the $p_j \in P$, each occuring with multiplicity one, and the multiplier at p_i is $(1 - w_i^{-1})$, and $1 - n$, for ϕ_P.*

In the limit where $w_i = 0$, ϕ_{PW} is well-defined, but since p_i is not a fixed-point , there is a discontinuity that arises–see section 2.1.

Proof. Using ((2.1)) to construct $\mathcal{C}$en, note that p_i are fixed points of f; $z = p_j \Rightarrow \frac{1}{y-z} = \infty \Rightarrow y = z$. Rewriting the r.h.s. with common denominator and taking reciprocals we see $y - z = -z + \frac{U_d(z)}{V_d(z)}$, using $\sum w_j = 1$, where $d = |P| - 1 = \deg U = \deg V$ so $\deg f = |P| - 1$ and P gives all fixed points of f by "accounting". In particular every fixed point of f is simple. Now letting $z \to p_j$, and using $y-z = (y-p_j)-(z-p_j)$, in $1 = \sum_j w_j \frac{y-z}{p_j-z}$, gives $1 = w_j(-f'(p_j)+1) \Rightarrow w_j = (1-m_j)^{-1}$. (Differentiating implicitly with respect to z at $z = p_j$ works via l'hopital, but is much messier.) $\square$

We thus get meromorphic maps: (between mapping spaces, restricting to degree d maps and configurations of size $d + 1$)

- Ev: SRat($\mathbb{C}$P^1) $\to SDiv_1$, by *evaluation*–restricted to the subspace of Rat, SRat, with *simple* fixed points; namely $r \mapsto$PW the fixed-points of r and their associated multipliers–giving weights as above.
- $\mathcal{C}$en: $SDiv_1 \to$ SRat, by the $\mathcal{C}$en construction as above. The latter is injective by (2.2), in fact

  Lemma 2.3. *Ev∘$\mathcal{C}$en is the identity map.*

Proof. (of (2.1)) SRat is Zariski *open* in an *irreducible* (algebrao-geometric sense of *unions*) variety; SRat is dense in $\mathbb{C}^{2d+1} \subset \mathbb{C}P^{2d+1}$, by using polynomial coefficients (of U, V, above) as coordinates. Also dim **Div** is $2d + 1$, by equation 1.2, so $\mathcal{C}$en must be onto (the unique irreducible component of*)* SRat and Ev is injective. $\square$

Remark. $\sum_j w_j = 1$ is the Lefschetz holomorphic fixed point thm for r. Applying this to $r \in Aut$ (with simple fixed-points) we get the well known

Corollary 2.4. *If r has exactly 2 fixed-points then $m_1 m_2 = 1$.*

Remark. (i) This leads to questions about the analogous interpretations of weights via Lefschetz for the meromorphic multi-dimensional $\mathcal{F}_{PW}$ ∈GRas discussed in section 3, and more generally, how broadly the weight-interpretation can be pushed? (ii) On S^N, with $|P| = 2$, $\mathcal{C}$en gives $r_{PW} = r_1 r_2$, with r_2 in the p part of the Lie group, (in terms of the $p + k$ decomposition of the Lie algebra) and $r_1 =$ a rotation by π radians along the axis of $r_2 \in Aut$.

The proof suggest the relevance of Runge expansions, indeed one can expand $1/(z - r(z))$ in terms of its poles and apply the holomorphic Lefschetz fixed point theorem to prove part of theorem 2.1. The proof would still use the calculations behind lemma 2.3 as above. This is the analytic alternative to the more geometric approach using $\mathcal{C}$en. The Runge theory for multiple poles then applies nicely to the case of degenerate fixed points.

2.1. Degenerate fixed points, higher multiplicity and holomorphic surgery.

Here we consider extending the $\mathcal{C}$en representation to maps with degenerate (ie multiple) fixed points. (One can skip directly to section 2.2.4 if only interested in the main theorem 3.12.) In its full complexity this involves a stratification of the possible degenerations, but we will only consider the simplest, or highest, strata here, so this is only a sketch of the theory as regards degeneration. In this case there are analogous formulae for the $\mathcal{C}$en representation: we consider a family of $r_t \in SRat_d(\mathbb{C}\mathrm{P}^1)$, $t \neq 0$ in the degenerate limit, $t = 0$, where a pair of fixed-points $p_i(t), p_{i+1}(t)$ of r_t collide; $p_1(0) = p_2(0)$, but with NO JUMP IN DEGREE, and we call this a *smooth* family. Checking smoothness in the U, V coordinates above, $(r = \frac{U}{V})$, is somewhat complicated (a resultant), but smoothness makes the choice of PW coordinates quite natural. We can choose the family r_t to be holomorphic and even algebraic, so the set V_k of k-periodic points is an algebraic variety, in particular for $k = 1$.

(1) Note that $dr_0(p_1) = m_1 = 1$; using difference quotients of the p_i.
(2) When $x \in V_1$ and $dr(x) \neq 1$ the implicit function theorem shows that V_1 is smooth, but when $dr(x) = 1$ singularities may arise. In this case P can have nontrivial monodromy, but is generically given by well-defined functions on the double cover, likewise for W. In what follows we could suppose that we have chosen a parametrization, (by passing to this double cover, if necessary) such that the p_i are well-defined functions across $t = 0$. In fact using $\gamma_t r_t$ with γ_t in $\mathrm{Aut}(\mathbb{C}\mathrm{P}^1)$, we can even suppose $p_1(t) = const = 0$ and $p_2(t) = t + O(t^2)$ near $t = 0$ (on the double cover in the generic case).
(3) Note that the condition $|dr(x)| \neq 1$ often arises in the study of convergence of normal forms, as well as bifurcation of attractors, but this shouldn't be confused with bifurcation of fixed-points .
(4) If $w_i(t_0) = 0$ then degree jumps and r_t is not smooth. Nevertheless $r_t(x)$ is smooth in t, in this case, unless $x = p_i(t_0)$, (check using $\mathcal{C}$en). On the other hand preimages $r_t^{-1}(y)$ jump at t_0 for any y. We call this the *bubbling* of the graph of r at t_0.
(5) At the degenerate p_i, $dr_0(p_i) = m_i = 1 \Rightarrow m_i, m_{i+1} \to 1 \Rightarrow w_i, w_{i+1} \to \infty$, (by (2.1)), as $t \to 0$, but

Lemma 2.5. *For a smooth family as above, $w_1 + w_2 = O(1)$ at the degeneracy, $t = 0$, ie the blow-up of the w_j occurs in pairs such that infinities cancel to give a finite net weight to p_i.*

Remark 2.6. The technique will be reused a couple of times here; calculate $r_t(x), x \neq p_i$ using $\mathcal{C}$en: choose coordinates using m_x as in ((2.1)) so $x \mapsto \infty$, but $x \neq p_i \implies |p_i| < \infty$ and we will use $2(p_1w_1 + p_2w_2) = (p_1 + p_2)(w_1 + w_2) + (p_1 - p_2)(w_1 - w_2)$, where $p_1w_1 + p_2w_2$ is "part of" the summation in ((2.1)) for $r_t(x)$. Hence to show that both summands on the r.h.s. are bounded in the limit, $t \to t_0$, it suffices to show that one is.

Proof. If there is just one degenerate pair, then $w_1 + w_2 = 1 - \sum_{i \neq 1,2} w_i = O(1)$ since $i \neq 1, 2 \implies w_i = O(1)$ by the hypothesis and smoothness, (and using the normalization to get the equality).

In case there are more degenerations at the same t_0, we'll check that each degenerating cluster has bounded total mass. The proof will follow directly from the smoothness of the family of maps, and the $\mathcal{C}$en representation thereof. Now we will derive a contradiction from the case where there are 2 *distinct* degenerating clusters, each having unbounded total mass in the limit $t \to 0$, ie where there are infinite weights cancelling "at a distance" (cancellation is again clear by the normalizations). The contradiction is that in this case the maps would degenerate; in applying (2.6) here, we will use a *single* w_i, p_i to denote the (weighted) contribution to $r_t(x)$ of each *cluster*, this approximation can be justified insofar as when $t \to t_0 = 0$, we just use it to show that $(p_1 + p_2)(w_1 + w_2) = O(1)$. But $(w_1 + w_2) = O(1)$ (this is the sum of weights over all points of both clusters) by the normalizations, and $(p_1 + p_2) = O(1)$ since $x \neq p_i$. Similarly, $(p_1 - p_2)(w_1 - w_2)$ must blow up; $(p_1 - p_2) \neq 0$ by the *distinctness* hypothesis, and $(w_1 + w_2) = O(1)$ but each w_i blows up by hypothesis, so $(w_1 - w_2)$ blows up. This and (2.6) now implies $r_t(x) = x$. Furthermore the same holds for all x in a nbhd ie $r_t(x)$ is the identity map, which is absurd. (2 degenerating clusters entails 4 fixed-points , so degree >2, but the identity map is degree 1, contradicting smoothness.) $\square$

(6) Observe now, that as $p_1 \to p_2$ (with p_i being single points) while staying away from x, $(p_1 + p_2) = O(1)$ (using notation as in (2.6)) and we have remarked that $(w_1 + w_2) = O(1)$ above, so $(p_1 + p_2)(w_1 + w_2) = O(1)$. Again $r_t(x) \neq x$ so by (2.6)

$$(p_1 - p_2)(w_1 - w_2) = O(1), \tag{2.2}$$

and we conclude that $|w_i| = O(t^{-1/2})$, or $O(t^{-1})$ on the double cover as discussed above. Thus, in the generic case of a double point, $w_i - w_{i+1}$ blows up, but

Lemma 2.7. $(p_i - p_{i+1})(w_i - w_{i+1}) = O(1)$

Proof. We give another proof here, based again on $(w_1 + w_2) \neq 0$, because the technique is useful in section 2.1.2. Grouping together the terms $i = 1, 2$ in ((2.1)), and rewriting with a common denominator, one gets the symmetric polynomials p_1p_2,$p_1 + p_2$, $p_1w_2 + p_2w_1$, $w_1 + w_2$, arising as coefficients. Smoothness implies that appropriate ratios of these are then all $O(1)$. In fact we can suppose that $w_1 + w_2 \neq 0$ is $O(1)$, and the same then follows for the rest of these "elementary polynomials". Simple algebra then gives $(p_i - p_{i+1})(w_i - w_{i+1}) = O(1)$. Note that this grouping can be done for higher order degeneracies as well and gives formulae such as ((2.3)). $\square$

2.1.1. SURGERY. This suggests that the SURGERY that transforms the compactified weighted configuration space (later to be identified with Local systems) to Rational maps is locally the same as the standard holomorphic surgery of ruled surfaces, (the case $\mathbb{CP}^1 \to E \to \mathbb{CP}^1$) as described in [32] , (p. 25). But this is not quite right; recall that we should pass to a double cover to define $p_i - p_{i+1}(t)$ as functions, so the surgery on the space of Rational maps is essentially the standard holomorphic surgery of ruled surfaces conjugated by a square-root map. (and this is only a preliminary step towards understanding the global relation of $\mathcal{L}$ to Rat.) But we also saw that the difference between compactifications of Local systems vs. Rational maps is that the latter involves a bubbling off of spheres in the graph of the map as divisors degenerate without the proper blow-up of weights. This also involves a kind of surgery. It would be interesting to see if the two are more directly related.

One motivation is potential applications of theorem 2.1 to analyzing the topology of mapping spaces. The obstructions involved in constructing retractions of SRat to a (circle or other) bundle over Poly, using retractions of weights to the unit circle for example, where one must avoid weight zero and respect the sum of weights, gives rise to nontrivial topological considerations (possibly related to linkages, an issue pursued no further here).

2.1.2. $\mathcal{C}$en *vs Rat with double points:* As in the proof of (2.7) one replaces $\frac{1}{p_j - z}$, in equation 2.1 by a polynomial in the latter, with no condition on any coefficient but the first;

$$\frac{1}{y-z} = \sum_j \sum_{k>0} u_{jk} (\frac{1}{p_j - z})^k; \quad y = \phi(z). \tag{2.3}$$

Remark 2.8. The following claims can be easily checked: (i) The only constraint is $\sum_j u_{j1} = 1$ (as for the case of simple fixed-points . (ii) The multiplicity of p_j is k_j, the largest k such that $u_{jk} \neq 0$; as in the proof of (2.2) one sees that $\frac{1}{y-z}$ blows up at $z = p_j$ as $(\frac{1}{p_j - z})^{k_j}$, and $1 = \sum_{kj} u_{jk} \frac{y-z}{(p_j-z)^k}$ implies $y - p_j = z - p_j + O(|z - p_j|^k)$. (iii) Furthermore, the coefficients u_{jk} are derived from the Taylor-series of ϕ at p_j. (iv) Note that these transform by $w_{jk} = a^{k-1} u_{jk}$ with respect to an affine transformation $az + b$ of $\mathbb{C}$, to get the right equivariance property, so they are not simply constants, (or masses) but tensors. This suggests that the appropriate generalization of the normal form for $r \in$SRat above, to very degenerate maps, would involve replacing the weights in divisors by distributions (in the sense of L. Schwartz). (v) Note that the representation ((2.3)) provides an easy construction of *smooth* families st for some j, $u_{j1}(0) = 0$, ie the total weight can "degenerate". This contrasts with the jump in degree that occurs at a fiber in which the weight of a *simple* fixed-point vanishes. In fact for multiple fixed-points it is the vanishing of the *highest* order coefficient u_{jk_j} mentioned above that corresponds to degeneration of a smooth family.

WARNING: the identity map is too degenerate as a rational map to have any nice representation of this sort! but it does sit in standard Zariski closures of the map space, evidently a limit as weights blow-up to infinity.

2.2. **Further developments for the One variable case.** We include a few more basic facts relevant to the one variable case here, before introducing the multivariable case. We then return to many other aspects of the one variable case after section 3.

2.2.1. *closed forms.* A homogeneous polynomial function $p : \mathbb{C}^2 \to \mathbb{C}$, $\deg p = d$ determines d roots, $s_i \in \mathbb{C}\mathrm{P}^1$ and supposing they're distinct, we can apply theorem 2.1 to construct the associated $r \in$Ras , oriented with symmetric weights, *i.e.* $w_i = \frac{1}{d}$ at s_i. There is an elegant underlying geometric construction, [9] , leading to an explicit closed form expression; restrict to an affine line $L \subset \mathbb{C}^2 - 0$, to get $p : \mathbb{C}\mathrm{P}^1 \to \bar{\mathbb{C}}$, $\deg p = d$ (d=degree)

$$R_p(x) = x - d(p(x)/p'(x)) \tag{2.4}$$

by (2.1) it suffices to check (easily) that the fixed-points and weights are correct (or a formal calculation using the derivative of $\log(p)$ and the factorization of p giving the $\mathcal{C}$en form directly). In fact this formula was derived by Doyle and McMullen (building on F. Kleins work on the icosahedron) by quite a different method, viewing it as a *conformally intrinsic* transformation of Poly to Rat; the relation to center-of-mass here is new. For an extension see section 4.1.2, also for an application, section 6, and the relation to plethysm section 7.

It may at first be surprising or puzzling that $\mathcal{C}$en can be rewritten using a derivative, but we'll provide quite a thorough explanation of this equivalence below. One explanation involves reinterpreting the $\mathcal{C}$en construction as a steepest descent method for a natural lyapunov function, see also (4.22) and section 4.2.1.

Remark 2.9. The construction of R_p can easily be extended from p polynomial to p a rational *function* f/g st (degree f – degree g) is nonzero. This is clear if one thinks of the associated divisor normalized to mass 1 as a signed measure. It can also easily be seen using either of the constructions of [9] . The interesting upshot is that the zeroes of f can be made superattracting by choosing degree f - degree g =1 while poles are always repelling, in fact $g(z) = z^{d-1}$ suffices.

2.2.2. *examples.* Letting $Z = Z^n$, denote the n-tuple of n-th roots of unity, $\phi_Z(x) = \mathcal{C}\text{en}(Z, x) = x^{1-n}$.

Letting $SZ = SZ^n$ denote the n-tuple of $(n-2)$-th roots of unity, augmented by $0, \infty$, $\phi_{SZ,W}(x) = x^{n-1}$ for some weights W, which are not symmetric or even positive. It will be useful (in (5.3)) to calculate $\phi_{SZ}(x)$ for symmetric weights, using (2.1) we get

$$\phi_{SZ}(x) = \frac{-x^{n-2} + n - 1}{(n-1)x^{n-2} - 1} x, \tag{2.5}$$

One should check directly that this is Z^{n-2} (multiplication) equivariant, and equivariant for the $\frac{1}{x}$ map, that 0 is a fixed-point as is $x = 1$, and these with their respective multipliers and the symmetries suffices to confirm the validity of the formula. (a direct derivation requires a slightly messier calculation).

2.2.3. *one variable; steepest descent vs energy as conformal factor etc.* We will soon see that the map $r \in$Rat, together with the choice of a round metric, $\mathbb{C}\mathrm{P}^1 = S^2$, or equivalently an antipodal map, $A_x, x \in H^3$, determines a *unique* energy function $\mathcal{E}_r$, in such a way that (i) $r(x)$ determines $\nabla \mathcal{E}_r(x)$ and (ii) fixed-points of r correspond to poles of $\mathcal{E}_r$. Simple O(3)–symmetry considerations show:

Proposition 2.10. *Given a correspondence with the properties (i) and (ii), a critical point x of $\mathcal{E}_r$* satisfies $r(x) = A(x)$, (A being the antipodal map).

This motivates the study of anticonformal maps (of the form $Ar(x)$, we call this space ARas) via Morse theory in section 5. Note especially the use of critical points of r in theorem 5.3.

The point of view we take in this article is to introduce energy and relate it to r by a steepest descent construction. We thus obtain fixed-points of $Ar(x)$ with an additional attractor property. There is a less natural, but shorter route to the proposition above; theorem 2.1 gives an explicit formula for a rational map with fixed-points and multipliers prescribed by a divisor WP. There is a very direct way to transform this into an explicit formula for a singular flat holomorphic connection defining the rank one local system (biholomorphic to $T\mathbb{C}\mathrm{P}^1$, singular on P) determined by WP, see theorem 4.18. We have put that material after the development of the energy function, but one can already appreciate that proposition 4.21 can be used as a shortcut to show the relevance of the PDE in proposition 4.1, insofar as it relates rational maps having divisors with $\mathbb{R}$–weights to $\mathcal{E}$.

2.2.4. *connections in one complex dimension.* (we assume the reader is familiar with this material, it is included to fix terminology only) that a *connection* D for $T\mathbb{C}\mathrm{P}^1$ at x is a linear map $D(x) : (T_x, J^2_x) \to T_x$ where J^2 represents 2-jets (one-jets of vector fields) at x. D is holomorphic if when extended naturally to complex vector fields, [31] , D sends holomorphic input data to holomorphic output data. Since a pair of connections D_i differs by a matrix valued one-form, and in one complex dimension, a pair of holomorphic connections D_i differs by a holomorphic scalar valued one-form, η, the holomorphic connections at x form a *one-dimensional complex affine space.*

Denoting by D_z the affine connection associated with A_z on $\mathbb{C}\mathrm{P}^1 - z$, it is not hard to check that the correspondence $z \in S^2 - x \mapsto D_z(x)$ is a holomorphic affine map, (use equivariance).

This doesn't directly give an existence or uniqueness proof for the A_x structures, but it induces a dynamical system on a larger class of connections, for which the desired structure gives at least one fixed-point.

The basic example of a holomorphic connection is ∂_z on $\mathbb{C}$, so every holomorphic connection on $U \subset \mathbb{C}$ is of the form $\partial_z + \eta$, η holomorphic . Furthermore, a holomorphic connection in one complex dimension is flat; (Froebenius integrability is vacuously satisfied in one complex dimension). Now it is possible to make sense of the statement that

Lemma 2.11. *A_x depends holomorphically on x.*

This just says that $A_x = A_0 + \eta_x$ and η_x depends holomorphically on x. Comparing to (2.1) this is a more intrinsic approach to seeing;

Corollary 2.12. *$\mathcal{C}\mathrm{en}(x, P)$ is meromorphic in $(x, P) \subset (\mathrm{CP}^1)^n$.*

2.2.5. *Remark: derived map.* Given $y = r(x) = r_{PW}(x)$ as above, we can define an averaged inverse map,

$$r^a(y) = \mathcal{C}\mathrm{en}(r^{-1}(y), y)$$

where averaging uses symmetric weights. One immediately sees that fixed-points of r are fixed-points of r^a.

Lemma 2.13. *r^a is a rational map of $\mathbb{C}P^1$ with the same fixed-points , P, as r. (This may fail if r has double fixed-points). The weights $w_i^a = \frac{1-w_i}{n}$ and multipliers $m_i^a = 1 - n + nm_i^{-1}$ where n=degree(r).*

Proof. r^a is clearly holomorphic up to the fixed-points of r, and continuous at the latter, hence holomorphic on $\mathbb{C}P^1$. The formula for the multipliers can be derived by a calculation similar to that in (2.1), using the proof of (but not applying) l'hopital. The formula for weights follows, note that there are $n + 1$ fixed-points so the total mass is indeed 1. □

A more conceptual proof might first check that $w_i^a = f(w_i)$ for some holomorphic f, and since the total mass of W^a is 1, f should be linear. Furthermore $w_i^a = 0$ is only possible at double points, where w_i=1. (Q: Do fixed-points of r^a jump, or does degree jump when r has double fixed-points ?) Note that the fixed-points of $r \mapsto r^a$ are precisely the maps with symmetric weights, a space isomorphic to configurations on $\mathbb{C}\mathrm{P}^1$. This is the space that $\mathcal{F}$, ((3.1)), acts on, as well as the image of ((2.4)).

More generally one wonders if this fits naturally into a general geometric theory of rational maps of spaces of rational maps or polynomials? We will present some evidence later that there is some such theory based on notions of geometric plethysm, [15], in section 7. This also could tie in neatly with the appearance of discriminants in the restriction above to SRat.

3. Multivariable case

This work began as an effort to understand a canonical rational map, $\mathcal{F}$, of configurations in $\mathbb{C}\mathrm{P}^1$, and was fueled by the unexpected relation to equidistribution problems. Let $\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n$, $n > 1$, be the meromorphic map defined by solving (here w is a variable, not a weight)

$$\frac{1}{w_i - z_i} = \frac{1}{n-1} \sum_{\{j:i \neq j\}} \frac{1}{z_j - z_i}; \ w = \mathcal{F}(z). \tag{3.1}$$

$\mathcal{F}$ has a natural extension to $(\mathrm{CP}^1)^n$, and at the risk of redundance the geometric interpretation is: given $z = (z_1, \ldots, z_n) \in (\mathrm{CP}^1)^n$, removing z_j from $\mathbb{C}\mathrm{P}^1$ gives an affine structure $A_j = A(z_j)$ that depends holomorphically on z_j, so the A_j center of mass w_j of *the remaining* points, z_i, in the set is well-defined and $w = (w_1, \ldots, w_n)$ is meromorphic in z. Note that the construction is conformally intrinsic. In particular it is antipodal equivariant, (identifying $S^2 = CP^1$). We now outline some of the MAIN PROPERTIES of $\mathcal{F}$;

3.1. Basic Examples:

(1) $N = 2, 3$ pts: $\mathcal{F}$ is trivial for $n = 2$, (it permutes the pair) and $\mathcal{F}$ is an involution for $n = 3$; check it on the 3rd roots of unity, and use Moebius-equivariance. But $\mathcal{F}$ is not smooth for $n = 3$; when all 3 points collide $\mathcal{F}$ is singular, *i.e.* $\mathcal{F}$ is meromorphic rather than holomorphic . These configurations always exhibit *period 2 behavior*, as do many symmetric configurations including k-th roots of unity. In fact, in each of these cases configurations (suitably normalized by automorphisms) get sent to their antipodal images (componentwise) as is easily seen using equivariance properties; nontrivial isotropy of a configuration at a point suffices. The same holds for Z^n the n-tuple of n-th roots of unity, and SZ^n the n-tuple of $(n-2)$-th roots of unity, augmented by $0, \infty$.

