Equivariant Rational maps and Configurations: spherical equidistribution and SO(N,1) contraction
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by

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EQUIVARIANT RATIONAL MAPS AND CONFIGURATIONS:
SPHERICAL EQUIDISTRIBUTION AND SO(N,1)
CONTRACTION

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Abstract. We build up a class of $O(N,1)$-intrinsic spherical rational maps, using only stereographic projections and affine centers of mass, and slightly extend it with antipodal maps. The geometric-analysis of their dynamics lends itself to applications to equidistribution of points on the sphere and to canonical global parametrizations of the rational maps of $\mathbb{C}P^1$. We construct geometrically natural examples of rational maps of $\mathbb{C}P^n$, and introduce a new approach, “suspension”, to producing iterative algorithms for factoring polynomials, and to finding the $k$-periodic points of rational maps of $\mathbb{C}P^2$.

Maps, $f$, are understood in terms of a discrete steepest descent method, involving, as Lyapunov functions, the log-chordal energy function associated to the fixed-points of $f$; i.e. the spherical Green’s function rather than Coulomb energy. A transformation of rational maps of $\mathbb{C}P^1$ which gives singular flat affine connections on $\mathbb{C}P^1$ (also known as local systems, a complexification of polyhedra) in a natural way, provides an $O(N,1)$-intrinsic analogue of the Lyapunov force-fields and suggests higher dimensional versions of Schwarz-Christoffel uniformization of polygonal regions.

Relations to the algebraic geometry of configuration and moduli spaces, discriminants and dual curves are touched on, and we begin a discussion of the relation to geometric plethysm—maps as $sl_2\mathbb{C}$-invariants or covariants. We note as well the connection to moment maps, and begin a study of the relation of these constructions to hyperbolic centers of mass (such as Douady-Earle).

A class of self maps $O(N,1)$-intrinsic for hyperbolic space is constructed in each dimension as restrictions of the spherical rational maps above with fixed-point parameters in a hemisphere, generalizing the class of holomorphic maps of the 2-dimensional disc, and an associated “Schwarz lemma” confirms that the maps have good geometric and topological properties.

0.0.1. Note on numbering: References to theorems, and all but sections and equation numbers, are of the form “theorem 2.1” or for short (2.1). Sections, sub-sections, are referred to as section 2.1, sometimes without the “sub’s,” or just 7.2.1, when they have 2 decimal points, while equations are sometimes called ((3.1)) for short. Insofar as the following summary is not comprehensive, the reader might supplement it by browsing section and sub-section headings. A list of some open problems is to be found at the end of the article.

1. Introduction

Given a point $x$ in $S^N$, the sphere, Stereographic-projection from $x$ provides an affine structure $A_x$ on its complement, $S^N - x$. We consider the Stereographic center-of-mass $\text{Cen}(P,x)$ of a configuration $P \subset S^N - x$. We focus on the case $N = 2$ in this article (up to section 8). $\text{Cen}(P,x)$ extends meromorphically to
$P \subset S^2 = CP^1$, it is even meromorphic in $(P, x) \subset (CP^1)^n$, we will allow weighted interactions, and see that \( \phi(x) = Cen(P, W, x) \) is defined by solving for \( y \) in

\[
y = \phi(z); \quad \frac{1}{y - z} = \sum_j w_j \frac{1}{p_j - z}
\]

We exploit the symmetries of this expression in section 3, to construct the geometrically natural rational map \( F : (CP^1)^n \to (CP^1)^n \) of configurations, as well as its quotients \( F : CP^n \to CP^n \), and even meromorphic \( F : M_{0,n} \to M_{0,n} \) on moduli space, in 3.1.1. We then begin a study of the dynamics of \( F \), leading to the first main theorem 3.12, relating rational maps and equidistribution, though this leaves open many problems about the global behavior of \( F \). These were our initial motivations for this work and as such they occupy the central role here.

In section 8 we undertake a study of \( Cen_{ PW} \), with \( P \) in a subsphere, \( S^N \subset S^{N+1} \), (or in a hemisphere, \( H^{N+1} \)), to obtain self-maps of hyperbolic space \( H^{N+1} \), and we prove a Schwarz lemma, (8.20, 8.21) which provides a complete description of their dynamics. This illustrates how constructions based on \( Cen \) have nicer properties than one might have otherwise expected.

A priori, one does expect \( F : CP^n \to CP^n \) to have a nice closed form expression, but its not clear how to produce one – computer packages did not help here. \( F \) provides an algebraically simple but geometrically rich behaviour that we consider from different viewpoints, and a number of equivalent constructions are provided; (3.1)), based on \( Cen \), (3.6), using fixed-points and equivariances, (3.9), which uses osculating maps, and finally section 7.1, which provides a closed form expression for \( F \), using the resultant.

Other related results spread throughout the text are less developed, the more exploratory parts being preceded by the advice that they are not to be used further on. We apologize for this, as well as the many questions and occasional speculation that can slow down the reader, but we include them in the hope that some of these directions will prove to be fruitful.

1.1. Overview. We begin section 3 with an analysis of standard elementary properties involving fixed-points and equivariances that leads to a relation to steepest descent methods–this would be of interest only because holomorphic maps normally cannot be constructed using geometric constructions over \( \mathbb{R} \) of this nature–moreover this interaction of real and complex geometry can be exploited to produce attractors for the holomorphic dynamical system.

A variety of relations of \( F \) to discriminants, in (3.6), dual curves, in 7.2.1 etc. arise that are surprising from the elementary viewpoint, but less so from the point of view of invariant or representation theory since \( F \) is also a covariant of \( sl(2, \mathbb{C}) \) in the classical sense. We make some attempt to understand \( F \) in this latter context, in section 7.3, which should provide more unity to the theory in the longer run, and to clarify what other such constructions might be possible.

The suspension construction, section 6, a slight generalization of \( F \) that admits some parameters, underlies a new approach to iterative algorithms for factoring polynomials, and for finding the k-periodic points of rational maps of \( CP^1 \). We first develop a one-variable theory, section 2, 4, 5.1, by fixing some points–treating them as parameters, at each stage of the development of the multi-variable theory. This is not only useful in developing some intuition, but can often be applied directly to establish key points of the multi-variable theory. In addition the one-variable
theory features a number of constructions and results interesting in their own right; we cite for example, (2.1), a general Normal form theorem for rational maps of \( \mathbb{CP}^1 \), (4.18), a natural transformation of rational maps of \( \mathbb{CP}^1 \) to singular flat affine connections on \( \mathbb{CP}^1 \) (also known as local systems, a complexification of polyhedra). We also discuss the relation of the latter to Schwartz-Christoffel uniformization of polygonal regions and hyperbolic centers-of-mass in section 8.

The one-variable (or “one-body”) theory turns out to be quite rich for higher dimensional spheres, particularly in relation to hyperbolic geometry—it provides an analogue of holomorphic self maps for hyperbolic spaces, \( H^N = \mathbb{R}^N \) (we specialize to real hyperbolic space here) which have both good algebraic and topological geometric behavior. The analogy to the Schwarz-Pick lemma is secured by (2.1).

(The conjunction of “Douady-Earle” and “Schwarz-lemma” probably brings to mind the work of Courtois et al, but as far as we can see there is no immediate relation to their work, the hypotheses and conclusions here being quite different.)

This work began as a study of \( \mathcal{F} \) as a dynamical system, and the relation to steepest descent dominates our presentation, but the other directions (algebraic, hyperbolic) that have emerged from their study could be of more interest in the long run. In any case the analysis relating to steepest descent is very useful as regards the latter.

1.1.1. steepest descent, equidistribution, moment map and equivariance. The relation of the rational map \( \mathcal{F} \) to a discrete steepest descent method, for an associated log-chordal energy function, \( \mathcal{E} \) (3.11); the spherical Green’s function rather than Coulomb energy, is one of the main themes, and it’s specialization to the one-variable theory which is already quite rich can be found in section (5.1). This is the energy of interest in problem 7 of Smale’s problem list for the 21st century, [37], but we do not consider computational complexity questions here. We do discuss the hessian of energy at Fej{\'e}r-Tsuji points, ie the global (or local) minimal energy configurations; in fact it seems that not only is nothing known about uniqueness of the global minimum, \( Z_0 \), but even nondegeneracy of the hessian at \( Z_0 \), (3.14), is non-trivial, and as far as we know, open.

The fundamental observation regarding equidistributed configurations is that there are very few solutions coming from finite symmetry groups, (though see [26]). The fact that there is an equivariant rational map that finds equidistributed configurations seems to provide a reasonable alternative, though specific applications will require different notions of equidistribution. Diverse energy functionals on the space of sets of \( k \) points, as well as the packing problem are among notions considered, [25, 34, 12], and our canonical rational map provides local minima of one of these. It seems that \( \mathcal{F} \) has not been considered elsewhere, and, in particular, not in relation to equidistribution.

The approach to the theory of rational maps presented here should notably be of interest with regards to the global structure of the space of rational maps, we briefly consider a holomorphic surgery construction in this context, section 2.1, though the emphasis here is on the construction of special maps, their geometry and dynamics.

We also emphasize that the theory here is naturally tied to the \( O(N+1,1) \) structure of the sphere. This is the natural equivariance of \( \mathcal{F} \). A variant of the moment map arises as a natural normalization of configurations; it reduces an \( O(N+1,1) \)-orbit of period 2 attractors for the natural rational map \( \mathcal{F} \), to an \( O(N+1) \)-orbit of local minima of the energy function. There is a strong coincidence in how well this
works for precisely the energy of interest here, and we make some attempt to find
an underlying reason for this coincidence. This leads us to also consider relations
of the moment map to hyperbolic centers of mass using these energy functions.
Moment maps are a standard tool for reducing from noncompact-holomorphic to
compact-riemannian-isometric group actions, but we do not know of previous cases
of its use in reducing properties of equivariant-holomorphic maps (of higher degree)
to properties of equivariant-(geo)metric maps. (The maps themselves do not com-
mute with this reduction and we observe that no reasonable reduction will commute
with the maps, but particular properties of the maps such as fixed-points are well
behaved under reduction.) The Schwarz lemma we prove exhibits the Schwarz-
Pick contraction phenomenon as an aspect of the $\text{O}(N+1,1)$ geometric structure,
in contrast to the usual holomorphic viewpoint.

The name “elliptic center-of-mass” is likely to win out over alternatives such as
stereographic center, not to mention stereognathy or stereography as a name for
our central construction, especially in view of its relation to hyperbolic centers of
mass, section 8, such as the well known construction of Douady-Earle.

1.1.2 natural correspondences. $\mathcal{E}$ is privileged with a multitude of seemingly un-
related connections to conformal maps and symplectic geometry, which arose here.
The two nice properties of $\mathcal{E}$ in (4.1) that make it useful in studying rational maps
are explained and unified by showing how $\mathcal{E}$ is related to homogeneous polynomials.
But there is a direct and natural relation of homogeneous polynomials to rational
maps in one variable, in [9]. This motivates us to look further at the underlying
unifying geometric structures, and a large part of section 4 involves this somewhat
exploratory material. It seems that the payoff in applications might come from the
connection to hyperbolic geometry in section 8, especially the relation of smooth
maps to curvature measures which is somewhat akin to quasiconformal distortion
measures. Various sections contain extended discussions of how these pieces fit to-
gether, in particular we systematically develop some natural correspondences, sec-
tion 4.2.1, providing a geometric explanation and tying together rational maps, local
systems, energy functions, etc. For example, (4.18) gives an easy correspondence
of Gauss-Bonnet for polyhedra to the holomorphic Lefschetz fixed point theorem.
We provide some of these natural correspondences, with a few applications, and
the hope that more will follow. One such correspondence is closely related to the
work of Doyle and McMullen, [9], on icosahedra and quintics, (as we near the 120th
anniversary of Klein’s 1884 book on the subject). There are also tempting links to
work described in [28], on moduli spaces of linkages, configurations, etc. The
local minima of the energy functions have been much studied numerically, in the
context of equidistribution, but it is difficult to give conceptual proofs of any of
their properties. Transforming to the context of holomorphic functions and sphered
harmonics, or rational maps may help, (5.13). While most of this work applies
directly to the main goal: relating rational maps and equidistribution, the relation
to hyperbolic geometry which grew partly out of the “coincidences” involving $\mathcal{E}$ and
the moment map provides an equally good motivation.

We have tried to strike a balance between comprehensibility of the proofs and
use of machinery (e.g. $\text{Kähler}$ geometry); more elementary approaches to some
points might rely only on trigonometry, and other constructions might benefit from
more powerful algebraic-geometric tools.
A summary of some open problems is provided at the end. Computer experiments will probably be useful in guiding the next stages of this subject.

We owe much thanks to Chris Connell for help with computer experiments and discussions leading to the initial conjectures, especially a version of conjecture 5.12 bearing some elements of theorem 3.12. The confirmation of the latter constitutes the core of this article. Most of this work was done in the summer of 1999, and part of it was presented, November of that year, in a seminar at Stanford. We also thank our host, MPI for the possibility to pursue this and other matters farther, and for their gracious hospitality. In the remainder of this section we introduce the main actors of the paper, with some clues as to their hidden characters.

1.2. Geometric structure of maps, GRas. The main ingredients used in constructing the geometric subclass of rational maps are described in 1.2.1–1.2.4 below, as well as compositions of maps, and we use the term GRas to refer quite generally to the class of all such possible constructions. This includes many things not studied here, such as $\text{Cen}([f_1(x), f_2(x), f_3(x)], x)$ where $f_1(x) = \text{Cen}_{pW}(x), f_2(x) = A\text{Cen}_{pW}(x), f_3(x) = g\text{Cen}_{pW}(x), g \in \text{Aut}$, combining holomorphic and anti-holomorphic objects, so we will specify some “tamer” classes below, (see 1.2.5).

We construct rational maps of the sphere $S^N = SO(N + 1)/SO(N) \subset \mathbb{R}^{N+1}$, and since an important component of what is used will be conformally intrinsic for $S^N$, i.e. intrinsic for the structure associated to $\partial H^{N+1} = S^N$, where $H^{N+1}$ = real hyperbolic space, with its isometry group of Automorphisms, including reflections that reverse orientation. [11], we denote $\text{Aut}(S^N) = SO(N + 1, 1) = \text{Aut}(H^{N+1})$.

(Note: we will use the geometric properties of the 2 groups, (of oriented or all isometries) in this article but not the algebraic properties of the matrix groups just alluded to. Subtleties regarding the exact identifications of these matrix groups to isometry groups, such as the use of $SO^+(2, 1)$ as (notation for) oriented isometries are not important in this article, so we do not strictly insist that the matrix groups act faithfully and our use of $SO(N+1, 1)$ as convenient notation might be off by a $\mathbb{Z}_2$ kernel as above. Most often the notation is simply a way to indicate the dimension, $N$, in use, as well as the associated Lie algebra that identifies the relevant geometric structure.)

Conformal structure usually refers to structures local in nature, (but we use this term for lack of better alternatives, the possible use of Conformal is suggested by the relation to center-of-mass) and it is more appropriate to think of $\text{Aut}(S^N)$ as preserving a global structure, for example, that provided by round circles, (see section 4.0.4), or cross ratios of round subshelves, i.e $S^2$’s.

1. Complete totally geodesic subspaces of $H^{N+1}$ limit to round spheres in $\partial H^{N+1} = S^N$, as follows by symmetry considerations such as those in the claims in section 4.0.4: any limiting sphere $S^k$ is invariant by an $O(k+1) \subset O(N + 1, 1)$. But $\text{Aut}(S^N)$ acts transitively on the space of all such $S^k$ (for each $k$),

2. On each such $S^2$, (with an orientation, thus a $\mathbb{C}P^1$), there is a $\mathbb{C}$-valued cross ratio, XR, and these are $\text{Aut}(S^N)$-invariant.

3. The space RC of round circles determines the Conformal structure of $S^N$ uniquely, by a passage to the limit giving round circles in $T_1S^N$.

Proposition 1.1. $\text{Aut}(S^N)$ is the largest subgroup of twice differentiable diffeomorphisms of $S^N$ preserving either XR or RC.
This is included for completeness, but isn’t essential in what follows. The structures XR or RC are typically studied in relation to geometrization of negatively curved spaces, and some related rigidity conjectures, [21]. These last remarks are relevant to section 8, if anything.

Rational maps of $S^N$ are often denoted $\text{Rat}_d S^N$, $d = \text{degree}$, they are maps whose components are quotients of polynomials, using co-ordinates (by Stereographic-projection from $p$) of $\mathbb{R}^N = S^N - p$. This is independent of $p$, since $\text{Ant}$ is itself rational (Moebius) in these co-ordinates. A natural alternative is to define $\text{Rat}$ using homogeneous coordinates. Certainly with this definition the antipodal map $A$ is in $\text{Rat}$, however in the $S^2 = \mathbb{CP}^1$ case, which is our main concern here, we prefer to reserve $\text{Rat}$ to denote the construction involving complex coordinates and holomorphic maps, so $A$ falls outside. We are generally considering maps of $\mathbb{CP}^1$, or $(\mathbb{CP}^1)^n$ to itself, rather than rational functions (which are of interest in control theory or ODEs, via the laplace transform). Rat is used somewhat ambiguously in the literature both for self-maps and functions. Note that the two are quite different insofar as the natural group-actions transform only the domain, for functions, or both domain and range, for self-maps. This is a common source of confusion when first looking at the $\text{Cen}$ construction (people tend to think of it as a function rather than a self map).

1.2.1 Center-of-mass:

**Proposition 1.2.** Given $x$ in $S^N$ there is a canonical (conformally intrinsic) affine structure $A_x$ on its complement, $S^N - x$; it is the affine structure produced by Stereographic-projection from $x$, and it is invariant by the (Borel) isotropy group of $x$, in $\text{SO}(N + 1, 1)$.

This leads us to consider configurations, $P \subset S^N - x$, of cardinality $|P| = n - 1$, and their Stereographic center-of-mass $\text{Cen}(P, x)$ with respect to $A_x$. $\text{Cen}(P, x)$ extends meromorphically to $P \subset S^2 = \mathbb{CP}^1$, it is even meromorphic in $(P, x) \subset (\mathbb{CP}^1)^n$, see equation 2.1, corollary 2.12.

**Uniqueness** of the affine structure determined by $x$ can be seen in a multitude of ways. We provide a few here quickly to get started, but others arise farther on that may be more interesting and intrinsic to the theory developed here, and might be preferable in the long run. To keep the discussion clear we will number the approaches to uniqueness by Aff(i). Each might be useful for different generalizations of GRa's, for example they will not all work over other fields, (a generalization not taken up in this article). The first couple, using the invariance of XR or RC introduced above, might be regarded as aside, (mentioned here in passing) whereas Aff(3,4) are more standard:

Aff(1) -fixing $p$, XR determines a well defined midpoint $m(x, y)$ of any $x, y \in S^N - p$ and the midpoints satisfy the necessary relation $m(m(x, y), m(x, z)) = m(x, m(y, z))$ in any $S^2 \ni p$ to get a bialike affine structure, [14].

Aff(2) -Likewise, RC provides a class of straight lines $S^1 \ni p$ in any plane $S^2 \ni p$ satisfying the parallel postulate etc. we will not elaborate here.

Aff(3) -In fact it can also be seen for $\mathbb{R}^N, N > 2$ by Darboux’s theorem, [20]. (a weak form—for global maps), it is standard textbook material for $N = 2$, a corollary of the uniformization theorem, the main point being the relation of degree to growth for holomorphic functions.
Aff(4) - Uniqueness can also be seen from a Lie-theoretic approach, noting that the Borel (isotropy) groups, in \( S^N = SO(N+1,1)/B_\rho \), are affine. In terms of Fractional-Linear-Transformations, fixing the point at infinity eliminates the denominator.

Aff(5) - If we choose a Moebius transform, \( M \) of \( \mathbb{C} \) such that \( x \to \infty, A_x \) is the pullback from \( \mathbb{C} \), if \( N=2 \), \( (\mathbb{R}^N, N > 2) \) of the usual affine structure and this is independent of the Moebius \( M \) chosen. This is how we use uniqueness in practice.

1.2.2 Automorphisms: \( Aut(S^N) \), as given above, is defined with respect to the invariant conformal structure. As just indicated, \( (\text{Aff}(5)) \) automorphisms are used to study \( \text{Cen} \) explicitly. On the other hand, many oriented automorphisms can be constructed directly using the center-of-mass, see theorem 2.1, but not so for the antipodal map, denoted \( A_x \), which plays a major role here. We use equivariance properties systematically to simplify proofs. We note that \( \text{antipodal maps}, A_x \) are parametrized by \( x \in H^{N+1} \) (we almost never use the subscript for antipodal maps) and affine structures, \( A_x \) or \( A(x) \) by \( x \in \partial H^{N+1} = S^N \). Note the obvious unifying property, that they are each centralized by the isotropy group in \( SO(N+1,1) \) of \( x \).

In fact we introduce a natural correspondence of affine connections to maps in (4.18), and one can apply isotropy equivariance to see that the Levi-Civita connection of the round metric on \( S^N \) associated to \( x \in H^{N+1} \) corresponds precisely to the antipodal maps, \( A_x \), see also section 8.2.

1.2.3 clamped and variable points: We distinguish clamped from variable points, in the construction of functions; clamped points are denoted \( p, q \), and configurations of clamped points are denoted \( P, Q \). They are constants in \( S^N \) that we use to parametrize self-maps of the configuration of variable points \( X \) in \( S^N \). Thus \( \text{Cen}(X, P, W, i) \) is the \( (W\text{-weighted}) \) center-of-mass of \( X - x_i, P \) with respect to the affine structure of \( x_i \), it is a function of \( X = (x_1, \ldots, x_k) \). We will see that \( p_j \in P \) are fixed-points of this map (viewed as a function of \( x_i \)). Given a configuration \( X \subset S^N \), one should ideally formalize it not as a set or a tuple, but rather, as a labelled (or marked) set \( X = \{ \ldots, x_j, \ldots \} \), thus \( j \) is not so much an index as a label or marker, though in the generic local case this doesn’t matter much. For degenerate configurations (double points) and global topology it is essential. (In section 5, we use \( Z \) for a configuration of variable points, but in the beginning we hold all but one fixed, calling it \( x \) to study the one variable case, so \( Z \) are clamped. Other such minor variations in notation are clarified locally in the relevant section.) Unless stated otherwise one should always suppose that the points in \( P \) are distinct. We use the prime, \( P' \) to denote the complementary space, \( \mathbb{CP}^1 - P \), while in configuration space we often refer to the complement \( D' \) of the diagonal \( D \) of degenerate configurations, see (3.3). \(|X|\) denotes the cardinality of a configuration, (and never a vector norm).