(2) $N = 4, 12, 20, etc.$ Standard symmetric configurations exhibit period 2 behavior, but computer experiments reveal that iterating $\mathcal{F}$ equidistributes points on the sphere! Certain standard symmetric configurations are in fact period 2 equidistributed attractors. In the computer experiments we initialize at random points, or *roots of unity*, Z^n. The latter are easily seen to be period 2, but their instability together with round off error rapidly leads them to local minima of a lyapunov energy function. Subsequent analysis proves that any (strong) local minimum X_{min} of the "log energy" $\mathcal{E}(X)$, subsection 3.3 is a period 2 attractor for $\mathcal{F}$. In fact, let A be **the antipodal map** with respect to a round metric on $\mathbb{C}\mathrm{P}^1$ such that pts of Z^n lie on the equator, then $A(x) = \frac{-1}{\bar{x}}$. For Z listing the vertices of any regular solid P, one sees from Moebius -equivariance of $\mathcal{F}$ and rotational symmetries of P that $F(Z) = A(Z)$.

(3) Appropriate notions of *attractor* and *antipodal* must take into account the equivariances of $\mathcal{F}$ discussed below. It suffices to normalize configurations by their *dipole moments*, subsection 4.3.

(4) $Z^n, n > 3$, is an unstable fixed set for $A\mathcal{F}$; one may calculate the linearization of $A\mathcal{F}$ at Z^n explicitly, by exploiting the cyclic group action. It will be easier to see this instability later, in terms of the associated energy function.

Example 3.1. Collapse: Consider $\mathcal{F}$ with $|Z| = 4$, and iterate $\mathcal{F}^n(Z_0)$ where Z_0 has a double point, $z_3 = z_4$. Note that the latter stay fixed. This illustrates the difference between weighted points and multiple points.

Iterating $\mathcal{F}$ the configuration *collapses i.e.* it converges to a single (quadruple) point. In fact there is a hyperbolic $g \in \mathrm{Aut}(S^2)$, such that $\mathcal{F}(Z_0) = g(Z_0)$, where $\{g^k, k \in \mathbb{N}\}$ is noncompact (in SO(N,1)).

We will see that there is a natural normalization of any Z, by the moment map, or Douady-Earle center, (associated to equivariances discussed below), but the example shows this can not give an $\mathcal{F}$ invariant slice.

Question: Does the same asymptotic collapsing phenomena obtain generically for large configurations with a single double point , (perhaps with total mass at least a half)? Collapse may even be generic in the Julia set. In fact, one may well find that generically, for Z in the Julia set, J, the Douady-Earle projections to $H^3 = SO(3,1)/SO(3)$, of iterates, $\mathcal{F}^n(Z)$, drift to infinity. It is too much to hope that a standard measure for J, [13], should behave like a Wiener measure, but one might say this is a topological version of random walk. One can observe this computationally, at least if display software does not automatically renormalize outputs.

Question: Describe the dynamics of $\mathcal{F}$, defined over the reals, (or other fields!), does it *exhibit generic collapse in a chaotic fashion*? (Recall remarks on the dipole moments in the disc and random walk.) Is $(\mathrm{RP}^1)^n$ repelling in $(\mathrm{CP}^1)^n$?

3.1.1. *Equivariance and diagonalization properties of $\mathcal{F}$:*

(1) The symmetric-group S_n: $\mathcal{F}$ is clearly equivariant with respect to permutations of the z_i; renaming the variables has no effect on their behavior. Quotienting, and noting that $(\mathrm{CP}^1)^n/S_n = \mathbb{C}P^n$, as follows by identifying a set of roots with its defining polynomial, $\mathcal{F}$ determines a *canonical* self-map of $\mathbb{C}P^n = Poly$(projectivized), (as promised in the abstract). Its geometric nature is further elaborated, in proposition 3.6, as a map of $(\mathrm{CP}^1)^n$, and in remark 3.9 as a map of $\mathbb{C}P^n$.

(2) PSL(2,C)-action: $\mathcal{F}$ is Moebius equivariant, since it is defined intrinsically with respect to the conformal structure. The G=PSL(2,C)-action descends to $(\mathrm{CP}^1)^n/S_n = \mathbb{C}P^n$, and the quotient map is well-defined on the standard (Fulton-MacPherson) blown-up compactifications of $\mathbb{C}P^n/G$, namely the moduli space, $\mathcal{M}_{0,n}$. It is not well-defined *into* the blown-up space, only to the singular space $\mathbb{C}P^n/G$. Nevertheless, blow-ups are meromorphic maps, and $\mathcal{F} : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n}$ thus gives a meromorphic map which is (almost**) smooth on the blow-up of $\mathcal{D}$, but which is singular on $C = \mathcal{F}^{-1}\mathcal{D} - \mathcal{D}$, (** and smoothness fails at $\mathcal{D} \cap \bar{C}$). We do not know if this qualifies as a rational map of $\mathcal{M}_{0,n}$, it seems the existence of such maps is of some interest.

(3) It is known in algebraic geometry that certain desingularizing blowups can be realized by passing to dual curves or varieties. This suggests the use of looking at the graph of $\mathcal{F}$ as a way of *constructing* desingularizing blowups in the construction of the compactification of the moduli space $\mathcal{M}_{0,n}$.

(4) Antipodal-symmetry. This follows as above, $\mathcal{F}$ is defined intrinsically with respect to the conformal structure, and it plays an important role, as the examples suggest. We will see below its role in relating metric to holomorphic constructs.

Problem 3.2. The map $\mathcal{F} : Poly \to Poly$, should be expressible directly in terms of the coefficients. The brute force method of (for each $|Z| = n$) grouping roots into symmetric polynomials, gives a huge and messy formulas. One would hope there is a more elegant closed form expression for any $|Z|$, or an expression with a clear geometric significance, or at least a nice algorithm. We comment on this more in section 7.1, where we provide an expression which gives a compromise of these qualities, with the hope that better expressions will be found.

3.1.2. *Diagonals and discriminants.*

Definition 3.3. The subvariety of configurations $Z = (z_1, \ldots, z_n)$ with a double point, *i.e.* $\exists i \neq j, z_i = z_j$ is called the *diagonal* and its S_n-quotient in $\mathbb{C}\mathrm{P}^n$ is the *discriminant locus*, they are (both) denoted $\mathcal{D}$. $\mathcal{D}$ has a stratification by subvarieties $\mathcal{D}_\alpha$, for multiindices α, listing the cardinalities of nontrivial multiple points. Some shorthand is indespensable; for example $\mathcal{D}_3$ consists of configurations with *at least* one point of multiplicity *at least* 3, ie the multiindex is (3,1,. . . 1,1).

$\mathcal{D}$ has an obvious, but interesting fixed-point property; diagonals are (componentwise) fixed points, $\forall X, i$,

$$\mathcal{F}_i(X) = X_i \iff \exists j \neq i, X_i = X_j.$$

We call the *only-if* part $\mathcal{D}$*-graph-invariance*, it is a property that is only well-defined for self-maps of configuration spaces.

Furthermore, one can check that these $X \in \mathcal{D}$ are repelling as fixed points in the x_i component, provided that they have weight less than half. Note the relation of this to the equidistribution problem. This uses the relation of weights to multipliers in (2.1);

Proposition 3.4. *The diagonal is an $\mathcal{F}$-invariant variety, with a* REPELLING PROPERTY *(see below) on the open dense $\mathcal{F}$-invariant subset, $\mathcal{D}^2 = \mathcal{D} - \mathcal{D}_3$, of configurations Z with at worst double points, and where $w_{ij}, w_{ji} < \frac{1}{2}$. $\mathcal{D}^2$ is also precisely the part of $\mathcal{D}$ on which $\mathcal{F}$ is smooth.*

In particular $\mathcal{D} \subset \mathcal{D}^- \subset \mathcal{J}$, the Julia set of $\mathcal{F}$, where $\mathcal{D}^-$ is the union of preimages of $\mathcal{D}$ under iteration.

Remark 3.5. The correct definition of *repelling* on the meromorphic part of $\mathcal{D}$ is not a priori clear, but we hope that the collapse phenomenon in example 3.1 will provide an appropriate notion for the case at hand; the collapse to p_3 is an explosion away from some other "condensed state", $\lim_{k \to -\infty} g^{-k} Z_0$. (In other words, we suggest that repelling points in $\mathcal{D}_3$ are fixed-points for $\mathcal{F}$ on $\mathcal{M}_{0,n}$ whose preimage in $\mathbb{C}\mathrm{P}^n$ includes a Zariski dense subset of a noncompact orbit.) Note also that sequences of configurations, $\gamma_i(Z)$, collapsing along a PSL(2,C)-orbit (3.1) give directions where the repelling property fails, but the degeneracy is in $\mathcal{D}_3$.

The definition of *repelling property* on $\mathcal{D}^2$, also requires explanation; essentially one pair of points is increasing its distance, but another pair could create a new double point, landing on a different diagonal component. Thus we can only hope to increase distance from that one component of $\mathcal{D}$ near which we start (this makes sense on $(\mathrm{CP}^1)^n$ but not $\mathbb{C}\mathrm{P}^n$).

3.2. **Geometric-Naturality of $\mathcal{F}$.** A full geometric characterization of $\mathcal{F}$ is possible using equivariance, meromorphicity and diagonals, we state a general version including the weighted versions of $\mathcal{F}$. Let $\mathbf{M}_x$ be Moebius transformations of $\mathbb{C}\mathrm{P}^1 = \bar{\mathbb{C}}$ such that $\mathbf{M}_x(x) = \infty$. Given a matrix U such that $\forall i \Sigma_{j \neq i} u_{ij} = 1$, consider the function

$$\mathcal{F}_{U,i}(z) = \mathbf{M}_{z_i}^{-1}(\Sigma_{j \neq i} u_{ij} \mathbf{M}_{z_i}(z_j)); \quad \mathcal{F}_U : (\mathrm{CP}^1)^n \to (\mathrm{CP}^1)^n.$$

Proposition 3.6. *Given U as above such that $i \neq j \implies u_{ij} \neq 0$,*

(1) *$\mathcal{F}_U$ is meromorphic in Z,*
(2) *$\mathcal{F}_U$ is Moebius (or better, $O(3,1)$) equivariant,*
(3) *$\mathcal{F}_{U,i}(Z) = Z_i \iff \exists r, j, i,$ st $j \neq i$,* and $Z \in \mathcal{D}_{ijr} = \{z : z_i = z_j = r\}$, *hence $D_{ijr}, \mathcal{D}$ are $\mathcal{F}_U$-invariant, and $\mathcal{F}_U$ is $\mathcal{D}$-graph-invariant,*
(4) *$\mathcal{F}_U$ is smooth at Z iff no three z_j are identical (no triple point).*

Furthermore given any $G : (\mathrm{CP}^1)^n \to (\mathrm{CP}^1)^n$ satisfying these properties there is a U as above such that $G = \mathcal{F}_U$. $\mathcal{F}_U$ is symmetric (i.e. S_n-equivariant for S_n the permutation group) iff $\forall j \neq i, u_{ij} = s_{ij} = \frac{1}{n-1}$, and in this paper $\mathcal{F}$ generally denotes $\mathcal{F}_S$ (S denoting the symmetric weights s_{ij}).

The graph-invariance of the 3rd point reduces the characterization to the following lemma; we can use graph-invariance to reduce to the study of $\mathcal{F}$ on the complement of a point, (call it ∞) in $\mathbb{C}\mathrm{P}^1$, hence to $\mathbb{C}^n$. For example fixing $x_1 = \infty$ and letting the other $x_i \in \mathbb{C}$ vary, we reduce to an analogous characterization of $\mathcal{C}$en. The affine group of M Moebius , fixing ∞, acts on $\mathbb{C}^n$ diagonally. Our characterization of a general weighted version of $\mathcal{F}$ is based on,

Lemma 3.7. *Consider the maps $f : \mathbb{C}^n \to \mathbb{C}$*

$$f_U(z) = \sum u_i z_i; \quad \sum u_i = 1,$$

with respect to weight vector U, f is holomorphic and affine-equivariant iff it is of this form. f is symmetric (i.e. S_n-equivariant for S_n the permutation group) iff $\forall i, u_i = s_i = \frac{1}{n}$, we may denote $f_S = f$ in this case.

Note that holomorphicity of f is a valid assumption because graph-invariance guarantees that we avoid ∞.

Proof. (of the nontrivial direction). By affine-equivariance, the small diagonal, $\mathcal{D}^s = \{(z_1, \ldots, z_n) : \forall i, j,\ z_i = z_j\}$ maps isomorphically to $\mathbb{C}$, and f is a smooth holomorphic fibration over $\mathbb{C}$. Dilation equivariance then gives that the full fibration is isomorphic to the bundle tangent to the fibration along D^s, and that the fibers are affine hyperplanes of $\mathbb{C}^n$, which must be parallel. It remains to identify U with the slopes of these planes. □

Problem 3.8. What is the simplest characterization of $\mathcal{F}$ on $(\mathbb{C}\mathrm{P}^1)^k$? It may be possible to substantially weaken the equivariance hypothesis; the fact that $\mathcal{F}$ is well-defined depends heavily on equivariance, and this suggests that under a very weak hypothesis $\mathcal{F}$ is forced to be equivariant. Similarly, what is the simplest natural property characterizing the induced map of $\mathcal{F}$ on $\mathbb{C}\mathrm{P}^n$? Note that graph-invariance is not directly definable on $\mathbb{C}\mathrm{P}^n$. It would be good to have a characterization in terms of the fixed-points of $\mathcal{F}$ (see below) and their multipliers, or just the symmetry property of the latter, noting the analogy to (2.1). If there is a unique map with the fixed-points and multipliers of $\mathcal{F}$, then it is clearly equivariant. Characterizations of $\mathcal{F}$ based on equivariance should be possible in connection with geometric plethysm, [15], see the discussion in section 7.

The introduction of weights is not just generalization for its own sake; they are necessary to produce attractors in certain algorithms, as in the *suspension* constructions below. They are also useful in producing spaces of maps that are naturally isomorphic to other interesting spaces as one sees in subsection 4.2.1. See also the application to degree $\mathcal{F}$, (7.1), where weights facilitate an inductive step, increasing the number of points, but with useful continuity properties.

Note that fixed-points of $\mathcal{F}$ correspond to configurations with every point occuring as a double-point, or polynomials where every linear factor occurs at least twice, denoted $\mathcal{D}_{sq}$, for *square.* In the notation of (3.3) $\mathcal{D}_{sq} = \mathcal{D}_\alpha$ with $\forall i, \alpha_i = 2$, a middle-dimensional variety, (with an extra *co*dimension one for odd $|Z|$; $\mathcal{D}_{sq}$ has $\alpha_i = 3$ for some i). Hence $\mathcal{F}$ is quite non-generic, even with weights. Adding clamped points to the $\mathcal{C}$en construction of $\mathcal{F}$ introduces associated fixed-point sets with dimensions reduced accordingly, but it does not affect $\mathcal{D}_{sq}$ which is still fixed. Note also that these fixed-points come up in questions disucussed here, in relation to holomorphic Lefschetz fixed-point theorem, and also with regards to possible characterizations of $\mathcal{F}$.

Remark 3.9. An alternative construction of $\mathcal{F}$, that works as well on $\mathbb{C}\mathrm{P}^n$, can be based on the notion of *osculation*: given a configuration, consider the associated polynomial, p. We saw, ((2.4)), there is an associated map R_p, and at a root, $p(z_i) = 0$, the 1-jet of p is determined by (2.1); it doesn't depend on any property of p aside from $p(z_i) = 0$.

We claim that for simple zeroes of p, the 2-jet of $R_p(z_i)$ is determined precisely by $\mathcal{F}_i(p)$, and equivalently by $p''(z_i)$, (or $\frac{d}{dz}\log p'(z_i)$ which is the same, in view of our remark on the 1-jet of p), in fact this is clear from ((7.2)). (For generic z, $R_p''(z_i)$ depends on the 3-jet, including p''', but in the special case $p(z) = 0$, $R_p''(z_i)$ depends only on the 2-jet, which just depends on p''. The shift is already evident insofar as $R_p(z)$ depends on p' generically, but $R_p(z_i)$ depends only on $p(z) = 0$.)

The 2-jet of $R_p(z_i)$ determines a unique osculating moebius transformation $M_z = M_{p,z_i}$ at z_i, and if we construct $\mathcal{F}^{osc} : (\mathrm{CP}^1)^n \to (\mathrm{CP}^1)^n$ which, by definition, takes each *root* $p(z) = 0$ to the unique opposite fixed-point of M_z, then one can easily verify that this satisfies the characterization (3.6). The main point, concerning the behavior of $\mathcal{F}$ on $\mathcal{D}$, is that $R_p'(z_i) = 1$ iff z_i is a double root of p (as well as meromorphicity and equivariance). As a corollary we can see that the formula for $g(z_i)$, in ((7.2)), must essentially be constructing the unique opposite fixed-point of M_z at each root $p(z) = 0$.

3.2.1. *Embedded dynamics of $\mathcal{F}$.* Fixing a weight vector W every $r = r_{PW} \in Ras$ of degree d is embedded in $\mathcal{F}_{2d+1}$, (with appropriately weighted interactions) in the sense that there is an invariant $\mathbb{C}\mathrm{P}^1 = L_r$ (in fact a *standard* $\mathbb{C}\mathrm{P}^1$, parallel to a coordinate axis) on which the action of $\mathcal{F}$ is conjugate to r, and L_r is specified by choosing *initial values* of $\mathcal{F}$ such that every fixed-point of r is represented by 2 variable points of $\mathcal{F}$. Initializing 2 variable points to p_i guarantees that these points stay fixed for any iteration of $\mathcal{F}$ (they stay *clamped*). Weights can be chosen for $\mathcal{F}$ that realize the weight vector of r on this restriction. (alternatively one could introduce nontrivial diagonal or self-weights to fix points). For rational weights it suffices to use $\mathcal{F}$ with symmetric weights if one allows more variable points to coincide at each clamped point. Noting that the image of a *standard* $\mathbb{C}\mathrm{P}^1$, under the canonical map to $\mathbb{C}\mathrm{P}^n$, is a line, (generically with $n-1$ double tangencies to $\mathcal{D}$) we get;

Theorem 3.10. *Fixing a degree, d, and the symmetric weight vector, S, every rational map of* $\mathbb{C}\mathrm{P}^1$ *of degree d, and weight vector S, is realized as the restriction, of* $\mathcal{F}_{2d+1}$ *acting on* $\mathbb{C}\mathrm{P}^n$ *, to a linearly embedded* $\mathbb{C}\mathrm{P}^1$ *in* $\mathbb{C}\mathrm{P}^n$ *. Furthermore, generically the line has* $n-1$ *quadruple tangencies to* $\mathcal{D}$*.*

These embedded dynamics might be useful in determining Hausdorff dimensions for the Julia set of $\mathcal{F}$, or just for showing that it is fractal, related issues have been considered in the literature, [13].

3.3. **The strong attractor property:** There are numerous notions of equidistribution on S^N. The one that interests us here is based on $G(x,y) = c\log(|x-y|)$, where $|x-y|$ is the *chordal* **Euclidean distance on** $\mathbb{R}^3$ restricted to $x, y \in S^2$, and $c < 0$ is a normalizing constant (fixed in (4.1)). G stands for Green's function , and $\mathcal{E}(x,P) = G(x,P) = c\sum_{y\in P}\log(|x-y|)$, where $\mathcal{E}$ stands for Energy, in this case using only the interactions of x with each p_i. (Generally G is a function of a pair of points or configurations and $\mathcal{E}$ a function of a configuration of variable points–with clamped points as parameters).

Definition 3.11. Equidistributed sets are global minima of

$$\mathcal{E}(Z) = c\Sigma_{\{(i,j):i\neq j\}}\log(|z_i - z_j|).$$

One should consider $\mathcal{E}$ to be an AVERAGE of pairwise energies, rather than a sum.

To generalize $\mathcal{E}$ to include real weights in the context of multi-variables with clamped points, first define the *partial energy* of z_i to be $\mathcal{E}_i(Z,P,W) = G(z_i,(Z - z_i)\cup P)$, which determines the force F_i on z_i, and the *internal energy* of Z to be $\mathcal{E}_W(Z)$, (for the latter the restricted weights do not sum to 1). Now if we take $\mathcal{E}(Z,P,W) = \sum\mathcal{E}_i + \theta\mathcal{E}_W(Z)$, first with $\theta = 0$, then the force vector components will double count the $\mathcal{E}_{ij}$ (or z_i, z_j) terms, so we must set $\theta = -1$. Note also that the effective weights of the force vectors are automatically reflexive, so we may as well choose them so to begin with. Thus, when applying $\mathcal{E}$ to study $\mathcal{F}$, the latter MUST have reflexive weights. (See remarks at the end of section 1.2.4).

Theorem 3.12. *Given* $\mathcal{F}$,$\mathcal{E}$ *as above, (i) Z is a critical point for* $\mathcal{E}$ *iff* $\mathcal{F}(Z) = A(Z)$*. Furthermore (ii) Z is a strong local minimum for* $\mathcal{E}$ *iff Z is a stable fixed point for* $A\mathcal{F}$*.*

Recall that A is the *antipodal* map, and that the metric enters via $\mathcal{E}$ on one side of the equation and via A on the other.

Corollary 3.13. *The period 2 property: Z is a critical point for* $\mathcal{E}$ *implies* $\mathcal{F}^2(Z) = Z$*. Furthermore* $dA\mathcal{F}(Z)$ *has real eigenvalues, as does* $d\mathcal{F}^{(2)}(Z)$*.*

Equivariance has to be considered to define *stable* and *strong local minimum* appropriately, see (5.10). The *reality* property is a byproduct of the relation to $\mathcal{E}$ as will be seen in the proof of (3.12)(ii) above. It should also follow from invariance of $\mathcal{F}$ by complex-conjugation with respect to the complex structure of $\mathbb{C}\mathrm{P}^1$, and weight reflexivity. (What is the most general statement about *reality* properties of the power series? perhaps in terms of symmetric polynomials invariant for a Lie group, note that there is an interesting implication for the dynamics, of nonrotation at attractors).

The hessian of energy at X critical has a simple functional relation to the linearization of $\mathcal{F}$ at X, using corollary 4.2. Furthermore, degenerate directions for the hessian correspond precisely to "*first-order* jets of configurations, staying period 2 to 2nd order". We do not see any easy approach to show,

Conjecture 3.14. *There does not exist a degenerate direction v for the hessian, $\nabla^2\mathcal{E}(Z)(v,v) = 0$ of $\mathcal{E}$ at a local minimum Z, such that v is orthogonal to the isometric (rigid motion) orbits.*

though this rigidity property should be true for any reasonable notion of equidistribution. On the other hand, the more general dictionary relating $\mathcal{E}$ to $\mathcal{F}$ may, at least, have a nice application to showing local minima are isolated, (5.13). As obvious as this sounds, we have not seen any such proof.

A better conjecture might propose effective lower bounds for the hessian. It would be interesting to have a geometric description of configuration perturbations corresponding to eigenvectors of the hessian; we would guess that they should be related to rigid motions of hemispherical sub-clusters of a configuration, ie small eigenvalues of the hessian should be related to "twist" maps rigid (as a first approximation) on hemispheres and shearing along the equator, "large eigenvalues" likewise correspond to the associated complex-conjugate flow; opposing dilations from the poles on each hemisphere. Since there are too many such shears, they should probably be averaged using eigenfunctions of the laplacian that are antipodally antisymmetric. The relation to twist maps is suggested by empirical observation (and not enough of it), but there is no theoretical mechanism relating hessians to twist maps that suggests itself.

Is there an interesting asymptotic behavior of these hessians as $n \to \infty$? For example do they become SO(n) invariant (ie round). This would diminish the potential importance of the basis just described, this seems unlikely in view of the incompatibility with the topology/symmetry of S^2. We'd expect them to be as round as possible subject to some natural geometric constraints.

3.3.1. *analytical aspects of equidistribution.* Although we do not discuss analytical aspects of the equidistribution problem in detail here, we mention some more classical problems that might relate to the material here. The survey [34], as well as other references below, provide more background and details.

Question: Rigorously estimate asymptotics for the number, $N(n)$ of local minima of $\mathcal{E}$, as a function of the size n, of configurations, and estimate the distribution of values of $\mathcal{E}$ at the local minima. It seems that $N(n) > 0$ is the only known rigorous lower bound; aside from numerical techniques, which don't treat $n \to \infty$.

Question: Relate the combinatorics of canonical triangulations associated to Z, (valences etc.) to the local minimum property. There is much work on this, mostly experimental, a recent reference being [4], but few rigorous results for large n. There are some proofs of uniqueness of minima for very small n, [10, 24]. Combinatorial structures of configurations, given a spherical metric, involve associated triangulations, the simplest of which are more naturally associated to the sphere or circle packing version of equidistribution. Using Morse-Smale complexes might be more pertinent for equidistribution associated to functions like $\mathcal{E}$. Conformally intrinsic combinatorial structures can be developed using Thurston's notion of hyperbolic convex hull.

4. Energy and the center-of-mass, connections and Polyhedra.

The full relation between $\mathcal{C}$en and $\mathcal{E}$ is a consequence of some basic properties of $G(x, y)$. We continue with the notation of section 3.3. In this section we establish this relation first in the one variable case, and most of our work is to establish this relation in the case of just 2 points, the rest follows essentially by averaging over all pairs. We achieve the proof of part (i) of theorem 3.12 in this section and lay some groundwork for part (ii).

$\triangle$ refers to the laplace beltrami operator (trace of hessian) on the round sphere, S^2, the normalizing constant factor is not important here, it is absorbed by an undetermined c factor in various equations below, and can be fixed later (somewhat arbitrarily). We have structured all proofs so as to avoid any explicit dependence on the normalizations. $\triangle_x$ indicates which variable the $\triangle$ applies to, when this might otherwise be ambiguous. The stereographic projection usually maps the plane to the sphere, but we find it convenient to denote this correpondence by $\Pi_x : S^N - x \to T_{Ax}S^N$.

We begin with the elementary properties of $\mathcal{E}$ that make it useful in studying $\mathcal{F}$.

Proposition 4.1. *(i)The log-energy is the Green's function,*

$$\forall x \in S^2 - z;\ c\triangle_x \log(|z - x|) = \triangle_x \mathcal{E}(z, x) = \frac{1}{2} > 0.$$

and (ii) the force $F_x(z) = -\nabla_x \mathcal{E}(z, x) : z \in S^2 - x \to T_x S^2$ gives a conformal isomorphism of $S^2 - \{x\}$ to $T_x S^2$, for each x.

Corollary 4.2. *For an appropriate choice of c and for each fixed x, $F_x(z)$ (or $(dA)F_x(z)$) **is isomorphic to** Π_x, **the stereographic projection** of x. In fact, $F_x(z)$ equals the stereographic projection from Ax; $\Pi_{Ax}Az$, of the antipodal Az.*

Remark 4.3. (i)It should be clear that some potential function will give a force vector isomorphic to stereographic projection, in view of radial symmetry. The particular function which works due to its first derivatives, has a second derivative with the nice properties discussed here, and this is what makes things interesting. In other words its not difficult to relate a map like $\mathcal{F}$ to discrete steepest descent, but its (at first sight) miraculous that the resulting steepest descent process actually has good convergence properties. (ii) One should expect that any maps intrinsic for the conformal structure should be constructible in terms of cross ratios. Though $\mathcal{C}$en doesn't have a direct definition of this sort, such a relation emerges here in section 4.0.5. (iii) The constant $\frac{1}{2}$ in the equation above is somewhat arbitrary; it effectively fixes a normalization of the laplacian; thus in all that follows we use this $\frac{1}{2}$, in place of a normalizing constant factor. (iv) part (i) will not hold for $S^N, N > 2$, though we recover an inequality in section 8, but (ii) is valid for any N. (v) We provide explicit formulae that can be used to give alternative proofs of both parts in ((8.2,8.3)).

The hidden relation of the 2 properties in proposition 4.1 will be clarified in the proof, via Kaehler geometry, [23], see theorem 4.6. Much of our use of the Kaehler formalism is just a natural way of treating a class of spherical harmonics, as in (4.10), by using line bundle terminology. A polar coordinate version of the laplacian would be a reasonable substitute.

4.0.2. *Proof of theorem 3.12, first steps.* By corollary 4.2, the total force exerted on x_i by the other clamped or variable points is now clearly just the weighted average of the individual forces, but this is also the weighted average of stereographic projections, which is just $\mathcal{C}$en, (up to stereographic projection $\Pi_{Ax_i}^{-1} : T_{x_i}S^2 \to S^2$ followed by an antipodal map A), summarizing,

Theorem 4.4. *Given a divisor PW with real weights, and the associated energy function $\mathcal{E}_{PW}$, with force $F_{PW}(x) = -\nabla_x \mathcal{E}_{PW}(x) \in T_x S^2$, $r_{PW}(x) = A\Pi_{Ax}^{-1}(x + F(x))$, ($F = F_{PW}$, as shorthand).*

r_{PW} is as in (2.1). The $x + F(x)$ is in terms of vector addition; it represents a tangent vector to S^2 in the embedded affine tangent space. The FIRST POINT of theorem 3.12 is an immediate consequence. (see also remarks at beginning of section 5.2).

Let us isolate one point from (4.4); given a section of the tangent bundle, $v \in \Gamma T\mathbb{C}\mathrm{P}^1$, (ie a vector field, such as $-\nabla \mathcal{E}_Z : \mathbb{C}\mathrm{P}^1 \to T\mathbb{C}\mathrm{P}^1$), define $\mathcal{R}_v : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ by

$$y = \mathcal{R}(x; v) = \mathcal{R}_v(x) \iff -c\nabla_x \log|x - y| = v(x) \tag{4.1}$$

$\mathcal{R}_v$ is well-defined by proposition 4.1. c is defined by (4.2). Note that the 1-1 correspondence of vector fields on $\mathbb{C}\mathrm{P}^1$ to self-maps of $\mathbb{C}\mathrm{P}^1$ thus extends from the special case of energy functions and holomorphic maps, (4.14), to arbitrary smooth vector fields and maps. Restating our result in these terms, (using ϕ_{ZU} rather than r_{PW} as notation);

Proposition 4.5.

$$\forall x \; \mathcal{R}(x; \nabla \mathcal{E}_{ZU}) = \phi_{ZU}(x), \quad \nabla \mathcal{E}_{ZU}(x) = 0 \iff A\phi_{ZU}(x) = x.$$

The SECOND POINT of theorem 3.12 will follow in section 5 using theorem 4.9 with a little more work, (continued at (4.14)).

4.0.3. *The proof of proposition 4.1 follows from;*

Theorem 4.6. *Consider $\mathcal{E} : X \to \mathbb{R}$, where $X = S^2 \times S^2$ with w the Kaehler form for the round metric on X. Then $c\sqrt{-1}\partial\bar{\partial} \log|x - y| = \frac{1}{2}w$ on $\mathcal{D}'$.*

where, as before we choose normalizations of $\sqrt{-1}\partial\bar{\partial}$ compatible with the 2-dimensional laplacian to justify the factor $\frac{1}{2}$. The Kaehler form w can be thought of as the curvature form of the dual to the tautological bundle, [18].

The first point of proposition 4.1 is an immediate consequence. For the second point, consider the mixed term; $\overline{\partial_y}\partial_x \log|x - y| = 0$, the inner derivative is the force, but complexified, (we use the S^2 metric to identify the one form with a vector field). The outer derivative says that the force is holomorphic ; one should check that the Cauchy-Riemann equations of the complexified vector field implies conformality of the real vector field, viewed as a map to the complex line.

Theorem 4.6 is proved, in turn, using;

Lemma 4.7. *There exists $\sigma \in End(T_x^*\mathbb{C}\mathrm{P}^1, T_y\mathbb{C}\mathrm{P}^1) = K^{-1}$, a Moebius -invariant (diagonal action) holomorphic section of the anticanonical bundle on $X = S^2 \times S^2$, and $|\sigma(x, y)| = c_1|x - y|$, where the norm $|\sigma|$ is with respect to the canonical constant curvature metric g (of K^{-1}).*

4.0.4. *Proof of lemma 4.7 and theorem 4.6:* We provide a purely geometric proof (ie one based on symmetry considerations, with no long calculations). A *round* $S^1 \subset S^2$ is one defined by a planar cross section.

CLAIM: Given $x, y \in S^2$ there is a round $S^1(x, y) \subset S^2$ such that $S^1(x, y) \ni x, y$ and x, y is a diameter of S^1. Choosing $S^1 \ni x, y$ with small a diameter as possible suffices. PROOF: Note that for any pair of points $x, y \in S^2$ the only invariant is their distance, ie there is an isometry taking x, y to x', y' iff $|x - y| = d(x, y) = d(x', y')$ (qed).

It follows that $|x-y| = c\times\mathrm{circum}(S^1(x,y))$ (i.e. circumference). This is the first step to relating the extrinsic distance $|x-y|$ to a more intrinsic distance.

CLAIM: $\exists\gamma \in$Aut=O(3,1) such that γ applied to N,S, (the poles north, south) gives x, y, and such that the great circle $S^1(N,S) \mapsto S^1(x,y)$.

Here we can suppose without loss of generality (by composition with an isometry applied to x, y, but NOT N, S) that the line $[x,y]$ is parallel to the line $[N,S]$, and we can choose γ to fix $\{E, W\}$ (east, west) so that γ transports the great circle $S^1(N,S)$ orthogonal to $[E,W]$ to the circle $S^1(x,y)$ orthogonal to $[E,W]$, and such that $d\gamma(E) \in \mathbb{R}$.

By rotational symmetry, $|d\gamma(N)| = |d\gamma(S)|$ (defined using the metric) is just the ratio of the $\mathbb{R}^3$-radii (or lengths of the circles, up to a constant factor); $\frac{1}{2}|x-y|$, of the circle $S^1(x,y)$ to that of $S^1(N,S)$.

This proves that

$$(|d\gamma(y)||d\gamma(x)|)^{1/2} = c'|x-y|$$

for this particular γ, but $(|d\gamma(y)||d\gamma(x)|)$ is independent of γ by corollary 2.4, showing:

Proposition 4.8. *If $\gamma \in$Aut=O(3,1) such that γ applied to N,S, (the poles north, south) gives x, y, respectively, then $(|d\gamma(N)||d\gamma(S)|)^{1/2} = c'|x-y|$, (for some constant c').*

Note that $d\gamma(N)e_N \otimes d\gamma(S)e_S = \sigma(x,y)$ (fixing vectors e_N, e_S at the poles) can be regarded as a Moebius -invariant holomorphic section σ of the anticanonical bundle $TS_1^2 \otimes TS_2^2 = End(T_x^*\mathbb{C}\mathrm{P}^1, T_y\mathbb{C}\mathrm{P}^1)$, (by corollary 2.4 again), and its norm, as above, can be taken in this bundle using the standard metrics. This proves the lemma. Now the theorem follows by noting that $-c\partial\bar\partial\log|\sigma|$ is (up to a factor) the curvature form of the standard metric. (σ is invariant for the diagonal action, while w is invariant for the full isometry group).

It follows by averaging that for the most general form of the energy function, (recall remarks in section 3.3),

Theorem 4.9. *$\partial\bar\partial\mathcal{E}_{PW}(x) = \frac{1}{2}w$ on P', in the one-variable case, and $\partial\bar\partial\mathcal{E}_{PW}(Z) = \frac{1}{2}w$ on $\mathcal{D}'$, in the multi-variable case, where w is the Kaehler form for the round metric on $X = (S^2)^N$.*

One checks, using the partial energies, that in the multi-variable case, the complex hessian in the component (z_i) basis, is diagonal with entries $\frac{1}{2}$.

4.0.5. *Remarks: cross-ratios, symplectic forms and energy as a hamiltonian.* This subsection gathers together interesting relations of the energy to other structures, but it is not essential to the main theorem and one can jump to (4.14) without harm.

σ could be defined by differentiating the cross-ratio twice to produce equivariant σ *naturally.* The cross ratio $X(x,y,z,w)$ is defined as $M(w)$ where M Moebius sends x, y, z to $0, 1, \infty$. Differentiating twice,

$$S_X = \partial_{x'}\partial_{z'}X(x,z',z,x') \text{ along } \{x = x', z = z'\}$$

determines a holomorphic Moebius -invariant 2-form; X vanishes to 2nd order near this diagonal, and the 2 derivatives specified pick out the leading order term, hence it is non-trivial.

This actually gives a meromorphic section of the dual bundle, K, in fact the real part corresponds to a *symplectic* form S which blows up on the diagonal, $x = z$.

This is essentially the canonical symplectic form, S, on the space of geodesics on H^3. Thus it generalizes to higher dimensional spheres, the essential point being (2.4) in any case. In fact, S is just the standard symplectic form for T^*S^2, where $S^2 \times S^2 - \mathcal{D} = T^*S^2$ reveals the Z_2 (anti-)symmetry interchanging factors. In this context $\mathcal{E}$, which depends on the choice of a metric, is none other than the hamiltonian for geodesic flow–the flow in T^*S^2 associated to the round metric on S^2. We wonder if the hamiltonian system given by an appropriately averaged lift of these structures to $(S^2)^n - \mathcal{D}$ may have interesting properties, dynamics?

Aff-6: Furthermore, σ actually *defines* the affine structure of $\mathbb{C}\mathrm{P}^1 - x$; given a vector v at y, σ provides a global holomorphic one form on $\mathbb{C}\mathrm{P}^1 - y$ and this determines an affine structure which is independent of v, (T_y being one dimensional). In fact even in higher dimensions we get from S a basis of one forms satisfying Frobenius, and determining A_x. A priori the affine structures are weaker than σ. Could σ similarly be *defined* using A_y? By adapting the remarks above, (to the real cross-ratio), to recover S it suffices to construct the cross-ratio from A_x, but this is trivial using $x = \infty$. To fully recover σ one needs to use something equivalent to the conformal structure or its automorphism group (this is possible, and implicit in (4.18, 8.7), possibly to be discussed elsewhere).

4.0.6. *Poly vs $\mathcal{E}$, energy as a homogeneous polynomial.* We recall some basic facts of geometry, [18]; a degree d polynomial on $\mathbb{C}^2$ is a section of the (-d)-th power of the tautological line bundle, $\tau\mathbb{C}\mathrm{P}^1$. Every holomorphic line bundle $L\mathbb{C}\mathrm{P}^1 = \tau^{-d}$ is determined by its degree, d, as a power of the tautological line bundle ($\mathrm{Pic}(\mathbb{C}\mathrm{P}^1)=\mathbb{Z}$; the Lefschetz-Grothendieck pencil theorem), in particular the tangent bundle, $T\mathbb{C}\mathrm{P}^1 = \tau^{(-2)}\mathbb{C}\mathrm{P}^1$ by topological considerations. The curvature of $L\mathbb{C}\mathrm{P}^1$ is $(-d)w = c_2\sqrt{-1}\partial\bar{\partial}\log|s|$ for any holomorphic section $s \neq 0$.

We have seen relations of energy to Ras , theorem 4.4 and of Ras to Poly, subsection 2.2.1. The relation of energy to Poly is even simpler (using homogeneous coordinates);

Proposition 4.10. $\log|x-y|$ *is equal to the function* $h = c_3 \log|(x_1y_2 - x_2y_1)|$ *on* $\mathbb{C}^2 \times \mathbb{C}^2$ *restricted to* $S^3 \times S^3$ *and pushed forward to* $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$.

Proof. $(x_1y_2 - x_2y_1)^2$ is a holomorphic section of the bundle $\mathcal{O}(2) \otimes \mathcal{O}(2) = TS^2_1 \otimes TS^2_2$ vanishing on the diagonal, and having the same divisor as σ, hence it is a multiple, $\lambda\sigma$. Furthermore with canonical metrics, $\mathcal{O}(2) \otimes \mathcal{O}(2) = TS^2_1 \otimes TS^2_2$ as *hermitian* bundles, by symmetry considerations. □

The push-forward is easily justified by S^1–invariance.

Theorem 4.11. *Given* W *defined over* $\mathbb{Q}$, $\exists\Phi, \mathcal{E}_{PW} = \alpha\log|\Phi|$ *for* $\alpha \in \mathbb{Q}$ *and* Φ *a polynomial on* $(\mathbb{C}^2)^n$ *as above.*

Remark 4.12. This means that extrema of energy are just the extrema of norms of homogeneous holomorphic polynomials restricted to $(S^3)^n$.

As an interesting aside relating Morse theory of plurisubharmonic functions to fixed-points of $A\mathcal{F}$ or $\mathcal{F}^2$;

Corollary 4.13. $\partial\bar{\partial}\mathcal{E}(Z) = \frac{1}{2}w$, $Z \in (S^2)^n - \mathcal{D}$; w *as above. In fact* $\mathcal{E}$ *is a strictly plurisubharmonic exhaustion function on* $(\mathrm{CP}^1)^n - \mathcal{D}$, *(the diagonal, or discriminant) as is its pushforward to the* S_n *quotient* $\mathbb{C}\mathrm{P}^n - \mathcal{D}$.

Note that we relate the Morse theory of $\mathcal{E}$ to the topological Lefschetz theorem for the one variable maps in (5.1). The question of a multivariable analog is thus suggested. The various relations of energy, Ras and Poly provided to this point clearly leave something to be desired and we will discuss below some generalizations that come closer to showing the full picture. It seems likely, in view of [36] and references therein, that the relation of Poly to $\mathcal{E}$ was known classically, and it is certain that (4.1)(i) was known classically, but the "geometric" proof we give is probably new.

4.1. **Connections encoding maps.** To pursue our proof of theorem 3.12 we will need (compare proposition 4.1 and theorem 4.4)

Proposition 4.14. *Given a smooth function $E : U \subset \mathbb{C}\mathrm{P}^1 \to \mathbb{R}$, with force $F_E(x) = -\nabla_x E(x) \in T_x S^2$, and its associated map, $r_E(x) = A\Pi^{-1}_{Ax}(x + F_E(x))$, r_E is holomorphic iff $\triangle E = \frac{1}{2}$.*

Note that the correspondence is local in S^2. In fact there is no need to restrict to functions here, and we will prove a generalized version,

Lemma 4.15. *Given a smooth (force) vector field $F : U \subset \mathbb{C}\mathrm{P}^1 \to \mathbb{R}$, its associated map, r_F , is holomorphic iff $DivF = -\frac{1}{2}$.*

As above, r_F is the map naturally associated to F, see ((4.1)). We identify as usual the space of (local) one-forms η such that $2d^*\eta = 1$ with the space of F as above, using the metric of S^2. We give two proofs of the lemma; first a simple calculation proving the 2 statements above, followed by some general observations summarizing the relations of maps to connections leading to an alternative, more geometric, proof. Note that the generalizations of (4.14,4.15) to holomorphic maps with fixed-points , allowing for isolated logarithmic poles of E, follows by removable singularities and continuity at fixed points.

Proof. Given a holomorphic map $h(x) = y$, on U, the corresponding form (or vector field) is $u(x) = -c\partial_{|y=h(x)} \log|x-y|$ where the derivative is taken with respect to x only (in $x - y$), the substitution of $h(x)$ coming after the differentiation. (Note that this is NOT a holomorphic form). Now $\bar\partial u(x) = -c\partial_{|y=h(x)}\bar\partial \log|x-y| = -\frac{1}{2}w$ because $\bar\partial h = 0$ and by theorem 4.1. This proves *only if,* but *if* follows directly because the correspondence is 1-1 and onto; as seen by localizing the construction, using well known global approximations, of holomorphic or harmonic functions by rational functions . □

One-forms correspond to connections using the Levi-Civita connection of S^2, and we get a nice geometric interpretation of the statements above in terms of connections, as well as an alternative proof . The rest of this section involves the geometry of connections, polyhedra, etc. and their relation to Ras , the reader wanting to see a proof of theorem 3.12 as quickly as possible may skip directly to the next section (with a quick glimpse at section 4.3).

We next consider locally flat holomorphic affine structures with singularities at z_i (we frequently use Z rather than P in the rest of this section, for fixed points, with x as a variable when necessary).

Lemma 4.16. *The space $\mathcal{L}$ of singular holomorphic (flat) affine connections on $T(\mathbb{C}\mathrm{P}^1 - Z)$, with complex monodromy, λ_i at singularities z_i is in canonical 1-1 correspondence to divisors WZ with mass one, where $\lambda_i = \sqrt{-1}2w_i$. They are thus denoted by D_{WZ}.*

Existence and uniqueness of such connections are discussed in [8] but can also be obtained from the following, (using theorem 2.1) and this provides *explicit expressions* for D_{WP} via r_{WP} in $\mathcal{C}$en form.

Definition 4.17. Given $D \in \mathcal{L}$, $L2R : \mathcal{L} \to$Rat is the meromorphic map defined by $r = L2R(D)$, such that $\forall x \in \mathbb{C}\mathrm{P}^1$, $A_{r(x)} = D(x)$ (as connections *at* x), identifying the affine structure A_y with the affine-connection on $\mathbb{C}\mathrm{P}^1 - y$.

This makes sense on S^N in any dimension, using connections over $\mathbb{R}$, but it is only for $N = 2$ that we can also apply the construction using connections over $\mathbb{C}$, and obtain the meromorphic structure.

Theorem 4.18. *$L2R$ is well-defined and it commutes with the canonical 1-1 correspondences of $\mathcal{L}$,Rat, to divisors, (as in the lemma above and (2.1)). $L2R$ is a holomorphic isomorphism between $\mathcal{L}$ and Rat away from the degenerate divisors.*

$r = L2R(aD_1 + (1-a)D_2)$ iff $r(x) = M_x^{-1}(aM_x r_1(x) + (1-a)M_x r_2(x))$ where $r_i = L2R(D_i)$, with M_x as in the lines preceeding (3.6).

For more on the last point see *suspension,* section *6*. We call this construction of r in terms of r_i the *combination principle*; it is used to break up divisors into pieces to facilitate proofs in section *6*, as well as in (8.5,8.7). The extension (by surgery) of $L2R$ at degenerate divisors was discussed in section 2.1.

Proof. $L2R$ is well-defined, holomorphic and 1-1, onto, by the remarks preceeding (2.11). A local version of the correspondence follows, $\mathcal{L}_B \to Rat_B$ for $B \subset \mathbb{C}\mathrm{P}^1$ a small nbhd, from connections on the nbhd to maps $r : B \to \mathbb{C}\mathrm{P}^1$ *from* the nbhd. (in the holomorphic or smooth categories).

To establish the correpondence to Div, it remains to verify that $\lambda_i = \sqrt{-1}2w_i$ under this correspondence. It suffices to show that there is an induced correspondence between λ_i, w_i, $l2r : \mathbb{C} \to \mathbb{C}$ which is 1-1 and holomorphic , recalling that the factor of $\sqrt{-1}2$ comes from normalizations of weights and Gauss-Bonnet. To see that the monodromy λ only depends on the weight w we refer to (4.21) which suffices for $w \in \mathbb{R}$, but analytic continuation for the holomorphic $L2R$ extends this to $w \in \mathbb{C}$. ☐

4.1.1. connection and force-matching. This *"connection-matching"* should be compared to *force-matching*; in theorem 4.4 we saw r_{PW} is isomorphic via $A\Pi_x^{-1}$, to discrete steepest descent by $F(x)$ we call this *force-jumping* insofar as one follows the force vector (note that this jumping construction also defined a transformation of smooth vector fields to smooth maps). By proposition 4.1:

Lemma 4.19. *$z = A\Pi_{Ax}^{-1}(x, v)$ iff $v = F_z(x)$. (here $(x, v) \in TS^2$, $v \in T_x S^2$)*

Here, F_z is the force on x due to z. The antipodal relation of matching to jumping is a special property of the functions $\mathcal{E}_{PW}$, as follows from (4.2) (or of vector fields as in (4.15)).

Corollary 4.20. *$r_{PW}(x) = y$ iff $F_{PW}(x) = F_y(x)$, and we call this* force-matching.

The correspondence in theorem 4.18 has an interesting specialization to real weights, (we use the standard formula for curvature under conformal rescaling of metrics in 2-dimensions, $K_{\phi g} = -c\triangle \log \phi + K_g$);

Proposition 4.21. *Using $exp(|\frac{2}{c}|\mathcal{E}_{ZU})$ as a conformal factor for the round metric g on S^2 gives a flat metric h on the complement Z' and the associated flat connection, D. An inverse correspondence is gotten likewise; a flat metric h on Z' determines a conformal factor $\phi = \frac{h}{g}$ and $-\frac{1}{2}\log\phi$ satisfies the same PDE as the energy $\mathcal{E}$.*

In particular note that metric connections correspond precisely to real weights. The singularities of h at Z give cone singularities with vertex angles precisely $K(S^2)u_i$ at z_i, ie proportional by the total curvature, to the weights, so that $\sum u_i = 1$ by Gauss-Bonnet (one can calculate the conformal metric at a cone point using the pullback of the metric $|dz|^2$, by $z \mapsto z^\alpha$). The singularities of D at Z give monodromy equal to $\sqrt{-1}u_i$ at z_i. (see [38] for much more on the polyhedral interpretation of D.) See section 8.4 for a subtle aspect of the relation of these metrics to equivariance.