1.2.4 weights: It is useful, (remarks after proposition 3.6) to generalize to the weighted center-of-mass using weights, \( W_{ij} \),

\[
\forall i, \sum_j W_{ij} = 1,
\]

with \( i \) indexing \( X \), and \( j \) indexing \( X \) and \( P \). If \( X \) is a single point we denote the associated center-of-mass, \( \text{Cen}(P, W, x) \), and \( W \) is a vector. In the very special case \( N=2 \), weights can be arbitrary complex numbers, and \( \text{Cen} \) is fully meromorphic in
X, P, W. Note that complex weights are justified because the intrinsic affine structures produced in proposition 1.2 have an additional complex structure preserved by O(2, 1), and that commutativity of C^* is essential here, so that generalization to the quaternionic case is not (obviously) possible.

We will see that \( w_{ij} \) essentially represents the multiplier i.e., the linearization at the fixed-point \( p_j \) of \( \text{Cen}(X, P, W, i) \). One naturally conceives \( X, P \) as the nodes of a directed graph \( G(X, P, W) \) and \( W \) as weights on its directed edges. We’ll assume throughout that every \( p \in P \) has at least one nonzero weight—(otherwise it can be left out), and that no \( x_i \) has a nonzero self-weight \( w_{ii} \) (otherwise \( x_i \) is constant under \( \text{Cen} \)). When discussing energy functions, \( \mathcal{E} \) the edge weights are always assumed reflexive, i.e. \( w_{ij} = w_{ji} \). As a general principle, weights of energy functions are dictated by the condition that the associated force vectors are determined by a matrix of weights, which must agree with the matrix used by \( \text{Cen} \) to construct the meromorphic map \( \mathcal{F} \). Thus one should also appreciate that there is a slight difference in handling the subsets in \( P \) and \( X \) when constructing \( \mathcal{E} \), (3.11).

1.2.5. \textit{Compositions, Ras classes.} There is not much to say about compositions, but formally one only can substitute a map for a variable, compositions come up notably in the construction of suspensions, section 6, as well as in the iteration of maps. In fact, for the most part we consider maps that are constructed using only the weighted center-of-mass, and we propose to use the shorthand term Ras (pronounced as the 1st syllable of \textit{rational} in your preferred language) to denote the geometric subclass of real rational maps consisting of self-maps of the form \( \text{Cen}_{PW} \) or \( A\text{Cen}_{PW} \), notably \( P \) with distinct points. In dimension 2 we will allow complex weights here and by (2.1) this just extends \text{Rat} by \( A \). But on \( S^N, N > 2 \), one only has real weights and this class will be much smaller than real-\text{Rat}, see (8.4). Furthermore ARas is the subclass of Ras of maps, \( A\text{Cen}(P, W, x) \), which reverse orientation.

\text{Ras} allows for compositions of maps in \text{Ras} , (see 8.1.1). \textit{GRas} includes the multivariable constructions, such as (6.1), viewed as a subclass of \text{Rat}. Finally we will use reflections in hyperspheres (codim 1) in section 8, to study a class of maps of hyperbolic space which we denote \( H\text{Ras} \). We often use \text{Rat} without explicitly indicating the space (it is usually \( \mathbb{C}\mathbb{P}^1 \)) or even whether maps are defined over \( \mathbb{R} \) or \( \mathbb{C} \), where this is clear from context.

It might seem that such simple ingredients wouldn’t produce anything of interest. While we leave this for the reader to judge, our first goal is thus to show that this is a non-trivial class, already in the case \( N = 2, S^2 = \mathbb{C}\mathbb{P}^1 \), which is our focus in most of this article. Note that for \( N = 2 \) we get a subclass of meromorphic maps, if one restricts to orientation preserving automorphisms, but the antipodal map turns out to play an important role even in this case (tying maps to steepest descent methods). In section 8 we begin to develop \( \text{GRas}(S^N) \), \( N > 2 \), but not beyond the one-variable case, and recover some of the the nice properties of holomorphic maps, such as a Schwarz lemma, as well as a related “tautness” property, 8.6.1.

1.3. \textbf{Divisors.} Many of the constructions in this article are parametrized by weighted configurations, and the set of pairs \( \{(p_i, w_i), 0 < i < n\} \) are best formalized as \textit{divisors}. For a divisor in \( \mathbb{C}\mathbb{P}^1 \), we allow coefficients in \( \mathbb{C} \). Since it is also useful to formalize \( P, W \), as \( n \)-tuples of points or weights, divisors are also denoted in terms of
these vectors by PW or WP, the pairing of \( \{(p_i, w_i), 0 < i < n\} \) being the essential structure; PW is an n-tuple of pairs quotiented by permutations.

**Definition 1.3.** A divisor in \( \mathbb{CP}^1 \) is a formal sum, denoted \( WP \in S\text{Div} = \{ \sum w_i p_i : p_i \neq p_j \in \mathbb{CP}^1, w_i \in \mathbb{C} \} \). The S prefix emphasizes the distinctness of the \( p_i \). \( S\text{Div} \) is the subspace with \( \sum w_i = 1 \). These are clearly complex manifolds.

Since we always assume \( \sum w_i = 1 \) in this article, and distinctness in \( P \) is our default mode, we generally drop the modifiers and simply write \( \text{Div} \) rather than \( S\text{Div} \). If \( P \) alone is used as a divisor, then weights are assumed symmetric, i.e., permutation invariant. Compactification of \( S\text{Div} \) is nontrivial; there are different compactifications, at the set of double points, of \( S\text{Div} \), corresponding to rational maps and local systems for example. We will consider their relation below in section 2.1.

2. The One variable case

We study maps of the form \( r_{PW}(x) = \text{Cen}(P, W, x) \) here, with \( N = 2, S^2 = \mathbb{CP}^1 \).

Following remarks in 1.2.1 and choosing the Moebius transform, \( m_z(w) = \frac{1}{w-z} \) of \( \mathbb{C} \), such that \( z \mapsto \infty \) to construct \( \text{Cen}, \phi(x) = \phi_{PW}(x) = \text{Cen}(P, W, x) \) is defined by solving for \( y \) in

\[
y = \phi(z): \quad \frac{1}{y-z} = \sum_j w_j \frac{1}{p_j - z}.
\]

This also shows that \( \phi \) is meromorphic in \( (z, P, W) \), (and that \( \text{Rat} \) is a subset of the extension by \( A \) of \( \text{Rat} \)).

We emphasize that this definition of \( \phi \) is independent of the choice of Moebius transform, \( m_z \), subject to \( z \mapsto \infty \), \( \text{Aff}(5) \) above; by direct calculation, \( \phi \) depends only on \( PW \).

**Theorem 2.1.** Every (holomorphic) Rational map of \( \mathbb{CP}^1 \) with simple fixed points \( \{p_i, 0 \leq i \leq d\} = P \) is of the form \( r_{PW}(x) = \text{Cen}(P, W, x) \), where \( w_i = (1 - m_i)^{-1} \), \( m_i = r'(p_i) \). In particular, \( r, (= r_{PW}) \), is uniquely determined by its fixed-points and multipliers, i.e, by the divisor \( PW = \{(w_i, p_i)\} \).

Note that oriented automorphisms are constructed by using \( |P| = 2 \), making the ingredient list above somewhat redundant. The \( \text{Cen} \) representation of \( r \) in this case is in effect a diagonalization (normal form) theorem. In low degrees one can simplify the form of \( r \in \text{Rat} \) alot, using automorphisms, by choosing special fixed-points , Milnor, [30] and Thurston, [39] (preprint), discuss degrees up to 4. Despite all efforts, we cannot find evidence that the general normal form theorem implicit in theorem 2.1 was already known. Although the proof we give here is technically simplest, there is a proof based on (4.18) which might be considered a better explanation of this phenomenon. The latter approach is taken further in section 8.2, where we extend the theorem here to smooth maps, using a symplectic construction to recover the measure (generalizing weights) which is also realized as the curvature of a natural connection. The holomorphic maps studied here are thus those whose graphs are Lagrangian with respect to a complexified symplectic form.

**Lemma 2.2.** The fixed points of \( \phi = r_{PW} \) are precisely the \( p_j \in P \), each occurring with multiplicity one, and the multiplier at \( p_i \) is \( (1 - w_i^{-1}) \), and \( 1 - n_i \), for \( \phi_p \).

In the limit where \( w_i = 0 \), \( \phi_{PW} \) is well-defined, but since \( p_i \) is not a fixed-point, there is a discontinuity that arises—see section 2.1.
Proof. Using ((2.1)) to construct $\text{Cen}$, note that $p_j$ are fixed points of $f$; $z = p_j \Rightarrow \frac{1}{y-z} = \infty \Rightarrow y = z$. Rewriting the r.h.s. with common denominator and taking reciprocals we see $y - z = -z + \frac{\nu_j(z)}{\nu_j(z)}$, using $\sum w_j = 1$, where $d = |P| - 1 = \deg U = \deg V$ so $\deg f = |P| - 1$ and $P$ gives all fixed points of $f$ by “accounting”.

In particular every fixed point of $f$ is simple. Now letting $z \to p_j$, and using $y-z = (y-p_j) - (z-p_j)$, in $1 = \sum_j w_j \frac{y-p_j}{z-p_j}$ gives $1 = w_j(-f'(p_j) + 1) \Rightarrow w_j = (1-m_j)^{-1}$.

(Differentiating implicitly with respect to $z$ at $z = p_j$ works via l’Hospital, but is much messier.) \hfill $\square$

We thus get meromorphic maps: (between mapping spaces, restricting to degree $d$ maps and configurations of size $d+1$)

- Ev: $\text{SRat}(\mathbb{P}^1) \to \text{SDiv}_1$, by evaluation—restricted to the subspace of $\text{Rat}$, $\text{SRat}$, with simple fixed points; namely $r \mapsto \text{PW}$ the fixed-points of $r$ and their associated multipliers—giving weights as above.
- $\text{Cen}$: $\text{SDiv}_1 \to \text{SRat}$, by the $\text{Cen}$ construction as above. The latter is injective by (2.2), in fact

Lemma 2.3. $\text{Ev} \circ \text{Cen}$ is the identity map.

Proof. (of (2.1) ) $\text{SRat}$ is Zariski open in an irreducible (algebrao-geometric sense of unions) variety; $\text{SRat}$ is dense in $\mathbb{C}_{\mathbb{P}^d+1} \subset \mathbb{C}_{\mathbb{P}^{2d+1}}$, by using polynomial coefficients (of $U, V$, above) as coordinates. Also $\dim \text{Div}$ is $2d+1$, by equation 1.2; so $\text{Cen}$ must be onto (the unique irreducible component of) $\text{SRat}$ and $\text{Ev}$ is injective. \hfill $\square$

Remark. $\sum j w_j = 1$ is the Lefschetz holomorphic fixed point thm for $r$. Applying this to $r \in \text{Aut}$ (with simple fixed-points ) we get the well known

Corollary 2.4. If $r$ has exactly 2 fixed-points then $m_1m_2 = 1$.

Remark. (i) This leads to questions about the analogous interpretations of weights via Lefschetz for the meromorphic multi-dimensional $\mathcal{F}_{\text{PW}} \in \text{GRas}$ discussed in section 3, and more generally, how broadly the weight-interpretation can be pushed?

(ii) On $S^N$, with $|P| = 2$, $\text{Cen}$ gives $r_{\text{PW}} = r_1r_2$, with $r_2$ in the $p$ part of the Lie group, (in terms of the $p+k$ decomposition of the Lie algebra) and $r_1$ is a rotation by $\pi$ radians along the axis of $r_2 \in \text{Aut}$.

The proof suggest the relevance of Runge expansions, indeed one can expand $1/(z-r(z))$ in terms of its poles and apply the holomorphic Lefschetz fixed point theorem to prove part of theorem 2.1. The proof would still use the calculations behind lemma 2.3 as above. This is the analytic alternative to the more geometric approach using $\text{Cen}$. The Runge theory for multiple poles then applies nicely to the case of degenerate fixed points.

2.1. Degenerate fixed points, higher multiplicity and holomorphic surgery.

Here we consider extending the $\text{Cen}$ representation to maps with degenerate (ie multiple) fixed points. (One can skip directly to section 2.2.4 if only interested in the main theorem 3.12.) In its full complexity this involves a stratification of the possible degenerations, but we will only consider the simplest, or highest, strata here, so this is only a sketch of the theory as regards degeneration. In this case there are analogous formulae for the $\text{Cen}$ representation: we consider a family of $r_t \in \text{SRat}_d(\mathbb{P}^1), t \neq 0$ in the degenerate limit, $t = 0$, where a pair of fixed-points $p_i(l), p_{i+1}(l)$ of $r_t$ collide; $p_i(0) = p_{i+1}(0)$, but with no jump in degree, and we call this a smooth family. Checking smoothness in the $U, V$ coordinates above, $(r = \frac{U}{V})$,
is somewhat complicated (a resultant), but smoothness makes the choice of \( PW \) coordinates quite natural. We can choose the family \( r_i \) to be holomorphic and even algebraic, so the set \( V_k \) of \( k \)-periodic points is an algebraic variety, in particular for \( k = 1 \).

1. Note that \( \frac{d r_0}{d t}(p_i) = m_i = 1 \); using difference quotients of the \( p_i \).
2. When \( x \in V_1 \) and \( \frac{d r(x)}{d t} = 1 \) the implicit function theorem shows that \( V_1 \) is smooth, but when \( \frac{d r(x)}{d t} = 1 \) singularities may arise. In this case \( P \) can have nontrivial monodromy, but is generically given by well-defined functions on the double cover, likewise for \( W \). In what follows we could suppose that we have chosen a parametrization, (by passing to this double cover, if necessary) such that the \( p_i \) are well-defined functions across \( t = 0 \). In fact using \( \gamma_i r_i \) with \( \gamma_i \in \text{Aut}(\mathbb{C}P^1) \), we can even suppose \( p_i(t) = \text{const} = 0 \) and \( p_2(t) = t + O(t^2) \) near \( t = 0 \) (on the double cover in the generic case).

3. Note that the condition \( |dr(x)| \neq 1 \) often arises in the study of convergence of normal forms, as well as bifurcation of attractors, but this shouldn’t be confused with bifurcation of fixed-points.

4. If \( w_i(t_0) = 0 \) then degree jumps and \( r_i \) is not smooth. Nevertheless \( r_i(x) \) is smooth in \( t \), in this case, unless \( x = p_i(t_0) \), (check using \( Cen \)). On the other hand preimages \( r_i^{-1}(y) \) jump at \( t_0 \) for any \( y \). We call this the **bubbling** of the graph of \( r \) at \( t_0 \).

5. At the degenerate \( p_i \), \( \frac{d r_0}{d t}(p_i) = m_i = 1 \Rightarrow m_i, m_{i+1} \to 1 \Rightarrow w_i, w_{i+1} \to \infty \), (by (2.1)) as \( t \to 0 \), but

\[ \textbf{Lemma 2.5.} \text{ For a smooth family as above, } w_1 + w_2 = O(1) \text{ at the degeneracy, } t = 0, \text{ ie the blow-up of the } w_j \text{ occurs in pairs such that infinities cancel to give a finite net weight to } p_i. \]

\[ \textbf{Remark 2.6.} \text{ The technique will be reused a couple of times here; calculate } r_i(x), x \neq p_i, \text{ using } Cen: \text{ choose coordinates using } m_x \text{ as in (2.1)} \text{ so } x \mapsto \infty, \text{ but } x \neq p_i \Rightarrow |p_i| < \infty \text{ and we will use } 2(p_1 w_1 + p_2 w_2) = (p_1 + p_2)(w_1 + w_2) + (p_1 - p_2)(w_1 - w_2), \text{ where } p_1 w_1 + p_2 w_2 \text{ is } \text{“part of”} \text{ the summation in (2.1)} \text{ for } r_i(x). \text{ Hence to show that both summands on the r.h.s. are bounded in the limit, } t \to t_0, \text{ it suffices to show that one is.} \]

\[ \textbf{Proof.} \text{ If there is just one degenerate pair, then } w_1 + w_2 = 1 - \sum_{i \neq 1, 2} w_i = O(1) \text{ since } i \neq 1, 2 \Rightarrow w_1 = O(1) \text{ by the hypothesis and smoothness, (and using the normalization to get the equality).} \]

In case there are more degenerations at the same \( t_0 \), we’ll check that each degenerating cluster has bounded total mass. The proof will follow directly from the smoothness of the family of maps, and the \( Cen \) representation thereof. Now we will derive a contradiction from the case where there are 2 distinct degenerating clusters, each having unbounded total mass in the limit \( t \to 0 \), i.e. where there are infinite weights cancelling “at a distance” (cancellation is again clear by the normalizations). The contradiction is that in this case the maps would degenerate; in applying (2.6) here, we will use a single \( w_1, p_i \) to denote the (weighted) contribution to \( r_i(x) \) of each *cluster*, this approximation can be justified insofar as when \( t \to t_0 = 0 \), we just use it to show that \( (p_1 + p_2)(w_1 + w_2) = O(1) \). But \( (w_1 + w_2) = O(1) \) (this is the sum of weights over all points of both clusters) by the normalizations, and \( (p_1 + p_2) = O(1) \) since \( x \neq p_i \). Similarly, \( (p_1 - p_2)(w_1 - w_2) \) must blow
up; $(p_1 - p_2) \neq 0$ by the distinctness hypothesis, and $(w_1 + w_2) = O(1)$
but each $w_i$ blows up by hypothesis, so $(w_1 - w_2)$ blows up. This and (2.6)
now implies $r_t(x)$ = $x$. Furthermore the same holds for all $x$ in a nbhd of $r_t(x)$ is the identity map, which is absurd. (2 degenerating clusters entail 4
fixed-points, so degree > 2, but the identity map is degree 1, contradicting
smoothness.)

(6) Observe now, that as $p_1 \to p_2$ (with $p_i$ being single points) while staying
away from $x$, $(p_1 + p_2) = O(1)$ (using notation as in (2.6) and we have
remarked that $(w_1 + w_2) = O(1)$ above, so $(p_1 + p_2)(w_1 + w_2) = O(1)$.
Again $r_t(x) \neq x$ so by (2.6)

\[(p_1 - p_2)(w_1 - w_2) = O(1),\]

and we conclude that $|w_i| = O(t^{-1/2})$, or $O(t^{-1})$ on the double cover as
discussed above. Thus, in the generic case of a double point, $w_i - w_{i+1}$
blows up, but

**Lemma 2.7.** $(p_i - p_{i+1})(w_i - w_{i+1}) = O(1)$

**Proof.** We give another proof here, based again on $(w_1 + w_2) \neq 0$, because the
technique is useful in section 2.1.2. Grouping together the terms $i = 1, 2$
in ((2.1)), and rewriting with a common denominator, one gets the symmetric polynomials $p_1 p_2, p_1 + p_2, p_1 w_2 + p_2 w_1, w_1 + w_2$, arising as coefficients.

Smoothness implies that appropriate ratios of these are then all $O(1)$. In
fact we can suppose that $w_1 + w_2 \neq 0$ is $O(1)$, and the same then follows
for the rest of these “elementary polynomials”. Simple algebra then gives
$(p_i - p_{i+1})(w_i - w_{i+1}) = O(1)$. Note that this grouping can be done for
higher order degeneracies as well and gives formulae such as ((2.3)).

2.1.1. **Surgery.** This suggests that the surgery that transforms the compactified
weighted configuration space (later to be identified with Local systems) to Rational
maps is locally the same as the standard holomorphic surgery of ruled surfaces, (the
case $\mathbb{CP}^1 \to E \to \mathbb{CP}^1$) as described in [32], (p. 25). But this is not quite right;
recall that we should pass to a double cover to define $p_i - p_{i+1}(t)$ as functions, so
the surgery on the space of Rational maps is essentially the standard holomorphic
surgery of ruled surfaces conjugated by a square-root map. (and this is only a
preliminary step towards understanding the global relation of $L$ to $\text{Rat}$.) But we
also saw that the difference between compactifications of Local systems vs. Rational
maps is that the latter involves a bubbling off of spheres in the graph of the map
as divisors degenerate without the proper blow-up of weights. This also involves a
kind of surgery. It would be interesting to see if the two are more directly related.

One motivation is potential applications of theorem 2.1 to analyzing the topology
of mapping spaces. The obstructions involved in constructing retractions of SRat to
a (circle or other) bundle over Poly, using retractions of weights to the unit circle
for example, where one must avoid weight zero and respect the sum of weights,
gives rise to nontrivial topological considerations (possibly related to linkages, an
issue pursued no further here).

2.1.2. **Cen vs Rat with double points:** As in the proof of (2.7) one replaces $\frac{1}{p_i - z}$, in
equation 2.1 by a polynomial in the latter, with no condition on any coefficient but
the first;

\[
\frac{1}{y-z} = \sum_j \sum_{k>0} u_{jk} \left( \frac{1}{p_j - z} \right)^k; \quad y = \phi(z).
\]

**Remark 2.8.** The following claims can be easily checked: (i) The only constraint is \(\sum_j u_{j1} = 1\) (as for the case of simple fixed-points). (ii) The multiplicity of \(p_j\) is \(k_j\), the largest \(k\) such that \(u_{jk} \neq 0\); as in the proof of (2.2) one sees that \(\frac{1}{y-z}\) blows up at \(z = p_j\) as \(\left( \frac{1}{p_j-z} \right)^{k_j}\), and \(1 = \sum_{k>0} u_{jk} y^{k} \cdot \frac{1}{(p_j-z)^k}\) implies \(y - p_j = z - p_j + O(|z - p_j|^k)\). (iii) Furthermore, the coefficients \(u_{jk}\) are derived from the Taylor-series of \(\phi\) at \(p_j\). (iv) Note that these transform by \(u_{jk} = a^{k-1} u_{jk}\) with respect to an affine transformation \(az + b\) of \(\mathbb{C}\), to get the right equivariance property, so they are not simply constants, (or masses) but tensors. This suggests that the appropriate generalization of the normal form for \(r \in \text{SRat} \) above, to very degenerate maps, would involve replacing the weights in divisors by distributions (in the sense of L. Schwartz). (v) Note that the representation (2.3) provides an easy construction of smooth families \(s_t\) for some \(j\), \(u_{j1}(0) = 0\), i.e total weight can "degenerate". This contrasts with the jump in degree that occurs at a fiber in which the weight of a simple fixed-point vanishes. In fact for multiple fixed-points it is the vanishing of the highest order coefficient \(u_{jk}\), mentioned above that corresponds to degeneration of a smooth family.

**Warning:** the identity map is too degenerate as a rational map to have any nice representation of this sort! but it does sit in standard Zariski closures of the map space, evidently a limit as weights blow-up to infinity.

### 2.2. Further developments for the One variable case.

We include a few more basic facts relevant to the one variable case here, before introducing the multivariable case. We then return to many other aspects of the one variable case after section 3.