One should by now expect that connection-matching is essentially the same thing as force-matching, and that the equivalence associates the one form η, such that $D = D_{S^2} + \eta$, to the (force) vector field F, in the usual way. There may be a constant factor intervening, based on normalizations, but otherwise this follows by equivariance properties, notably invariance with respect to affine reflection in a line. It is good to know that a metric relation between holomorphic objects just barely hides a purely holomorphic relation between these objects.

4.1.2. *Informal summary of Poly to $\mathcal{L}$ correspondence.* The homogeneous polynomial, $p : \mathbb{C}^2 \to \mathbb{C}$ with $n = \deg(p)$ is a section of $\tau^{(-n)}\mathbb{C}P^1 = T^{*(\bigotimes n/2)}\mathbb{C}P^1$. Thus $p^{2/n}$ provides local holomorphic sections of $T^*\mathbb{C}P^1$, ie holomorphic one forms, *well-defined up to locally constant factors* away from the roots $r_i \in Z$ of p, and they can be *regarded as locally flat holomorphic affine structures.* Furthermore there is no need to insist on integer exponents (ie multiplicities of factors) for the polynomials; a standard use of logs shows that for any divisor WP with *complex* weights of total mass 1, $\Pi(z - p_i)^{w_i}$ (with $z - p_i \leftrightsquigarrow z_1\xi_2 - z_2\xi_1$ in homogeneous form) provides *well-defined* locally flat holomorphic affine structures as above, with singularities at the roots p_i.

The natural transformation ((2.4)) from polynomials to rational maps becomes the special case, for maps with symmetric weights, of a correspondence of $\mathcal{L}$ to Rat. In fact theorem 4.18 extends this to a transformation from local systems to rational maps, which is ONTO SRat (simple fixed-points), see also subsection 2.1. The correpondence $p^{2/n}$ of Poly to $\mathcal{L}$ here is based on the *abstract* isomorphism $\tau^{(-n)}\mathbb{C}P^1 = T^{*(\bigotimes n/2)}\mathbb{C}P^1$ (via the pencil theorem) and we do not know of a concrete geometric realization of the latter, even for $n = 2$ (it is well defined *only* up to a factor in $\mathbb{C}^*$). Thus it is nice to get the simple expression in proposition 4.22. Passing from one-forms to connections kills this $\mathbb{C}^*$ ambiguity and makes possible such formulae as we proceed to show.

4.1.3. *homogeneous polynomials and flat affine connections:* Given a homogeneous polynomial, $p : \mathbb{C}^2 \to \mathbb{C}$ with $n = \deg(p)$, the roots of p determine a connection $D_p \in \mathcal{L}$, on $\mathbb{C}P^1$ with symmetric weights as above. Let L be an affine complex line $0 \notin L \subset \mathbb{C}^2$. In view of the chain of correspondences above, we should express $D_p = D_L + \eta$ where D_L is the standard flat affine connection of L, and η is of the form $C(d\log p_L)$ where p_L is the restriction of p to L, and C is to be determined. Considering the simplest case, $p(x) = (x_1y_2 - x_2y_1)$ and $L = \{x : (x_1\lambda_2 - x_2\lambda_1) = 1\}$ choose C st the poles of $D_p = D_L + \eta$ cancel at $x = \infty$, (ie such that D_p extends smoothly across $x = \infty$ in L), noting that p_L blows up at ∞ in L (unless $p_L = const$, in which case $x = \infty$ should be the singularity of D_p, and $D_p = D_L$; $x = \infty$ is the root of p in this case). The general case follows likewise, (by factorization, and averaging, as above) with $c_n = \frac{c_1}{n}, c_1 = C$, and we get the formula:

Proposition 4.22. *The local system for a divisor Z with symmetric weights, $\frac{1}{n}$, has connection $D_Z = D_L + \frac{c_1}{n} d \log p_Z$ where p_Z is a homogeneous polynomial with simple roots Z.*

Remark 4.23. A more direct approach to this formula might be to solve $D_p p = 0$, ie p is parallel wrt D_p.

4.2. **Discussion: Natural correspondences.** To summarize the structure that has developed: we have studied analytic objects which correspond to divisors:

analytic object	type of weights
polynomials	symmetric
energy functions	real
singular flat-connections and Rat	complex

The divisors are formal algebraic objects, and they determine formal correspondences or embeddings between the associated analytic objects, but in each case there is a more direct geometric construction of the correspondences between analytic objects which leads to localized constructions and extension to smooth categories etc. It is at this level that the correspondences have some interesting applications (see remark below on symmetry breaking). It is essential that the analytic objects are conformally intrinsic (or almost so, in the case of energy–see section 8.4); they are determined by LINEAR elliptic PDEs with the divisors as forcing terms (except Rat). Note also that passing to the smooth categories amounts, in some but not all cases, to passing to the limit of divisors converging weakly to smooth measures. This issue arises many times in section 8.

There is a 3rd level, a sort of meta-theorem (not meant to be formalized) which explains these geometric constructions and in particular the relation of elliptic centrums to differentiation, this is the *matching* construction: given such an analytic object, f, the germ of f at a point p in the support of its divisor is represented by a Taylor series. The lowest order terms might depend only on self-interactions of p, (the multiplier at a fixed-point for example), but the next term, τ, in each case depends on the AVERAGE influence of the other p_j. *Matching* amounts to comparing this value of τ to that due to a single point q; in an object of the same class, f_2, whose associated divisor has support exactly p, q, we get the same value of τ at p. Because these objects are conformally intrinsic, they are well behaved with respect to automorphisms , and this provides a 1-1 correspondence of points $q \in \mathbb{C}\mathrm{P}^1 - p$ to values of τ. Thus there is a unique q with the same τ as the "average influence of the other p_j".

Aff(7): We can recover the affine structure $S^2 - z$ by embedding $S^2 - z$ canonically to affine subspaces of algebraic functions or one forms; (i) embed $S^2 - z$ to degree 1 Poly canonically; $x \in S^2 - z \rightsquigarrow p_{xz}$ with one root, x, and normalized st $p_{xz}(\bar{z}) = 1$, where $\bar{z} \in \mathbb{C}^2 - 0 \to S^2 = \mathbb{C}\mathrm{P}^1$ is a fixed representative of z. For the analogous embedding to one forms, choose p_{xz} st $p_{xz}(s) = 0$ iff $s = x, z$, and normalize, in addition, $dp_{xz}(\bar{z}) = v \neq 0$. Note invariance of the normalization under averaging. This induces an affine structure on $S^2 - z$ in an equivariant manner. The essential point is linearity of the PDE (d-bar) and holomorphic dependence on the divisor. Though this only works in dimension 2, one might exploit the fact that higher dimensional affine structures can be characterized in terms of their 2-dimensional affine subspaces (closely related to characterizing flatness in terms of 2-dimensional affine subspaces) to obtain a general version of this construction.

4.2.1. *Applications of natural correspondences.* We list some observations, results and potential applications of the existence of these correspondences:

(1) Recall that Gauss-Bonnet is "conjugate", (2.1, 4.18) under such a transformation to the holomorphic Lefschetz fixed-point theorem for Rat. Similarly we will see an equivalence of topological Lefschetz fixed-point theorem to the Morse theory equality for the Euler class, for maps of degree $-d$ on $\mathbb{C}\mathrm{P}^1$, in section 5.1. The question of multivariable analogs arises.
(2) Compactifications of Rat and of $\mathcal{L}$ as roots collide are quite different. We briefly considered bubbling and surgery in this context, section 2.1. Each space has a different class of canonical algebraic subvarieties, it may be interesting to compare them. Similarly the natural invariants in each space are quite different; spectra for polyhedra, or zeta functions for Rat, for example. One wonders if there are interesting relations of some such invariants?
(3) We will show, in section 8, how to identify smooth self maps of the sphere with certain singular measures on the sphere. This might be useful in studying the topology of mapping spaces.
(4) Extracting roots of polynomials involves a symmetry breaking process, which obstruct existence of formulae for roots, while transforming polynomials to rational maps or potential-energy functions can be done without any symmetry breaking in a very simple way. We will see below, section 6, how this gives iterative methods for extracting roots.
(5) The local minima of the energy functions have been much studied numerically, but its difficult to give conceptual proofs of any of their properties. Transforming to the context of holomorphic functions and spherical harmonics, or rational maps may help.

One more observation to mention in passing; there is a well-defined, but singular holomorphic quadratic differential ξ_{ZU}, representing the distortion of projective structure of the conformal map, (identity map) from the round projective $\mathbb{C}\mathrm{P}^1$ to the locally affine-flat hence projectively flat $(\mathbb{C}\mathrm{P}^1 - Z, D)$. It is also holomorphic in the parameters (Z, U). It is easy to see that,

Proposition 4.24. *The zeroes of ξ_{ZU} correspond to the critical points of r_{ZU} .*

4.3. **A Moment for $\mathcal{E}$.** Notice that we never claim that local minima of energy are in 1-1 correspondence with period 2 attractors for $\mathcal{F}$, but only with fixed-points of $A\mathcal{F}$. This because the period 2 attractors have different equivariances, they are O(n,1)–orbits, whereas local minima etc are O(n)–orbits. It is reasonable to expect that the orbits are in 1-1 correspondence; we have essentially shown that an O(n)–orbit of local minima of energy is the reduction of an O(n,1)–orbit of period 2 attractors of $\mathcal{F}$, but the converse is not treated here–we haven't shown that every period 2 *attractor* of $\mathcal{F}$ is of this type (having antipodal behavior etc.) though we'd readily conjecture it is so. (On the other hand there may well be unstable period 2 sets, possibly even with support on S^1, that don't correspond to higher index critical points of $\mathcal{E}$.)

It turns out that there is a particularly nice way to characterize the O(n)–sub-orbits of energy-minimizing configurations; the *vanishing dipole moment property* for equilibria is a special relation between $\mathcal{E}$ and the Moebius action:

Definition 4.25. The dipole moment of a set of n *vectors,* $z_i \in Z \subset S^{N-1} \subset R^N$, is $\Updownarrow(Z) = \frac{1}{n}\Sigma z_i \in \mathbb{R}^N$, (vector addition). (the latex code for $\Updownarrow$ is { \mathcal m} !). This, for $N = 3$, is the standard moment map, for the standard SO(3) action preserving the standard volume form on the 2-sphere.

By a calculation observed in [2], which generalizes easily to the weighted case, and arbitrary dimension,

Proposition 4.26. *If Z is a critical point for $\mathcal{E}_U$, then $\Updownarrow(Z) = 0$.*

Proof. The total of all forces at z_i from all pairs is clearly zero (Newtonian action-reaction). But in equilibrium, forces F_i at z_i are orthogonal to the sphere, hence $F_i{=}\lambda_i z_i$, and it suffices to show that $\forall i, j, \lambda_i = \lambda_j$, but this is clear from the following lemma. □

Lemma 4.27. $x \cdot \nabla \log|x-y| = \frac{1}{2}$.

Proof. Rewrite the l.h.s. as $x \cdot (x-y)/|x-y|^2 = (1 - x \cdot y)/(2 - 2x \cdot y)$. □

4.3.1. *dipole moment in terms of $\mathcal{E}$, and hyperbolic center of mass.* (The rest of this section is "optional", ie it is not applied in the rest of this article). We recently came across the article [28] which surveys many structures that involve configurations similar to those studied here (albeit only with a vector of weights–weighted nodes rather than a matrix of edge-weights) and it may be interesting to compare further the structures there with those presented here. In particular they includes remarks about the *moment map* in relation to the hyperbolic center of mass (which had been exploited in earlier work) and a similar application to normalization of Z by $\Updownarrow(Z)$. Note that the moment map is by definition symplectically intrinsic whereas the hyperbolic center of mass is conformally intrinsic. (The group action implicitly provides conformal structure to the symplectic theory). We have defined $\Updownarrow(Z)$ here using the extrinsic euclidean structure.

We know of no simple analogue of the property (4.26) for other energies. It is a bit mysterious at first that the function $\mathcal{E}$ has the properties of (4.1) as well as (4.26), with no apparent relation of their derivations. Recall that (4.6) explains how the properties of (4.1) fit together, but the relation here doesn't have any apparent connection to them. We proceed to discuss the geometry underlying this coincidence. In the process we will find some interesting relations of the hyperbolic center of mass to $\mathcal{E}$, here and in section 8.

First note,

Theorem 4.28. *$\mathcal{E}$ restricted to an SO(N,1) orbit is well-defined, proper and convex on SO(N,1)/SO(N) $= H^N$. The local minima form a single SO(N) orbit.*

Comparing the exhaustion function $\mathcal{E}$ used here, to other treatments of the Douady-earle centre, it is interesting to notice that $\mathcal{E}$ is proper on all of H^N, and not just a totally geodesic subspace, even when Z is in the boundary of such a subspace.

Proof. First consider the case $N = 3$ where we can use the complex structure on S^2; $\mathcal{E}(Z)$ is strictly plurisubharmonic in Z, and well-defined along an orbit as a function of SO(N,1)/SO(N) $= H^N$, using isometry invariance of $\mathcal{E}$. Now for $N = 3$ strict plurisubharmonicity along with the vanishing of both the gradient and the hessian along SO(N) orbits, which are totally real in configuration space, implies positivity of the hessian along H^N, hence strict convexity. Note that this uses that the $\underline{p} + \underline{k}$

decomposition satisfies $\dim \underline{p} = \dim \underline{k}$, which only holds for $N = 3$, and that the complex structure J satisfies $J\underline{p} = \underline{k}$ in this case, using the equivalence to psl(2,$\mathbb{C}$). We now observe that $\mathcal{E}$ restricted to an SO(N,1) orbit is proper; as $\gamma \in$SO(N,1) goes to infinity, subconfigurations collapse at the attracting fixed-point on ∂H^N, and the last statement of the theorem follows immediately. The case $N = 2$ follows by restriction to the associated subspaces.

The general case, $N \geq 3$, follows because we can reduce both to (a) convexity of $\mathcal{E}$ in the case of 2 points, $x_i \in X, |X| = 2$, (by linearity; $\mathcal{E}$ as a sum over pairs) and (b) along one geodesic γ in H^N. Furthermore we can suppose γ is stabilized by a *nonrotational* 1-parameter group called λ_t. Now the 2 fixed-points $\lambda_\pm \in \partial H^N$ of λ_t, and $\{x_i\}$ give 4 points which are all in some round $S^2 \subset S^{N-1}$, which is furthermore, (using *nonrotational)* λ_t–invariant, so we can apply the preceeding arguement, ($N = 3$). □

Comparing the exhaustion function $\mathcal{E}$ used here, to other treatments of the Douady-earle centre, (using Busemann functions, see also (8.20) and related discussion below) its interesting to notice that $\mathcal{E}$ is proper on all of H^N, and not just a totally geodesic subspace, even when Z is in the boundary of such a subspace. This reflects the introduction of the auxiliary round metric in the construction of $\mathcal{E}$.

This motivates us to next establish the identification of $\Updownarrow(Z)$ with $\nabla\mathcal{E}_{|GZ}(Z)$, where G is SO(N,1), and GZ is the G–orbit of Z. This will strengthen the existence of normalizations, and it shows that

Proposition 4.29. *In the O(N,1)–orbit of a critical point for $\mathcal{E}_U$, $\Updownarrow(Z) = 0$ iff Z is a critical point.*

Note that this implies existence and uniqueness of the hyperbolic center of mass as in [7].

To this end we identify the Lie algebra, $\underline{g} = \underline{p} + \underline{k}$ of SO(3,1) with vector fields v on S^2, and on $(S^2)^n$, $|Z| = n$, by the diagonal action, and restricting v to $T_Z = T_Z(S^2)^n$, we get $\underline{g} \to T_Z$ and the orthogonal projection $\pi_Z : T_Z \to g$. Our goal is to relate $\pi_Z(F(Z))$, where $F = -c\nabla\mathcal{E}$ is the usual force, to $\Updownarrow(Z)$. Noting that $F \perp \underline{k}$, (by isometry invariance of $\mathcal{E}$) we may as well define $\pi_Z : T_Z \to \underline{p}$.

We denote by $\pi_{ave}F = \sum_i F_i(Z)$, ie the average, in $\mathbb{R}^3$, of net forces $F_i \in T_{z_i}S^2 \subset \mathbb{R}^3$, and note that by the proof of (4.26) including (4.27) or the more intrinsic approach below (4.32),

Lemma 4.30. $\pi_{ave}F(Z) = c\Updownarrow(Z)$.

But note that π_{ave} is also well-defined on $\underline{p} \subset T_Z$, and we claim that π_{ave} is an isomorphism of $\underline{p} \to \mathbb{R}^3$, up to the trivial exceptional case where $|Z| = 2$, of an antipodal pair. But given $\underline{v} \in \underline{p}$, $\exists v \in \mathbb{R}^3$, st $\forall x \in S^2, \underline{v}(x) = \pi_x v$ using the orthogonal projection to T_xS^2. Now $\langle \pi_{ave}\underline{v}, v\rangle \geq 0$ (each summand is clearly nonnegative) and can vanish only in the exceptional case noted above which we can exclude. Thus we get a nondegenerate quadratic form, and conclude that π_{ave} is an isomorphism.

Our goal is to show that $\nabla\mathcal{E}_{|GZ} = \pi_Z(F(Z)) = 0 \Leftrightarrow \Updownarrow(Z) = 0$, and it now clearly suffices to check that $\pi_{ave}\pi_Z(F(Z)) = \pi_{ave}F(Z) = c\Updownarrow(Z)$.

Once more the key is to consider the case of a pair of points; $|Z| = 2$, and average. The averaging step is just a bit more subtle here; we write $F = \sum F_{ij}$, F_{ij} being the contribution only from the interaction of the pair z_i, z_j, so $\pi_Z F = \sum \pi_Z F_{ij}$ and

$\pi_Z F_{ij} = \pi_Z \pi_{Z_{ij}} F_{ij}$ since $F_{ij} \in T_{Z_{ij}} \subset T_Z$ (more to the point, $F_{ij} \in \underline{p} \cap T_{Z_{ij}} \subset T_Z$ by ((4.2))). $T_{Z_{ij}}$ is the $\mathbb{C}^2-$ subspace of the pair z_i, z_j. Thus it suffices to study $\pi_{Z_{ij}} F_{ij}$, ie to show $\pi_{ave}\pi_Z(F(Z)) = \pi_{ave}F(Z)$ for $|Z| = 2$.

Furthermore, by symmetry considerations (or equivariance of $\mathcal{F}$ in (4.32)), for $|Z| = 2$ it suffices to consider the S^1 case, and in the S^1 case, with $|Z| = 2$,

$$\pi_Z(F(Z)) = F(Z), \tag{4.2}$$

so we are finished. In fact,

Theorem 4.31. *$\pi_{ave}^{-1}\Updownarrow(Z) \in \underline{g}$ is equal to $\nabla\mathcal{E}_{|GZ}(Z)$ up to a constant factor.*

The upshot is that our normalization criterion has an expression intrinsic to the theory previously developed.

Kirwan discusses in great depth the relation of $\nabla|\Updownarrow|^2$ to the structure of configuration space, [22].

Remark 4.32. Now in the interest of proving (4.30) more intrinsically note that we already know this from the relation of $\mathcal{E}, \mathcal{F}$; for $|Z| = 2$, one reads off the forces at Z, from $\mathcal{F}(Z)$, to explicitly describe F in terms of A, Π as in section 4 (for example (4.4)); note thus that $-2\Updownarrow(Z) = Az_2 - z_1 = Az_1 - z_2 = \frac{1}{4}(F_1(Z) + F_2(Z)) = \frac{1}{2}\pi_{ave}F$. This gives the same conclusion , in general, for $|Z| = n$, by averaging.

5. Energy and Stability, stable attractors for $\mathcal{F}$.

Our main goal in this section is to produce attracting periodic cycles for $\mathcal{F}$ using the minima of $\mathcal{E}$.

5.1. one-variable case and energy.

We summarize some basic facts about the space, ARas , of anticonformal maps of $\mathbb{C}\mathrm{P}^1$ (ϕ_{ZU} is the notation used here instead of r_{PW}, $A(x)$ is antipodal to x). Note that the hessian $\nabla^2\mathcal{E}_{ZU}(x)$ is symmetric with trace$=\frac{1}{2}$, while $dA\phi_{ZU}(x)$ is symmetric with trace=0 using the conjugation. This suggests a simple correspondence between the two. We will discuss eigenvalues and eigenvectors of the hessian, though the term characteristic values may be more precise. It may be better to think of $\nabla^2\mathcal{E}_{ZU}(x)$ as a map $H : v \in T_x \to T_x$, defined by $\nabla_v(\nabla\mathcal{E})$.

Theorem 5.1. *Given a rational map $\phi_{ZU} : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1 = S^2$, with real weights,*

1. *x is a critical point for $\mathcal{E}_{ZU}(x) = c\Sigma u_i \log(|x - z_j|)$ iff $\phi_{ZU}(x) = A(x)$.*
2. *x is a strong local minimum for the energy $\mathcal{E}_{ZU}$ iff x is a* STABLE *fixed point for the conformal (but antiholomorphic) map $A\phi_{ZU}$. In fact the hessian $\nabla^2\mathcal{E}_{ZU}(x)$ determines the linearization $d(A\phi_{ZU})(x)$ and vice vs. The eigenvalues $\pm\lambda$ of $dA\phi_{ZU}(x)$ are real, and*
3. *at a local minimum, (or more generally any critical point) the hessian has eigenvalue zero iff $A\phi_{ZU}$ has a parabolic (multiplicity > 1) fixed-point at x, and $\lambda = 1$,*
4. *If x is a degenerate local minimum (one eigenvalue $= 0$) for the energy $\mathcal{E}_{ZU}$ then $A\phi_{ZU}$ has a nontrivial open parabolic basin of attraction (in fact the same holds for any degenerate critical point).*
5. *the hessian has a double eigenvalue, iff $A\phi_{ZU}$ has a superattracting (critical) fixed-point at x.*
6. *If $|Z| = n > 3$, then x (a local minimum or degenerate critical point) must attract a critical point of ϕ_{ZU}, ie there is a critical point p of ϕ_{ZU}, such that the forward orbit of p by $A\phi_{ZU}$ converges to x.*

We have already proved the first statement above, see (4.5), also (2.10). The 2nd iterate $A\phi_Z^{(2)}$ is holomorphic , so to prove the fourth statement above,we can apply the Leau-Fatou flower theorem, [29] .

$A\phi$ is a steepest-descent method for minimizing $\mathcal{E}$, but with large step size, so this does not imply stability. The key points underlying stability are thus (4.6), and (5.7). In fact, one can extend the constructions of ϕ and $\mathcal{E}$ to higher dimensional spheres, and the correspondence of fixed-points of $A\phi$ to critical points of $\mathcal{E}$ still holds for log-energy, but we will need extra work to determine higher dimensional stability.

5.1.1. *Critical point structure.* Our goal here is to apply the preceeding theorem, together with some Morse theory applied to $\mathcal{E}$, to clarify the dynamics and fixed-point structure of $A\phi_{ZU}$. Two interesting things arise as side-benefits; we can apply the critical point structure of ϕ_{ZU} to get extra information on the critical points of $\mathcal{E}$, and the Morse theory turns out to be equivalent to the topological Lefschetz fixed-point theorem for $A\phi_{ZU}$.

(1) To get the Morse theory started it is very convenient to be able to restrict to the case of nondegenerate hessians. This is standard in the category of smooth functions , where generically h is Morse. In the finite dimensional space of degree n maps $ARas_n$, it is not quite clear that generic fixed-points are nondegenerate, but it suffices to find a single nondegenerate fixed-point in each connected component of the fixed-point variety; using real analyticity, the nondegenerate maps are then open and dense. Transforming back from ARas to $\mathcal{E}$, (as above) we have the analogous fact for critical points of $\mathcal{E}$, it suffices to find a single nondegenerate critical point in each connected component of the critical point variety . But given a critical point x of $\mathcal{E}$, x is critical for any $PW \in Div$ with the same center of mass $0 \in \mathbb{CP}^1 - x$ without loss of generality . This is clearly a connected set, (assuming positive weights) and it is easy to pick a configuration with nondegenerate hessian at x, for example at roots of unity, ie with cyclic symmetry around $x, 0$. In particular the generic local minimum is a strong local minimum, and generically $\mathcal{E}_{ZU}$ is Morse. We will suppose that $\mathcal{E}_{ZU}$ is Morse in what follows.

(2) By subharmonicity, the poles at Z are (or we consider them to be) the only local maxima of $\mathcal{E}_Z$, (ASSUMING POSITIVE WEIGHTS). Let m_i be the number of critical points of $\mathcal{E}$ of index i, (Morse numbers), then ($\mathcal{E}$ is a Morse function and) $n - m_1 + m_0 = 2$, and $m_0 > 0$ so $m_1 \geq n-1$. (Note that this is the topological Lefschetz fixed-point theorem for the corresponding map). But m_1 counts repelling fixed-points of $A\phi$.

Proposition 5.2. *$A\phi$ has at least $n-1$ repelling fixed-points . (assuming positive weights)*

(3) Having 3 fixed-points of $A\phi$ in the complement of a basin of attraction implies the last statement of (5.1) by hyperbolicity [3], this aspect of the theory of rational maps generalizing to the (antiholomorphic) conformal branched cover at hand. (or use the holomorphic 2nd iterate).

(4) One can generalize (5.1) to the case of complex weights, so that certain sinks of the vector-field F_{ZU} correspond to attractors of $A\phi_{ZU}$, this is pursued further in the multivariable stability discussion (remark 3).

By the last statement of (5.1),

Theorem 5.3. *For a given $Z \in (\mathrm{CP}^1)^n - \mathcal{D}$, there are at most $2n-4$ local minima of $\mathcal{E}_Z$, (supposing $n > 3$) and this is sharp.* (*assuming positive weights*)

This is the number of critical points available. It coincides exactly with the number of triangles in any triangulation of S^2 with vertices Z. SZ^n, $n > 7$ ((2.5)), realizes the maximum as can be seen by symmetry considerations; one needs enough points on the equator to rule out equatorial local minima. In cases where $\exp \mathcal{E}$ is a polynomial, this may be re-expressed in terms of properties of local maxima of spherical harmonics.

One can establish that the local minima of $\mathcal{E}_{SZ^n}$ are nondegenerate by equating this to nondegeneracy of (attracting) fixed points for the antiholomorphic $A\phi_{SZ^n}$, now one can qualitatively construct the trace of the map along a triangle (triangulation as above) with adjacent vertices in SZ^n, and from this one calculates the degree to be -1.

Using the Morse theory above, and (5.1),

Corollary 5.4. *$A\phi_{ZU}$ has at most $3n-6 = 3|d|-3$ repelling fixed-points , and $\mathcal{E}$ has at most $3n-6$ saddles.* $(n = d+1 = |Z| = \deg(\phi)+1 = -\deg(A\phi)+1$ *)*

5.1.2. *one variable stability, proof of (5.1).* By theorem 4.9 , we immediately get;

Lemma 5.5. $0 < \nabla^2 \mathcal{E}_{ZU}(x) \iff \nabla^2 \mathcal{E}_{ZU}(x) < \frac{1}{2} I$.

By differentiating the construction in (4.4),

Lemma 5.6. *$d(A\phi_{ZU})(x)$ (recall, notation ϕ_{ZU} here, instead of r_{PW}) is determined at its fixed points by the (real) Hessian, $H = \nabla^2 \mathcal{E}_{ZU}(x)$ and vice vs.*

We present the main features of this relation here for the one variable case and in the next section for $\mathcal{F}$. It will be useful to have a more explicit formula; take $x = N$, $Ax = S$, (at the poles) without loss of generality , and fix the bases e_i at T_N, T_S, from $\mathbb{R}^3$, so $dA = -I$. The Hessian H of $\mathcal{E}$ at N is symmetric hence diagonal over $\mathbb{R}$ with respect to this basis without loss of generality . Noting that Π_x, Π_{Ax} and the Hessian of $\mathcal{E}$ are simultaneously diagonalized, by theorem 4.4, we now show $(dA)d\phi(x)$ is also diagonalized. Noting that

$$A\phi(x) = \Pi_{Ax}^{-1}(x + F_Z(x)) = \Pi_{Ax}^{-1}(x - \nabla \mathcal{E}_Z(x)),\ \textit{and}\ d[\Pi_{Ax}^{-1}(x)] = I,$$

applying the chain rule, first note that the term due to ∇(Ax) vanishes; $\nabla \mathcal{E} = 0$, so we are projecting $x \in S^2$ to S^2, we can extend the projection maps to $\mathbb{R}^3 - Ax = \mathbb{R}^3 - y$ canonically, and as y varies, the image $\Pi_y x = x$, trivially stays fixed. Thus,

$$dAd\phi(x) = d[\Pi_{Ax}^{-1}(x - \nabla \mathcal{E}_Z(x))] = I - \lambda \nabla^2 \mathcal{E}_Z(x) \tag{5.1}$$

the factor λ coming indirectly from the curvature of S^2 and various unspecified (not explicitly) normalizations. Note that (5.1) gives a monotonicity property for diagonal entries, α_i, the eigenvalues of $d\phi(x)(\xi) \cdot dA\xi$, as a function of diagonal entries, the "eigenvalues" β_i of $\nabla_\xi^2 \mathcal{E}_{ZU}(x)$, and we conclude that $-1 < \alpha_i < 1$, ie $dAd\phi(x)$ is contracting, precisely when β_i lies in an interval which we proceed to determine.

Proposition 5.7. *Given a fixed-point x of $A\phi_{ZU} = A\phi$ as above, $A\phi$ is attracting at x iff $0 < \nabla^2 \mathcal{E}_{ZU}(x)$, x is also a critical point of ϕ (superattracting at x) iff $\nabla^2 \mathcal{E}_{ZU}(x) = \frac{1}{4} I$.*

Proof. Note that the following uses the localization of the correspondence of vector fields to maps, defined in the extension of $\mathcal{R}$ from (4.1),

Lemma 5.8. *Given x as above, and $\xi \in T_x$ such that $\nabla^2_\xi \mathcal{E}_{ZU}(x) = 0$, then $d(A\phi)(x)\xi = \xi$.*

Proof. First consider the stronger case of a curve γ such that $\mathcal{E}$ is constant on γ. Using theorem 4.4 $A\phi$, is clearly the identity map on γ. But in the case at hand, $\nabla_\xi \mathcal{E}_{ZU}(x) = \nabla^2_\xi \mathcal{E}_{ZU}(x) = 0$, so $A\phi$ is the identity at least to 1st-order, implying $d(A\phi)(x) = 1$. □

Lemma 5.9. *Given x as above, if $\frac{1}{2}$ is an eigenvalue of H, and $\xi \in T_x$ such that $\nabla^2_\xi \mathcal{E}_{ZU}(x) = \frac{1}{2}$, then $dAd\phi(\xi) = -\xi$.*

Proof. This follows immediately from the preceeding lemma, the orientation reversing property, and (4.9,4.14). □

Now we can use the lemmas to fix the sought for β_i interval, and we conclude that $dAd\phi(x)$ is contracting precisely when $0 < \nabla^2_\xi \mathcal{E}_{ZU}(x) < \frac{1}{2}$. A *strong local minimum* is one where the hessian is non-degenerate and we've shown that a strong local minimum, $\mathcal{E}_{ZU}(x)$, determines a smooth map $A\phi$ whose linearization at x is attracting. The converse follows by (5.6). Also, $\nabla^2 \mathcal{E}_{ZU}(x) = \frac{1}{4}I$ gives a critical point by symmetry considerations, or ((5.1)). □

Finally we return to the...

5.2. **MULTIVARIABLE CASE; stability for $\mathcal{F}$.** For the first-order theory, relating $A\mathcal{F}(Z) = Z$ to $\nabla\mathcal{E}(Z) = 0$, the single variable theory is better than a mere analogy; we can simply apply it componentwise to each $z_i \in Z$, in fact componentwise force-matching (the construction in (4.4)), provides a natural correspondence relating the force, $-\nabla\mathcal{E}_U(Z)$ whose components are force vectors $F_i \in T_{z_i}S^2$, to the components of $\mathcal{F}$.

But for the 2nd-order theory, relating $dA\mathcal{F}(Z)$ to $\nabla^2\mathcal{E}(Z)$ the task is (a priori) more complicated because the components, z_i of Z, and the diagonalization of $\nabla^2\mathcal{E}(Z)$ are not necessarily aligned. Nevertheless, differentiating the construction in (4.4), we see that, as in (5.6), the 1-1 correspondence, of $dA\mathcal{F}(Z)$ to $\nabla^2\mathcal{E}(Z)$, for configurations $Z \subset \mathbb{C}\mathrm{P}^1$ holds in the multivariable case.

As with ϕ_Z we see that $\mathcal{F}$ is a discretized gradient flow for $\mathcal{E}$, (i.e. a steepest descent method) but again, this does not directly imply stability.

Another complication is the SO(3,1) equivariance of $\mathcal{F}$, and the associated SO(3) equivariance of $\mathcal{E}$, but this is easily handled using section 4.3:

Definition 5.10. The notions of (i) strong local minimum for $\mathcal{E}$, i.e. nondegenerate $\nabla^2\mathcal{E}(Z)$, (ii) stable fixed point for $A\mathcal{F}$, and (iii) basins of attraction for $A\mathcal{F}$ or $\mathcal{F}^2$, should all be interpreted with the natural equivariant actions quotiented out, i.e. they allow for degeneracies, along orbits by these actions.

Our goal is now to prove (3.12)(ii), in particular (generalizing to the weighted case);

Theorem 5.11. *Given $\mathcal{F}$, $\mathcal{E}$ as above, Z is a critical point for $\mathcal{E}_W$ iff $\mathcal{F}(Z) = A(Z)$. Furthermore Z is a strong local minimum for $\mathcal{E}_W$ iff Z is a stable fixed point for $A\mathcal{F}$.*

Note that $A\mathcal{F} = \mathcal{F}A$, so $(A\mathcal{F})^{(2)} = \mathcal{F}^{(2)}$ is holomorphic . A version of theorem 3.12 with any real weights follows similarly. One needs positive weights to guarantee the existence of local minima.

Proof. The first statement follows from the first part of (5.1) and the remarks above, preceeding the theorem.

For the second statement, we extend the correspondence of positive hessians to attracting $dA\mathcal{F}$, from the single variable to the multivariable case. In fact the same technique of proof goes through, (for mixed derivatives) and one gets, $dAd\phi(Z) = I - \lambda\nabla^2\mathcal{E}(Z)$, where the derivative of the Az_i term vanishes as before. (One should check that the constant λ will be the same for mixed derivatives, this is also clear from remarks below on neutral directions.) □

The latter, the derivative of the Az_i term, which represents the nonlinearity due to curvature of the sphere, could show up in higher derivatives, for example in certain approaches to the case of degenerate fixed-points .

At a local minimum of $\mathcal{E}$ the eigenvalues of the hessian H are all weakly positive, so as in (5.5) they are always in $[0, \frac{1}{2}]$. Furthermore *the proofs of* lemmas (5.8,5.9) applied in every complex direction provide a correspondence between neutral directions for the hessian of $\mathcal{E}$, ie those with $\beta_i = 0, \frac{1}{2}$, and neutral directions for $d\mathcal{F}$, $\alpha_i = \pm 1$, (notation as above). Note that a neutral direction is *extremal* in the class of weakly positive hessians, so it must correspond to an eigenvector (using the Rayleigh-Ritz criterion), likewise for neutral directions in weakly attracting $dA\mathcal{F}$. We can immediately conclude that the connected subset of hessians $\mathcal{H}^+$ with all eigenvalues in $[0, \frac{1}{2}]$, must map to the connected subset of linearized maps, Λ^+ of $\mathrm{End}(T_x, T_x)$ (of the type of $dA\mathcal{F}$) with all eigenvalues in $[-1, 1]$, and likewise for the inverse direction.

In fact the boundaries, such as $\partial\mathcal{H}^+$, the neutral regions, decompose into connected components which are completely determined by the *number* of eigenvalue ± 1's they have (likewise for $0, \frac{1}{2}$); this uses a $k + p$ type decomposition of the associated lie algebras, and groups, and the fact that the hessians and maps have full bases of eigenvectors. They thus cut the full matrix spaces up into connected pieces characterized by the number of eigenvalues of each type.

(1) There is a generalized correspondence between the signature of fixed-points of $A\mathcal{F}$ and index of critical points of $\mathcal{E}$, for any index.
(2) The conjecture (3.14) might well be extended to the case of positive weights, or even to claim that $\mathcal{E}_W$ is Morse, in particular that all local minima are strong. Almost nothing, beyond numerical work, is known of the classification of all local minima. We thus discuss weak local minima below. They are bound to occur in families as weights vary over $\mathbb{C}$.
(3) We sketch modifications that provide sufficient criteria for existence of attractors for the more general case of complex weights. As in the single variable $\mathbb{C}\mathrm{P}^1$ case there is no global energy function, but a force-field F, which is locally exact off the singular set, where it has periods. Decompose the matrix ∇F as a sum of S, the symmetric and Q, the skew(-hermitian) form; diagonalize over $\mathbb{C}$ and separate real $\Re$, and imaginary $\Im$ parts of the eigenvalues λ_i. The necessary and sufficient condition for an attractor is $|\lambda_i - \frac{1}{4}| < \frac{1}{4}$, by (5.1) which by (4.9) is equivalent (for $\Re\lambda_i < \frac{1}{4}$) to the "one-sided condition" $\Re\lambda_i > \frac{1}{4} - (\frac{1}{16} - (\Im\lambda_i)^2)^{\frac{1}{2}}$, generalizing the real case.

5.2.1. *Appendix: Weak local minima.* This appendix contains a somewhat detailed description of problems involved in establishing the existence of a Fatou set.

Conjecture 5.12. *The Fatou set of $\mathcal{F}$ is nontrivial.*

where the *Fatou set* is an open set defined to be the maximal domain of normality, (ie where iterates form a normal family). Note that Fatou sets can have periodic dynamics, but we actually expect a *strong* Fatou set, ie one whose domain of normality has a stronger characterization; the convergence of iterates to constant maps. This is based on the observation that eigenvalues of the linearization are real (3.13).

Various different approaches to this conjecture are evident,

(1) ruling out degenerate local minima as in conjecture (3.14), we will not have more to say on this here, or
(2) extending the analysis of nondegenerate hessians to the degenerate case. In its full generality, normal form theory for degenerate maps in higher dimensions is very technical and difficult, a reserve of the specialist, but one can expect our special assumptions to simplify things somewhat. There is probably a good chance of ruling out degenerate local minima in the case of real weights and no clamped points, but these singularities can arise for other maps, and it is certainly interesting to extend to the nondegenerate case.
(3) We note also that our use of the term *"lyapunov function of $\mathcal{F}$", for $\mathcal{E}$,* is somewhat unconventional insofar as we haven't shown the lyapunov property holds a priori, and not even that it holds on full basins of attraction a posteriori, though we do use $\mathcal{E}$ to produce these basins. (The term *"lyapunov"* is jsutified by the steepest descent involved). If we could prove the lyapunov property without first producing a fixed-point then the distinction of degenerate and nondegenerate hessians would not arise. This brings up the question of convexity of $\mathcal{E}$ at degenerate local minima; a degenerate local minimal variety of $\mathcal{E}$ is totally geodesic if $\mathcal{E}$ is convex, but one can create functions (similar to $\mathcal{E}$) locally with minimal variety lying on a circle (in $\mathbb{C}$) for example. Convexity of $\mathcal{E}$ at degenerate local minima is likely important, if not the main point in approach (2) above.

The specific characteristic (direction) along which one should look for contraction is defined by choosing a real-analytic curve $\gamma(t)$ with endpoint x for which $\mathcal{E}$ increases *as slowly as possible*; we break the Taylor series expansions of $\mathcal{E} = \sum \mathcal{E}^i$, into its homogeneous parts, in the inductive step choose γ_k , to pass through a local minimum ξ of $\mathcal{E}^k$ on the unit sphere. If $\mathcal{E}^k(\xi) \neq 0$ we can stop; the connected local minimal set will be an attractor in some blow-up (roughly as in the nondegenerate case). If not, the curve $\gamma(t)$ will attract in directions transversal to $T_\xi^{\mathbb{C}}\gamma$, but we have to look at higher k to get retraction in $T_\xi^{\mathbb{C}}\gamma$ ie along the complexification of γ (which might have larger dimension if the local minimal set is large). To see that this procedure stops, we need to show there is some maximal $k(\gamma)$ for which $\nabla_t^{2+i}\mathcal{E}(x) = 0,\ i < k$. The lemma below will only reduce this to another conjecture. Note that the study of dynamics near such a curve $\gamma(t)$ would normally involve blowing-up and applying center-manifold theorems by induction on degree in the Taylor series expansions of $\mathcal{E}$ and $\mathcal{F}$, at x.

The technical aspect will certainly be reduced if we can establish that local minima are isolated. By analyticity, they come in smooth families, so it suffices to show;

Lemma 5.13. *(*CONJECTURAL*) If γ is a real-analytic curve with endpoint x (x as above) and $\mathcal{E}$ is constant along γ, then γ is in an $SO(3)$-orbit.*

Proof. In some nbhd of x, every $y \in \gamma$ is a local minimum, hence period 2 for $\mathcal{F}$, so there is a compact *holomorphic* curve, V of period 2 points extending γ. γ is not in an $SO(3)$-orbit iff V is not in an $SL(2, C)$-orbit, by (4.29). But $V \cap \mathcal{D}_2 = \emptyset$, in $(\mathrm{CP}^1)^n$, because $\mathcal{F}$ is repelling at $\mathcal{D}_2$, (as in (3.4)), contradicting $Z \in V$ being period 2 for $\mathcal{F}$.

Thus $V \cap \mathcal{D}_3$ is nontrivial. This entails the existence of *degenerate period 2 configurations* for $\mathcal{F}$ as limiting cases at $\mathcal{D}_3$, so we have reduced the problem to the following: □

Conjecture 5.14. *There are no degenerate period 2 configurations for $\mathcal{F}$.*

Now we describe *degenerate period 2 configurations.* In the limit where a part $Z_1 \subset Z$ of a configuration degenerates, the map $\mathcal{F}$ degenerates into what can be described as a weakly coupled pair of rational maps, where *the coupling just keeps track of the conservation of mass*, and that in an (almost**–see below) arbitrary way, as far as we are concerned here. In full generality we should consider a finite set of configurations, analyzed along the lines of the Fulton-MacPherson compactification of moduli space, with subclusters nested more deeply, but for the essential points it suffices to consider the case where Z breaks up into 2 configurations, corresponding to $Z_1, Z_2 = Z - Z_1$, which we call $C_* = C_N, C_S$ and which, without loss of generality live at or near the poles, $N, S \in S^2$.

There is a special (one might say *virtual*) point, $s \in C_N \supset Z_1$, which is not a point of Z, but which represents all of Z_2, ie the subset in C_S (so $s = S$ in effect). This being (for Z_1) the "missing part" of Z, the weights, $W(C_N)$, are inherited accordingly, and vice vs, for $n \in C_S$, etc. So $C_N - s$ is in a small nbhd of N, without loss of generality .

$\mathcal{F}$ is defined on $C_* = C_N, C_S$ using the weights $W(C_N)$,$W(C_S)$, as usual, and we could call the restrictions $\mathcal{F}_N, \mathcal{F}_S$. But the images D_N, D_S, may have a different subconfiguration structure, which we now outline.

$C_N - s = C_{NN} \cup C_{NS}$ where $\mathcal{F}_N$ sends C_{NS} to S, ("the evicted part" or "defectors" if one likes) and $C_{NN} \to D_N \subset S^2 - S$, and vice vs. for $N \longleftrightarrow S$. Note that the evicted part is generically going to be empty, but there are certainly cases where it is nontrivial. Also there are maps $\mathcal{F}^* : C_{NS} \to D_S \subset S^2 - N$ so the image $\mathcal{F}(C_*) = D_* = D_N, D_S$, where D_N is the union of the 2 pieces, $\mathcal{F}_N(C_{NN}) \cup \mathcal{F}_S(C_{SN})$ just given, as well as the special point, s, representing the *updated* missing part of Z. The $\mathcal{F}$ image of D_* is defined likewise.

(** There is at least one obvious restriction on $\mathcal{F}_N$ on C_{NS}; it must vary holomorphically, though the space of configurations for which C_{NS} is nontrivial will be of very small dimension. We hope (and expect) that the conjecture above does not require any special knowledge of this latter map.)

In brief the D_N image of C is determined by $\mathcal{F}$ on C_N, as usual, together with an (essentially) arbitrary piece corresponding to the part of C_S that $\mathcal{F}$ evicted.

6. Suspension

We use the term *suspension* to refer quite generally to an operator from a space of maps, such as $Ras \subset Maps(\mathbb{C}\mathrm{P}^1 \circlearrowleft)$ (the arrow indicating self-maps), to an enriched space of maps, such as $GRas \subset Maps((\mathrm{CP}^1)^n \circlearrowleft)$. We discuss a very specific such construction here, based on $\mathcal{C}$en, with applications to algorithms for finding roots of polynomials of one variable, or periodic points of $r \in Rat$. It doesn't have any special relation to other suspension-notions in dynamical systems; for Kleinian groups or Riemann surface laminations. But we can begin by noting that the diagonal lifting $r^{\triangle}_{PW} : (\mathrm{CP}^1)^n \circlearrowleft$ of r_{PW} is defined by $r^{\triangle}_{PW}(z_1, \ldots, z_n) = (r_{PW}(z_1), \ldots, r_{PW}(z_n))$, and that the *suspension will combine* $r^{\triangle}_{PW}$ with $\mathcal{F}_U$, acting on the same space, where *combination* is the $\mathcal{C}$en based construction in (4.18). In fact, by combining each component $\mathcal{F}_i$ with $r = r_{PW}$ with appropriate weights, W, we get a map, denoted $S(r, W) : (\mathrm{CP}^1)^n \circlearrowleft$ which under iteration is attracted to repelling fixed-points of r. This doesn't *explicitly* allow for clamped points, they could be allowed also, but note that the fixed-points of $r(x)$ are implicitly playing the role of clamped points here.

We use notation as above, but with special weights $w_{i,r} = w_{ir}$ for r;

Theorem 6.1. *Let*

$$S(r, W) = S(r, k; W)(x_1, \ldots, x_k)_i = \mathbf{M}_{x_i}^{-1}[w_{i,r}\mathbf{M}_{x_i}(r(x_i)) + \sum_{j \neq i} w_{ij} M_{x_i}(x_j)]$$

where M_x is Moebius as in (3.6). Then $S(r, W) = \mathcal{F}_{PW'}$ where $r = \phi_{PU}$, w_{ij} for indices i, j, of X are the same for both $S, \mathcal{F}$, and $w_{ik}(\mathcal{F}) = w_{ir}u_k$ for indices i of X, and k of P, where $w(F)$ indicates a weight of the map F.

Suppose that $k \le \deg(\phi) + 1 = n + 1$ and that $\phi = \phi_P$ has symmetric weights, then

$$S(\phi, k)(x_1, \ldots, x_k)_i = \mathbf{M}_{x_i}^{-1}[n\mathbf{M}_{x_i}(\phi(x_i)) + \sum_{j \neq i} \frac{1-n}{k-1} M_{x_i}(x_j)]$$

has $P_k \subset P$ as an attractor, for any part with $|P_k| = k$.

In particular, if $\phi = R_p$, as in ((2.4)) then S has a basin of attraction, convergent to a superattracting configuration of roots of p.

Proof. The weight formulae follow directly from definitions, we are just taking affine linear combinations of divisors under the equivalence in (4.18), w_{ir} represents a rescaling factor for the weights of the divisor of r. Recall that a clamped point q_i in $\mathcal{F}(X; Q; W)$ is superattracting for x_j iff their joint weight is one. This is the case in the *weighted suspension* of R_p constructed above. Note that weights W of the x_i, x_j interactions have been chosen to set the weight-sums equal to one, as required, (using the known weights of R_p). □

Note also: x_i, x_j are mutually repelling (near the diagonal).

These seem to provide good candidates for generally convergent purely iterative algorithms (on a large dimensional space) for finding roots of polynomials of one variable, [27]. With a little more effort the technique generalizes to finding repelling (and all) fixed or periodic pts of any $r(x)$ in Rat of $\mathbb{C}\mathrm{P}^1$; we need a rough approximation of the multiplier of the periodic point to be able to set weights for S appropriately. Most such points lie in the Julia set, J, and their multiplier is close to some central value, determined by the Hausdorff dimension of J, and the period.

Doyle and McMullen constructed R_P in their work on 1-variable purely iterative Rats. They noted explicitly that R_P doesn't converge to roots; they are *repelling* fixed points. They were interested in the possibility of iterating a single-variable map to solve polynomial equations, while respecting certain group actions, see also [5]. We have shown that using the averaging technique of $\mathcal{C}$en it is easy to overcome this problem in the multivariable context, with appropriate weights for $S(r, k; W)$.