#### 2.2.1. closed forms.

A homogeneous polynomial function \(p : \mathbb{C}^2 \to \mathbb{C}\), \(\text{deg } p = d\) determines \(d\) roots, \(s_i \in \mathbb{CP}^1\) and supposing they're distinct, we can apply theorem 2.1 to construct the associated \(r \in \text{Rat}\), oriented with symmetric weights, i.e. \(w_i = \frac{1}{d}\) at \(s_i\). There is an elegant underlying geometric construction, [9], leading to an explicit closed form expression; restrict to an affine line \(L \subset \mathbb{C}^2 \to 0\) to get \(p : \mathbb{CP}^1 \to \mathbb{C}\), \(\text{deg } p = d\) (d=degree)

\[
R_p(x) = x - d(p(x)/p'(x))
\]

by (2.1) it suffices to check (easily) that the fixed-points and weights are correct (or a formal calculation using the derivative of \(\log(p)\) and the factorization of \(p\) giving the \(\text{Cen}\) form directly). In fact this formula was derived by Doyle and McMullen (building on F. Klein's work on the icosahedron) by quite a different method, viewing it as a conformally intrinsic transformation of Poly to Rat; the relation to center-of-mass here is new. For an extension see section 4.1.2, also for an application, section 6, and the relation to plethysm section 7.

It may at first be surprising or puzzling that \(\text{Cen}\) can be rewritten using a derivative, but we'll provide quite a thorough explanation of this equivalence below. One explanation involves reinterpreting the \(\text{Cen}\) construction as a steepest descent method for a natural lyapunov function, see also (4.22) and section 4.2.1.
Remark 2.9. The construction of $R_p$ can easily be extended from $p$ polynomial to $p$ a rational function $f/g$ st (degree $f$ - degree $g$) is nonzero. This is clear if one thinks of the associated divisor normalized to mass 1 as a signed measure. It can also easily be seen using either of the constructions of [9]. The interesting upshot is that the zeroes of $f$ can be made super attracting by choosing degree $f$ - degree $g$ =1 while poles are always repelling, in fact $g(z) = z^{d-1}$ suffices.

2.2.2. examples. Letting $Z = \mathbb{Z}^n$, denote the n-tuple of n-th roots of unity, $\phi_z(x) = \text{Cen}(Z, x) = x^1/n$.

Letting $SZ = S\mathbb{Z}^n$ denote the n-tuple of $(n-2)$-th roots of unity, augmented by $0, \infty$, $\phi_{SZ,W}(x) = x^{n-1}$ for some weights $W$, which are not symmetric or even positive. It will be useful (in (5.3)) to calculate $\phi_{SZ}(x)$ for symmetric weights, using (2.1) we get

\begin{equation}
\phi_{SZ}(x) = \frac{-x^{n-2} + n - 1}{(n-1)x^{n-2} - 1} x,
\end{equation}

One should check directly that this is $\mathbb{Z}^{n-2}$ (multiplication) equivariant, and equivariant for the $\frac{1}{x}$ map, that 0 is a fixed-point as is $x = 1$, and these with their respective multipliers and the symmetries suffice to confirm the validity of the formula. (A direct derivation requires a slightly messier calculation).

2.2.3. one variable; steepest descent vs energy as conformal factor etc. We will soon see that the map $r \in \text{Rat}$, together with the choice of a round metric, $CP^1 = S^2$, or equivalently an antipodal map, $A_x, x \in H^3$, determines a unique energy function $E_r$, in such a way that (i) $r(x)$ determines $\nabla E_r(x)$ and (ii) fixed-points of $r$ correspond to poles of $E_r$. Simple O(3)-symmetry considerations show:

**Proposition 2.10.** Given a correspondence with the properties (i) and (ii), a critical point $x$ of $E_r$ satisfies $r(x) = A(x)$, (A being the antipodal map).

This motivates the study of anticonformal maps (of the form $Ar(x)$, we call this space ARAs) via Morse theory in section 5. Note especially the use of critical points of $r$ in theorem 5.3.

The point of view we take in this article is to introduce energy and relate it to $r$ by a steepest descent construction. We thus obtain fixed-points of $Ar(x)$ with an additional attractor property. There is a less natural, but shorter route to the proposition above; theorem 2.1 gives an explicit formula for a rational map with fixed-points and multipliers prescribed by a divisor $WP$. There is a very direct way to transform this into an explicit formula for a singular flat holomorphic connection defining the rank one local system (biholomorphic to $TCP^1$, singular on $P$) determined by $WP$, see theorem 4.18. We have put that material after the development of the energy function, but one can already appreciate that proposition 4.21 can be used as a shortcut to show the relevance of the PDE in proposition 4.1, insofar as it relates rational maps having divisors with $\mathbb{R}$-weights to $E$.

2.2.4. connections in one complex dimension. (we assume the reader is familiar with this material. it is included to fix terminology only) that a connection $D$ for $TCP^1$ at $x$ is a linear map $D(x) : (T_x, J_2) \to T_x$ where $J^2$ represents 2-jets (one-jets of vector fields) at $x$. $D$ is holomorphic if when extended naturally to complex vector fields, [31] . $D$ sends holomorphic input data to holomorphic output data. Since a pair of connections $D_i$ differs by a matrix valued one-form, and in one complex dimension, a pair of holomorphic connections $D_i$ differs by a holomorphic
scalar valued one-form, \( \eta \), the holomorphic connections at \( x \) form a \textit{one-dimensional complex affine space}.

Denoting by \( D_x \) the affine connection associated with \( A_x \) on \( \mathbb{C}P^1 - z \), it is not hard to check that the correspondence \( z \in S^2 - x \mapsto D_z(x) \) is a holomorphic affine map, (use equivariance).

This doesn’t directly give an existence or uniqueness proof for the \( A_z \) structures, but it induces a dynamical system on a larger class of connections, for which the desired structure gives at least one fixed-point.

The basic example of a holomorphic connection is \( \partial_z \) on \( \mathbb{C} \), so every holomorphic connection on \( U \subset \mathbb{C} \) is of the form \( \partial_z + \eta \), \( \eta \) holomorphic. Furthermore, a holomorphic connection in one complex dimension is flat; (Frobenius integrability is vacuously satisfied in one complex dimension). Now it is possible to make sense of the statement that

**Lemma 2.11.** \( A_x \) depends holomorphically on \( x \).

This just says that \( A_x = A_0 + \eta_x \) and \( \eta_x \) depends holomorphically on \( x \). Comparing to (2.1) this is a more intrinsic approach to seeing.

**Corollary 2.12.** \( \text{Cen}(x, P) \) is meromorphic in \( (x, P) \subset (\mathbb{C}P^1)^n \).

2.2.5. **Remark: derived map.** Given \( y = r(x) = r_{PW}(x) \) as above, we can define an averaged inverse map,

\[
r^a(y) = \text{Cen}(r^{-1}(y), y)
\]

where averaging uses symmetric weights. One immediately sees that fixed-points of \( r \) are fixed-points of \( r^a \).

**Lemma 2.13.** \( r^a \) is a rational map of \( \mathbb{C}P^1 \) with the same fixed-points \( , P \), as \( r \). This may fail if \( r \) has double fixed-points \( . \) The weights \( w^a_i = \frac{1-w_i}{n} \) and multipliers \( m^a_i = 1 - n + nm_i^{-1} \) where \( n = \text{degree}(r) \).

**Proof.** \( r^a \) is clearly holomorphic up to the fixed-points of \( r \), and continuous at the latter, hence holomorphic on \( \mathbb{C}P^1 \). The formula for the multipliers can be derived by a calculation similar to that in (2.1), using the proof of (but not applying) l’Hospital. The formula for weights follows, note that there are \( n+1 \) fixed-points so the total mass is indeed 1. \( \square \)

A more conceptual proof might first check that \( w^a_i = f(w_i) \) for some holomorphic \( f \), and since the total mass of \( W^a \) is 1, \( f \) should be linear. Furthermore \( w_i^a = 0 \) is only possible at double points, where \( w_i = 1 \). (Q: Do fixed-points of \( r^a \) jump, or does degree jump when \( r \) has double fixed-points \?) Note that the fixed-points of \( r \mapsto r^a \) are precisely the maps with symmetric weights, a space isomorphic to configurations on \( \mathbb{C}P^1 \). This is the space that \( \mathcal{F} \), ((3.1)), acts on, as well as the image of ((2.4)).

More generally one wonders if this fits naturally into a general geometric theory of rational maps of spaces of rational maps or polynomials? We will present some evidence later that there is some such theory based on notions of geometric plethysm, [13], in section 7. This also could tie in neatly with the appearance of discriminants in the restriction above to \( \text{SRat} \).

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3. Multivariable case

This work began as an effort to understand a canonical rational map, $\mathcal{F}$, of configurations in $\mathbb{CP}^1$, and was fueled by the unexpected relation to equidistribution problems. Let $\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n$, $n > 1$, be the meromorphic map defined by solving (here $w$ is a variable, not a weight)

$$
\frac{1}{w_i - z_i} = \frac{1}{n-1} \sum_{j \neq i} \frac{1}{z_j - z_i}; \quad w = \mathcal{F}(z).
$$

$\mathcal{F}$ has a natural extension to $(\mathbb{CP}^1)^n$, and at the risk of redundancy the geometric interpretation is: given $z = (z_1, \ldots, z_n) \in (\mathbb{CP}^1)^n$, removing $z_j$ from $\mathbb{CP}^1$ gives an affine structure $A_j = A(z_j)$ that depends holomorphically on $z_j$, so the $A_j$ center of mass $w_j$ of the remaining points, $z_i$, in the set is well-defined and $w = (w_1, \ldots, w_n)$ is meromorphic in $z$. Note that the construction is conformally intrinsic. In particular it is antipodal equivariant, identifying $S^2 = CP^1$. We now outline some of the main properties of $\mathcal{F}$.

3.1. Basic Examples:

1. $N = 2, 3$ pts: $\mathcal{F}$ is trivial for $n = 2$, (it permutes the pair) and $\mathcal{F}$ is an involution for $n = 3$; check it on the 3rd roots of unity, and use Moebius-equivariance. But $\mathcal{F}$ is not smooth for $n = 3$; when all 3 points collide $\mathcal{F}$ is singular, i.e. $\mathcal{F}$ is meromorphic rather than holomorphic. These configurations always exhibit period 2 behavior, as do many symmetric configurations including k-th roots of unity. In fact, in each of these cases configurations (suitably normalized by automorphisms) get sent to their antipodal images (componentwise) as is easily seen using equivariance properties; nontrivial isotropy of a configuration at a point suffices. The same holds for $Z^n$ the n-tuple of n-th roots of unity, and $SZ^n$ the n-tuple of $(n-2)$-th roots of unity, augmented by 0, $\infty$.

2. $N = 4, 12, 20$, etc. Standard symmetric configurations exhibit period 2 behavior, but computer experiments reveal that iterating $\mathcal{F}$ equidistributes points on the sphere! Certain standard symmetric configurations are in fact period 2 equidistributed attractors. In the computer experiments we initialize at random points, or roots of unity, $Z^n$. The latter are easily seen to be period 2, but their instability together with round off error rapidly leads them to local minima of a lyapunov energy function. Subsequent analysis proves that any (strong) local minimum $X_{\text{min}}$ of the “log energy” $\mathcal{E}(X)$, subsection 3.3 is a period 2 attractor for $\mathcal{F}$. In fact, let $A$ be the antipodal map with respect to a round metric on $\mathbb{CP}^1$ such that pts of $Z^n$ lie on the equator, then $A(x) = \frac{-x}{x}$. For $Z$ listing the vertices of any regular solid $P$, one sees from Moebius -equivariance of $\mathcal{F}$ and rotational symmetries of $P$ that $F(Z) = A(Z)$.

3. Appropriate notions of attractor and antipodal must take into account the equivariances of $\mathcal{F}$ discussed below. It suffices to normalize configurations by their dipole moments, subsection 4.3.

4. $Z^n$, $n > 3$, is an unstable fixed set for $A\mathcal{F}$; one may calculate the linearization of $A\mathcal{F}$ at $Z^n$ explicitly, by exploits the cyclic group action. It will be easier to see this instability later, in terms of the associated energy function.
Example 3.1. Collapse: Consider $\mathcal{F}$ with $|Z| = 4$, and iterate $\mathcal{F}^n(Z_0)$ where $Z_0$ has a double point, $z_3 = z_4$. Note that the latter stay fixed. This illustrates the difference between weighted points and multiple points.

Iterating $\mathcal{F}$ the configuration collapses i.e. it converges to a single (quadruple) point. In fact there is a hyperbolic $g \in \text{Aut}(S^2)$, such that $\mathcal{F}(Z_0) = g(Z_0)$, where $\{g^k, k \in \mathbb{N}\}$ is noncompact (in $SO(N,1)$).

We will see that there is a natural normalization of any $Z$, by the moment map, or Douady-Earle center, (associated to equivariances discussed below), but the example shows this can not give an $\mathcal{F}$ invariant slice.

Question: Does the same asymptotic collapsing phenomena obtain generically for large configurations with a single double point , (perhaps with total mass at least a half)? Collapse may even be generic in the Julia set. In fact, one may well find that generically, for $Z$ in the Julia set, $J$, the Douady-Earle projections to $H^3 = SO(3,1)/SO(3)$, of iterates, $\mathcal{F}^n(Z)$, drift to infinity. It is too much to hope that a standard measure for $J$, [13], should behave like a Wiener measure, but one might say this is a topological version of random walk. One can observe this computationally, at least if display software does not automatically renormalize outputs.

Question: Describe the dynamics of $\mathcal{F}$, defined over the reals, (or other fields!). does it exhibit generic collapse in a chaotic fashion? (Recall remarks on the dipole moments in the disc and random walk.) Is $(\mathbb{RP}^1)^n$ repelling in $(\mathbb{CP}^1)^n$?

3.1.1. Equivariance and diagonalization properties of $\mathcal{F}$:

1. The symmetric-group $S_4$: $\mathcal{F}$ is clearly equivariant with respect to permutations of the $z_i$; renaming the variables has no effect on their behavior. Quotienting, and noting that $(\mathbb{CP}^1)^n/S_n = \mathbb{CP}^n$, as follows by identifying a set of roots with its defining polynomial, $\mathcal{F}$ determines a canonical self-map of $\mathbb{CP}^n = \text{Poly}(\text{projectivized})$, (as promised in the abstract). Its geometric nature is further elaborated, in proposition 3.6, as a map of $(\mathbb{CP}^1)^n$, and in remark 3.9 as a map of $\mathbb{CP}^n$.

2. PSL(2, $C$)-action: $\mathcal{F}$ is Moebius equivariant, since it is defined intrinsically with respect to the conformal structure. The $G=\text{PSL}(2,\mathbb{C})$-action descends to $(\mathbb{CP}^1)^n/S_n = \mathbb{CP}^n$, and the quotient map is well-defined on the standard (Fulton-MacPherson) blow-up compactifications of $\mathbb{CP}^n/G$, namely the moduli space, $\mathcal{M}_{0,n}$. It is not well-defined into the blown-up space, only to the singular space $\mathbb{CP}^n/G$. Nevertheless, blow-ups are meromorphic maps, and $\mathcal{F} : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n}$ thus gives a meromorphic map which is (almost**) smooth on the blow-up of $D$, but which is singular on $C = \mathcal{F}^{-1}D - D$. (** and smoothness fails at $\mathcal{D} \cap \mathcal{C}$). We do not know if this qualifies as a rational map of $\mathcal{M}_{0,n}$, it seems the existence of such maps is of some interest.

3. It is known in algebraic geometry that certain desingularizing blowups can be realized by passing to dual curves or varieties. This suggests the use of looking at the graph of $\mathcal{F}$ as a way of constructing desingularizing blowups in the construction of the compactification of the moduli space $\mathcal{M}_{0,n}$.

4. Antipodal-symmetry. This follows as above. $\mathcal{F}$ is defined intrinsically with respect to the conformal structure, and it plays an important role, as the examples suggest. We will see below its role in relating metric to holomorphic constructs.
**Problem 3.2.** The map $F : Poly \to Poly$, should be expressible directly in terms of the coefficients. The brute force method of (for each $|Z| = n$) grouping roots into symmetric polynomials, gives a huge and messy formulas. One would hope there is a more elegant closed form expression for any $|Z|$, or an expression with a clear geometric significance, or at least a nice algorithm. We comment on this more in section 7.1, where we provide an expression which gives a compromise of these qualities, with the hope that better expressions will be found.

3.1.2. **Diagonals and discriminants.**

**Definition 3.3.** The subvariety of configurations $Z = (z_1, \ldots, z_n)$ with a double point, i.e. $\exists i \neq j, z_i = z_j$ is called the diagonal and its $S_n$-quotient in $CP^n$ is the discriminant locus, they are both denoted $D$. $D$ has a stratification by subvarieties $D_\alpha$, for multiindices $\alpha$, listing the cardinalities of nontrivial multiple points. Some shorthand is indespensable; for example $D_3$ consists of configurations with at least one point of multiplicity at least 3, i.e the multiindex is $(3,1,1,1)$.

$D$ has an obvious, but interesting fixed-point property; diagonals are (componentwise) fixed points, $\forall X, i,$

$$F_i(X) = X_i \iff \exists j \neq i, X_i = X_j.$$ 

We call the only-if part $D$-graph-invariance, it is a property that is only well-defined for self-maps of configuration spaces.

Furthermore, one can check that these $X \in D$ are repelling as fixed points in the $x_i$ component, provided that they have weight less than half. Note the relation of this to the equidistribution problem. This uses the relation of weights to multipliers in (2.1);

**Proposition 3.4.** The diagonal is an $F$-invariant variety, with a repelling property (see below) on the open dense $F$-invariant subset, $D^2 = D - D_3$, of configurations $Z$ with at worst double points, and where $w_{ij}, w_{ji} < \frac{1}{2}$. $D^2$ is also precisely the part of $D$ on which $F$ is smooth.

In particular $D \subset D^+ \subset J$, the Julia set of $F$, where $D^+$ is the union of preimages of $D$ under iteration.

**Remark 3.5.** The correct definition of repelling on the meromorphic part of $D$ is not a priori clear, but we hope that the collapse phenomenon in example 3.1 will provide an appropriate notion for the case at hand; the collapse to $p_0$ is an explosion away from some other “condensed state”, $\lim_{k \to \infty} g^{-k}Z_0$. (In other words, we suggest that repeating points in $D_3$ are fixed-points for $F$ on $M_{0,n}$, whose preimage in $CP^n$ includes a Zariski dense subset of a noncompact orbit.) Note also that sequences of configurations, $\gamma_i(Z)$, collapsing along a $PSL(2, C)$-orbit (3.1) give directions where the repelling property fails, but the degeneracy is in $D_3$.

The definition of repelling property on $D^2$, also requires explanation; essentially one pair of points is increasing its distance, but another pair could create a new double point, landing on a different diagonal component. Thus we can only hope to increase distance from that one component of $D$ near which we start (this makes sense on $(CP^1)^n$ but not $CP^n$).

3.2. **Geometric-Naturality of $F$.** A full geometric characterization of $F$ is possible using equivariance, meromorphicity and diagonals, we state a general version including the weighted versions of $F$. Let $\mathcal{M}_\alpha$ be Möbius transformations...
of $\mathbb{C}P^1 = \mathbb{C}$ such that $M_x(z) = \infty$. Given a matrix $U$ such that $\forall i \Sigma_{j \neq i} u_{ij} = 1$, consider the function
\[ F_{U,i}(z) = M_x^{-1}(\Sigma_{j \neq i} u_{ij} M_x(z_j)) ; \quad F_U : (\mathbb{C}P^1)^n \rightarrow (\mathbb{C}P^1)^n. \]

**Proposition 3.6.** Given $U$ as above such that $i \neq j \implies u_{ij} \neq 0$,

1. $F_U$ is meromorphic in $Z$,
2. $F_U$ is Möbius (or better, $O(3, 1)$) equivariant,
3. $F_{U,i}(Z) = Z_i \iff \exists r, j, i, st j \neq i$, and $Z \in D_{ijr} = \{z : z_i = z_j = r\}$, hence $D_{ijr}$, $D$ are $F_U$-invariant, and $F_U$ is $D$-graph-invariant,
4. $F_U$ is smooth at $Z$ iff no three $z_j$ are identical (no triple point).

Furthermore given any $G : (\mathbb{C}P^1)^n \rightarrow (\mathbb{C}P^1)^n$ satisfying these properties there is an $U$ as above such that $G = F_U$. $F_U$ is symmetric (i.e. $S_n$-equivariant for $S_n$ the permutation group) iff $\forall j \neq i, u_{ij} = s_{ij} = \frac{1}{n-1}$, and in this paper $F$ generally denotes $F_S$ ($S$ denoting the symmetric weights $s_{ij}$).

The graph-invariance of the 3rd point reduces the characterization to the following lemma; we can use graph-invariance to reduce to the study of $F$ on the complement of a point, (call it $\infty$) in $\mathbb{C}P^1$, hence to $\mathbb{C}^n$. For example fixing $x_1 = \infty$ and letting the other $x_i \in \mathbb{C}$ vary, we reduce to an analogous characterization of $\mathbb{C}en$. The affine group of $M$ Möbius, fixing $\infty$, acts on $\mathbb{C}^n$ diagonally. Our characterization of a general weighted version of $F$ is based on.

**Lemma 3.7.** Consider the maps $f : \mathbb{C}^n \rightarrow \mathbb{C}$
\[ f_U(z) = \sum u_i z_i ; \quad \sum u_i = 1, \]
with respect to weight vector $U$, $f$ is holomorphic and affine-equivariant iff it is of this form. $f$ is symmetric (i.e. $S_n$-equivariant for $S_n$ the permutation group) iff $\forall i, u_i = s_i = \frac{1}{n}$, we may denote $f_S = f$ in this case.

Note that holomorphicity of $f$ is a valid assumption because graph-invariance guarantees that we avoid $\infty$.