The existence of an algorithm for putting $r \in Ras(\mathbb{C}\mathrm{P}^1)$ in normal form, (2.1), by constructing the suspended $S \in Ras((\mathrm{CP}^1)^n)$ to find its fixed-points , is also interesting wrt the Smale model of computation over the reals, insofar as it is viewed as a self-contained model of computation; the suspension provides a construction using only rational functions, (in several variables) that finds normal forms of one-variable rational maps. Note that the suspension technique treats the input r as a "blackbox"; we never need to formally compose or manipulate it, but only use evaluations of $r(x)$.

7. Algebraic-geometry of $\mathcal{F}$.

It should be possible to find an explicit expression for $\mathcal{F}$ as a self-map of Poly, since no symmetry breaking occurs at this level. We will provide a closed-form expression of this type, using resultants, sylvester matrices etc. We also find the (generic topological) degree of $\mathcal{F}$ below, on $(\mathrm{CP}^1)^n$, (though our proof is not really algebraic) and it is clearly the same on $\mathbb{C}\mathrm{P}^n$; preimages generically won't have permuted pairs. An easy remark relating $\mathcal{C}$en to dual curves is also included for motivation–our other goal here is to begin a discussion relating constructions here to representation theory; many coincidences that have so far been observed involving notably $\mathcal{C}$en, differentiation, discriminants, duality, R_p of ((2.4)), will seem more natural in this light.

7.1. F **as resultant .** Our goal here is to explain and prove the following formula for $\mathcal{F}$ as a transformation of Poly

$$\mathcal{F}(p(z)) = q(\lambda) = RSLT_z(p(z), \lambda p''(z) - zp''(z) - 2(1-n)p'(z)), \tag{7.1}$$

where the sylvester determinant realization of the resultant (RSLT) has to be constructed to reflect the natural degrees of the polynomial entries; the leading order terms in the summands of the second entry *do not cancel* due to the factor of 2, (the *non-cancellation* should probably be related to dilation equivariance of $\mathcal{F}$) but one can cancel leading order terms between the two resultant entries (as for the standard discriminant construction) and we discuss below the correct total degree in the coefficients proposing that it is 2n-3. Note that this is not explicitly an equivariant expression; there is a choice of coordinates both in taking derivatives and calculating resultants. Nevertheless we provide more intrinsic descriptions of this map below, and in (3.9).

We first give a closed-form expression for $\mathcal{F}(Z)_i$ in terms of the polynomial, p, and z_i such that $\forall i, p(z_i) = 0$, but it does not satisfy the symmetry demands stated above; we use the individual roots z_i in Z, explicitly,

$$\mathcal{F}(Z)_i = z_i + 2(1-n)p'(z_i)/p''(z_i) (=^{def} g(z_i)). \tag{7.2}$$

The easiest proof is to rewrite p in terms of factors, and formally use the sum $\frac{d}{dz}\log(p'(z_i))$ which simplifies alot since $p(z_i) = 0$.

Note that p, h have a common zero iff

$$(7.3) \qquad RSLT(p,h) = \Pi_{p(z)=0}(h(z)) = 0,$$

so the explicit regrouping of this product expression is just the usual sylvester determinant expression of the resultant. Now the polynomial $q = \mathcal{F}(p) = \Pi(\lambda - \mathcal{F}(Z)_i)$ has the form of a quotient of resultants;

$$\Pi_{p(z)=0}(\lambda - g(z)) = \Pi_{p(z)=0}(\lambda p''(z) - zp''(z) - 2(1-n)p'(z))/\Pi_{p(z)=0}p''(z) =$$

$$= RSLT(p, \lambda p''(z) - zp''(z) - 2(1-n)p'(z))/RSLT(p,p'').$$

In the RHS, λ should be regarded as part of the coefficients of a polynomial in z, and it enters the sylvester determinant as such. The polynomial $q(\lambda)$ sought doesn't depend on the denominator–the latter gives just an irrelevant $\mathbb{C}^*$ factor of q, and dropping it greatly simplifies computation of q. On the other hand this factor seems necessary if one is interested in $sl_2\mathbb{C}$ equivariance as opposed to $psl_2\mathbb{C}$ equivariance (see remarks below on such lifts).

Thus

$$\mathcal{F}(p(z)) = q(\lambda) = RSLT_z(p(z), \lambda p''(z) - zp''(z) - 2(1-n)p'(z)).$$

$\deg p = d$ guarantees that $\deg q = d$ as is clear from ((7.3)).

7.2. **Topological Degree of $\mathcal{F}$.**

Theorem 7.1. *The degree of $\mathcal{F}^n$ ($= \mathcal{F}$ on $(\mathrm{CP}^1)^n$) is $(n-2)!$.*

Proof. We calculate the degree on $(\mathrm{CP}^1)^n$, by induction on the number, n, of points. Since the map is meromorphic the generic point has degree d preimages. Recall that $(n-2)!$ has been checked (section 3.1) for $n = 2, 3$. Suppose $\mathcal{F}^n(Z) = Y$ has $(n-2)!$ solutions. For the inductive step, we introduce a generic new variable point y_{n+1} in the configuration Y, and count solutions of $\mathcal{F}_t^{n+1}(Z, z_{n+1}) = (Y, y_{n+1})$, using a continuity (homotopy) method, with t parametrizing weights $W(t, n+1)$, chosen such that the new point is introduced with weight close to zero, *i.e.* $w_{i,n+1}(t, n+1) = t/n$, and other weights close to those of $\mathcal{F}^n$, so for $t << 1$ solutions are generically (see section 2.1 remarks on this) only perturbed slightly. Since y_{n+1} is degree $n-1$ in z_{n+1}, (2.1), $\mathcal{F}_t^{n+1}(Z) = Y$ has $(n-1)!$ solutions for t small. We homotope $\mathcal{F}_t^{n+1}$ to $\mathcal{F}^{n+1}$, $t = 1$, by choosing W and Y so as to avoid the real codimension 2 set on which $\mathcal{F}_t$ can degenerate. (ie $\mathcal{D}_3$ which is complex codimension 2 is blown up and the image is complex codimension 1, see section 3.6). $\square$

It is interesting to compare to Bezout's theorem; this is not just the product of degrees of components, suggesting some analytic dependency among the component equations.

7.2.1. *duality.* We present here a natural application of $\mathcal{C}$en to the geometry of dual curves, and though the result itself is not particularly new, we include it because there are probably more interesting constructions along these lines–and we hope that duality might shed more light on the geometry of $\mathcal{F}$.

Given a curve $V \subset \mathbb{C}\mathrm{P}^2$ of degree d, one can enrich the standard dual map $f : V \to V^*$ by providing a canonical rational map $f_V^* : \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^{2^*}$ which extends f to $\mathbb{C}\mathrm{P}^2$.

Let $p \in \mathbb{C}\mathrm{P}^2$, let L_{tp} be the associated pencil of lines through p, t being the angular coordinate. For each p, t let $L_{tp} \cap V = Z_{tp}$, and let $\phi_{tp} : L_{tp} \to L_{tp}$ be the map $\mathcal{C}\text{en}_Z; Z = Z_{tp}$. Recall that this is well-defined even though the degree of ϕ_{tp} can jump when Z_{tp} degenerates. Now setting $W(V,p) \subset \mathbb{C}\mathrm{P}^2$ to be $\{\phi_{tp}(p) : t \in \mathbb{C}\mathrm{P}^1\}$, W defines a compact curve, with one point for each t, hence,

Lemma 7.2. *$W(V,p)$ is a line, depending holomorphically on V,p.*

Thus we have canonically associated to V a map $\mathcal{G}_V(p) = W(V,p)$ of the complement $V' \subset \mathbb{C}\mathrm{P}^2$ to the dual space, $(\mathbb{C}P^2)^*$.

Theorem 7.3. *$\mathcal{G}_V(p)$ extends across V except possibly at singularities and flex points, such that $\mathcal{G}_V(p)$ restricted to V is precisely the standard dual map $f : V \to V^*$. Furthermore $\mathcal{G}_V$ determines V uniquely.*

"Sketch of proof": Given p close to $x \in V$ such that x is not a flex point, let L_1 be the line through p and x ("orthogonal to V") and L_2 be the line through p but parallel to T_xV. The image p_1 of p associated to L_1 lies close to x, hence close to T_xV, while the image p_2 of p associated to L_2 lies far from x, as the 2 pts of $L_2 \cap V$ near p will cancel, but it is close to T_xV. Thus $L(p)$ which is determined by the p_i is close to T_xV. This proves that not only does the extension exist, but that it is precisely the map of V to its dual curve V^*. "qed"

In fact, the map we construct is well known, it coincides with the one-form obtained by *differentiating* the homogeneous defining function of V. Compare ((2.4))!

7.3. **Representation theory of rational maps via geometric plethysm.** Many of the intrinsic constructions of algebraic geometry are clarified when viewed from the point of view of representation theory, and the term *geometric plethysm* [15] refers to the aspects of this involved in decomposing symmetric powers into irreducibles. Since these projections provide multilinear maps they can be used to construct equivariant rational maps– this is implicit in the treatment of geometric plethysm in [15] for example. In our case the relevant Lie algebra is $sl_2\mathbb{C}$, and homogeneous degree d polynomials $Poly_d\mathbb{C}\mathrm{P}^1 = Sym^d Poly_1\mathbb{C}\mathrm{P}^1 = \mathcal{O}(1)\mathbb{C}P^{d*}$ (up to a blow-up at 0) are the irreducibles, $V = Poly_1\mathbb{C}\mathrm{P}^1$ denotes the standard/fundamental irreducible. Thus discriminants, derivatives and dual maps arise quite naturally as invariants or covariants of the group and a variety of identities that have arisen here can be considered from this viewpoint– and some can be seen as consequences of the machinery. Note though that $psl_2\mathbb{C}$ equivariant maps of $\mathbb{C}P^{d*}$ are not quite the same as covariants of $sl_2\mathbb{C}$; there is a meromorphic factor (a multiplicative 1-cocycle with values in meromorphic functions) that intervenes in lifting from maps of $\mathbb{C}P^{d*}$to maps of $\mathbb{C}^{d+1}$, which it seems should vanish in cohomology for topological reasons. While we will not discuss this in general here, one can remark it in the examples.

The powerful counting/combinatorial techniques available from representation theory reduce certain identities to systematic (but not always easy) dimension counts, though it is not clear to us to what extent they provide explicit algebraic or geometric identifications in the process (in practice...this is tied up in the shift from 19th century "classical" invariant theory to modern representation theory). This approach thus complements the geometric understanding achieved so far, but we do not think it subsumes it. We will only give a few examples and illustrations of its relevance to our constructions here (while pleading lack of expertise on the general theory).

Note that preimages of discriminants by $\mathcal{F}$ and its iterations $\mathcal{F}^{(-k)}\mathcal{D}$ define invariants quite easily, (we have no idea how this relates to the classical literature) also forward images of the (blow-ups of) more degenerate higher discriminants, eg $\mathcal{D}_k$,k>2, by $\mathcal{F}$ provide other invariant constructions relevant elsewhere in the paper, (such as (5.14)). It is natural to ask then, what generalizations or extensions of the constructions presented so far, here, are possible, and representation theory/geometric plethysm should be an indispensable guide. Recall also the questions about characterization of $\mathcal{F}$ in section 3 as another source of motivation.

As a preliminary (and primordial) example, notice that the correspondence $R : Poly_d \to Rat_k, k = d-1$ can be viewed as a map $P(R) : Sym^dV \otimes Sym^kV \to V$ by first lifting a degree k map $r : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ to $r^h : V \to V$, in homogeneous coordinates, then r^h is the restriction of a k-multilinear map $p(r) : Sym^kV \to V$, (the "polarization") to the diagonal (a veronese type construction). But $Sym^dV \otimes Sym^kV \otimes V$ is dimension 1 iff $k = d \pm 1$ and dimension 0 otherwise, (see the introduction to [33] and the formula for $Sym^kV \otimes Sym^lV$ in [15]). Thus R_p which corresponds to $k = d-1$ is singled out as the only possible such construction *linear* in $Poly_d$. $k = d+1$ it turns out, corresponds to a trivial construction once projectivized; $z \mapsto p(z)z$ projectivized gives the identity map. This also provides a systematic way to look for higher degree, equivariant rational map constructions, ie nonlinear $Poly_d \to Rat$.

Notice also that this effectively isolates/characterizes the $\mathcal{C}$en construction; the $d = 2$ case gives a midpoint construction, ie evaluating R_p at x gives the midpoint of the roots of p with respect to $A(x)$. In fact, in the product $Sym^dV \otimes Sym^kV \otimes V$ the first factor gives the configuration P, the 2nd factor corresponds to x using a diagonal restriction, and the 3rd factor is the midpoint. Recall that *affine structure* is characterized by functional equations (Aff(1) after (1.2)) and this could be related to the uniqueness above for $Sym^dV \otimes Sym^kV \otimes V$, d=4, thus providing a first step in translating every $sl_2\mathbb{C}$ intrinsic construction of the article to the present language.

One of the main goals should obviously be to situate $\mathcal{F}$ (as a map on $\mathbb{C}\mathrm{P}^n$) in $(Sym^kSym^nV) \otimes Sym^nV$, and in the process to pin down the algebraic degree of $\mathcal{F}$. It seems to us likely that the latter is $k = 2n-3$ and furthermore that $\mathcal{F}$ belongs to an irreducible factor F of $Sym^kSym^nV \otimes Sym^nV$ where F corresponds to the Young diagram/partition $2n-3 = (n-1)+(n-2)$ via the Littlewood-Richardson or Pieri enumeration of the factors of $(Sym^nV)^{\otimes k}$. (One might use the Schur functor associated to this Young diagram to furnish a multilinear map, whose restriction to the diagonal in $(Sym^nV)^{\otimes k}$ is the polynomial map desired). This is in part motivated by the explicit formula ((7.1)) above . It would also be interesting to know if there is a nice way to describe $\mathcal{F}$ in the context of the theory of [17].

A final observation along these lines is motivated by some "numerology"; comparing the appearance of $2n-4$ critical points of $R_p = f/g$ in (5.3), $n = \deg(p)$, to the 2nd irreducible factor $Sym^2Sym^nV \to Sym^{2n-4}V$. It is obvious that the degree 2 rational map of $Poly_n \to Poly_{2n-4}$ has to be the one that takes $p \mapsto q$ where $q(z) = 0 \iff R_p'(z) = 0$, as this satisfies the required equivariance etc. In particular q is quadratic in p insofar as it depends on the vanishing of $gf' - fg'$. Symmetry is implicit; the full projection is recovered by polarization which gives symmetric maps by definition–the key point is nontriviality on the diagonal. Note also that we introduce an *auxiliary variable* z to define $p \mapsto q$ *implicitly* in terms of p. This suggest that such constructions are not to be had by any easy systematic means.

It would seem to be a difficult challenge to find analogous constructions for the $Sym^{2n-8}V$ factor not to mention the rest of the sequence for Sym^kSym^nV,k=2 –this begins to touch on classical invariant theory [6], already for $n = 4$. In fact the difficulty of systematically producing these geometric constructions leads us to suspect that the appearance of the relatively simple construction of $\mathcal{F}$ is more the exception than the rule. For higher k the combinatorics of the decomposition become more complicated, and though the extra flexibility of large k makes it easy to find some intuitive geometric constructions of factors, we would guess (very superficially) that these are likely to represent a vanishingly small fraction of the full decomposition.

8. The hyperbolic case

8.1. Divisors in the equator; extending maps and hyperbolic centrum.

We will consider maps whose associated divisor is contained in a subsphere, $S^N \subset S^{N+k}$, they are especially interesting in relation with hyperbolic centrums.

8.1.1. *Extending maps and degree.* We first define embeddings of mapping spaces $Ras^N \to Ras^{N+k}$, or more specifically, extensions of maps, obtained simply by pushing forward the associated divisors, we will assume that weights are real;

Lemma 8.1. *Given $r \in Ras^N$, and a round, (section 4.0.4) embedding $S^N \to S^{N+k}$, let PW be the associated divisor, ie $r(x) = \mathcal{C}\text{en}(PW, x)$ on S^N. Then using the embedding to pushforward PW, define $r = r^{N+k}(x) = \mathcal{C}\text{en}(PW, x)$ on S^{N+k}. This* extends *r to S^{N+k}. If $k = 1$, and weights are real and positive, then r maps each complementary hemisphere $H_\pm = H^{N+1}$ of S^{N+1} to its opposite hemisphere.*

The last point follows easily using convexity properties, S^N is A_x–convex to the opposite hemisphere of x. Note that $r(x) = \mathcal{C}\text{en}(PW, x)$ does determine PW uniquely, the correspondence of weights to multipliers does extend to higher dimension even though the maps are not everywhere conformal, (8.4). Thus the extension is *somewhat* canonical; it's not clear that a composition of such maps gives the same extension when definable, ie composition and extension may not commute. This is reminiscent of the quasiconformal reflection situation. In our case, for N=k=1 though they do commute, by analytic continuation. For N=2 we don't know, but (2.1) assures us that at least the composition has a divisor and, if the associated divisor has real weights, its associated extension exists. For $N = 3$ compositions rarely have a divisor; the fixed-points do not have conformal linearizations, (compare (8.9)), and the question becomes mute.

It is not quite trivial to calculate the degree of a map in Ras^N unless $N = 1, 2$ (N is the dimension, and it amounts to calculating the degree of $\mathcal{C}\text{en}(PW, x)$); already for $N = 3$ the orientation information at preimages of a point might differ from point to point. For positive real weights, W, standard homotopy arguements show that degree only depends on the cardinality of the configuration P, $\deg(r_{PW}) = d(|P|)$. Similarly if there are exactly κ positive weights, then $\deg(r_{PW}) = d(|P|, \kappa)$. It is only for $N = 2$ that we may use $W \in \mathbb{C}^d$ to homotope around $w_i = 0$.

Proposition 8.2. *Given $r \in Ras^N$ with positive real weights W,* $\deg(r_{PW}) = d(|P|) = (-1)^N(|P| - 1)$.

Proof. Using (8.1) with $N = 2, k = 1$, we see that by homotoping P into $S^2 \subset S^3$, $r^{-1}(x) \subset S^2$ for $x \in S^2$ and that the orientation at each preimage has *negative* orientation; along S^2 it's positive, but the last point of (8.1) provides a negative factor transverse to S^2. By induction on dimension, $S^N \subset S^{N+1}$ the general case follows similarly. $\square$

Problem 8.3. What is $d(|P|, \kappa, N)$?

Applying the topological lefschetz fixed-point theorem should suffice in view of (8.4).

8.1.2. *Dimension > 2: the weight-multiplier relation, properties of $\mathcal{E}$, etc.;*

Proposition 8.4. *Let $\mathcal{E}_{PW} : S^N \to \mathbb{R}$ as above, and $\phi(x) = \mathcal{C}\mathrm{en}_{PW}(x)$ be the associated self-map. Note that (4.5,4.1(ii)) generalize directly to $N > 2$. At poles x of $\mathcal{E}$, $d\phi(x)$ is conformal, and the weight to multiplier relation is the same as in (2.1).*

Note though, that for $N > 2$, f is certainly not conformal on open sets (by Darboux's theorem).

Proof. For $N = 2$, $\mathcal{E}$ is of the form $\lambda \log|z| + smooth$ where λ is the weight at $p = 0$, thus it is the residue λ of $\mathcal{E}$ that determines the multiplier of $\phi(x)$, as in (4.4,2.1). But this residue to multiplier relation can be applied in the higher dimensional case, (in the context of (4.5)) along any 2-dimensional subspace spanned by eigenvectors of $d\phi(x)$. Since it provides the same eigenvalues on each such slice, $d\phi(x)$ is conformal, and the N-dimensional weight to multiplier relation follows.

Another way of seeing this is to break up the measure $PW = p_1 w_1 + P'W'$ to 2 parts, and apply the combination principle (4.18); this reduces $d\phi(p_1)$ to the case of $|P| = 2$, $P = \{p_1, \mathcal{C}\mathrm{en}_{P'W'}(p_1)\}$, hence, for a given $X \in T_{p_1}S^N$, to the S^2 case; $P \subset S^2$ and $X \in T_{p_1}S^2$. $\square$

We next present the higher dimensional extensions of (4.1)(i) and its consequences that will be of use later.

Proposition 8.5. *Let $\mathcal{E}_{PW} : S^N \to \mathbb{R}$ as above, and let $\mathcal{G}$ be the grassmannian of 2-planes in TS^N, then the hessian of $\mathcal{E}_{PW}$ on $S^N - P$ has trace $tr(\nabla^2_{|U}\mathcal{E}_{PW}) \leq \frac{1}{2}$ for every $U \in \mathcal{G}$.*

Proof. It suffices to prove this for P having one point, $P = \{p\}$, by calculating $\Delta\mathcal{E} \leq \frac{1}{2}$ along any "great 2-sphere" ie round 2-sphere $S^2 \subset S^N$, of maximal radius. A useful trick here is to exploit the positive curvature *form* of S^2 as in (4.18,4.21); $\mathcal{E}$ is a conformal factor for a *flat* metric on $S^N - p$, and $\Delta\mathcal{E}$ determines the curvature *form,* which restricted to S^2 gives either a flat plane when $p \in S^2$, or a round sphere when $p \notin S^2$, in a euclidean space $E^N = S^N - p$.

Furthermore, the map from the round $S^2 \subset S^N$ to the image, in E^N, is a moebius transformation M (conformal automorphism) which enjoys an S^1 symmetry around the extremal points of the S^N distance $d(x,p) = d_p(x)$ along S^2. We thus claim that the extremal points of $\Delta\mathcal{E}$ along S^2 are the same pair; the S^1 symmetry reduces this to monotonicity of $\Delta\mathcal{E}$ in d_p, ie the monotonicity of $(M^*(vol) - vol)/vol$ where vol is the volume (and curvature) form of each S^2. Thus it suffices to check this for $M : \mathbb{C} \to \mathbb{C}, M(z) = \lambda z, \lambda \in \mathbb{R}$, with $vol = \frac{|dz|^2}{(1+|z|^2)^2}$ which boils down to $\log(1 + |\lambda z|^2) - \log(1 + |z|^2)$ monotone in $|z|$, an exercise.

But it is easy to see that the minimum, x_-, of distance to p is in the p-hemisphere, and $\Delta\mathcal{E}(x_-) \leq 0$, and the maximum, x_+, being in the opposite hemisphere, $\Delta\mathcal{E}(x_+) \leq \frac{1}{2}$; since S^2 is totally geodesic it suffices to note that at the extremal points TS^2 is in an eigenspace for the hessian and to use facts about the $N = 2$ case. Using polar coordinates one easily sees the eigenvalue for the radial direction $e_r(t)$ is monotone decreasing from ∞ to $\frac{1}{4}$ and the eigenvalue for the orthogonal direction is $e_n(t) = \frac{1}{2} - e_r(t)$ and $e_n(t) = 0$ at the boundary of the p-hemisphere which is clearly a totally geodesic level set. $\square$

Proposition 8.6. *$A\phi_{PW}(x) = x$ iff $\nabla\mathcal{E}_{PW}(x) = 0$ and x is a local minimum iff $A\phi_{PW}$ is attracting at x.*

Proof. This now follows as for $N = 2$, (section 5.1, in fact with a better upper bound). $\square$

8.2. **Measures associated to maps.** To begin the generalization of $\mathcal{C}$en from divisors PW to measures, note;

Proposition 8.7. *Given smooth $f : S^2 \to S^2$, with isolated fixed-points , there is a probability measure μ (complex-valued, $\int \mu = 1$), which is smooth up to a finite set of atoms, st $f(x) = \mathcal{C}\mathrm{en}_\mu(x)$ iff the fixed-points of f are simple and df at fixed-points is (oriented) conformal, and the atoms correspond to the fixed-points , as in (). If μ is smooth then* $\deg(f) = -1$.

Note that the $\deg(f) = -1$ property follows from absence of fixed-points by the Lefschetz fixed-point theorem. In particular such maps cannot be holomorphic . Note that $\mathcal{C}\mathrm{en}_\mu(x)$ is meromorphic in $\mu(y), x$, away from $\mathcal{D}$ ie $\{(x, y) : x = y\}$, much like the case of $\mathcal{C}\mathrm{en}_{PW}(x)$, but with $\mathcal{C}\mathrm{en}_\mu(x)$ not being holomorphic in x, this property fails on $\mathcal{D}$ (in view of separate holomorphicity etc) in contrast to the case of $\mathcal{C}\mathrm{en}_{PW}(x)$. One should also note here that

Lemma 8.8. *Given a smooth probability measure μ on S^2, $f(x) = \mathcal{C}\mathrm{en}_\mu(x)$ is absolutely convergent, hence well-defined. $f(x) = \mathcal{C}\mathrm{en}_\mu(x)$ is also well-defined for μ smooth with isolated atoms.*

and that this distinguishes (in the smooth case) $N = 1$ from $N = 2$. See also remarks before (8.20). The case with isolated atoms is clear by continuity and removeable singularities. It can also be treated by the combination principle, (4.18). Note also that the latter implies

Lemma 8.9. *Given μ smooth with isolated atoms, the weight to multiplier relation for $f(x) = \mathcal{C}\mathrm{en}_\mu(x)$ at fixed-points is the same as in (2.1).*

In particular $df(x)$ at fixed-points is independent of the smooth part of μ, and it is conformal.