**Proof.** (of the nontrivial direction). By affine-equivariance, the small diagonal, $D^a = \{\{z_1, \ldots, z_n\} : \forall i, j, z_i = z_j\}$ maps isomorphically to $\mathbb{C}$, and $f$ is a smooth holomorphic fibration over $\mathbb{C}$. Dilatation equivariance then gives that the full fibration is isomorphic to the bundle tangent to the fibration along $D^a$, and that the fibers are affine hyperplanes of $\mathbb{C}^n$, which must be parallel. It remains to identify $U$ with the slopes of these planes. \(\square\)

**Problem 3.8.** What is the simplest characterization of $F$ on $(\mathbb{C}P^1)^k$? It may be possible to substantially weaken the equivalence hypothesis; the fact that $F$ is well-defined depends heavily on equivariance, and this suggests that under a very weak hypothesis $F$ is forced to be equivariant. Similarly, what is the simplest natural property characterizing the induced map of $F$ on $\mathbb{C}^n$? Note that graph-invariance is not directly definable on $\mathbb{C}^n$. It would be good to have a characterization in terms of the fixed-points of $F$ (see below) and their multipliers, or just the symmetry property of the latter, noting the analogy to (2.1). If there is a unique map with the fixed-points and multipliers of $F$, then it is clearly equivariant. Characterizations of $F$ based on equivariance should be possible in connection with geometric plethysm, [13], see the discussion in section 7.
The introduction of weights is not just generalization for its own sake; they are necessary to produce attractors in certain algorithms, as in the suspension constructions below. They are also useful in producing spaces of maps that are naturally isomorphic to other interesting spaces as one sees in subsection 4.2.1. See also the application to degree \(\mathcal{F}, (7.1)\), where weights facilitate an inductive step, increasing the number of points, but with useful continuity properties.

Note that fixed-points of \(\mathcal{F}\) correspond to configurations with every point occurring as a double-point, or polynomials where every linear factor occurs at least twice, denoted \(D_{sq}\) for square. In the notation of (3.3) \(D_{sq} = D_\alpha\) with \(\forall \alpha, \alpha_i = 2\), a middle-dimensional variety, (with an extra codimension one for odd \(|\alpha|\), \(D_{sq}\) has \(\alpha_i = 3\) for some \(i\). Hence \(\mathcal{F}\) is quite non-generic, even with weights. Adding clamped points to the Cen construction of \(\mathcal{F}\) introduces associated fixed-point sets with dimensions reduced accordingly, but it does not affect \(D_{sq}\) which is still fixed.

Note also that these fixed-points come up in questions discussed here, in relation to holomorphic Lefschetz fixed-point theorem, and also with regards to possible characterizations of \(\mathcal{F}\).

Remark 3.9. An alternative construction of \(\mathcal{F}\), that works as well on \(\mathbb{CP}^n\), can be based on the notion of osculation: given a configuration, consider the associated polynomial, \(p\). We saw, ((2.4)), there is an associated map \(R_p\) and at a root, \(p(z) = 0\), the 1-jet of \(p\) is determined by (2.1); it doesn’t depend on any property of \(p\) aside from \(p(z) = 0\).

We claim that for simple zeroes of \(p\), the 2-jet of \(R_p(z)\) is determined precisely by \(\mathcal{F}(p)\), and equivalently by \(p''(z)\), (or \(\frac{d}{dz}\log p'(z)\)) which is the same, in view of our remark on the 1-jet of \(p\), in fact this is clear from ((7.2)). (For generic \(z\), \(R_p''(z)\) depends on the 3-jet, including \(p''\), but in the special case \(p(z) = 0\), \(R_p''(z)\) depends only on the 2-jet, which just depends on \(p''\). The shift is already evident insofar as \(R_p(z)\) depends on \(p''\) generically, but \(R_p(z)\) only depends on \(p(z) = 0\).)

The 2-jet of \(R_p(z)\) determines a unique osculating mobius transformation \(M_z = M_{p(z)}\), at \(z\), and if we construct \(\mathcal{F}^{osc} : (\mathbb{CP}^1)^n \to (\mathbb{CP}^1)^n\) which, by definition, takes each root \(p(z) = 0\) to the unique opposite fixed-point of \(M_z\), then one can easily verify that this satisfies the characterization (3.6). The main point, concerning the behavior of \(\mathcal{F}\) on \(\mathcal{D}\), is that \(R_p''(z) = 1\) iff \(z\) is a double root of \(p\) (as well as meromorphy and equivariance). As a corollary we can see that the formula for \(g(z)\) in ((7.2)), must essentially be constructing the unique opposite fixed-point of \(M_z\) at each root \(p(z) = 0\).

3.2.1. Embedded dynamics of \(\mathcal{F}\). Fixing a weight vector \(W\) every \(r = r_{PW} \in \text{Ras}\) of degree \(d\) is embedded in \(\mathcal{F}_{2d+1}\), (with appropriately weighted interactions) in the sense that there is an invariant \(\mathbb{CP}^1 = L_r\) (in fact a standard \(\mathbb{CP}^1\), parallel to a coordinate axis) on which the action of \(\mathcal{F}\) is conjugate to \(r\), and \(L_r\) is specified by choosing initial values of \(\mathcal{F}\) such that every fixed-point of \(r\) is represented by 2 variable points of \(\mathcal{F}\). Initializing 2 variable points to \(p_i\) guarantees that these points stay fixed for any iteration of \(\mathcal{F}\) (they stay clamped). Weights can be chosen for \(\mathcal{F}\) that realize the weight vector of \(r\) on this restriction, (alternatively one could introduce nontrivial diagonal or self-weights to fix points). For rational weights it suffices to use \(\mathcal{F}\) with symmetric weights if one allows more variable points to coincide at each clamped point. Noting that the image of a standard \(\mathbb{CP}^1\), under the canonical map to \(\mathbb{CP}^n\), is a line, (generically with \(n-1\) double tangencies to \(\mathcal{D}\)) we get,

\(20\)
Theorem 3.10. Fixing a degree, \( d \), and the symmetric weight vector, \( S \), every rational map of \( \mathbb{CP}^1 \) of degree \( d \), and weight vector \( S \), is realized as the restriction, of \( F_{2d+1} \) acting on \( \mathbb{CP}^n \), to a linearly embedded \( \mathbb{CP}^1 \) in \( \mathbb{CP}^n \). Furthermore, generically the line has \( n-1 \) quadruple tangencies to \( D \).

These embedded dynamics might be useful in determining Hausdorff dimensions for the Julia set of \( F \), or just for showing that it is fractal, related issues have been considered in the literature, [13].

3.3. The strong attractor property: There are several notions of equidistribution on \( S^2 \). The one that interests us here is based on \( G(x, y) = c \log(|x - y|) \), where \(|x - y|\) is the choral Euclidean distance on \( \mathbb{R}^2 \) restricted to \( x, y \in S^2 \), and \( c < 0 \) is a normalizing constant (fixed in (4.1)). \( G \) stands for Green's function, and \( E(x, P) = G(x, P) = c \sum_{y \in P} \log(|x - y|) \), where \( E \) stands for Energy, in this case using only the interactions of \( x \) with each \( p_i \). (Generally \( G \) is a function of a pair of points or configurations and \( E \) a function of a configuration of variable points—with clamped points as parameters).

Definition 3.11. Equidistributed sets are global minima of
\[ E(Z) = c \sum_{(i, j) : i \neq j} \log(|z_i - z_j|). \]

One should consider \( E \) to be an average of pairwise energies, rather than a sum.

To generalize \( E \) to include real weights in the context of multi-variables with clamped points, first define the partial energy of \( z_i \) to be \( E_i(Z, P, W) = G(z_i, (Z - z_i) \cup P) \), which determines the force \( F \) on \( z_i \), and the internal energy of \( Z \) to be \( E_W(Z) \), (for the latter the restricted weights do not sum to 1). Now if we take \( E(Z, P, W) = \sum E_i + \theta E_W(Z) \), first with \( \theta = 0 \), then the force vector components will double count the \( E_{ij} \) (or \( z_i, z_j \) ) terms, so we must set \( \theta = -1 \). Note also that the effective weights of the force vectors are automatically reflexive, so we may as well choose them so to begin with. Thus, when applying \( E \) to study \( F \), the latter must have reflexive weights. (See remarks at the end of section 1.2.4).

Theorem 3.12. Given \( F, E \) as above, (i) \( Z \) is a critical point for \( E \) iff \( F(Z) = A(Z) \). Furthermore (ii) \( Z \) is a strong local minimum for \( E \) iff \( Z \) is a stable fixed point for \( AF \).

Recall that \( A \) is the antipodal map, and that the metric enters via \( E \) on one side of the equation and via \( A \) on the other.

Corollary 3.13. The period 2 property: \( Z \) is a critical point for \( E \) implies \( F^2(Z) = Z \). Furthermore \( dAF(Z) \) has real eigenvalues, as does \( dF(Z) \).

Equivalence has to be considered to define stable and strong local minimum appropriately, see (5.10). The reality property is a byproduct of the relation to \( E \) as will be seen in the proof of (3.12)(ii) above. It should also follow from invariance of \( F \) by complex-conjugation with respect to the complex structure of \( \mathbb{CP}^1 \), and weight reflexivity. (What is the most general statement about reality properties of the power series? perhaps in terms of symmetric polynomials invariant for a Lie group, note that there is an interesting implication for the dynamics, of nonrotation at attractors).

The hessian of energy at \( X \) critical has a simple functional relation to the linearization of \( F \) at \( X \), using corollary 4.2. Furthermore, degenerate directions for the hessian correspond precisely to "first-order jets of configurations, staying period 2 to 2nd order". We do not see any easy approach to show,
Conjecture 3.14. There does not exist a degenerate direction \( v \) for the hessian, \( \nabla^2 \mathcal{E}(Z)(v, v) = 0 \) of \( \mathcal{E} \) at a local minimum \( Z \), such that \( v \) is orthogonal to the isometric (rigid motion) orbits.

Though this rigidity property should be true for any reasonable notion of equidistribution. On the other hand, the more general dictionary relating \( \mathcal{E} \) to \( \mathcal{F} \) may, at least, have a nice application to showing local minima are isolated, (5.13). As obvious as this sounds, we have not seen any such proof.

A better conjecture might propose effective lower bounds for the hessian. It would be interesting to have a geometric description of configuration perturbations corresponding to eigenvectors of the hessian; we would guess that they should be related to rigid motions of hemispherical sub-clusters of a configuration, i.e. small eigenvalues of the hessian should be related to “twist” maps (as a first approximation) on hemispheres and shearing along the equator. “large eigenvalues” likewise correspond to the associated complex-conjugate flow; opposing dilations from the poles on each hemisphere. Since there are too many such shears, they should probably be averaged using eigenfunctions of the Laplacian that are antipodally antisymmetric. The relation to twist maps is suggested by empirical observation (and not enough of it), but there is no theoretical mechanism relating Hessians to twist maps that suggests itself.

Is there an interesting asymptotic behavior of these Hessians as \( n \to \infty \)? For example do they become \( \text{SO}(n) \) invariant (ie round). This would diminish the potential importance of the basis just described, this seems unlikely in view of the incompatibility with the topology/symmetry of \( S^2 \). We’d expect them to be as round as possible subject to some natural geometric constraints.

3.3.1. analytical aspects of equidistribution. Although we do not discuss analytical aspects of the equidistribution problem in detail here, we mention some more classical problems that might relate to the material here. The survey [34], as well as other references below, provide more background and details.

Question: Rigorously estimate asymptotics for the number, \( N(n) \) of local minima of \( \mathcal{E} \), as a function of the size \( n \), of configurations, and estimate the distribution of values of \( \mathcal{E} \) at the local minima. It seems that \( N(n) > 0 \) is the only known rigorous lower bound; aside from numerical techniques, which don’t treat \( n \to \infty \).

Question: Relate the combinatorics of canonical triangulations associated to \( Z \), (valences etc.) to the local minimum property. There is much work on this, mostly experimental, a recent reference being [4], but few rigorous results for large \( n \). There are some proofs of uniqueness of minima for very small \( n \), [10, 24]. Combinatorial structures of configurations, given a spherical metric, involve associated triangulations, the simplest of which are more naturally associated to the sphere or circle packing version of equidistribution. Using Morse-Smale complexes might be more pertinent for equidistribution associated to functions like \( \mathcal{E} \). Conformally intrinsic combinatorial structures can be developed using Thurston’s notion of hyperbolic convex hull.


The full relation between \( \mathcal{C} \)en and \( \mathcal{E} \) is a consequence of some basic properties of \( G(x, y) \). We continue with the notation of section 3.3. In this section we establish this relation first in the one variable case, and most of our work is to establish this
relation in the case of just 2 points, the rest follows essentially by averaging over all pairs. We achieve the proof of part (ii) of theorem 3.12 in this section and lay some groundwork for part (ii).

\( \triangle \) refers to the laplace beltrami operator (trace of hessian) on the round sphere, \( S^2 \), the normalizing constant factor is not important here, it is absorbed by an undetermined \( c \) factor in various equations below, and can be fixed later (somewhat arbitrarily). We have structured all proofs so as to avoid any explicit dependence on the normalizations. \( \Delta_x \) indicates which variable the \( \Delta \) applies to, when this might otherwise be ambiguous. The stereographic projection usually maps the plane to the sphere, but we find it convenient to denote this correspondence by \( \Pi_x : S^N - x \to T_{Ax} S^N \).

We begin with the elementary properties of \( \mathcal{E} \) that make it useful in studying \( \mathcal{F} \).

**Proposition 4.1.** (i) The log-energy is the Green’s function,

\[
\forall x \in S^2 - z; \quad c \Delta_x \log(|z - x|) = \Delta_x \mathcal{E}(z, x) = \frac{1}{2} > 0.
\]

and (ii) the force \( F_x(z) = -\nabla_x \mathcal{E}(z, x) : z \in S^2 - x \to T_x S^2 \) gives a conformal isomorphism of \( S^2 - \{x\} \) to \( T_x S^2 \), for each \( x \).

**Corollary 4.2.** For an appropriate choice of \( c \) and for each fixed \( x \), \( F_x(z) \) (or \( (dA)F_x(z) \)) is isomorphic to \( \Pi_x \), the stereographic projection of \( x \). In fact, \( F_x(z) \) equals the stereographic projection from \( Ax; \Pi_{Ax} Az \), of the antipodal \( Az \).

Remark 4.3. (i) It should be clear that some potential function will give a force vector isomorphic to stereographic projection, in view of radial symmetry. The particular function which works due to its first derivatives, has a second derivative with the nice properties discussed here, and this is what makes things interesting. In other words its not difficult to relate a map like \( \mathcal{F} \) to discrete steepest descent, but its (at first sight) miraculous that the resulting steepest descent process actually has good convergence properties. (ii) One should expect that any maps intrinsic for the conformal structure should be constructible in terms of cross ratios. Though \( Cen \) doesn’t have a direct definition of this sort, such a relation emerges here in section 4.0.5. (iii) The constant \( \frac{1}{2} \) in the equation above is somewhat arbitrary; it effectivley fixes a normalization of the laplacian; thus in all that follows we use this \( \frac{1}{2} \), in place of a normalizing constant factor. (iv) part (i) will not hold for \( S^N, N > 2 \), though we recover an inequality in section 8, but (ii) is valid for any \( N \). (v) We provide explicit formulae that can be used to give alternative proofs of both parts in ((8.2,8.3)).

The hidden relation of the 2 properties in proposition 4.1 will be clarified in the proof, via Kaehler geometry, [23], see theorem 4.6. Much of our use of the Kaehler formalism is just a natural way of treating a class of spherical harmonics, as in (4.10), by using line bundle terminology. A polar coordinate version of the laplacian would be a reasonable substitute.

4.0.2. **Proof of theorem 3.12, first steps.** By corollary 4.2, the total force exerted on \( x_i \) by the other clamped or variable points is now clearly just the weighted average of the individual forces, but this is also the weighted average of stereographic projections, which is just \( Cen \), (up to stereographic projection \( \Pi_{Ax}^{-1} : T_x S^2 \to S^2 \) followed by an antipodal map \( A \)), summarizing.
Theorem 4.4. Given a divisor $PW$ with real weights, and the associated energy function $E_{PW}$, with force $F_{PW}(x) = -\nabla_x E_{PW}(x) \in T_x S^2$, $r_{PW}(x) = A\Pi^{-1}_x(x + F(x))$, $(F = F_{PW}$, as shorthand).

$r_{PW}$ is as in (2.1). The $x + F(x)$ is in terms of vector addition; it represents a tangent vector to $S^2$ in the embedded affine tangent space. The first point of theorem 3.12 is an immediate consequence. (see also remarks at beginning of section 5.2).

Let us isolate one point from (4.4); given a section of the tangent bundle, $v \in \Gamma T\mathbb{C}P^1$, (ie a vector field, such as $-\nabla E_Z : \mathbb{C}P^1 \to T\mathbb{C}P^1$), define $R_v : \mathbb{C}P^1 \to \mathbb{C}P^1$

by

(4.1) \[ y = R(x; v) = R_v(x) \iff -c\nabla_x \log |x - y| = v(x) \]

$R_v$ is well-defined by proposition 4.1. $c$ is defined by (4.2). Note that the 1-1 correspondence of vector fields on $\mathbb{C}P^1$ to self-maps of $\mathbb{C}P^1$ thus extends from the special case of energy functions and holomorphic maps, (4.14), to arbitrary smooth vector fields and maps. Restating our result in these terms, (using $\phi_{ZU}$ rather than $r_{PW}$ as notation):

Proposition 4.5.

\[ \forall x \ R(x; \nabla E_{ZU}) = \phi_{ZU}(x), \ \nabla E_{ZU}(x) = 0 \iff A\phi_{ZU}(x) = x. \]

The second point of theorem 3.12 will follow in section 5 using theorem 4.9 with a little more work, (continued at (4.14)).

4.0.3 The proof of proposition 4.1 follows from;

Theorem 4.6. Consider $E : X \to \mathbb{R}$, where $X = S^2 \times S^2$ with $w$ the Kaehler form for the round metric on $X$. Then $c\sqrt{-1}\partial\bar{\partial} \log |x - y| = \frac{1}{2}w$ on $\mathcal{D}^\prime$.

where, as before we choose normalization of $\sqrt{-1}\partial\bar{\partial}$ compatible with the 2-dimensional laplacian to justify the factor $\frac{1}{2}$. The Kaehler form $w$ can be thought of as the curvature form of the dual to the tautological bundle, [18].

The first point of proposition 4.1 is an immediate consequence. For the second point, consider the mixed term; $\partial_y \partial_x \log |x - y| = 0$, the inner derivative is the force, but complexified, (we use the $S^2$ metric to identify the one form with a vector field). The outer derivative says that the force is holomorphic; one should check that the Cauchy-Riemann equations of the complexified vector field implies conformality of the real vector field, viewed as a map to the complex line.

Theorem 4.6 is proved, in turn, using;

Lemma 4.7. There exists $\sigma \in \text{End}(T_x \mathbb{C}P^1, T_y \mathbb{C}P^1)$ is $K^{-1}$, a Moebius -invariant (diagonal action) holomorphic section of the anticanonical bundle on $X = S^2 \times S^2$, and $|\sigma(x, y)| = c_1|x - y|$, where the norm $|\sigma|$ is with respect to the canonical constant curvature metric $g$ (of $K^{-1}$).

4.0.4 Proof of lemma 4.7 and theorem 4.6: We provide a purely geometric proof (lie one based on symmetry considerations, with no long calculations). A round $S^1 \subset S^2$ is one defined by a planar cross section.

Claim: Given $x, y, z \in S^2$ there is a round $S^1(x, y) \subset S^2$ such that $S^1(x, y) \ni x, y$ and $x, y$ is a diameter of $S^1$. Choosing $S^1 \ni x, y$ with small a diameter as possible suffices. Proof: Note that for any pair of points $x, y \in S^2$ the only invariant is their distance, ie there is an isometry taking $x, y$ to $x', y'$ iff $|x - y| = d(x, y) = d(x', y')$ (qed).
It follows that \(|x-y| = cx \cdot \text{circum}(S^1(x,y))\) (i.e., circumference). This is the first step to relating the extrinsic distance \(|x-y|\) to a more intrinsic distance.

**Claim:** \(\exists \gamma \in \text{Aut} = O(3,1)\) such that \(\gamma\) applied to \(N,S\) (the poles north, south) gives \(x,y\), and such that the great circle \(S^1(N,S) \mapsto S^1(x,y)\).

Here we can suppose without loss of generality (by composition with an isometry applied to \(x,y\), but not \(N,S\)) that the line \([x,y]\) is parallel to the line \([N,S]\), and we can choose \(\gamma\) to fix \([E,W]\) (east, west) so that \(\gamma\) transports the great circle \(S^1(N,S)\) orthogonal to \([E,W]\) to the circle \(S^1(x,y)\) orthogonal to \([E,W]\), and such that \(d\gamma(E) \in \mathbb{R}\).

By rotational symmetry, \(|d\gamma(N)| = |d\gamma(S)|\) (defined using the metric) is just the ratio of the \(\mathbb{R}^3\)-radii (or lengths of the circles, up to a constant factor); \(\frac{1}{2}|x-y|\), of the circle \(S^1(x,y)\) to that of \(S^1(N,S)\).

This proves that
\[
(|d\gamma(y)||d\gamma(x)|)^{1/2} = c|x-y|
\]
for this particular \(\gamma\), but \((|d\gamma(y)||d\gamma(x)|)\) is independent of \(\gamma\) by corollary 2.4, showing:

**Proposition 4.8.** If \(\gamma \in \text{Aut} = O(3,1)\) such that \(\gamma\) applied to \(N,S\), (the poles north, south) gives \(x,y\), respectively, then \((|d\gamma(N)||d\gamma(S)|)^{1/2} = c|x-y|\), (for some constant \(c')\).

Note that \(d\gamma(N)e_N \otimes d\gamma(S)e_S = \sigma(x,y)\) (fixing vectors \(e_N, e_S\) at the poles) can be regarded as a Moebius -invariant holomorphic section \(\sigma\) of the anticanonical bundle \(TS^2_1 \otimes TS^2_2 = \text{Eul}(T^* \mathbb{C}P^1, T_0 \mathbb{C}P^1)\), (by corollary 2.4 again), and its norm, as above, can be taken in this bundle using the standard metrics. This proves the lemma. Now the theorem follows by noting that \(-c\partial \bar{\partial} \log |\sigma|\) is (up to a factor) the curvature form of the standard metric. (\(\sigma\) is invariant for the diagonal action, while \(w\) is invariant for the full isometry group).

It follows by averaging that for the most general form of the energy function, (recall remarks in section 3.3),

**Theorem 4.9.** \(\partial \bar{\partial} E_{PW}(x) = \frac{1}{2}w\) on \(P'\), in the one-variable case, and \(\partial \bar{\partial} E_{PW}(Z) = \frac{1}{2}w\) on \(D'\), in the multi-variable case, where \(w\) is the Kaehler form for the round metric on \(X = (S^2)^N\).

One checks, using the partial energies, that in the multi-variable case, the complex hessian in the component \((z_i)\) basis, is diagonal with entries \(\frac{1}{2}\).

4.0.5. **Remarks:** cross-ratios, symplectic forms and energy as a hamiltonian. This subsection gathers together interesting relations of the energy to other structures, but it is not essential to the main theorem and one can jump to (4.14) without harm.

\(\sigma\) could be defined by differentiating the cross-ratio twice to produce equivariant \(\sigma\) naturally. The cross ratio \(X(x,y,z,w)\) is defined as \(M(w)\) where \(M\) Moebius sends \(x,y,z\) to 0,1, \(\infty\). Differentiating twice,

\[
S_X = \partial_x \partial_{x'} X(x,z',z)\] along \(\{x = x', z = z'\}\) determines a holomorphic Moebius -invariant 2-form: \(X\) vanishes to 2nd order near this diagonal, and the 2 derivatives specified pick out the leading order term, hence it is non-trivial.

This actually gives a meromorphic section of the dual bundle, \(K\), in fact the real part corresponds to a symplectic form \(S\) which blows up on the diagonal, \(x = z\).
This is essentially the canonical symplectic form, \( S \), on the space of geodesics on \( H^3 \). Thus it generalizes to higher dimensional spheres, the essential point being (2.4) in any case. In fact, \( S \) is just the standard symplectic form for \( T^*S^2 \), where \( S^2 \times S^2 - \mathcal{D} = T^*S^2 \) reveals the \( Z_2 \) (anti-)symmetry interchanging factors. In this context \( \mathcal{E} \), which depends on the choice of a metric, is none other than the hamiltonian for geodesic flow—the flow in \( T^*S^2 \) associated to the round metric on \( S^2 \). We wonder if the hamiltonian system given by an appropriately averaged lift of these structures to \( (S^2)^n - \mathcal{D} \) may have interesting properties, dynamics? 

Aff-6: Furthermore, \( \sigma \) actually defines the affine structure of \( \mathbb{C}P^1 - x \); given a vector \( v \) at \( y \), \( \sigma \) provides a global holomorphic one form on \( \mathbb{C}P^1 - y \) and this determines an affine structure which is independent of \( v \). (\( T_y \) being one dimensional). In fact even in higher dimensions we get from \( S \) a basis of one forms satisfying Frobenius, and determining \( A_y \). A priori the affine structures are weaker than \( \sigma \). Could \( \sigma \) similarly be defined using \( A_y \)? By adapting the remarks above, (to the real cross-ratio), to recover \( S \) it suffices to construct the cross-ratio from \( A_y \), but this is trivial using \( x = \infty \). To fully recover \( \sigma \) one needs to use something equivalent to the conformal structure or its automorphism group (this is possible, and implicit in (4.18, 8.7), possibly to be discussed elsewhere).

4.0.6 Poly vs \( \mathcal{E} \), energy as a homogeneous polynomial. We recall some basic facts of geometry, [18]: a degree \( d \) polynomial on \( \mathbb{C}^2 \) is a section of the \((-d)\)-th power of the tautological line bundle, \( \tau \mathbb{C}P^1 \). Every holomorphic line bundle \( L \mathbb{C}P^1 = \tau^{-d} \) is determined by its degree, \( d \), as a power of the tautological line bundle (Pic(\( \mathbb{C}P^1 \)) \( = \mathbb{Z} \); the Lefschetz-Grothendieck pencil theorem), in particular the tangent bundle, \( T \mathbb{C}P^1 = \tau^{-1} \mathbb{C}P^1 \) by topological considerations. The curvature of \( L \mathbb{C}P^1 \) is \( -(d)w = c_2(\mathbb{C}P^1, 1) \log |s| \) for any holomorphic section \( s \neq 0 \).

We have seen relations of energy to Ras , theorem 4.4 and of Ras to Poly, subsection 2.2.1. The relation of energy to Poly is even simpler (using homogeneous coordinates):

**Proposition 4.10.** \( \log |x - y| \) is equal to the function \( h = c_3 \log |(x_1y_2 - x_2y_1)| \) on \( \mathbb{C}^2 \times \mathbb{C}^2 \) restricted to \( S^3 \times S^3 \) and pushed forward to \( \mathbb{C}P^1 \times \mathbb{C}P^1 \).

**Proof.** \( (x_1y_2 - x_2y_1)^2 \) is a holomorphic section of the bundle \( \mathcal{O}(2) \otimes \mathcal{O}(2) = TS^2 \otimes TS^2 \) vanishing on the diagonal, and having the same divisor as \( \sigma \), hence it is a multiple, \( \sigma \). Furthermore with canonical metrics, \( \mathcal{O}(2) \otimes \mathcal{O}(2) = TS^1 \otimes TS^1 \) as hermitian bundles, by symmetry considerations. \( \square \)

The push-forward is easily justified by \( S^1 \)-invariance.

**Theorem 4.11.** Given \( W \) defined over \( \mathbb{Q} \), \( 3\Phi, \mathcal{E}_{PW} = \alpha \log |\Phi| \) for \( \alpha \in \mathbb{Q} \) and \( \Phi \) a polynomial on \( \mathbb{C}^2 \) as above.

**Remark 4.12.** This means that extrema of energy are just the extrema of norms of homogeneous holomorphic polynomials restricted to \( (S^2)^n \).

As an interesting aside relating Morse theory of plurisubharmonic functions to fixed-points of \( \mathcal{A}F \) or \( \mathcal{F}^2 \):

**Corollary 4.13.** \( \partial \mathcal{E}(Z) = \frac{1}{2}w \), \( Z \in (S^2)^n \) - \( \mathcal{D} \); \( w \) as above. In fact \( \mathcal{E} \) is a strictly plurisubharmonic exhaustion function on \( (\mathbb{C}P^1)^n - \mathcal{D} \), (the diagonal, or discriminant) as is its pushforward to the \( S_n \) quotient \( \mathbb{C}P^n - \mathcal{D} \).

Note that we relate the Morse theory of \( \mathcal{E} \) to the topological Lefschetz theorem for the one variable maps in (5.1). The question of a multivariable analog is thus
suggested. The various relations of energy, Ras and Poly provided to this point clearly leave something to be desired and we will discuss below some generalizations that come closer to showing the full picture. It seems likely, in view of [35] and references therein, that the relation of Poly to $\mathcal{E}$ was known classically, and it is certain that (4.1)(i) was known classically, but the “geometric” proof we give is probably new.

4.1. Connections encoding maps. To pursue our proof of theorem 3.12 we will need (compare proposition 4.1 and theorem 4.4)

**Proposition 4.14.** Given a smooth function $E : U \subset \mathbb{C}P^1 \to \mathbb{R}$, with force $Fe(x) = -\nabla_x E(x) \in T_x S^2$, and its associated map, $r_E(x) = A\Pi^{-1}_{Ae}(x + Fe(x))$, $r_E$ is holomorphic iff $\Delta E = \frac{1}{2} x$.

Note that the correspondence is local in $S^2$. In fact there is no need to restrict to functions here, and we will prove a generalized version,

**Lemma 4.15.** Given a smooth (force) vector field $F : U \subset \mathbb{C}P^1 \to \mathbb{R}$, its associated map, $r_F$, is holomorphic iff $\text{Div} F = -\frac{1}{2} y$.

As above, $r_F$ is the map naturally associated to $F$, see (4.1)). We identify as usual the space of (local) one-forms $\eta$ such that $2d^*\eta = 1$ with the space of $\mathcal{F}$ as above, using the metric of $S^2$. We give two proofs of the lemma; first a simple calculation proving the 2 statements above, followed by some general observations summarizing the relations of maps to connections leading to an alternative, more geometric, proof. Note that the generalizations of (4.14,4.15) to holomorphic maps with fixed-points, allowing for isolated logarithmic poles of $E$, follows by removable singularities and continuity at fixed points.

**Proof.** Given a holomorphic map $h(x) = y$, on $U$, the corresponding form (or vector field) is $u(x) = -c \partial_{y=b(x)} \log |x - y|$ where the derivative is taken with respect to $x$ only (in $x - y$), the substitution of $h(x)$ coming after the differentiation. (Note that this is not a holomorphic form). Now $\partial u(x) = -c \partial_{y=b(x)} \partial \log |x - y| = -\frac{1}{2} w$ because $\partial y = 0$ and by theorem 4.1. This proves only if, but if follows directly because the correspondence is 1-1 and onto; as seen by localizing the construction, using well known global approximations, of holomorphic or harmonic functions by rational functions.

One-forms correspond to connections using the Levi-Civita connection of $S^2$, and we get a nice geometric interpretation of the statements above in terms of connections, as well as an alternative proof. The rest of this section involves the geometry of connections, polyhedra, etc. and their relation to Ras, the reader wanting to see a proof of theorem 3.12 as quickly as possible may skip directly to the next section (with a quick glimpse at section 4.3).

We next consider locally flat holomorphic affine structures with singularities at $z_i$ (we frequently use $Z$ rather than $P$ in the rest of this section, for fixed points, with $x$ as a variable when necessary).

**Lemma 4.16.** The space $\mathcal{L}$ of singular holomorphic (flat) affine connections on $T(\mathbb{C}P^1 - Z)$, with complex monodromy, $\lambda_i$ at singularities $z_i$ is in canonical 1-1 correspondence to divisors $WZ$ with mass one, where $\lambda_i = \sqrt{-12 w_i}$. They are thus denoted by $D_{WZ}$. 27
Existence and uniqueness of such connections are discussed in [8] but can also be obtained from the following, (using theorem 2.1) and this provides explicit expressions for $D_{WP}$ via $r_{WP}$ in Cen form.

**Definition 4.17.** Given $D \in \mathcal{L}$, $L2R : \mathcal{L} \rightarrow \text{Rat}$ is the meromorphic map defined by $r = L2R(D)$, such that $\forall x \in \mathbb{CP}^1$, $A_{r(x)} = D(x)$ (as connections at $x$), identifying the affine structure $A_{y}$ with the affine connection on $\mathbb{CP}^1 - y$.

This makes sense on $S^N$ in any dimension, using connections over $\mathbb{R}$, but it is only for $N = 2$ that we can also apply the construction using connections over $\mathbb{C}$, and obtain the meromorphic structure.

**Theorem 4.18.** $L2R$ is well-defined and it commutes with the canonical 1-1 correspondences of $\mathcal{L}, \text{Rat}$, to divisors, (as in the lemma above and (2.1) ). $L2R$ is a holomorphic isomorphism between $\mathcal{L}$ and $\text{Rat}$ away from the degenerate divisors.

$$r = L2R(aD_1 + (1 - a)D_2) \iff r(x) = M_z^{-1}(aM_x r_1(x) + (1 - a)M_x r_2(x))$$

where $r_i = L2R(D_i)$, with $M_x$ as in the lines preceding (3.6).

For more on the last point see *suspenion*, section 6. We call this construction of $r$ in terms of $r_i$ the **combination principle**; it is used to break up divisors into pieces to facilitate proofs in section 6, as well as in (8.5.8.7). The extension (by surgery) of $L2R$ at degenerate divisors was discussed in section 2.1.

**Proof.** $L2R$ is well-defined, holomorphic and 1-1, onto, by the remarks preceding (2.11). A local version of the correspondence follows, $\mathcal{L}_B \rightarrow \text{Rat}_B$ for $B \subset \mathbb{CP}^1$ a small nbhd, from connections on the nbhd to maps $r : B \rightarrow \mathbb{CP}^1$ from the nbhd. (in the holomorphic or smooth categories).

To establish the correspondence to Div, it remains to verify that $\lambda_i = \sqrt{-1}2w_i$ under this correspondence. It suffices to show that there is an induced correspondence between $\lambda_i, w_i, L2r : \mathbb{C} \rightarrow \mathbb{C}$ which is 1-1 and holomorphic, recalling that the factor of $\sqrt{-1}2$ comes from normalizations of weights and Gauss-Bonnet. To see that the monodromy $\lambda$ only depends on the weight $w$ we refer to (4.21) which suffices for $w \in \mathbb{R}$, but analytic continuation for the holomorphic $L2R$ extends this to $w \in \mathbb{C}$.  

4.1.1. connection and force-matching. This “connection-matching” should be compared to force-matching; in theorem 4.4 we saw $r_{PW}$ is isomorphic via $\text{All}_{x^{-1}}$, to discrete steepest descent by $F(x)$ we call this *force-jumping* insofar as one follows the force vector (note that this jumping construction also defines a transformation of smooth vector fields to smooth maps). By proposition 4.1:

**Lemma 4.19.** $z = \text{All}_{x}^{-1}(x, v) \iff v = F_z(x)$. (here $(x, v) \in T\mathbb{S}^2, v \in T_x\mathbb{S}^2$)

Here, $F_z$ is the force on $x$ due to $z$. The antipodal relation of matching to jumping is a special property of the functions $E_{PW}$, as follows from (4.2) (or of vector fields as in (4.15)).

**Corollary 4.20.** $r_{PW}(x) = y \iff F_{PW}(x) = F_y(x)$, and we call this force-matching.

The correspondence in theorem 4.18 has an interesting specialization to real weights, (we use the standard formula for curvature under conformal rescaling of metrics in 2-dimensions, $K_\phi = -\Delta \log \phi + K$).

** Proposition 4.21.** Using $\text{exp}((\frac{1}{2}) E_{Z^2})$ as a conformal factor for the round metric $g$ on $\mathbb{S}^2$ gives a flat metric $h$ on the complement $Z'$ and the associated flat connection, $D$. An inverse correspondence is gotten likewise; a flat metric $h$ on $Z'$ determines a conformal factor $\phi = \frac{\lambda}{2}$ and $-\frac{1}{2}\log \phi$ satisfies the same PDE as the energy $E$.  

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In particular note that metric connections correspond precisely to real weights. The singularities of \( h \) at \( Z \) give cone singularities with vertex angles precisely \( K(S^2)u_i \) at \( z_i \), i.e., proportional by the total curvature, to the weights, so that \( \sum u_i = 1 \) by Gauss-Bonnet (one can calculate the conformal metric at a cone point using the pullback of the metric \( |dz|^2 \), by \( z \to z^\nu \)). The singularities of \( D \) at \( Z \) give monodromy equal to \( \sqrt{-1}u_i \) at \( z_i \), (see [38] for much more on the polyhedral interpretation of \( D \).) See section 8.4 for a subtle aspect of the relation of these metrics to equivariance.

One should by now expect that connection-matching is essentially the same thing as force-matching, and that the equivalence associates the one form \( \eta \), such that \( D = D_{\nu} + \eta \), to the (force) vector field \( F \), in the usual way. There may be a constant factor intervening, based on normalizations, but otherwise this follows by equivariance properties, notably invariance with respect to affine reflection in a line. It is good to know that a metric relation between holomorphic objects just barely hides a purely holomorphic relation between these objects.

4.1.2 Informal summary of Poly to \( \mathcal{L} \) correspondence. The homogeneous polynomial, \( p : \mathbb{C}^2 \to \mathbb{C} \) with \( n = \text{deg}(p) \) is a section of \( \tau^{-n}\mathbb{C}P^1 = T^*(\mathbb{S}^n/2)\mathbb{C}P^1 \). Thus \( p^{2/n} \) provides local holomorphic sections of \( T^*\mathbb{C}P^1 \), i.e., holomorphic one forms, well-defined up to locally constant factors away from the roots \( r_1 \in Z \) of \( p \), and they can be regarded as locally flat holomorphic affine structures. Furthermore there is no need to insist on integer exponents (i.e. multiplicities of factors) for the polynomials; a standard use of logs shows that for any divisor WP with complex weights of total mass 1, \( \Pi(z - p_i)^{n_1} \) (with \( z - p_i \to z_1 \xi_2 - z_2 \xi_1 \) in homogeneous form) provides well-defined locally flat holomorphic affine structures as above, with singularities at the roots \( p_i \).

The natural transformation \((2.4)\) from polynomials to rational maps becomes the special case, for maps with symmetric weights, of a correspondence of \( \mathcal{L} \) to \( \text{Rat} \). In fact theorem 4.18 extends this to a transformation from local systems to rational maps, which is onto \( \text{SRat} \) (simple fixed-points), see also subsection 2.1. The correspondence \( p^{2/n} \) of Poly to \( \mathcal{L} \) here is based on the abstract isomorphism \( \tau^{-n}\mathbb{C}P^1 = T^*(\mathbb{S}^n/2)\mathbb{C}P^1 \) (via the pencil theorem) and we do not know of a concrete geometric realization of the latter, even for \( n = 2 \) (it is well defined only up to a factor in \( \mathbb{C}^* \)). Thus it is nice to get the simple expression in proposition 4.22. Passing from one-forms to connections kills this \( \mathbb{C}^* \) ambiguity and makes possible such formulae as we proceed to show.

4.1.3 Homogeneous polynomials and flat affine connections: Given a homogeneous polynomial, \( p : \mathbb{C}^2 \to \mathbb{C} \) with \( n = \text{deg}(p) \), the roots of \( p \) determine a connection \( D_p \in \mathcal{L} \), on \( \mathbb{C}P^1 \) with symmetric weights as above. Let \( L \) be an affine complex line \( 0 \not\in L \subset \mathbb{C}^2 \). In view of the chain of correspondences above, we should express \( D_p = D_L + \eta \) where \( D_L \) is the standard flat affine connection of \( L \) and \( \eta \) is of the form \( C(d \log(p_L)) \) where \( p_L \) is the restriction of \( p \) to \( L \), and \( C \) is to be determined. Considering the simplest case, \( p(x) = (x_1y_2 - x_2y_1) \) and \( L = \{(x : (x_1\lambda_2 - x_2\lambda_1) = 1)\} \) choose \( C \) so the poles of \( D_p = D_L + \eta \) cancel at \( x = \infty \), (i.e such that \( D_p \) extends smoothly across \( x = \infty \) in \( L \)), noting that \( p_L \) blows up at \( \infty \) in \( L \) (unless \( p_L = \text{const} \), in which case \( x = \infty \) should be the singularity of \( D_p \) and \( D_p = D_L \); \( x = \infty \) is the root of \( p \) in this case). The general case follows likewise, (by factorization, and averaging, as above) with \( c_n = \frac{\xi_1}{\xi_2}, c_1 = C \), and we get the formula:
Proposition 4.22. The local system for a divisor $Z$ with symmetric weights, $\frac{1}{n}D_Z = D_L + \frac{1}{n} \log p_Z$ where $p_Z$ is a homogeneous polynomial with simple roots $Z$.

Remark 4.23. A more direct approach to this formula might be to solve $Dp\eta = 0$, i.e., $p$ is parallel wrt $Dp$.

4.2. Discussion: Natural correspondences. To summarize the structure that has developed: we have studied analytic objects which correspond to divisors:

<table>
<thead>
<tr>
<th>analytic object</th>
<th>type of weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>polynomials</td>
<td>symmetric</td>
</tr>
<tr>
<td>energy functions</td>
<td>real</td>
</tr>
<tr>
<td>singular flat-connections and Rat</td>
<td>complex</td>
</tr>
</tbody>
</table>

The divisors are formal algebraic objects, and they determine formal correspondences or embeddings between the associated analytic objects, but in each case there is a more direct geometric construction of the correspondences between analytic objects which leads to localized constructions and extension to smooth categories etc. It is at this level that the correspondences have some interesting applications (see remark below on symmetry breaking). It is essential that the analytic objects are conformally intrinsic (or almost so, in the case of energy—see section 8.4); they are determined by linear elliptic PDEs with the divisors as forcing terms (except Rat). Note also that passing to the smooth categories amounts, in some but not all cases, to passing to the limit of divisors converging weakly to smooth measures. This issue arises many times in section 8.

There is a 3rd level, a sort of meta-theorem (not meant to be formalized) which explains these geometric constructions and in particular the relation of elliptic cen-
trums to differentiation, this is the matching construction: given such an analytic object, $f$, the germ of $f$ at a point $p$ in the support of its divisor is represented by a Taylor series. The lowest order terms might depend only on self-interactions of $p$, (the multiplier at a fixed-point for example), but the next term, $\tau$, in each case depends on the average influence of the other $p_j$. Matching amounts to comparing this value of $\tau$ to that due to a single point $q$: in an object of the same class, $f_2$, whose associated divisor has support exactly $p, q$, we get the same value of $\tau$ at $p$. Because these objects are conformally intrinsic, they are well behaved with respect to automorphisms, and this provides a 1-1 correspondence of points $q \in \mathbb{CP}^1 - p$ to values of $\tau$. Thus there is a unique $q$ with the same $\tau$ as the “average influence of the other $p_j$”.

Aff(7): We can recover the affine structure $S^2 - z$ by embedding $S^2 - z$ canonically to affine subspaces of algebraic functions or one-forms; (i) embed $S^2 - z$ to degree 1

Poly canonically; $x \in S^2 - z \mapsto px$ with one root, $x$, and normalized st $px(z) = 1$, where $z \in \mathbb{C}^2 - 0 \to S^2 = \mathbb{CP}^1$ is a fixed representative of $z$. For the analogous embedding to one-forms, choose $px$ st $px(s) = 0$ iff $s = x, z$, and normalize, in addition, $dp_x(z) = v \neq 0$. Note invariance of the normalization under averaging. This induces an affine structure on $S^2 - z$ in an equivariant manner. The essential point is linearity of the PDE (d-bar) and holomorphic dependence on the divisor. Though this only works in dimension 2, one might exploit the fact that higher dimensional affine structures can be characterized in terms of their 2-dimensional affine subspaces (closely related to characterizing flatness in terms of 2-dimensional affine subspaces) to obtain a general version of this construction.
4.2.1. Applications of natural correspondences. We list some observations, results and potential applications of the existence of these correspondences:

(1) Recall that Gauss-Bonnet is “conjugate”, (2.1, 4.18) under such a transformation to the holomorphic Lefschetz fixed-point theorem for Rat. Similarly we will see an equivalence of topological Lefschetz fixed-point theorem to the Morse theory equality for the Euler class, for maps of degree \(-d\) on \(\mathbb{CP}^1\), in section 5.1. The question of multivariable analogs arises.

(2) Compactifications of Rat and of \(\mathcal{L}\) as roots collide are quite different. We briefly considered bubbling and surgery in this context, section 2.1. Each space has a different class of canonical algebraic subvarieties, it may be interesting to compare them. Similarly the natural invariants in each space are quite different; spectra for polyhedra, or zeta functions for Rat, for example. One wonders if there are interesting relations of some such invariants?

(3) We will show, in section 8, how to identify smooth self maps of the sphere with certain singular measures on the sphere. This might be useful in studying the topology of mapping spaces.

(4) Extracting roots of polynomials involves a symmetry breaking process, which obstruct existence of formulae for roots, while transforming polynomials to rational maps or potential-energy functions can be done without any symmetry breaking in a very simple way. We will see below, section 6, how this gives iterative methods for extracting roots.

(5) The local minima of the energy functions have been much studied numerically, but its difficult to give conceptual proofs of any of their properties. Transforming to the context of holomorphic functions and spherical harmonics, or rational maps may help.