Proof. (of (8.7) First f determines a connection D_f on S^2, which as in (4.18), is well-defined and smooth away from fixed-points . The curvature form (or measure) of D_f (constructed using monodromies of D_f as measures of opens) provides a probability measure μ_f, which, under the hypotheses above on fixed-points of f, includes atoms at the fixed-points . In fact one can separate the smooth and atomic parts of the measure by first decomposing the map using the combination principle

$$f(x) = \mathcal{C}\mathrm{en}(P(x), x), P(x) = \{f_a(x), f_s(x)\} \tag{8.1}$$

as in (8.4) into (i) a part $f_a(x)$ (*the atomic part*) with the same fixed-points as f, and in view of (8.9) weights determined by multipliers as in (2.1) up to a normalizing factor (total mass one) and (ii) $f_s(x)$ (*the smooth part*) being the solution to ((8.1)), which will have no fixed-points . In fact $f_s(x)$ can be constructed directly from $f_a(x)$ and $f(x)$ using the combination principle (with a negative weight on $f_a(x)$

to kill the atomic part–continuity has to be checked using (8.8,8.9) and removeable singularities applied where the fixed-points were "removed").

By the linear relation of measures to connections in (4.18) underlying the combination principle, it thus suffices to establish (8.7) in the fixed-point free case, $f = f_s$ and assume μ_f smooth. Also (4.18,4.21), already show that for holomorphic maps $\mathcal{C}\mathrm{en}(\mu_f, x) = f$.

That this holds generally essentially follows from the correspondences in section 4; recall first that smooth maps f with no fixed-points correspond 1-1 to smooth vector fields, v_f, ((4.1)), or smooth connections D_f (4.18) (and these are equivalent as noted after (4.21)). Thus it remains, for example, to show that smooth connections correspond 1-1 to smooth (*complex-valued*) curvature *forms* (it is the curvature form–not the function which one is prescribing, otherwise the PDEs involved would be nonlinear) satisfying the Gauss-Bonnet normalization, but we have seen that these connections correspond faithfully to 1-forms, and the correspondence becomes a standard application of the complex-*one*-dimensional d-bar theory. □

We will now also sketch the relation to Laplace's equation as in section 4. First, we fix an auxiliary round metric on S^2, vol =its volume form, as in section 4 and 5, and restricting to the subclass defined by $curl(v_f) = 0$ there is a 1-1 correspondence of smooth vector fields, v_f, to smooth energy functions $\mathcal{E}_f$ (mod constant summands), and to $div(v_f) = \triangle\mathcal{E}_f$, which is in 1-1 correspondence with the (curvature-form) $= \mu_f = h_f vol = -c\Delta\mathcal{E} vol + \kappa vol$, with κ the curvature of the round metric as in (4.21), and we restrict to the subclass of curvature forms with values in $\mathbb{R}$, as in (4.21), satisfying Gauss-Bonnet. Here we use a density function, $\mu_f = h_f vol$. This provides a proof for the special case of maps $f(x) = \mathcal{C}\mathrm{en}_\mu(x)$ determined by real measures μ.

The proof also shows that the theorem is not only analogous to existence and uniqueness theory for laplace's equation on the 2-sphere, but it is formally equivalent , via adding a round metric in the background. In fact the theorem gives a conformal (metric free) version where the flat affine structures play the role of green's functions. Note that we implicitly use the uniqueness theory for $\triangle\mathcal{E} = \frac{1}{2} - \mu$ to prove that for a given measure there is at most one map. Compare to the technique used in (2.1) where we could easily reduce to finite dimensional spaces.

This leaves some questions involved in weakening the condition on conformality at fixed-points . Note that a smooth map st the linearization at a fixed-point is not conformal will have an associated measure whose smooth part is not absolutely convergent, hence some work is necessary to have a well-defined inversion $\mathcal{C}\mathrm{en}$ associating maps to such measures. These measures do have other interesting properties (integrals vanishing on fundamental domains of the linearization) and could be worth considering.

It is not at all straightforward (not even rectifiable) to prove the correspondence above for smooth measures using discrete approximations; the latter have too many fixed-points , and too large a degree to converge smoothly! It would be interesting to know if there is any geometrically well motivated notion of convergence of maps that corresponds to convergence of measures (ie finitely supported, converging to smooth) in this context? This is related in spirit to bubbling phenomena.

In higher dimensions, maps of S^N determine connections, hence curvature and monodromy at fixed-points . But the inversion that would recover a map, from curvature and monodromy, is less evident, owing essentially to the noncommutativity of the Lie algebra associated to the connections. For $N = 2$ we have seen that one doesn't need any elliptic PDE theory for (stating the) solution for the connection given the curvature form; affine center-of-mass and stereographic projection suffices. On the other hand the $Sym^2 \bigwedge^2$–structure of curvature tensors suggests a possible generalization of the use of complex weights to $N > 2$; the weights can be assigned to every pair of oriented 2-planes, with appropriate bianchi identities in case of sufficient regularity. While we do not pursue this here, we now consider a further development of μ_f in the 2-D case that suggests an easier way to recover the measure from the map.

Recalling that the liouville symplectic form S on $S^2 \times S^2 - \mathcal{D}$ in section 4.0.5 is $SO^+(3,1)$ invariant, it is natural to try to recover μ_f as S evaluated on the graph $Gf \subset S^2 \times S^2 - \mathcal{D}$, up to fixed-points , where the intersection with $\mathcal{D}$ provides atomic contributions. In fact this is almost correct, the point though is that μ_f is a *complex* measure, so we extend to a *complex* valued symplectic form $\omega = \omega_{\mathbb{C}}(X,Y) = S(X,Y) + \sqrt{-1}S(X,JY)$ using the complex structure J on S^2. This is the ("anti-holomorphic ") complex conjugate of the holomorphic extension σ discussed in 4.0.5. Thus given a map f as above,

Theorem 8.10. *$\mu_f = c\omega_{\mathbb{C}}(Gf)$ on the smooth part (up to fixed-points , c is some normalizing constant factor) and this is also the curvature of the canonical connection associated to f.*

Proof. By conformal invariance of both ω and μ_f it suffices to check this for f, x normalized, st $x = n$, (north) and $f(n) = An = s$, by fixing an auxiliary round metric as in section 4,5. Thus the restriction $S_{|Gf} = C \times tr(d(Af))$ for some constant C. On the other hand, recall that in section 5.1 we saw that (at least for maps $f(x) = \mathcal{C}\mathrm{en}_\mu(x)$ determined by *real* measures μ) $d(Af) = I - 4\nabla^2\mathcal{E}$, so $tr(d(Af)) = 2 - 4\Delta\mathcal{E}$, and, in (4.21), that the (curvature-form) $= \mu_f = h_f vol = -c\Delta\mathcal{E}vol + \kappa vol$, so $tr(d(Af)) = a + bh_f$, for some constants a, b. We also remarked in section 5.1 that $tr(d(Af)) = 0$ in the holomorphic case, and in (4.21), that $h_f = 0$ in the holomorphic case, so $a = 0$. But this now gives $S_{|Gf} = Cbh_f$ as desired, (in the case $\mu_f = h_f vol$ real). The proof can be systematically translated to extend to the general case, replacing $\nabla\mathcal{E}$ by the connection-1-form η, $\Delta\mathcal{E}$ by $\overline{\partial}\eta$, etc.

In fact all one really needs, as in section 5, is to remark the existence of an affine relation between $d(Af)$ and $\nabla\eta$, (the latter being the same as $\nabla^2\mathcal{E}$ in the special case). By linearity, it now suffices to find the contraction of $d(Af)$ that corresponds to $h_f = \sqrt{-1}$. But symmetry and uniqueness reduce it to the 2 possibilities, $\pm tr(\langle d(Af)(X), JY\rangle) = \pm S(X, JY)$. Finally, one uses the vanishing of h_f associated to any complex-linear $d(Af)$, as in (4.15), to fix the choice of sign to get the ("anti-holomorphic ") ω as claimed. □

Corollary 8.11. *Holomorphic maps are defined by $(\omega_{\mathbb{C}})_{|Gf} = 0$ ie pointwise vanishing of the pullback form, ie their graphs are Lagrangian for the (anti-holomorphic) complexified Liouville form.*

8.3. **Schwarz-Christoffel type uniformization.**

Definition 8.12. A subvariety V of a manifold is said to be *invariant* for a connection D if $\forall X, Y \in TV, D_X Y \in TV$, we also say V is *totally geodesic* or has the *D–convexity property,* or CP(D), or just CP .

If $p \in S^N \subset S^{N+k}$ then S^N is totally geodesic for A_p, and since

Lemma 8.13. *CP is preserved under averaging of connections. (Averaging must use real weights W, not necessarily positive though.)*

we have shown,

Proposition 8.14. Given any divisor PW in $S^N \subset S^{N+k}$, with real weights W, S^N has the D_{PW}–convexity property. *(*D_{PW} as in (4.18))

One can immediately conclude that for $N = k = 1$, D_{PW} gives a Schwarz-Christoffel type uniformization of the polygonal region whose angles correspond, (4.18), to the weights of W; D_{PW} is flat on each complementary component of S^1 and each arc of S^1 complementary to P is (totally) geodesic for D_{PW}.

The higher dimensional versions similarly give polyhedra with totally geodesic faces, though the regions and faces *are not flat* except when their dimension is N=1,2.

Schwartz-Christoffel uniformization gives a conformally intrinsic center of mass constructions for divisors PW in $S^1 \subset S^2$; the flat convex structure on the disc allows one to assign the D_{PW} affine center of mass to PW. This doesn't generalize easily to higher dimension. We call these *hyperbolic centers of mass.* The Douady-Earle center of PW is another such hyperbolic center, and we do not know if the two are equal (probably not) or somehow related?

8.4. **Polyhedral volume and hyperbolic center of mass.** Note that the connection provided in theorem 4.18 is conformally intrinsic, but the metric in proposition 4.21 DEPENDS a priori on the choice of a METRIC ON THE SPHERE, S^2 and these are parametrized by H^3. In fact it is easy to see that the metric is determined up to a scale factor, $v(h)$ in $\mathbb{R}^+$, (which stands for volume), $h \in SO^+(3,1)/SO(3) = H^3$. More generally we can define $v(Z)$ in $\mathbb{R}^+$on configurations in S^2, assuming some metric, so that $v(h) = v(hZ)$, and show it is non-trivial;

Lemma 8.15. *$v(h) \to \infty$ as $h \to \infty$ in H^3.*

Proof. To see this, note that $v(h)$ is finite iff $\forall i, w_i < 1/2$ and use continuity of v *on configurations* Z in S^2 into $\overline{\mathbb{R}}$; as $h \to \infty$, Z tends to a configuration with 1 or 2 points, hence $\forall i, w_i < 1/2$ fails in the limit. □

Theorem 8.16. *v along an orbit is well-defined and convex on H^3, and the local minimum of v exists and is unique.*

So this also defines a hyperbolic center of mass. The proof uses:

Lemma 8.17. *Given fixed positive weights $v(Z)$ is strictly plurisubharmonic on SDiv, the nondegenerate configurations on S^2.*

Again one wonders if this has any nice relation to the Douady-Earle center? Note the relation of the proof here to that of (4.28), and in particular the method there for extending to higher dimension.

Proof. $v(Z)$ is an S^2–integral of $e^{\mathcal{E}}$ as in proposition 4.21. Fixing a point x at which we evaluate $\mathcal{E}(x; P)$ and varying the configuration, $\mathcal{E}$ is strictly plurisubharmonic in P. Thus v is strictly plurisubharmonic in P. v is well-defined along an orbit as a function of $SO^+(3,1)/SO(3)$= H^3 using isometry invariance of v. Now strict plurisubharmonicity and the vanishing of the gradient and hessian along SO(3), which is totally real in configuration space, implies positivity along H^3, hence convexity. □

8.5. **A sketchy form of a Projective Paley-Wiener formalism.** Our goal here is to introduce a decomposition, of self maps g of S^1, using the combination principle (4.18): $g(t) = \mathcal{C}\mathrm{en}(\{g_+(t), g_-(t)\}, t)$, where $g_\pm$ are themselves maps of S^1 to S^2 which extend to holomorphic fixed-point free maps $g_\pm : H_\pm \to \mathbb{C}\mathrm{P}^1$, of each hemisphere. Perhaps the most natural viewpoint though, is to consider the *connection* D^g on S^1 associated to g as in (4.18). Then the decomposition here provides holomorphic connections $D^g_\pm$ on the hemispheres $H_\pm$, st $D^g = D^g_+ + D^g_-$.

The term *Paley-Wiener* usually refers to decomposition of real or complex functions on S^1 to summands which extend holomorphically (or harmonically) to either hemisphere $H_\pm$ of S^2. In fact this gives one way of decomposing self maps g of a metric space S^1; consider the first derivative as a function and apply Paley-Wiener. The construction here is different in that it is projective intrinsic and requires no metric. The basic idea, which we sketch here, is to associate a divisor in S^2 as we have been doing throughout the paper, to a map g of S^1. Here we must extend g from S^1 to S^2 appropriately, and then split the divisor into its hemispherical pieces.

Given a sufficiently regular self map g of S^1, g is associated to an energy function E_g, in section 4; by this we intend ((4.1)), over $S^1 = \mathbb{R}P^1$, rather than $\mathbb{C}\mathrm{P}^1$. One can extend approximations E_i of E_g to a nbhd (in $\mathbb{C}\mathrm{P}^1$) st $\triangle E_i = c$, but there are well-known global approximation methods that allow us to assume without loss of generality that E_i is defined globally with simple logarithmic poles, which by the same, reversed, $\mathcal{E} \leftrightarrow g$ correspondence , give approximations of g by self maps of $\mathbb{C}\mathrm{P}^1$, g_i, that extend globally as holomorphic maps. Furthermore, reflection symmetry of g implies existence of global reflection symmetric approximations.

We note that the associated divisors PW^i are reflection symmetric, where reflection symmetry of weights means that *weights are complex-conjugate under reflection*; we write $PW^i = PW^i_+ + PW^i_-$ to express the decomposition of PW into pieces in each hemisphere (one can divide weights in half for $p_i \in S^1$). Finally, each part $PW_\pm$ defines a map $g_\pm = \mathcal{C}\mathrm{en}(PW_\pm, x)$ which is well-defined *in the limit* and holomorphic on the hemisphere *opposite the support of its divisor*, and g_i is the combination of $g^i_\pm$ in the sense of (4.18), since this corresponds to the sum of divisors.

In the case of positive real weights we get in addition that $g^i_\pm$ maps each hemisphere to itself, but we cannot expect this in general.

To finish we should pass back to the limit $g_i \to g$, and take limits of the associated PW_i. This leads to the question of the nature of the limiting divisor; whether it is a measure or perhaps a distribution, and how this depends on regularity hypotheses for the original circle map g? We do not discuss these questions here, which is why we referred above to a "formalism". Using a Paley-Wiener decomposition on the level of connections or one-forms, as suggested above, should provide some ready made regularity statements.

8.6. **Schwarz-Pick for $\mathcal{C}$en.** The following is standard;

Lemma 8.18. *Given $S^N \subset S^{N+1}$ there is a canonical reflection map, $\rho : S^N :\circlearrowleft$, (self-map) , ie a conformal involution of S^{N+1} fixing S^N, and equivariant for $Aut(S^N) \subset Aut(S^{N+1})$,* [1, 7].

e.g. use canonical boundary measures associated to the H^{N+1} structures. Applying the canonical reflection ρ to the maps described in the last point of (8.1) provides *self maps* $\rho\mathcal{C}\mathrm{en}_{PW}$ of H^{N+1} and (continuing with notation as in (8.1), $P \subset S^N$)

Lemma 8.19. *If $\forall i, 0 < w_i < 1/2$, then there is a unique fixed-point in H^{N+1} of $\rho\mathcal{C}\mathrm{en}_{PW}$ and it is the Douady-Earle center of PW.*

This follows immediately from the existence and uniqueness of the Douady-Earle center, [7] , especially using the characterization in terms of measures on the unit sphere of T_xH, which is exactly the equilibrium condition involving the dipole moment, in section 4.3. We have given a self-contained proof, using $\mathcal{E}$, in section 4.3.

It is natural to ask then if the Douady-Earle center is an attracting fixed-point for $\rho\mathcal{C}\mathrm{en}_{PW}$, and whether there is an extension of the Schwarz lemma which would prove this? Our goal here is to show this is so, and to discuss some of its consequences –or further questions that this suggests.

Note also that if $\forall i, w_i < 1/2$, then the H^{N+1}-boundary fixed-points of $\rho\mathcal{C}\mathrm{en}_{PW}(x)$ are repelling in $S^{N+1} \supset H^{N+1}$.

This affine construction makes sense for *other symmetric spaces H* of noncompact type; reflection of $x \in H$ can be generalized; it gives a sphere or other space (at ∞), rather than a point, but this still determines an affine structure A_{x^*} on H, (by Harish-Chandra embedding theory, [19]) thus a divisor PW on ∂H determines a well-defined map denoted $\mathcal{C}\mathrm{en}^*_{PW}(x)$ for any H of noncompact type, but we do not know if the extension of the Schwarz lemma is valid in such generality–we only consider real hyperbolic space here.

We restrict ourselves to the real hyperbolic space, and measures with finite support. But in general, any measure can be handled using finite approximations, (using $x \in H^{N+1}$,$P \subset \partial H^{N+1}$) since $\mathcal{C}\mathrm{en}$ is continuous in this larger class. Two extreme cases of the latter are worth noting:

(1) The case of standard spherical measure, μ; for the H^2 case this determines a constant map, it suffices to approximate by measures with support on the roots of unity, and to note that $z^n \to 0$ as $n \to \infty$. One should use this to see that for any *smooth* μ, the image of $\mathcal{C}\mathrm{en}^*_\mu(x)$ is bounded in H^2. For the H^{N+1} case equivariance shows that $\mathcal{C}\mathrm{en}^*_\mu(x) = \lambda_N(|x|)x$ for some function λ. But for $N \geq 2$, one can define the associated map $\mathcal{C}\mathrm{en}(x,\mu) : S^N \to S^N$, see (8.8), and by continuity of extensions this implies that $\mathcal{C}\mathrm{en}^*_\mu(x)$ is proper! A simple calculation shows that the critical dimension is $\frac{3}{2}$, so we would guess that the property of unboundedness holds even for very singular boundary measures in the case $N \geq 2$. It is also interesting to relate this to existence of the affine connection on S^N associated to singular μ, see (8.7) for the smooth case.

(2) At the opposite extreme, the possibility exists that (with $N = 1$) for very singular measures one can hope to construct inner functions using finite approximations, and the fixed-point property of the support of μ in the finite case.

Theorem 8.20. *("Schwarz lemma") Suppose $P \subset S^N$ and $\forall i, 0 < w_i$, then $f = \rho\mathcal{C}\mathrm{en}_{PW} : H^{N+1}_\pm \circlearrowleft$ is (weakly) distance decreasing with respect to the hyperbolic metric.*

(1) In fact the map is strongly contracting (but not uniformly) unless the support P only *spans*** a totally geodesic subspace, but even in this case a

reflection property in the transverse directions (see below) ensures the existence and uniqueness of the fixed-points desired. (**the *span* being the *smallest* totally geodesic subspace containing (possibly in the ideal boundary) the set.)

(2) Observe that $N = 1$ involves the case, $H^2 \to H^2$, and for any P this is a standard fact of complex analysis; essentially the sharp form of the Schwarz lemma, but *our statement*, as it stands, only applies to the case of maps with positive real weights and with all fixed-points in the boundary, in the correspondence established in (2.1). We will overcome these limitations, to some extent, further below. On the other hand our proof is quite different from the usual (known?) proofs even in this case.

(3) In higher dimensions $N > 1$, for |P|=3; we will exhibit a fibration st $\mathcal{C}\mathrm{en}_{PW}$ is handled using the $H^2 \to H^2$ case along fibers, and is isometric along the base. This thus describes the *extremal* case where weak contraction is in fact isometry. P spans an $H^2 = H^2_P$, which is invariant for f, so $H^2_P \subset S^2_P$ is $\mathcal{C}\mathrm{en}_{PW}$ invariant. Note that the isotropy G_P of P in $O(N+2,1)$ is compact–it fixes the Douady-Earle center of P in H^{N+2}, and $\mathcal{C}\mathrm{en}_{PW}$ is G_P–equivariant, so $\forall g \in G_P, gS^2_P$ is $\mathcal{C}\mathrm{en}_{PW}$ invariant. The contraction property follows *along* each $gS^2_P \cap H^{N+1}$ noting the cancellation of the *conformal* factors (in domain and range) which relate the hyperbolic metrics on the fiber to that of the ambient space. Also G_P–orbits, G_Px are orthogonal to $x \in S^2_P$ by symmetry, clearly $G_Px \cap S^2_P = x$, hence conformal invariance implies G_Px is orthogonal to gS^2_P, $\forall g \in G_P$. These orbits are spheres, as one can see that $P \subset S^1$ and G_Px correspond to fibers associated to an embedding $SO(2)\times SO(N) \to SO(N+2)$. Since $S^N = \partial H^{N+1}$ is fibered by gS^2_P's, G_Px is orthogonal to ∂H^{N+1} so $G_Px \cap H^{N+1}$ is totally geodesic (wrt the hyperbolic metric) in H^{N+1}. Thus the fibration, with fibers $gS^2_P, g \in G_P$, induces mutual isometries of the orbits $G_Px \cap H^{N+1}, x \in H^{N+1}$ which are thus isometric copies of the base space of this fibration. Transverse to the fibers the map $\rho\mathcal{C}\mathrm{en}_{PW}$ is now clearly isometric along the base, hence only weakly contracting. In fact it corresponds to a map of the base of the fibration, which is reflection through a point, s, where s is the (class of the) fiber H^2 spanned by P. In particular, there is a unique fixed-point. By an obvious extension to other codimensions, we can assume without loss of generality that P spans H^{N+1}, so $|P| > N+1$, this being useful for intuition but not strictly necessary in our proof.

(4) Note also that for |P|=2 $\rho\mathcal{C}\mathrm{en}_{PW}$ is an isometry and it follows that for any |P| finite, $x \to p \in P$, nontangential approach entails asymptotically isometric $\rho\mathcal{C}\mathrm{en}_{PW}$.

(5) The proof here is augmented below to work also for measures with support in $\overline{H^N_+}$, in (8.21,8.22) allowing support in the interior (of the complementary *target* space). It is not clear if the Douady-Earle or other proofs cover this generalization. One can weaken the hypotheses to allow negative weights in the (interior of) H^N_+, and ask if $\rho\mathcal{C}\mathrm{en}_{PW}$ being a self map still suffices for the contraction property? (See also 8.6.2 below.)

(6) The idea of the proof is to use the correspondence of minima to attractors (8.6). Critical points are denoted $crit(x')$, in terms of a point $x' \in H^{N+2}$ used to parametrize the round metrics on S^{N+1}. We show that for enough

of the associated energy functions interior critical points, $crit(x')$ have to be local minima. It will be useful to now consider a couple of special situations where this is easy to establish; we consider the boundary behaviour of $crit(x')$ as well as an obvious case where $crit(x')$ has a local minimum in $H^{N+1}_{\pm}$;

(7) For the latter, choose P to be the vertices of a regular simplex inscribed in $S^N \subset S^{N+1}$, with symmetric weights, $|P| = N + 2$. The energy $\mathcal{E}_P$ and it's hessian, H, are invariant by the symmetries, which fix the centrum $x_c \in crit(\mathcal{E}) \cap H^{N+1}_+$, so H is either strictly positive or strictly negative, or zero. For $N = 1$ it has positive trace, (4.1), so it is strictly positive. For $N = 1$ with larger $|P|$, any $|P| > N + 1$, one can perturb the simplex, replacing each vertex by a cluster, and the hessian perturbed slightly is strictly positive. For larger N, it is easy to see that at x_c, H has *positive* trace; x_c is at radial distance $\pi/2$ from each $p \in P$ by symmetry, so the hessian of each $\mathcal{E}_p$ is positive in the radial direction and zero along the totally geodesic level set. In fact this works for energy $\mathcal{E}_P(x)$ of *any* $P \subset S^N$, at x the pole in S^{N+1} (ie the center of the hemisphere).