One more observation to mention in passing: there is a well-defined, but singular holomorphic quadratic differential \(\xi_{Z,U}\), representing the distortion of projective structure of the conformal map, (identity map) from the round projective \(\mathbb{CP}^1\) to the locally affine-flat hence projectively flat \((\mathbb{CP}^1 - Z, D)\). It is also holomorphic in the parameters \((Z, U)\). It is easy to see that,

**Proposition 4.24.** The zeroes of \(\xi_{Z,U}\) correspond to the critical points of \(r_{Z,U}\).

4.3. A Moment for \(\mathcal{E}\). Notice that we never claim that local minima of energy are in 1-1 correspondence with period 2 attractors for \(\mathcal{F}\), but only with fixed-points of \(A\mathcal{F}\). This because the period 2 attractors have different equivariances, they are \(O(n,1)\)-orbits, whereas local minima etc are \(O(n)\)-orbits. It is reasonable to expect that the orbits are in 1-1 correspondence; we have essentially shown that an \(O(n)\)-orbit of local minima of energy is the reduction of an \(O(n,1)\)-orbit of period 2 attractors of \(\mathcal{F}\), but the converse is not treated here—we haven’t shown that every period 2 attractor of \(\mathcal{F}\) is of this type (having antipodal behavior etc.) though we’d readily conjecture it is so. (On the other hand there may well be unstable period 2 sets, possibly even with support on \(S^1\), that don’t correspond to higher index critical points of \(\mathcal{E}\).)

It turns out that there is a particularly nice way to characterize the \(O(n)\)-sub-orbits of energy-minimizing configurations; the **vanishing dipole moment property** for equilibria is a special relation between \(\mathcal{E}\) and the Moebius action:
Definition 4.25. The dipole moment of a set of \( n \) vectors, \( z_i \in Z \subseteq S^{N-1} \subseteq \mathbb{R}^N \), is \( \mathcal{Z}(Z) = \frac{1}{n} \sum z_i \in \mathbb{R}^N \) (vector addition). (The latex code for \( \mathcal{Z} \) is \{ \text{mathcal Z} \}). This, for \( N = 3 \), is the standard moment map, for the standard \( \text{SO}(3) \) action preserving the standard volume form on the 2-sphere.

By a calculation observed in [2], which generalizes easily to the weighted case, and arbitrary dimension, Proposition 4.26. If \( Z \) is a critical point for \( \mathcal{E}_U \), then \( \mathcal{Z}(Z) = 0 \).

Proof. The total of all forces at \( z_i \) from all pairs is clearly zero (Newtonian action-reaction). But in equilibrium, forces \( F_i \) at \( z_i \) are orthogonal to the sphere, hence \( F_i = \lambda_i z_i \), and it suffices to show that \( \forall i, j, \lambda_i = \lambda_j \), but this is clear from the following lemma. \( \Box \)

Lemma 4.27. \( x \cdot \nabla \log |x - y| = \frac{1}{2} \).

Proof. Rewrite the l.h.s. as \( x \cdot (x - y)/|x - y|^2 = (1 - x \cdot y)/(2 - 2x \cdot y) \). \( \Box \)

4.3.1. dipole moment in terms of \( \mathcal{E} \), and hyperbolic center of mass. (The rest of this section is “optional”, i.e. it is not applied in the rest of this article.) We recently came across the article [28] which surveys many structures that involve configurations similar to those studied here (albeit only with a vector of weights–weighted nodes rather than a matrix of edge-weights) and it may be interesting to compare further the structures there with those presented here. In particular they includes remarks about the moment map in relation to the hyperbolic center of mass (which had been exploited in earlier work) and a similar application to normalization of \( Z \) by \( \mathcal{Z}(Z) \). Note that the moment map is by definition symplectically intrinsic whereas the hyperbolic center of mass is conformally intrinsic. (The group action implicitly provides conformal structure to the symplectic theory.) We have defined \( \mathcal{Z}(Z) \) here using the extrinsic euclidean structure.

We know of no simple analogue of the property (4.26) for other energies. It is a bit mysterious at first that the function \( \mathcal{E} \) has the properties of (4.1) as well as (4.26), without apparent relation of their derivations. Recall that (4.6) explains how the properties of (4.1) fit together, but the relation here doesn’t have any apparent connection to them. We proceed to discuss the geometry underlying this coincidence. In the process we will find some interesting relations of the hyperbolic center of mass to \( \mathcal{E} \), here and in section 8.

First note,

Theorem 4.28. \( \mathcal{E} \) restricted to an \( \text{SO}(N,1) \) orbit is well-defined, proper and convex on \( SO(N,1)/SO(N) = H^N \). The local minima form a single \( SO(N) \) orbit.

Comparing the exhaustion function \( \mathcal{E} \) used here, to other treatments of the Denny–earle centre, it is interesting to notice that \( \mathcal{E} \) is proper on all of \( H^N \), and not just a totally geodesic subspace, even when \( Z \) is in the boundary of such a subspace.

Proof. First consider the case \( N = 3 \), where we can use the complex structure on \( S^2 \); \( \mathcal{E}(Z) \) is strictly plurisubharmonic in \( Z \), and well-defined along an orbit as a function of \( SO(N,1)/SO(N) = H^N \), using isometry invariance of \( \mathcal{E} \). Now for \( N = 3 \) strict plurisubharmonicity along with the vanishing of both the gradient and the hessian along \( SO(N) \) orbits, which are totally real in configuration space, implies positivity of the hessian along \( H^N \), hence strict convexity. Note that this uses that the \( p + k \)
decomposition satisfies $\dim p = \dim \mathbb{R}^k$, which only holds for $N = 3$, and that the complex structure $J$ satisfies $J p = \mathbb{R}^k$ in this case, using the equivalence to $\text{psl}(2, \mathbb{C})$.

We now observe that $\mathcal{E}$ restricted to an $\text{SO}(N,1)$ orbit is proper; as $\gamma \in \text{SO}(N,1)$ goes to infinity, subconfigurations collapse at the attracting fixed-point on $\partial H^N$, and the last statement of the theorem follows immediately. The case $N = 2$ follows by restriction to the associated subspaces.

The general case, $N \geq 3$, follows because we can reduce both to (a) convexity of $\mathcal{E}$ in the case of 2 points, $x_i \in X$, $|X| = 2$, (by linearity; $\mathcal{E}$ as a sum over pairs) and (b) along one geodesic $\gamma$ in $H^N$. Furthermore we can suppose $\gamma$ is stabilized by a nonrotational 1-parameter group called $\lambda_t$. Now the 2 fixed-points $\lambda_{t \pm} \in \partial H^N$ of $\lambda_t$, and $\{x_i\}$ give 4 points which are all in some round $S^2 \subset S^{N-1}$, which is furthermore, (using nonrotational) $\lambda_t$-invariant, so we can apply the preceding argument, $(N = 3)$. □

Comparing the exhaustion function $\mathcal{E}$ used here, to other treatments of the Donally-earl centre, (using Bussemann functions, see also (8.20) and related discussion below) it is interesting to notice that $\mathcal{E}$ is proper on all of $H^N$, and not just a totally geodesic subspace, even when $Z$ is in the boundary of such a subspace. This reflects the introduction of the auxiliary round metric in the construction of $\mathcal{E}$.

This motivates us to next establish the identification of $\mathfrak{J}(Z)$ with $\nabla \mathcal{E}_{GZ}(Z)$, where $G$ is $\text{SO}(N,1)$, and $GZ$ is the $G$-orbit of $Z$. This will strengthen the existence of normalizations, and it shows that

**Proposition 4.29.** In the $O(N,1)$-orbit of a critical point for $\mathcal{E}_U$, $\mathfrak{J}(Z) = 0$ iff $Z$ is a critical point.

Note that this implies existence and uniqueness of the hyperbolic center of mass as in [7].

To this end we identify the Lie algebra, $\mathfrak{g} = p + \mathbb{R}^k$ of $\text{SO}(3,1)$ with vector fields $v$ on $S^2$, and on $(S^2)^n$, $|Z| = n$, by the diagonal action, and restricting $v$ to $T_Z = T_Z(S^2)^n$, we get $\mathfrak{g} \to T_Z$ and the orthogonal projection $\pi_Z : T_Z \to \mathfrak{g}$. Our goal is to relate $\pi_Z(F(Z))$, where $F = -c\nabla \mathcal{E}$ is the usual force, to $\mathfrak{J}(Z)$. Noting that $F \perp \mathbb{R}^k$ (by isometry invariance of $\mathcal{E}$) we may as well define $\pi_Z : T_Z \to \mathbb{R}^k$.

We denote by $\pi_{ave} F = \sum_i F_i(Z)$, $i$, the average, in $\mathbb{R}^3$, of net forces $F_i \in T_z S^2 \subset \mathbb{R}^3$ and note that by the proof of (4.26) including (4.27) or the more intrinsic approach below (4.32).

**Lemma 4.30.** $\pi_{ave} F(Z) = c \mathfrak{J}(Z)$.

But note that $\pi_{ave}$ is also well-defined on $p \subset T_Z$, and we claim that $\pi_{ave}$ is an isomorphism of $p \to \mathbb{R}^3$, up to the trivial exceptional case where $|Z| = 2$, of an antipodal pair. But given $p \in p$, $\exists v \in \mathbb{R}^3$, s.t. $\forall x \in S^2, \pi_Z(x) = \pi_Z v$ using the orthogonal projection to $T_z S^2$. Now $\langle \pi_{ave} F, v \rangle \geq 0$ (each summand is clearly nonnegative and can vanish only in the exceptional case noted above which we can exclude. Thus we get a nondegenerate quadratic form, and conclude that $\pi_{ave}$ is an isomorphism.

Our goal is to show that $\nabla \mathcal{E}_{GZ} = \pi_Z(F(Z)) = 0 \leftrightarrow \mathfrak{J}(Z) = 0$, and it now clearly suffices to check that $\pi_{ave} \pi_Z F(Z) = c \mathfrak{J}(Z)$.

Once more the key is to consider the case of a pair of points, $|Z| = 2$, and average. The averaging step is just a bit more subtle here; we write $F = \sum F_{ij}$, $F_{ij}$ being the contribution only from the interaction of the pair $z_i, z_j$, so $\pi_Z F = \sum \pi_Z F_{ij}$ and
\[ \pi_Z f_{ij} = \pi_Z \pi_{Z_j} f_{ij} \] since \( f_{ij} \in T_{Z_j} \subset T_Z \) (more to the point, \( f_{ij} \in \mathcal{P} \cap T_{Z_j} \subset T_Z \) by \((4.2)\)) \( T_{Z_j} \) is the \( C^2 \)– subspace of the pair \( z_i, z_j \). Thus it suffices to study \( \pi_{Z_j} f_{ij} \), i.e., to show \( \pi_{\text{ave}} \pi_Z (F(Z)) = \pi_{\text{ave}} F(Z) \) for \( |Z| = 2 \).

Furthermore, by symmetry considerations (or equivariance of \( F \) in \((4.3)\)), for \( |Z| = 2 \) it suffices to consider the \( S^1 \) case, and in the \( S^1 \) case, with \( |Z| = 2 \),

\[ \pi_Z (F(Z)) = F(Z), \]

so we are finished. In fact,

**Theorem 4.31.** \( \pi_{\text{ave}}^{-1} \mathcal{P}(Z) \in \mathcal{G} \) is equal to \( \nabla \mathcal{E}_{\mathcal{G}Z}(Z) \) up to a constant factor.

The upshot is that our normalization criterion has an expression intrinsic to the theory previously developed.

Kirwan discusses in great depth the relation of \( \nabla \mathcal{E} \) to the structure of configuration space. [22].

**Remark 4.32.** Note in the interest of proving \((4.3)\) more intrinsically note that we already know this from the relation of \( \mathcal{E}, \mathcal{F} \) for \( |Z| = 2 \), one reads off the forces at \( Z \), from \( \mathcal{F}(Z) \), to explicitly describe \( F \) in terms of \( A, \Pi \) as in section 4 (for example \((4.4)\)). Note thus that \(-2 \dot{\phi}_Z(Z) = A_{Z_2} - z_1 = A_{Z_1} - z_2 = \frac{1}{2} (F_1(Z) + F_2(Z)) = \frac{1}{2} \pi_{\text{ave}} F \).

This gives the same conclusion, in general, for \( |Z| = n \), by averaging.

5. **Energy and Stability, Stable Attractors for \( F \).**

Our main goal in this section is to produce attracting periodic cycles for \( F \) using the minima of \( \mathcal{E} \).

5.1. **one-variable case and energy.** We summarize some basic facts about the space, \( \mathcal{A} \mathcal{R}_c \), of anticonformal maps of \( \mathbb{C} P^1 \) (\( \phi_{\mathcal{Z}U} \) is the notation used here instead of \( \tau_{PU} \), \( A(x) \) is antipodal to \( x \)). Note that the hessian \( \nabla^2 \mathcal{E}_{\mathcal{Z}U}(x) \) is symmetric with trace \( \frac{3}{2} \), while \( dA \phi_{\mathcal{Z}U}(x) \) is symmetric with trace \( \frac{1}{2} \) using the conjugation. This suggests a simple correspondence between the two. We will discuss eigenvalues and eigenvectors of the hessian, though the term characteristic values may be more precise. It may be better to think of \( \nabla^2 \mathcal{E}_{\mathcal{Z}U}(x) \) as a map \( H : v \in T_x \rightarrow T_x \), defined by \( \nabla \nabla \mathcal{E} \).

**Theorem 5.1.** Given a rational map \( \phi_{\mathcal{Z}U} : \mathbb{C} P^1 \rightarrow \mathbb{C} P^1 = S^2 \), with real weights, \( x \) is a critical point of \( \mathcal{E}_{\mathcal{Z}U}(x) = c \sum_i \log(|x - z_i|) \) iff \( \phi_{\mathcal{Z}U}(x) = A(x) \).

(2) \( x \) is a strong local minimum for the energy \( \mathcal{E}_{\mathcal{Z}U} \) iff \( x \) is a stable fixed point for the conformal (but antiholomorphic) map \( A \phi_{\mathcal{Z}U} \). In fact the hessian \( \nabla^2 \mathcal{E}_{\mathcal{Z}U}(x) \) determines the linearization \( d(A \phi_{\mathcal{Z}U})(x) \) and vice versa. The eigenvalues \( \pm \lambda \) of \( dA \phi_{\mathcal{Z}U}(x) \) are real, and

(3) at a local minimum, (or more generally any critical point) the hessian has eigenvalue zero iff \( A \phi_{\mathcal{Z}U} \) has a parabolic (multiplicity \( 
abla \)) fixed point at \( x \), and \( \lambda = 1 \),

(4) If \( x \) is a degenerate local minimum (one eigenvalue \( = 0 \)) for the energy \( \mathcal{E}_{\mathcal{Z}U} \) then \( A \phi_{\mathcal{Z}U} \) has a nontrivial open parabolic basin of attraction (in fact the same holds for any degenerate critical point).

(5) the hessian has a double eigenvalue, \( 0 \) if \( A \phi_{\mathcal{Z}U} \) has a superattracting (critical) fixed point at \( x \).

(6) If \( |Z| = n > 3 \), then \( x \) (a local minimum or degenerate critical point) must attract a critical point of \( \phi_{\mathcal{Z}U} \), i.e. there is a critical point \( p \) of \( \phi_{\mathcal{Z}U} \), such that the forward orbit of \( p \) by \( A \phi_{\mathcal{Z}U} \) converges to \( x \).
We have already proved the first statement above, see (4.5), also (2.10). The
2nd iterate $A^2\phi_z$ is holomorphic, so to prove the fourth statement above, we can
apply the Léau-Fatou flower theorem, [29].

$A\phi$ is a steepest-descent method for minimizing $E$, but with large step size,
so this does not imply stability. The key points underlying stability are thus (4.6), and
(5.7). In fact, one can extend the constructions of $\phi$ and $E$ to higher dimensional
spheres, and the correspondence of fixed-points of $A\phi$ to critical points of $E$ still
holds for log-energy, but we will need extra work to determine higher dimensional
stability.

5.1.1. Critical point structure. Our goal here is to apply the preceding theorem,
together with some Morse theory applied to $E$, to clarify the dynamics and fixed-
point structure of $A\phi_{zu}$. Two interesting things arise as side-benefits: we can apply
the critical point structure of $\phi_{zu}$ to get extra information on the critical points
of $E$, and the Morse theory turns out to be equivalent to the topological Lefschetz
fixed-point theorem for $A\phi_{zu}$.

(1) To get the Morse theory started it is very convenient to be able to restrict
to the case of nondegenerate Hessians. This is standard in the category of
smooth functions, where generically $h$ is Morse. In the finite dimensional
space of degree $n$ maps $ARas_n$, it is not quite clear that generic fixed-
points are nondegenerate, but it suffices to find a single nondegenerate fixed-
point in each connected component of the fixed-point variety; using real
analyticity, the nondegenerate maps are then open and dense. Transforming
back from ARas to $E$, (as above) we have the analogous fact for critical
points of $E$, it suffices to find a single nondegenerate critical point in each
connected component of the critical point variety. But given a critical
point $x$ of $E$, $x$ is critical for any $PW \in Div$ with the same center of
mass $0 \in \mathbb{CP}^1 - x$ without loss of generality. This is clearly a connected
set, (assuming positive weights) and it is easy to pick a configuration with
nondegenerate hessian at $x$, for example at roots of unity, i.e with cyclic
symmetry around $x, 0$. In particular the generic local minimum is a strong
local minimum, and generically $E_{zu}$ is Morse. We will suppose that $E_{zu}$
is Morse in what follows.

(2) By subharmonicity, the poles at $Z$ are (or we consider them to be ) the
only local maxima of $E_Z$, (assuming positive weights). Let $m_i$ be the
number of critical points of $E$ of index $i$, (Morse numbers), then ($E$ is a
Morse function and) $n - m_1 - m_0 = 2$, and $m_0 > 0$ so $m_1 \geq n - 1$. (Note that
this is the topological Lefschetz fixed-point theorem for the corresponding
map). But $m_1$ counts repelling fixed-points of $A\phi$.

**Proposition 5.2.** $A\phi$ has at least $n - 1$ repelling fixed-points, (assuming
positive weights)

(3) Having 3 fixed-points of $A\phi$ in the complement of a basin of attraction
implies the last statement of (5.1) by hyperbolicity [3], this aspect of the
theory of rational maps generalizing to the (antiholomorphic) conformal
branched cover at hand. (or use the holomorphic 2nd iterate).

(4) One can generalize (5.1) to the case of complex weights, so that certain
sinks of the vector-field $F_{zu}$ correspond to attractors of $A\phi_{zu}$, this is pursued
further in the multivariable stability discussion (remark 3).
By the last statement of (5.1),

**Theorem 5.3.** For a given $Z \in (\mathbb{C}P^1)^n - D$, there are at most $2n - 4$ local minima of $\mathcal{E}_Z$, (supposing $n > 3$) and this is sharp. (assuming positive weights)

This is the number of critical points available. It coincides exactly with the number of triangles in any triangulation of $S^2$ with vertices $Z$, $SZ^n$, $n > 7 (2, 5i)$, realizes the maximum as can be seen by symmetry considerations; one needs enough points on the equator to rule out equatorial local minima. In cases where $\exp \mathcal{E}$ is a polynomial, this may be re-expressed in terms of properties of local maxima of spherical harmonics.

One can establish that the local minima of $\mathcal{E}_{SZ^n}$ are nondegenerate by equating this to nondegeneracy of (attracting) fixed points for the antiholomorphic $A\phi_{SZ^n}$, now one can qualitatively construct the trace of the map along a triangle (triangulation as above) with adjacent vertices in $SZ^n$, and from this one calculates the degree to be $-1$.

Using the Morse theory above, and (5.1),

**Corollary 5.4.** $A\phi_{ZU}$ has at most $3n - 6 = 3|d| - 3$ repelling fixed points, and $\mathcal{E}$ has at most $3n - 6$ saddles. $(n = d + 1 = |Z| = \deg(\phi) + 1 = -\deg(A\phi) + 1)$

5.1.2. One variable stability, proof of (5.1). By theorem 4.9, we immediately get:

**Lemma 5.5.** $0 < \nabla^2 \mathcal{E}_{ZU}(x) \iff \nabla^2 \mathcal{E}_{ZU}(x) < \frac{1}{2}I$.

By differentiating the construction in (4.4),

**Lemma 5.6.** $d(A\phi_{ZU})(x)$ (recall, notation $\phi_{ZU}$ here, instead of $\tau_{PW}$) is determined at its fixed points by the (real) Hessian, $H = \nabla^2 \mathcal{E}_{ZU}(x)$ and vice versa.

We present the main features of this relation here for the one variable case and in the next section for $\mathcal{F}$. It will be useful to have a more explicit formula; take $x = N$, $Ax = S$, (at the poles) without loss of generality, and fix the bases $e_i$ at $T_N, T_S$, from $\mathbb{R}^3$, so $dA = -I$. The Hessian $H$ of $\mathcal{E}$ at $N$ is symmetric hence diagonal over $\mathbb{R}$ with respect to this basis without loss of generality. Noting that $\Pi_N, \Pi_A$, and the Hessian of $\mathcal{E}$ are simultaneously diagonalized, by theorem 4.4, we now show $(dA)d\phi(x)$ is also diagonalized. Noting that

$$A\phi(x) = \Pi^{-1}_N(x + E_Z(x)) = \Pi^{-1}_A(x - \nabla \mathcal{E}_Z(x)),$$

and $d\Pi^{-1}_A(x) = I$, applying the chain rule, first note that the term due to $\nabla(Ax)$ vanishes; $\nabla \mathcal{E} = 0$, so we are projecting $x \in S^2$ to $S^2$, we can extend the projection maps to $\mathbb{R}^3 - Ax = \mathbb{R}^3 - y$ canonically, and as $y$ varies, the image $\Pi_N x = x$, trivially stays fixed. Thus,

$$dA\phi(x) = d[\Pi^{-1}_N(x - \nabla \mathcal{E}_Z(x))] = I - \lambda \nabla^2 \mathcal{E}_Z(x)$$

the factor $\lambda$ coming indirectly from the curvature of $S^2$ and various unspecified (not explicitly) normalizations. Note that (5.1) gives a monotonicity property for diagonal entries, $\alpha_i$, the eigenvalues of $d\phi(x)(\xi) \cdot dA\xi$, as a function of diagonal entries, the “eigenvalues” $\beta_i$ of $\nabla^2 \mathcal{E}_{ZU}(x)$, and we conclude that $-1 < \alpha_i < 1$, i.e. $dA\phi(x)$ is contracting, precisely when $\beta_i$ lies in an interval which we proceed to determine.

**Proposition 5.7.** Given a fixed-point $x$ of $A\phi_{ZU} = A\phi$ as above, $A\phi$ is attracting at $x$ iff $0 < \nabla^2 \mathcal{E}_{ZU}(x)$, $x$ is also a critical point of $\phi$ (superattracting at $x$) iff $\nabla^2 \mathcal{E}_{ZU}(x) = \frac{1}{2}I$. 

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Proof. Note that the following uses the localization of the correspondence of vector fields to maps, defined in the extension of $R$ from (4.1).

**Lemma 5.8.** Given $x$ as above, and $\xi \in T_x$ such that $\nabla^2_{\xi} E_{ZU}(x) = 0$, then $d(A\phi)(x)\xi = \xi$.

**Proof.** First consider the stronger case of a curve $\gamma$ such that $E$ is constant on $\gamma$. Using theorem 4.4, $A\phi$ is clearly the identity map on $\gamma$. But in the case at hand, $\nabla_{\xi} E_{ZU}(x) = \nabla^2_{\xi} E_{ZU}(x) = 0$, so $A\phi$ is the identity at least to 1st-order, implying $d(A\phi)(x) = 1$. \qed

**Lemma 5.9.** Given $x$ as above, if $\lambda$ is an eigenvalue of $H$, and $\xi \in T_x$ such that $\nabla^2_{\xi} E_{ZU}(x) = \frac{1}{\lambda} \xi$, then $dA\phi(\xi) = -\xi$.

**Proof.** This follows immediately from the preceding lemma, the orientation reversing property, and (4.9,4.14). \qed

Now we can use the lemmas to fix the sought for $\beta_i$ interval, and we conclude that $dA\phi(x)$ is contracting precisely when $0 < \nabla^2_{\xi} E_{ZU}(x) < \frac{1}{\lambda}$. A strong local minimum is one where the hessian is non-degenerate and we've shown that a strong local minimum, $E_{ZU}(x)$, determines a smooth map $A\phi$ whose linearization at $x$ is contracting. The converse follows by (5.6). Also, $\nabla^2 E_{ZU}(x) = \frac{1}{\lambda} I$ gives a critical point by symmetry considerations, or ((5.1)). \qed

Finally we return to the...

5.2. **Multivariable Case; stability for $F$.** For the first-order theory, relating $A\mathcal{F}(Z) = Z$ to $\nabla E(Z) = 0$, the single variable theory is better than a mere analogy; we can simply apply it componentwise to each $z_i \in Z$, in fact componentwise force-matching (the construction in (4.4)), provides a natural correspondence relating the force, $-\nabla E(U)(Z)$ whose components are force vectors $F_i \in T_{z_i} S^2$, to the components of $\mathcal{F}$.

But for the 2nd-order theory, relating $dA\mathcal{F}(Z)$ to $\nabla^2 E(Z)$ the task is (a priori) more complicated because the components, $z_i$, of $Z$, and the diagonalization of $\nabla^2 E(Z)$ are not necessarily aligned. Nevertheless, differentiating the construction in (4.4), we see that, as in (5.6), the 1-1 correspondence of $dA\mathcal{F}(Z)$ to $\nabla^2 E(Z)$, for configurations $Z \subset \mathbb{CP}^1$ holds in the multivariable case.

As with $\phi_Z$ we see that $\mathcal{F}$ is a discretized gradient flow for $E$, (i.e. a steepest descent method) but again, this does not directly imply stability.

Another complication is the SO(3,1) equivariance of $\mathcal{F}$, and the associated SO(3) equivariance of $E$, but this is easily handled using section 4.3.

**Definition 5.10.** The notions of (i) strong local minimum for $E$, i.e. nondegenerate $\nabla^2 E(Z)$, (ii) stable fixed point for $A\mathcal{F}$, and (iii) basins of attraction for $A\mathcal{F}$ or $\mathcal{F}^2$, should all be interpreted with the natural equivariant actions quotiented out, i.e. they allow for degeneracies, along orbits by these actions.

Our goal is now to prove (3.12)(ii), in particular (generalizing to the weighted case):

**Theorem 5.11.** Given $\mathcal{F}$, $E$ as above, $Z$ is a critical point for $E_W$ iff $\mathcal{F}(Z) = A(Z)$. Furthermore $Z$ is a strong local minimum for $E_W$ iff $Z$ is a stable fixed point for $A\mathcal{F}$.
Note that $\mathcal{A}F = FA$, so $(\mathcal{A}F)^{(2)} = F^{(2)}$ is holomorphic. A version of theorem 3.12 with any real weights follows similarly. One needs positive weights to guarantee the existence of local minima.

Proof. The first statement follows from the first part of (5.1) and the remarks above, preceding the theorem.

For the second statement, we extend the correspondence of positive hessians to attracting $d\mathcal{A}F$, from the single variable to the multivariable case. In fact the same technique of proof goes through, (for mixed derivatives) and one gets, $d\mathcal{A}\phi(Z) = I - \lambda \nabla^2 \mathcal{E}(Z)$, where the derivative of the $A_z$ term vanishes as before. (One should check that the constant $\lambda$ will be the same for mixed derivatives, this is also clear from remarks below on neutral directions.)

The latter, the derivative of the $A_z$ term, which represents the nonlinearity due to curvature of the sphere, could show up in higher derivatives, for example in certain approaches to the case of degenerate fixed-points.

At a local minimum of $\mathcal{E}$ the eigenvalues of the hessian $H$ are all weakly positive, so as in (5.5) they are always in $[0, \frac{1}{2}]$. Furthermore, the proofs of lemmas (5.8, 5.9) applied in every complex direction provide a correspondence between neutral directions for the hessian of $\mathcal{E}$, i.e. those with $\beta_i = 0, \frac{1}{2}$, and neutral directions for $d\mathcal{F}$, $\alpha_i = \pm 1$. (Notation as above.) Note that a neutral direction is extremal in the class of weakly positive hessians, so it must correspond to an eigenvector (using the Rayleigh-Ritz criterion), likewise for neutral directions in weakly attracting $d\mathcal{A}F$.

We can immediately conclude that the connected subset of hessians $\mathcal{H}^+$ with all eigenvalues in $[0, \frac{1}{2}]$, must map to the connected subset of linearized maps, $\Lambda^+$ of $\text{End}(T_z, T_z)$ (of the type of $d\mathcal{A}F$) with all eigenvalues in $[-1, 1]$, and likewise for the inverse direction.

In fact the boundaries, such as $\partial \mathcal{H}^+$, the neutral regions, decompose into connected components which are completely determined by the number of eigenvalue $\pm 1$’s they have (likewise for $0, \frac{1}{2}$); this uses a $k + p$ type decomposition of the associated lie algebras, and groups, and the fact that the hessians and maps have full bases of eigenvectors. They thus cut the full matrix spaces up into connected pieces characterized by the number of eigenvalues of each type.

(1) There is a generalized correspondence between the signature of fixed-points of $\mathcal{A}F$ and index of critical points of $\mathcal{E}$, for any index.

(2) The conjecture (3.14) might well be extended to the case of positive weights, or even to claim that $\mathcal{E}_W$ is Morse, in particular that all local minima are strong. Almost nothing, beyond numerical work, is known of the classification of all local minima. We thus discuss weak local minima below. They are bound to occur in families as weights vary over $\mathbb{C}$.

(3) We sketch modifications that provide sufficient criteria for existence of attractors for the more general case of complex weights. As in the single variable $\mathbb{C}P^1$ case there is no global energy function, but a force-field $F$, which is locally exact off the singular set, where it has periods. Decompose the matrix $\nabla F$ as a sum of $S$, the symmetric and $Q$, the skew-(hermitian) form; diagonalize over $\mathbb{C}$ and separate real $\Re$, and imaginary $\Im$ parts of the eigenvalues $\lambda_i$. The necessary and sufficient condition for an attractor is $|\lambda_i - \frac{1}{2}| < \frac{1}{2}$, by (5.1) which by (4.9) is equivalent (for $\Re \lambda_i < \frac{1}{2}$) to the “one-sided condition” $\Re \lambda_i > \frac{1}{2} - (\frac{1}{16} - (\Im \lambda_i)^2)^{\frac{1}{2}}$, generalizing the real case.
5.2.1. **Appendix:** *Weak local minima.* This appendix contains a somewhat detailed description of problems involved in establishing the existence of a Fatou set.

**Conjecture 5.12.** The Fatou set of $\mathcal{F}$ is nontrivial.

where the Fatou set is an open set defined to be the maximal domain of normality, (ie where iterates form a normal family). Note that Fatou sets can have periodic dynamics, but we actually expect a strong Fatou set, ie one whose domain of normality has a stronger characterization; the convergence of iterates to constant maps. This is based on the observation that eigenvalues of the linearization are real (3.13).

Various different approaches to this conjecture are evident,

1. ruling out degenerate local minima as in conjecture (3.14), we will not have more to say on this here, or
2. extending the analysis of nondegenerate hessians to the degenerate case. In its full generality, normal form theory for degenerate maps in higher dimensions is very technical and difficult, a reserve of the specialist, but one can expect our special assumptions to simplify things somewhat. There is probably a good chance of ruling out degenerate local minima in the case of real weights and no clamped points, but these singularities can arise for other maps, and it is certainly interesting to extend to the nondegenerate case.
3. We note also that our use of the term “lyapunov function of $\mathcal{F}$”, for $\mathcal{E}$, is somewhat unconventional insofar as we haven’t shown the lyapunov property holds a priori, and not even that it holds on full basins of attraction a posteriori, though we do use $\mathcal{E}$ to produce these basins. (The term “lyapunov” is justified by the steepest descent involved). If we could prove the lyapunov property without first producing a fixed-point then the distinction of degenerate and nondegenerate hessians would not arise. This brings up the question of convexity of $\mathcal{E}$ at degenerate local minima; a degenerate local minimal variety of $\mathcal{E}$ is totally geodesic if $\mathcal{E}$ is convex, but one can create functions (similar to $\mathcal{E}$) locally with minimal variety lying on a circle (in $\mathbb{C}$) for example. Convexity of $\mathcal{E}$ at degenerate local minima is likely important, if not the main point in approach (2) above.

The specific characteristic (direction) along which one should look for contraction is defined by choosing a real-analytic curve $\gamma(t)$ with endpoint $x$ for which $\mathcal{E}$ increases as slowly as possible; we break the Taylor series expansions of $\mathcal{E} = \sum \mathcal{E}_k$, into its homogeneous parts, in the inductive step choose $\gamma_k$, to pass through a local minimum $\xi$ of $\mathcal{E}_k$ on the unit sphere. If $\mathcal{E}_k(\xi) \neq 0$ we can stop; the connected local minimal set will be an attractor in some blow-up (roughly as in the nondegenerate case). If not, the curve $\gamma(t)$ will attract in directions transversal to $T^*\mathcal{E}_{\xi}$, but we have to look at higher $k$ to get retraction in $T^*\mathcal{E}_{\xi}$ along the complexification of $\gamma$ (which might have larger dimension if the local minimal set is large). To see that this procedure stops, we need to show there is some maximal $k(\gamma)$ for which $\nabla^2 \mathcal{E}(x) = 0, i < k$. The lemma below will only reduce this to another conjecture. Note that the study of dynamics near such a curve $\gamma(t)$ would normally involve blowing-up and applying center-manifold theorems by induction on degree in the Taylor series expansions of $\mathcal{E}$ and $\mathcal{F}$, at $x$. 

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The technical aspect will certainly be reduced if we can establish that local minima are isolated. By analyticity, they come in smooth families, so it suffices to show:

**Lemma 5.13.** (Conjectural.) If \( \gamma \) is a real-analytic curve with endpoint \( x \) (as above) and \( E \) is constant along \( \gamma \), then \( \gamma \) is in an \( SO(3) \)-orbit.

**Proof.** In some nbhd of \( x \), every \( y \in \gamma \) is a local minimum, hence period 2 for \( \mathcal{F} \), so there is a compact holomorphic curve, \( V \) of period 2 points extending \( \gamma \). \( \gamma \) is not in an \( SO(3) \)-orbit if and only if \( V \) is not in an \( SL(2, C) \)-orbit, by (4.29). But \( V \cap D_2 = \emptyset \), in \( (CP^1)^n \), because \( \mathcal{F} \) is repelling at \( D_2 \), (as in (3.4)), contradicting \( Z \in V \) being period 2 for \( \mathcal{F} \).

Thus \( V \cap D_3 \) is nontrivial. This entails the existence of degenerate period 2 configurations for \( \mathcal{F} \) as limiting cases at \( D_3 \), so we have reduced the problem to the following:

**Conjecture 5.14.** There are no degenerate period 2 configurations for \( \mathcal{F} \).

Now we describe degenerate period 2 configurations. In the limit where a part \( Z_1 \subset Z \) of a configuration degenerates, the map \( \mathcal{F} \) degenerates into what can be described as a weakly coupled pair of rational maps, where the coupling just keeps track of the conservation of mass, and that in an (almost** – see below) arbitrary way, as far as we are concerned here. In full generality we should consider a finite set of configurations, analyzed along the lines of the Fulton-MacPherson compactification of moduli space, with subclusters nested more deeply, but for the essential points it suffices to consider the case where \( Z \) breaks up into 2 configurations, corresponding to \( Z_1 \), \( Z_2 = Z - Z_1 \), which we call \( C_1 = C_N, C_S \) and which, without loss of generality live at \( r \) or near the poles, \( N, S \in S^2 \).

There is a special (one might say virtual) point, \( s \in C_N \supset Z_1 \), which is not a point of \( Z \), but which represents all of \( Z_2 \), ie the subset in \( C_S \) (so \( s = Z \) in effect). This being (for \( Z_1 \)) the “missing part” of \( Z \), the weights, \( W(C_N) \), are inherited accordingly, and vice versa, \( n \in C_S \), etc. \( C_N - s \in S \) in a small nbhd of \( N \), without loss of generality.

\( \mathcal{F} \) is defined on \( C_* = C_N, C_S \) using the weights \( W(C_N), W(C_S) \), as usual, and we could call the restrictions \( \mathcal{F}_N, \mathcal{F}_S \). But the images \( D_N, D_S \), may have a different subconfiguration structure, which we now outline.

\( C_N - s = C_{NS} \cup C_{NS} \) where \( \mathcal{F}_N \) sends \( C_{NS} \) to \( S \), (“the evicted part” or “defectors” if one likes) and \( C_{NS} \rightarrow D_{NS} \subset S^2 - S \), and vice versa, for \( N \leftrightarrow S \) Note that the evicted part is generically going to be empty, but there are certainly cases where it is nontrivial. Also there are maps \( \mathcal{F}: C_{NS} \rightarrow D_{NS} \subset S^2 - N \) so the image \( \mathcal{F}(C_*) = D_* = D_N, D_S \), where \( D_N \) is the union of the 2 pieces, \( \mathcal{F}_N(C_{NS}) \cup \mathcal{F}_S(C_{SN}) \) just given, as well as the special point, \( s \), representing the updated missing part of \( Z \).

The \( \mathcal{F} \) image of \( D_\) is defined likewise.

(** There is at least one obvious restriction on \( \mathcal{F}_N \) on \( C_{NS} \); it must vary holomorphically, though the space of configurations for which \( C_{NS} \) is nontrivial will be of very small dimension. We hope (and expect) that the conjecture above does not require any special knowledge of this latter map.)

In brief the \( D_N \) image of \( C \) is determined by \( \mathcal{F} \) on \( C_N \), as usual, together with an (essentially) arbitrary piece corresponding to the part of \( C_S \) that \( \mathcal{F} \) evicted.

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6. Suspension

We use the term suspension to refer quite generally to an operator from a space of maps, such as \( \text{Ras} \subset \text{Maps}(\mathbb{C}P^1 \circlearrowright) \) (the arrow indicating self-maps), to an enriched space of maps, such as \( \text{GRas} \subset \text{Maps}(\mathbb{C}P^1)^n \circlearrowright \). We discuss a very specific such construction here, based on \( \text{Cen} \), with applications to algorithms for finding roots of polynomials of one variable, or periodic points of \( r \in \text{Rat} \). It doesn’t have any special relation to other suspension-notions in dynamical systems; for Kleinian groups or Riemann surface laminations. But we can begin by noting that the diagonal lifting \( r_{PW}^\Delta : (\mathbb{C}P^1)^n \circlearrowright \) of \( r_{PW} \) is defined by \( r_{PW}^\Delta(z_1, \ldots, z_n) = (r_{PW}(z_1), \ldots, r_{PW}(z_n)) \), and that the suspension will combine \( r_{PW}^\Delta \) with \( \mathcal{F}_U \), acting on the same space, where combination is the \( \text{Cen} \) based construction in (4.18). In fact, by combining each component \( \mathcal{F}_i \) with \( r = r_{PW} \) with appropriate weights, \( W \), we get a map, denoted \( S(r, W) : (\mathbb{C}P^1)^n \circlearrowright \) which under iteration is attracted to repelling fixed-points of \( r \). This doesn’t explicitly allow for clamped points, they could be allowed also, but note that the fixed-points of \( r(x) \) are implicitly playing the role of clamped points here.

We use notation as above, but with special weights \( w_{i,r} = w_{r,i} \) for \( r \).

**Theorem 6.1.** Let

\[
S(r, W) = S(r, k; W)(x_1, \ldots, x_k)_i = M_{x_i}^{-1}[w_{i,r}M_{x_i}(r(x_i)) + \sum_{j \neq i} w_{i,j}M_{x_i}(x_j)]
\]

where \( M_z \) is Moebius as in (3.6). Then \( S(r, W) = \mathcal{F}_{PW} \) where \( r = \phi_{PW} \), \( w_{i,j} \) for indices \( i, j \) of \( X \) are the same for both \( S, F \), and \( w_{i,k}(F) = w_{i,r}u_k \) for indices \( i \) of \( X \), and \( k \) of \( P \), where \( w(F) \) indicates a weight of the map \( F \).

Suppose that \( k \leq \text{deg}(\phi) + 1 = n + 1 \) and that \( \phi = \phi_P \) has symmetric weights, then

\[
S(\phi, k)(x_1, \ldots, x_k)_i = M_{x_i}^{-1}[nM_{x_i}(\phi(x_i)) + \sum_{j \neq i} \frac{1 - n}{k}M_{x_i}(x_j)]
\]

has \( P_k \subset P \) as an attractor, for any part with \( |P_k| = k \).

In particular, if \( \phi = R_p \), as in ((24)) then \( S \) has a basin of attraction, convergent to a superattracting configuration of roots of \( p \).

**Proof.** The weight formulas follow directly from definitions, we are just taking affine linear combinations of divisors under the equivalence in (4.18). \( w_{i,r} \) represents a rescaling factor for the weights of the divisor of \( r \). Recall that a clamped point \( q_i \) in \( \mathcal{F}(X; Q; W) \) is superattracting for \( x_j \) iff their joint weight is one. This is the case in the weighted suspension of \( R_p \) constructed above. Note that weights \( W \) of the \( x_i, x_j \) interactions have been chosen to set the weight-sums equal to one, as required, (using the known weights of \( R_p \)). \( \square \)

Note also: \( x_i, x_j \) are mutually repelling (near the diagonal).

These seem to provide good candidates for generally convergent purely iterative algorithms (on a large dimensional space) for finding roots of polynomials of one variable, [27]. With a little more effort the technique generalizes to finding repelling (and all) fixed or periodic pts of any \( r(x) \) in \( \text{Rat} \) of \( \mathbb{C}P^1 \); we need a rough approximation of the multiplier of the periodic point to be able to set weights for \( S \) appropriately. Most such points lie in the Julia set, \( J \), and their multiplier is close to some central value, determined by the Hausdorff dimension of \( J \), and the period.
Doyle and McMullen constructed \( R_p \) in their work on 1-variable purely iterative \( \text{Ras} \). They noted explicitly that \( R_p \) doesn’t converge to roots; they are repelling fixed points. They were interested in the possibility of iterating a single-variable map to solve polynomial equations, while respecting certain group actions, see also [5]. We have shown that using the averaging technique of Cen it is easy to overcome this problem in the multivariable context, with appropriate weights for \( S(r, k; W) \).

The existence of an algorithm for putting \( r \in \text{Ras}(\mathbb{CP}^1) \) in normal form, (2.1), by constructing the suspended \( S \in \text{Ras}((\mathbb{CP}^1)^n) \) to find its fixed-points , is also interesting with the Smale model of computation over the reals, insofar as it is viewed as a self-contained model of computation; the suspension provides a construction using only rational functions, (in several variables) that finds normal forms of one-variable rational maps. Note that the suspension technique treats the input \( r \) as a "blackbox"; we never need to formally compose or manipulate it, but only use evaluations of \( r(x) \).

7. Algebraic-geometry of \( \mathcal{F} \).

It should be possible to find an explicit expression for \( \mathcal{F} \) as a self-map of \( \text{Poly} \), since no symmetry breaking occurs at this level. We will provide a closed-form expression of this type, using resultants, sylvester matrices etc. We also find the (generic topological) degree of \( \mathcal{F} \) below, on \( (\mathbb{CP}^1)^n \), (though our proof is not really algebraic) and it is clearly the same on \( \mathbb{CP}^n \); preimages generically won’t have permuted pairs. An easy remark relating Cen to dual curves is also included for motivation—our other goal here is to begin a discussion relating constructions here to representation theory; many coincidences that have so far been observed involving notably Cen, differentiation, discriminants, duality, \( R_p \) of ((2.4)), will seem more natural in this light.

7.1. \( F \) as resultant . Our goal here is to explain and prove the following formula for \( \mathcal{F} \) as a transformation of \( \text{Poly} \)

\[
\mathcal{F}(p(z)) = q(\lambda) = \text{RSLT}_z(p(z), \lambda p''(z) - zp''(z) - 2(1 - n)p'(z)),
\]

where the sylvester determinant realization of the resultant (RSLT) has to be constructed to reflect the natural degrees of the polynomial entries; the leading order terms in the summands of the second entry do not cancel due to the factor of 2, (the non-cancellation should probably be related to dilation equivariance of \( \mathcal{F} \) but one can cancel leading order terms between the two resultant entries (as for the standard discriminant construction) and we discuss below the correct total degree in the coefficients proposing that it is \( 2n-3 \). Note that this is not explicitly an equivariant expression: there is a choice of coordinates both in taking derivatives and calculating resultants. Nevertheless we provide more intrinsic descriptions of this map below, and in (3.9).