(8) The boundary behaviour of $crit(\mathcal{E})$; as we approach the case where P is clustered at one point Aq "opposite" to q, $\mathcal{E}_{PW}$ is a small perturbation of the function $\mathcal{E}_{Aq}$ (case |P|=1), but $\mathcal{E}_{Aq}$ has a unique local minimum at q, and this is the only critical point of $\mathcal{E}_{PW}$ *in the limit* as the size of perturbations vanishes. One can see directly then that the perturbation doesn't change the critical point structure outside a nbhd of Aq; in a nbhd of Aq it must create saddle points, but they clearly stay near Aq in S^{N+1}. We claim these must stay in S^N for *sufficiently small* perturbations; it suffices to note that for p_i near Aq the $\nabla \mathcal{E}_{p_i}$ are all transverse to the concentric spheres S^N_ϵ (perturbations of S^N) near Aq. The version of this last point which is relevant below involves perturbing the choice of round metric rather than P, ie $x' \to \partial H^{N+2}$.

Proof. (of (8.20)). To simplify notation and facilitate the comparison to the usual Schwarz lemma for holomorphic maps in $H^2 \subset \mathbb{C}$, we will consider $\mathcal{C}\mathrm{en}_{PW}$: $H^{N+1}_- \to H^{N+1}_+$, ie we don't use the reflection map here. This isometric change doesn't affect contraction properties.

Given $x \in H^{N+1}_-$, as in (8.1), $\mathcal{C}\mathrm{en}_{PW}(x) \in H^{N+1}_+ \subset S^{N+1}$. Now S^{N+1} bounds a copy of H^{N+2} and $S^N = \partial H^{N+1}_-$ bounds a totally geodesic copy of $H^{N+1}_0 \subset H^{N+2}$, and the geodesic in H^{N+2} from x to $\mathcal{C}\mathrm{en}_{PW}(x)$ crosses this H^{N+1}_0 at exactly one point $x' = n(x)$, the *normalizing projection*. We can assume without loss of generality that $x' = 0 \in H^{N+2}$, so that x and $\mathcal{C}\mathrm{en}_{PW}(x)$ are antipodal (with respect to the round metric determined by x'). The antipodal property implies $\mathcal{E}_{PW}$ is critical at x, and we claim that $\mathcal{E}_{PW}$ has a local minimum at x.

But first, taking into account equality of the conformal factors of the hyperbolic metrics at x and $\mathcal{C}\mathrm{en}_{PW}(x)$ with respect to the round metric, (by symmetry; both are antipodal invariant), proposition 8.6 implies the distance decreasing property at x, a local minimum. It remains to verify the claim.

To confirm the claim now, we will need some explicit formulae for derivatives of $\mathcal{E}_p(x) = -\log(|p - x|)$, (we do not need to worry about the positive constant normalizing factors here) ; by explicit calculation, for the single point p

$$(8.2) \qquad -\nabla_u \mathcal{E}_p(x) = \frac{(((p,x)x-p)\,,u)}{1-(p,x)},$$

where p, x are points on the sphere $|p|^2 = |x|^2 = 1$ as well as vectors in $\mathbb{R}^{N+2}$, we use inner products (p, x) multiplying vectors as usual, and $\nabla \mathcal{E}_p(x) \in T_x S^{N+1}$ is the usual *projection.* We record also

$$(8.3) \qquad \nabla_v^2 \mathcal{E}_p(x) = (1-(p,x))^{-2}((p,v)^2 + (p,x)^2 - (p,x)),$$

$|v|^2 = 1, v \perp x$ and one can check easily it has constant trace for $N = 1$.

Now the critical point condition is

$$(8.4) \qquad 0 = \nabla \mathcal{E}_{PW}(x) = -\sum w_i (\frac{(p_i,x)x-p_i}{1-(p_i,x)}) = \xi(x,PW) - (\xi(x,PW),x)x,$$

where we write

$$(8.5) \qquad \xi(x,PW) = \sum \frac{w_i}{1-(p_i,x)} p_i.$$

But $P \subset S^N \subset \mathbb{R}^{N+1} \implies \xi(x,PW) \subset \mathbb{R}^{N+1}$, and $x \in \mathbb{R}^{N+2} - \mathbb{R}^{N+1}$ so ((8.4)) implies $(\xi(x,PW),x) = 0$, and moreover, $\xi(x,PW) = 0$.

Now it is easy to see that

$$\nabla_v^2 \mathcal{E}_{PW}(x) = \sum w_i (1-(p_i,x))^{-2}((p_i,v)^2 + (p_i,x)^2 - (p_i,x)) =$$

$$= [\sum w_i (1-(p_i,x))^{-2}(p_i,v)^2] - (\xi(x,PW),x) \geq 0,$$

finishing the proof. $\square$

An obvious candidate for generalizing (8.20) is to allow divisors PW with support in $H_{\pm}^{N+1}$ rather than S^N, and just as well, in $\overline{H_+^{N+1}}$. We can easily adapt the proof above to cover this case; we pick up in the preceeding proof from ((8.5)), and rather than $\xi(x,PW) = 0$, we will show $(\xi(x,PW),x) \leq 0$ which clearly suffices. But the p_i all have positive components in the e_{N+2} direction, where zero components in the e_{N+2} direction cuts out S^N, and x has negative component in the e_{N+2} direction. Thus ((8.4)) implies $(\xi(x,PW),x) \leq 0$.

Finally we can apply (8.20) to $\mathcal{E}_\mu$ for any positive measure, regarding μ as a limit of divisors with finite support in $\overline{H_+^{N+1}}$, and using continuity with respect to H_- and Div (noting disjointness) to get the decreasing property in the limit.

Theorem 8.21. *Suppose $P \subset \overline{H_+^{N+1}}$ and $\forall i, 0 < w_i$, then $\mathcal{C}\mathrm{en}_{PW} : H_-^{N+1} \to H_+^{N+1}$ is (weakly) distance decreasing with respect to the hyperbolic metric. The same holds for $\mathcal{C}\mathrm{en}_\mu$ for any (nontrivial positive probability) measure μ with support in $\overline{H_+^{N+1}}$.*

As a consequence, and using remark 3 following (8.20), we can extend the Douady-Earle center;

Corollary 8.22. *Given $P \subset \overline{H_+^{N+1}}$ and weights st $\forall i, 0 < w_i < 1/2$, or a measure μ as above but without atoms of mass $\geq 1/2$, there is a unique fixed-point in H_+^{N+1} of $\rho\mathcal{C}\mathrm{en}_{PW}$ and it agrees with the Douady-Earle center where the latter is well defined.*

8.6.1. *unique critical point for $\mathcal{E}_{PW}$ in H_-^{N+1}.* (This subsection is just a technical follow-up to things touched upon in the proof.) Note that there is a natural candidate for the left–inverse of $x' = n(x)$, from the proof above, namely the map $crit(x')$ that assigns to each $x' \in H_0^{N+1}$ the critical points x_i of $\mathcal{E}_{PW}$ associated to this normalization. This is a priori multivalued, but it is natural to conjecture

Conjecture 8.23. *If P spans H_-^{N+1} then $\mathcal{E}_{PW}$ has at most one critical point for $\mathcal{E}_{PW}$ in H_-^{N+1} (assuming positive weights).*

We do not expect there to be a natural metric in which the functions are convex though. In the degenerate case where P spans a subspace, we expect a unique linear slice of the sphere to correspond to the critical set. We also remark that there is an interesting relation between degeneration of the hessian at a critical point and reversal of orientation of the map n which we will not be able to discuss here. The conjecture is easily motivated by a continuity method arguement; as long as every critical point is a local minimum no bifurcation is possible. One can use real analyticity of $\mathcal{E}$ as well. We give a linear-algebra arguement below in a special case.

The functions $\mathcal{E}_{PW}$ are thus conjectured to be *perfect* in the sense of Morse theory in these hemispheres (our proof of (8.20) has even ruled out degenerate local minima). If the geometry of $\mathcal{E}_{PW}$ reflects a global topological property we will say that it is *topologically adapted* (in fact we would prefer to say "taut", but the term is already taken). It would be interesting to know how well $\mathcal{E}_{PW}$ is topologically adapted for other classes of divisors as well. The motivation comes from complex analysis, where holomorphic functions have well known perfect intersection theoretic properties. One is always interested in finding other classes of functions or maps with analogous behaviour.

We now reduce to a simplified setting; noting that ((8.5)) is linear in W, as is $\nabla_v^2\mathcal{E}_{PW}(x)$ it would have sufficed to check positivity of the latter at extreme points of the compact convex space of $\{W : \xi(x, PW) = 0\}$, and by standard convexity theory this reduces to weights with at most $N+2$ nonzero entries. We can also write $\xi(x, PW) = \sum w_i \hat{p_i}$, with $\hat{p_i} = \frac{1}{1-(p_i,x)} p_i$. By fixing a moebius map M sending $x \mapsto \infty, Ax \mapsto 0$, and letting $\hat{p_i} = Mp_i$, the latter lie in a fixed sphere $\hat{S}$ and a similar formula can be derived from $\mathcal{C}\mathrm{en}_{PW}(x) = Ax$.

So it is of interest to consider the case $|P| = N+2$, and fixing a generic such P, x now determines W uniquely by $\xi(x, PW) = 0$.

We will now show that W determines $x \in H^{N+1}$ uniquely assuming $|P| = N+2$. But for generic P the $N+1$ by $N+2$ matrix of p_i is of rank $N+1$, so it has a one dimensional kernel, say $0 = \sum w_i(P) p_i$. Clearly $\xi = 0 \implies \frac{w_i}{1-(p_i,x)} = \lambda w_i(P) \implies w_i = \lambda w_i(P)(1-(p_i,x))$. Rewriting $(p_i, x) = Px$ as a matrix product, we need to check that the image in $\mathbb{R}^{N+2}$ of the unit ball B^{N+1} by $e - Px, e = (1, \ldots, 1)$ intersects any line ℓ through 0 at most once (no 2 x's give proportional $e - Px$). (or the image projects 1-1 into the sphere or simplex). If the contrary were true then the affine linear space $V =$ range of $e - Px$ on E^{N+1} (alias $\mathbb{R}^{N+1}$) contains $\ell \ni 0$ which implies that the range of Py on $y \in E^N$contains $Py_0 = e$ for some y_0.

Since $w_i, 1-(p,x) > 0$, $\sum \frac{w_i}{1-(p_i,x)} p_i = \sum a_i p_i = 0$ has positive coefficients, showing that 0 is in the convex hull of p_i. (Note also that if 0 is not in the convex hull of p_i, this implies there is no critical point in H^{N+1} and it is clear in this case that the local minimum should be in the boundary, this is an intuitively satisfying analysis of the case where P lies in a hemisphere.) So $0 < \sum a_i = (\sum a_i p_i, y_0) = 0$ and by the contradiction, W determines x uniquely for generic P.

Proposition 8.24. *If P spans H_-^{N+1} and $|P| = N+2$ then $\mathcal{E}_{PW}$ has at most one critical point for $\mathcal{E}_{PW}$ in H_-^{N+1}.*

Motivated by the continuity method suggested above, we discuss, in the case $|P| = N+2$, the set of all solutions, Crit, in x, PW, its connectedness properties, with attention to it's interior $Crit^o$, and the relation of the boundary to the degenerate set (nonmaximal rank of $\hat{P}$).

We consider Crit fibered over $x \in H^{N+1}$ and parametrize the fiber by $\hat{P}$, ie configurations of $\hat{p}_i \in \hat{S}$ as above. W exists iff $0 \in Hull(\hat{p}_i)$ and the $\hat{p}_i$ determine W uniquely if the the $\hat{p}_i$ have maximal rank, this holds if $0 \in Hull^o(\hat{p}_i)$, the interior. One would like to view this as the interior of a solution-space, and we expect that modulo permutations of the indices $i \in$I, there is a unique open connected set P_0 of configurations $\hat{P}$, in the complement of the degenerate set (nonmaximal rank ie dim(hull))whose closure is Crit.

8.6.2. *the classical Schwarz lemma and divisors .* The comparison to the 2-D case, ie the classical Schwarz lemma, should be completed by (i) weakening the hypotheses to allow any weights in the (interior of) H_+^2, and extending the technique here to show that $\rho\mathcal{C}\mathrm{en}_{PW}$ being a self map still suffices for the contraction property, (recall as motivation the case of negative weights in H_+^N) and (ii) analyzing which divisors/measures are associated to self-maps of H^2. The case of atomic measures with support $P \subset S^1$ on the boundary is easy; the fixed-points $p \in P$ must have, for $f : H_-^2 \to H_+^2$, negative real multipliers, hence, by (2.1), $0 < w \le 1$. This is not at all clear for smooth support: note that the fixed-point property is lost in the smooth limit of atomic support, so a different approach is necessary. In fact we remarked that lebesgue measure on S^1 gives a constant map, so small *complex* perturbations still give self-maps of H^2. If the perturbation involves a smooth measure μ on S^1, then one must be careful about uniformity, up to the boundary, of the resulting perturbation of $\rho\mathcal{C}\mathrm{en}_\mu$ over H_-^2. But one can perturb by a pair of atoms in H_+^2 with small imaginary weights, and total weight real, to avoid the uniformity issue. Then using the "S^2-harmonic measure" construction, which we proceed to discuss, given any $P \subset H_-$, we produce μ on S^1 with smooth support, which determines the same map on H_-^2, and one thus gets a perturbation by a nontrivial *imaginary* smooth measure μ on S^1, which induces a self-map.

Suppose PW has support in $\overline{H_+^2}$, then for each $p_i \in P$, the restriction of $\mathcal{E}_{p_i}$ to $\overline{H_-^2}$ is by potential theory equal to $\mathcal{E}_\mu$ where $\mu = \mu_{p_i}$ is the "S^2-harmonic measure" of p_i on S^1; this is just the jump of the normal derivative $\partial_\nu \widehat{\mathcal{E}_{p_i}}$ at each $x \in S^1$, (or equivalently a distributional laplacian) where we construct $\widehat{\mathcal{E}_{p_i}}$ by continuously extending $\mathcal{E}_{p_i}$ from $\overline{H_+^2}$ to H_-^2 as an S^2–harmonic function. This can also be described using the standard euclidean harmonic measure of (p_i+ an additional mass) where the additional mass corresponds to the forcing term in the definition of $\mathcal{E}$ (coming from the curvature of the sphere). The main point is that

Lemma 8.25. *μ_{p_i} is positive on S^1 ,*

despite this forcing term. This is best seen using $\partial_\nu \widehat{\mathcal{E}_{p_i}}$; the maximum principle implies that $\mathcal{E}_{p_i|H_-}$ is everywhere smaller than the (S^2–)harmonic function with the same boundary values, and this implies an inequality on the normal derivatives at the boundary. But the reflection of this harmonic function is the extension used to define S^2–harmonic measure. Note that we use the isometry property of the reflection map here, this being a special feature of the hemisphere case.

Since an atom PW in H_+^2 and its S^2-harmonic measure induce the same energy $\mathcal{E}$ on H_-^2, the $\mathcal{C}$en construction is not 1-1 from measures to maps in this context.

8.6.3. *more questions:* Q: It is natural to consider existence of a.e. boundary limits for $\rho\mathcal{C}\mathrm{en}_{PW}$ as well as growth of $\rho\mathcal{C}\mathrm{en}_\mu$ in relation to Hausdorff dimension, or *entropy numbers,* associated to the boundary measure.

Q: Are there notions of convexity associated to the distance decreasing property? It is tempting to use a convexity arguement to get the unique local minimum in H^{N+1}, but we don't see how.

Q: The Schwarz lemma is a powerful tool for studying holomorphic maps as dynmical systems. In the general context of maps constructed using $\mathcal{C}$en form, this requires a Poincare or Kobayashi metric type construction on the complement of the Julia set compatible with associated generalized Schwarz lemmas.

9. PROBLEMS

We summarize some but not all problems from the text, some not yet mentioned, and some relations to those already mentioned.

(1) (3.14) diagonalizing hessians geometrically. Note the relation to discriminants implicit in 4.0.6.
(2) The conjectures in section 5.2.1 on degenerate attractors.
(3) General convergence of the maps in section 6. Classify all periodic-*attractors* for $\mathcal{F}$ and for the root-finding algorithm. Relate the combinatorics of triangulations to attractors for $\mathcal{F}$. This includes the case of collapsing attractors, 3.1, and the problems there.
(4) possible characterizations of $\mathcal{F}$, (3.8).
(5) (3.2).
(6) Morse-theory of $\mathcal{E}$ vs dynamics of $\mathcal{F}$:The energy functions we study provide strictly plurisubharmonic exhaustions with interesting equivariance properties and relations to the moment map. It should be interesting to compare the Morse-Smale complex of $\mathcal{E}$ to dynamics of $\mathcal{F}$. Does Z^n (roots of unity) play a special role in the dynamics of $\mathcal{F}$, or $\nabla\mathcal{E}$? Are all local minima of $\mathcal{E}$ in the drainage basin of Z^n? Note that this is the grad flow analogue of the question as to whether every basin of attraction for $\mathcal{F}$ has Z^k in its closure? Is Z^n the only critical point of index n, for $\mathcal{E}$?
(7) The holomorphic surgery analysis of Rat; section 2.1.1.
(8) The holomorphic Lefschetz fixed-point theorem in terms of weights, after (2.1) vs generalizations to the multivariable case including $\mathcal{F}$ and suspensions?
(9) (8.6.3) problems following up on the Schwarz lemma for $\mathcal{C}\mathrm{en}_{PW}$.
(10) Regularity theory for section 8.5. The problem would be better motivated with some specific applications in mind.
(11) Deformation and rigidity theory for the $\mathcal{C}\mathrm{en}_{PW}$ in higher dimension; topological conjugacy vs smooth(er) conjugacy. Qualitative descriptions of the branching behavior of $\mathcal{C}\mathrm{en}_{PW}$.

References

[1] Ahlfors, Lars V. Complex analysis. An introduction to the theory of analytic functions of one complex variable. Third edition. McGraw-Hill Book Co., New York, 1978. xi+331 pp.

[2] B. Bergersen, D. H. Boal and P. Palffy-Muhoray, Equilibrium configurations of particles on a sphere: the case of logarithmic interactions, J. Phys. A27:2579-2586 (1994). and from http://www.phys.sfu.ca/research/workarea/boal/papers.html number 85.

[3] Blanchard, Paul Complex analytic dynamics on the Riemann sphere. Bull. Amer. Math. Soc. (N.S.) 11 (1984), no. 1, 85–141.

[4] M. Bowick, A. Cacciuto, D. R. Nelson, A. Travesset, Crystalline Order on a Sphere and the Generalized Thomson Problem,Phys. Rev. Lett. 89 (2002) 185502, or cond-mat/0206144

[5] Crass, Scott; Doyle, Peter Solving the sextic by iteration: a complex dynamical approach. Internat. Math. Res. Notices 1997, no. 2, 83–99.

[6] L. E. Dickson, *Algebraic theories*, Dover, New York, 1959

[7] A. Douady and C. J. Earle, Acta Math. **157** (1986), no. 1-2, 23–48; MR 87j:30041

[8] Deligne, P.; Mostow, G. D. Monodromy of hypergeometric functions and nonlattice integral monodromy. Inst. Hautes Études Sci. Publ. Math. No. 63 , (1986), 5–89.

[9] Doyle, Peter; McMullen, Curt Solving the quintic by iteration. Acta Math. 163 (1989), no. 3-4, 151–180.

[10] PD Dragnev, DA Legg and DW Townsend, Discrete logarithmic energy on the sphere, Pacific Journal of Mathematics Vol. 207, No. 2 , 2002

[11] P. Eberlein and B. O'Neill, Pacific J. Math. **46** (1973), 45–109; MR **49** #1421

[12] Erber, T.; Hockney, G. M. Complex systems: equilibrium configurations of N equal charges on a sphere ($2 \leq N \leq 112$). Adv. Chem. Phys., Vol. XCVIII, 495–594, , Wiley, New York, 1997.

[13] J. E. Fornæss and N. Sibony, in *Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998)*, 47–85, Contemp. Math., 269, Amer. Math. Soc., Providence, RI,2001;

[14] S. Frankel, in *Recent developments in geometry (Los Angeles, CA, 1987)*, 263–286, Contemp. Math., 101, Amer. Math. Soc., Providence, RI, 1989; MR 91b:32020

[15] W. Fulton and J. Harris, *Representation theory*, Springer, New York, 1991

[16] Fulton, William; MacPherson, Robert A compactification of configuration spaces. Ann. of Math. (2) 139 (1994), no.

[17] Gelfand, I. M. Kapranov, M. M. Zelevinsky, A. V. Discriminants, resultants, and multidimensional determinants. Mathematics: Theory and Applications. Birkhäuser Boston, Inc., Boston, MA, 1994. x+523 pp.

[18] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Reprint of the 1978 original, Wiley, New York, 1994.

[19] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Corrected reprint of the 1978 original, Amer. Math. Soc., Providence, RI, 2001; MR 2002b:53081

[20] N. J. Hicks, *Notes on differential geometry*, D. Van Nostrand Co., Inc., Princeton, N.J., 1965; MR **31** #3936

[21] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge Univ. Press, Cambridge, 1995; and
B. Hasselblatt, Hyperbolic dynamical systems. in *Handbook of dynamical systems, Vol. 1A*, 239–319, North-Holland, Amsterdam, 2002; MR 1 928 520

[22] F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Princeton Univ. Press, Princeton, NJ, 1984, see also;
D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, Third edition, Springer, Berlin, 1994; MR 95m:14012

[23] K. Kodaira, *Complex manifolds and deformation of complex structures*, Translated from the Japanese by Kazuo Akao, Springer, New York, 1986; MR 87d:32040

[24] Eduardo S. G. Leandro:Finiteness and Bifurcations of some Symmetrical Classes of Central Configurations, Arch. Rational Mech. Anal. 167 (2003) 2, 147-177)

[25] Kuijlaars, A. B. J.; Saff, E. B. Asymptotics for minimal discrete energy on the sphere. Trans. Amer. Math. Soc. 350 (1998), no. 2, 523–538.

[26] Lubotzky, A.; Phillips, R.; Sarnak, P. Hecke operators and distributing points on the sphere. I. Frontiers of the mathematical sciences: 1985 (New York, 1985). Comm. Pure Appl. Math.

39 (1986), no. S, suppl., S149–S186. see also part II. Comm. Pure Appl. Math. 40 (1987), no. 4, 401–420.

[27] C. McMullen, Ann. of Math. (2) **125** (1987), no. 3, 467–493; MR 88i:58082

[28] J. J. Millson and J. A. Poritz, Comm. Math. Phys. **218** (2001), no. 2, 315–331; MR 2002b:53132

[29] J. Milnor, *Dynamics in one complex variable*, Vieweg, Braunschweig, 1999; MR 2002i:37057,

[30] Milnor, J. Geometry and dynamics of quadratic rational maps. Experiment. Math. 2 (1993), no. 1, 37–83.

[31] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Ann. of Math. Stud., 76, Princeton Univ. Press, Princeton, N. J., 1974; MR **55** #13428

[32] J. Morrow and K. Kodaira, *Complex manifolds*, Holt, Rinehart and Winston, Inc., New York, 1971; MR **46** #2080

[33] J. Roberts, Classical $6j$-symbols and the tetrahedron. Geom. Topol. **3** (1999), 21–66 (electronic); MR 2000k:22020

[34] Saff, E. B.; Kuijlaars, A. B. J. Distributing many points on a sphere. Math. Intelligencer 19 (1997), no. 1, 5–11.

[35] Segal, Graeme The topology of spaces of rational functions. Acta Math. 143 (1979), no. 1-2, 39–72.

[36] Shub, Michael; Smale, Steve Complexity of Bezout's theorem. III. Condition number and packing. Festschrift for Joseph F. Traub, Part I. J. Complexity 9 (1993), no. 1, 4–14

[37] Smale, Steve Mathematical problems for the next century. Math. Intelligencer 20 (1998), no. 2, 7–15.

[38] W. Thurston, Shapes of polyhedra and triangulations of the sphere, preprint,math.GT/9801088 xxx.lanl.gov

[39] W. Thurston, The dynamics and combinatorics of iterated rational maps, preprint