We first give a closed-form expression for \( \mathcal{F}(Z) \), in terms of the polynomial, \( p \), and \( z_i \) such that \( \forall i, p(z_i) = 0 \), but it does not satisfy the symmetry demands stated above; we use the individual roots \( z_i \) in \( Z \), explicitly,

\[
\mathcal{F}(Z)_i = z_i + 2(1 - n)p'(z_i)/p''(z_i) (= \text{def} \ g(z_i)).
\]

The easiest proof is to rewrite \( p \) in terms of factors, and formally use the sum \( \frac{d}{dz} \log(p'(z_i)) \) which simplifies a lot since \( p(z_i) = 0 \).
Note that \( p, h \) have a common zero iff
\[
RSLT(p, h) = \Pi_{p(z)=0}(h(z)) = 0,
\]
so the explicit regrouping of this product expression is just the usual sylvestor determinant expression of the resultant. Now the polynomial \( q = \mathcal{F}(p) = \Pi(\lambda - \mathcal{F}(Z)) \) has the form of a quotient of resultants:
\[
\Pi_{p(z)=0}(\lambda - g(z)) = \frac{\Pi_{p(z)=0}(\lambda p''(z) - zp''(z) - 2(1-n)p'(z))}{\Pi_{p(z)=0}(\lambda p''(z))} = RSLT(p, \lambda p''(z) - zp''(z) - 2(1-n)p'(z))/RSLT(p, p')
\]
In the RHS, \( \lambda \) should be regarded as part of the coefficients of a polynomial in \( z \), and it enters the sylvestor determinant as such. The polynomial \( q(\lambda) \) sought doesn’t depend on the denominator—the latter gives just an irrelevant \( \mathbb{C}^s \) factor of \( q \), and dropping it greatly simplifies computation of \( q \). On the other hand this factor seems necessary if one is interested in \( sl_2 \mathbb{C} \) equivariance as opposed to \( psl_2 \mathbb{C} \) equivariance (see remarks below on such lifts).

Thus
\[
\mathcal{F}(p(z)) = q(\lambda) = RSLT_z(p(z), \lambda p''(z) - zp''(z) - 2(1-n)p'(z)).
\]
deg \( p = d \) guarantees that deg \( q = d \) as is clear from (7.3).

7.2. Topological Degree of \( \mathcal{F} \).

**Theorem 7.1.** The degree of \( \mathcal{F}^n \) \( (= \mathcal{F} \text{ on } \mathbb{C}P^1)^n \) is \( (n-2)! \).

**Proof.** We calculate the degree on \( \mathbb{C}P^1 \), by induction on the number, \( n \), of points. Since the map is meromorphic the generic point has degree \( d \) preimages. Recall that \((n-2)!\) has been checked (section 3.1) for \( n = 2, 3 \). Suppose \( \mathcal{F}^n(Z) = Y \) has \( (n-2)! \) solutions. For the inductive step, we introduce a generic new variable point \( y_{n+1} \) in the configuration \( Y \), and count solutions of \( \mathcal{F}^{n+1}_t(Z, z_{n+1}) = (Y, y_{n+1}) \), using a continuity (homotopy) method, with \( t \) parametrizing weights \( W(t, n+1) \), chosen such that the new point is introduced with weight close to zero, i.e., \( w_{t,n+1}(t, n+1) = 0 \), and other weights close to those of \( \mathcal{F}^n \), so for \( t << 1 \) solutions are generically (see section 2.1 remarks on this) only perturbed slightly. Since \( y_{n+1} \) is degree \( n - 1 \) in \( z_{n+1} \), (2.1), \( \mathcal{F}^{n+1}_t(Z) = Y \) has \( (n-1)! \) solutions for \( t \) small. We homotope \( \mathcal{F}^{n+1}_t \) to \( \mathcal{F}^{n+1}, \ t = 1 \), by choosing \( W \) and \( Y \) so as to avoid the real codimension 2 set on which \( \mathcal{F}_t \) can degenerate. (ie \( D_3 \) which is complex codimension 1, see section 3.6). \( \Box \)

It is interesting to compare to Bezout’s theorem; this is not just the product of degrees of components, suggesting some analytic dependency among the component equations.

7.2.1. **Duality.** We present here a natural application of Chen to the geometry of dual curves, and though the result itself is not particularly new, we include it because there are probably more interesting constructions along these lines—and we hope that duality might shed more light on the geometry of \( \mathcal{F} \). Given a curve \( V \subset \mathbb{C}P^2 \) of degree \( d \), one can enrich the standard dual map \( f : V \to V^* \) by providing a canonical rational map \( f^\vee : \mathbb{C}P^2 \to \mathbb{C}P^2* \) which extends \( f \) to \( \mathbb{C}P^2 \).

Let \( p \in \mathbb{C}P^2 \), let \( L_{tp} \) be the associated pencil of lines through \( p \), \( t \) being the angular coordinate. For each \( p, t \) let \( L_{tp} \cap V = Z_{tp} \), and let \( \phi_{tp} : L_{tp} \to L_{tp} \) be the
map $\mathcal{C}_{\text{eu}}: Z = Z_{tp}$. Recall that this is well-defined even though the degree of $\phi_{tp}$ can jump when $Z_{tp}$ degenerates. Now setting $W(V, p) \subset \mathbb{CP}^2$ to be $\{ \phi_{tp}(p) : t \in \mathbb{CP}^1 \}$, $W$ defines a compact curve, with one point for each $t$; hence,

**Lemma 7.2.** $W(V, p)$ is a line, depending holomorphically on $V, p$.

Thus we have canonically associated to $V$ a map $\mathcal{G}_V(p) = W(V, p)$ of the complement $V' \subset \mathbb{CP}^2$ to the dual space, $(\mathbb{CP}^2)^*$.

**Theorem 7.3.** $\mathcal{G}_V(p)$ extends across $V$ except possibly at singularities and flex points, such that $\mathcal{G}_V(p)$ restricted to $V$ is precisely the standard dual map $f : V \to V^*$. Furthermore $\mathcal{G}_V$ determines $V$ uniquely.

“Sketch of proof”: Given $p$ close to $x \in V$ such that $x$ is not a flex point, let $L_1$ be the line through $p$ and $x$ (orthogonal to $V^\perp$) and $L_2$ be the line through $p$ but parallel to $T_x V$. The image $p_1$ of $p$ associated to $L_1$ lies close to $x$, hence close to $T_x V$, while the image $p_2$ of $p$ associated to $L_2$ lies far from $x$, as the 2 pts of $L_2 \cap V$ near $p$ will cancel, but it is close to $T_x V$. Thus $L(p)$ which is determined by the $p_1$ is close to $T_x V$. This proves that not only does the extension exist, but that it is precisely the map of $V$ to its dual curve $V^*$, "qed"

In fact, the map we construct is well known, it coincides with the one-form obtained by differentiating the homogeneous defining function of $V$. Compare (2.4)!!

7.3. **Representation theory of rational maps via geometric plethysm.**

Many of the intrinsic constructions of algebraic geometry are clarified when viewed from the point of view of representation theory, and the term geometric plethysm [15] refers to the aspects of this involved in decomposing symmetric powers into irreducibles. Since these projections provide multilinear maps they can be used to construct equivariant rational maps; this is implicit in the treatment of geometric plethysm in [15] for example. In our case the relevant Lie algebra is $\mathfrak{sl}_2 \mathbb{C}$, and homogeneous degree $d$ polynomials $\text{Poly}_d \mathbb{CP}^1 = \text{Sym}^d \text{Poly}_1 \mathbb{CP}^1 = O(1)\mathbb{CP}^{d+1}$ (up to a blow-up at 0) are the irreducibles, $V = \text{Poly}_1 \mathbb{CP}^1$ denotes the standard fundamental irreducible. Thus discriminants, derivatives and dual maps arise quite naturally as invariants or covariants of the group and a variety of identities that have arisen here can be considered from this viewpoint— and some can be seen as consequences of the machinery. Note though that $\mathfrak{psl}_2 \mathbb{C}$ equivariant maps of $\mathbb{CP}^{d+1}$ are not quite the same as covariants of $\mathfrak{sl}_2 \mathbb{C}$; there is a meromorphic factor (a multiplicative 1-cocycle with values in meromorphic functions) that intervenes in lifting from maps of $\mathbb{CP}^{d+1}$ to maps of $\mathbb{C}^{d+1}$, which it seems should vanish in cohomology for topological reasons. While we will not discuss this in general here, one can remark it in the examples.

The powerful counting/combinatorial techniques available from representation theory reduce certain identities to systematic (but not always easy) dimension counts, though it is not clear to us to what extent they provide explicit algebraic or geometric identifications in the process (in practice...this is tied up in the shift from 19th century “classical” invariant theory to modern representation theory). This approach thus complements the geometric understanding achieved so far, but we do not think it subsumes it. We will only give a few examples and illustrations of its relevance to our constructions here (while pleading lack of expertise on the general theory).

Note that preimages of discriminants by $F$ and its iterations $F^{-k} \mathcal{D}$ define invariants quite easily, (we have no idea how this relates to the classical literature)
also forward images of the (blow-ups of) more degenerate higher discriminants, eg $D_{k, k-2}$, by $F$ provide other invariant constructions relevant elsewhere in the paper, (such as (5.14)). It is natural to ask then, what generalizations or extensions of the constructions presented so far, here, are possible, and representation theory/geometric plethysm should be an indispensable guide. Recall also the questions about characterization of $F$ in section 3 as another source of motivation.

As a preliminary (and primordial) example, notice that the correspondence $R : \text{Poly}_d \rightarrow \text{Rat}_k$, $k = d - 1$ can be viewed as a map $P(R) : \text{Sym}^d V \circledast \text{Sym}^k V \rightarrow V$ by first lifting a degree $k$ map $r : \mathbb{C}P^k \rightarrow \mathbb{C}P^d$ to $r^h : V \rightarrow V$ in homogeneous coordinates, then $r^h$ is the restriction of a $k$-multilinear map $p(r) : \text{Sym}^h V \rightarrow V$, (the “polarization”) to the diagonal (a veronese type construction). But $\text{Sym}^d V \circledast \text{Sym}^k V \otimes V$ is dimension 1 if $k = d \pm 1$ and dimension 0 otherwise, (see the introduction to [33] and the formula for $\text{Sym}^k V \circledast \text{Sym}^l V$ in [15]). Thus $R_p$ which corresponds to $k = d - 1$ is singled out as the only possible such construction linear in $\text{Poly}_d$, $k = d + 1$ it turns out, corresponds to a trivial construction once projectivized; $z \mapsto p(z)$ projectivized gives the identity map. This also provides a systematic way to look for higher degree, equivariant rational map constructions, ie nonlinear $\text{Poly}_d \rightarrow \text{Rat}$.

Notice also that this effectively isolates/characterizes the $\text{Cen}$ construction: the $d = 2$ case gives a midpoint construction, ie evaluating $R_p$ at $x$ gives the midpoint of the roots of $p$ with respect to $A(x)$. In fact, in the product $\text{Sym}^d V \circledast \text{Sym}^k V \otimes V$ the first factor gives the configuration $P$, the 2nd factor corresponds to $x$ using a diagonal restriction, and the 3rd factor is the midpoint. Recall that affine structure is characterized by functional equations (Aff(1) after (1.2) ) and this could be related to the uniqueness above for $\text{Sym}^d V \circledast \text{Sym}^k V \otimes V$, $d=4$, thus providing a first step in translating every $\text{sl}_2 \mathbb{C}$ intrinsic construction of the article to the present language.

One of the main goals should obviously be to situate $F$ (as a map on $\mathbb{C}P^n$) in $(\text{Sym}^k \text{Sym}^n V) \circledast \text{Sym}^n V$, and in the process to pin down the algebraic degree of $F$. It seems to us likely that the latter is $k = 2n-3$ and furthermore that $F$ belongs to an irreducible factor $F$ of $\text{Sym}^k \text{Sym}^n V \circledast \text{Sym}^n V$ where $F$ corresponds to the Young diagram partition $2n-3 = (n-1) + (n-2)$ via the Littlewood-Richardson or Pieri enumeration of the factors of $(\text{Sym}^n V)^{\otimes k}$. (One might use the Schur functor associated to this Young diagram to furnish a multilinear map, whose restriction to the diagonal in $(\text{Sym}^n V)^{\otimes k}$ is the polynomial map desired). This is in part motivated by the explicit formula ((7.1)) above. It would also be interesting to know if there is a nice way to describe $F$ in the context of the theory of [17].

A final observation along these lines is motivated by some “numerology”: comparing the appearance of $2n-4$ critical points of $R_p = f/g$ in (5.3), $n = \deg(p)$, to the 2nd irreducible factor $\text{Sym}^2 \text{Sym}^n V \rightarrow \text{Sym}^{2n-4} V$. It is obvious that the degree 2 rational map of $\text{Poly}_d \rightarrow \text{Poly}_{2n-4}$ has to be the one that takes $p \mapsto q$ where $q(z) = 0 \iff R_p(z) = 0$, as this satisfies the required equivariance etc. In particular $q$ is quadratic in $p$ insofar as it depends on the vanishing of $g' - f g'$. Symmetry is implicit; the full projection is recovered by polarization which gives symmetric maps by definition—the key point is nontriviality on the diagonal. Now also that we introduce an auxiliary variable $z$ to define $p \mapsto q$ implicitly in terms of $p$. This suggest that such constructions are not to be had by any easy systematic means.
It would seem to be a difficult challenge to find analogous constructions for the \(\text{Sym}^{2n-k}V\) factor not to mention the rest of the sequence for \(\text{Sym}^k\text{Sym}^nV, k=2\) - this begins to touch on classical invariant theory [6], already for \(n=4\). In fact the difficulty of systematically producing these geometric constructions leads us to suspect that the appearance of the relatively simple construction of \(\mathcal{F}\) is more the exception than the rule. For higher \(k\) the combinatorics of the decomposition become more complicated, and though the extra flexibility of large \(k\) makes it easy to find some intuitive geometric constructions of factors, we would guess (very superficially) that these are likely to represent a vanishingly small fraction of the full decomposition.

8. The hyperbolic case

8.1. Divisors in the equator; extending maps and hyperbolic centrum.

We will consider maps whose associated divisor is contained in a subsphere, \(S^N \subset S^{N+k}\), they are especially interesting in relation with hyperbolic centra.

8.1.1. Extending maps and degree. We first define embeddings of mapping spaces \(\text{Ras}^N \to \text{Ras}^{N+k}\), or more specifically, extensions of maps, obtained simply by pushing forward the associated divisors, we will assume that weights are real;

**Lemma 8.1.** Given \(r \in \text{Ras}^N\), and a round, (section 4.0.4) embedding \(S^N \to S^{N+k}\), let \(PW\) be the associated divisor, ie \(r(x) = \text{Cen}(PW, x)\) on \(S^N\). Then using the embedding to pushforward \(PW\), define \(r = r^{N+k}(x) = \text{Cen}(PW, x)\) on \(S^{N+k}\). This extends \(r\) to \(S^{N+k}\). If \(k = 1\), and weights are real and positive, then \(r\) maps each complementary hemisphere \(H_\pm = H^{N+1}\) of \(S^{N+1}\) to its opposite hemisphere.

The last point follows easily using convexity properties, \(S^N\) is \(A_a\)-convex to the opposite hemisphere of \(x\). Note that \(r(x) = \text{Cen}(PW, x)\) does determine \(PW\) uniquely, the correspondence of weights to multipliers does extend to higher dimension even though the maps are not everywhere conformal, (8.4). Thus the extension is somewhat canonical; it’s not clear that a composition of such maps gives the same extension when definable, ie composition and extension may not commute. This is reminiscent of the quasiconformal reflection situation. In our case, for \(N=1\) though they do commute, by analytic continuation. For \(N=2\) we don’t know, but (2.1) assures us that at least the composition has a divisor and, if the associated divisor has real weights, its associated extension exists. For \(N=3\) compositions rarely have a divisor; the fixed-points do not have conformal linearizations, (compare (8.9)), and the question becomes mute.

It is not quite trivial to calculate the degree of a map in \(\text{Ras}^N\) unless \(N = 1, 2\) (\(N\) is the dimension, and it amounts to calculating the degree of \(\text{Cen}(PW, x)\)); already for \(N = 3\) the orientation information at preimages of a point might differ from point to point. For positive real weights, \(W\), standard homotopy arguments show that degree only depends on the cardinality of the configuration \(P\), \(\deg(r_{PW}) = d(|P|)\). Similarly if there are exactly \(\kappa\) positive weights, then \(\deg(r_{PW}) = d(|P|, \kappa)\). It is only for \(N = 2\) that we may use \(W \in \mathbb{C}^d\) to homotope around \(w_i = 0\).

**Proposition 8.2.** Given \(r \in \text{Ras}^N\) with positive real weights \(W\), \(\deg(r_{PW}) = d(|P|) = (-1)^N(|P| - 1)\).

**Proof.** Using (8.1) with \(N = 2, k = 1\), we see that by homotoping \(P\) into \(S^2 \subset S^3\), \(r^{-1}(x) \subset S^2\) for \(x \in S^2\) and that the orientation at each preimage has negative orientation; along \(S^2\) it’s positive, but the last point of (8.1) provides a negative
factor transverse to $S^2$. By induction on dimension, $S^N \subset S^{N+1}$ the general case follows similarly. □

**Problem 8.3.** What is $d([P], \kappa, N)$?

Applying the topological lefschetz fixed-point theorem should suffice in view of (8.4).

**8.1.2. Dimension > 2: the weight-multiplier relation, properties of $E$, etc.**

**Proposition 8.4.** Let $E_{PW} : S^N \to \mathbb{R}$ as above, and $\phi(x) = \text{Cent}_{PW}(x)$ be the associated self-map. Note that (4.5.4.1(ii)) generalize directly to $N > 2$. At poles $x$ of $E$, $d\phi(x)$ is conformal, and the weight to multiplier relation is the same as in (2.1).

Note though, that for $N > 2$, $f$ is certainly not conformal on open sets (by Darboux’s theorem).

**Proof.** For $N = 2$, $E$ is of the form $\lambda \log |z| + \text{smooth}$ where $\lambda$ is the weight at $p = 0$, thus it is the residue $\lambda$ of $E$ that determines the multiplier of $\phi(x)$, as in (4.4.2.1). But this residue to multiplier relation can be applied in the higher dimensional case, (in the context of (4.5)) along any 2-dimensional subspace spanned by eigenvectors of $d\phi(x)$. Since it provides the same eigenvalues on each such slice, $d\phi(x)$ is conformal, and the N-dimensional weight to multiplier relation follows.

Another way of seeing this is to break up the measure $PW = p_1w_1 + PW'$ to 2 parts, and apply the combination principle (4.18); this reduces $d\phi(p_1)$ to the case of $|P| = 2, P = \{p_1, \text{Cent}_{PW}(p_1)\}$, hence, for a given $X \in T_p S^N$, to the $S^2$ case; $P \subset S^2$ and $X \in T_p S^2$. □

We next present the higher dimensional extensions of (4.1)(i) and its consequences that will be of use later.

**Proposition 8.5.** Let $E_{PW} : S^N \to \mathbb{R}$ as above, and let $G$ be the grassmannian of 2-planes in $TS^N$, then the hessian of $E_{PW}$ on $S^N - P$ has trace $tr(\nabla^2_{PW} E_{PW}) \leq \frac{1}{2}$ for every $U \in G$.

**Proof.** It suffices to prove this for $P$ having one point, $P = \{p\}$, by calculating $\Delta E \leq \frac{1}{2}$ along any “great 2-sphere” ic round 2-sphere $S^2 \subset S^N$, of maximal radius. A useful trick here is to exploit the positive curvature form of $S^2$ as in (4.18.4.21); $E$ is a conformal factor for a flat metric on $S^N - p$, and $\Delta E$ determines the curvature form, which restricted to $S^2$ gives either a flat plane when $p \in S^2$, or a round sphere when $p \notin S^2$, in a euclidean space $E^N = S^N - p$.

Furthermore, the map from the round $S^2 \subset S^N$ to the image, in $E^N$, is a mobius transformation $M$ (conformal automorphism) which enjoys an $S^1$ symmetry around the extremal points of the $S^N$ distance $d(x, p) = d_p(x)$ along $S^2$. We thus claim that the extremal points of $\Delta E$ along $S^2$ are the same pair; the $S^1$ symmetry reduces this to monotonicity of $\Delta E$ in $d_p$, ie the monotonicity of $(M^*(\text{vol}) - \text{vol})/\text{vol}$ where $\text{vol}$ is the volume (and curvature) form of each $S^2$. Thus it suffices to check this for $M : \mathbb{C} \to \mathbb{C}, M(z) = \lambda z, \lambda \in \mathbb{R}$, with $\text{vol} = \frac{|dz|^2}{(1 + |z|^2)^2}$ which boils down to $\log(1 + |\lambda z|^2) - \log(1 + |z|^2)$ monotone in $|z|$, an exercise.

But it is easy to see that the minimum, $x_-$, of distance to $p$ is in the $p$-hemisphere, and $\Delta E(x_-) \leq 0$, and the maximum, $x_+$, being in the opposite hemisphere, $\Delta E(x_+) \leq \frac{1}{2}$; since $S^2$ is totally geodesic it suffices to note that at the extremal points $TS^2$ is in an eigenspace for the hessian and to use facts about
the $N = 2$ case. Using polar coordinates one easily sees the eigenvalue for the radial direction $e_r(t)$ is monotone decreasing from $\infty$ to $\frac{1}{2}$ and the eigenvalue for the orthogonal direction is $e_n(t) = \frac{1}{2} - e_r(t)$ and $e_n(t) = 0$ at the boundary of the $p$-hemisphere which is clearly a totally geodesic level set.

**Proposition 8.6.** $A\phi_{PW}(x) = x$ iff $\nabla E_{PW}(x) = 0$ and $x$ is a local minimum iff $A\phi_{PW}$ is attracting at $x$.

**Proof.** This now follows as for $N = 2$, (section 5.1, in fact with a better upper bound). \qed

### 8.2. Measures associated to maps.

To begin the generalization of $\text{Cen}$ from divisors $PW$ to measures, note:

**Proposition 8.7.** Given smooth $f : S^2 \to S^2$, with isolated fixed-points, there is a probability measure $\mu$ (complex-valued, $\int \mu = 1$), which is smooth up to a finite set of atoms, $st f(x) = \text{Cen}_\mu(x)$ iff the fixed-points of $f$ are simple and $df$ at fixed-points is (oriented) conformal, and the atoms correspond to the fixed-points, as in (i). If $\mu$ is smooth then $\deg(f) = -1$.

Note that the $\deg(f) = -1$ property follows from absence of fixed-points by the Lefschetz fixed-point theorem. In particular such maps cannot be holomorphic. Note that $\text{Cen}_\mu(x)$ is meromorphic in $\mu(y), x$, away from $D$ i.e. \{$(x, y) : x = y$, much like the case of $\text{Cen}_{PW}(x)$, but with $\text{Cen}_\mu(x)$ not being holomorphic in $x$, this property fails on $D$ (in view of separate holomorphicity etc) in contrast to the case of $\text{Cen}_{PW}(x)$. One should also note here that

**Lemma 8.8.** Given a smooth probability measure $\mu$ on $S^2$, $f(x) = \text{Cen}_\mu(x)$ is absolutely convergent, hence well-defined. $f(x) = \text{Cen}_\mu(x)$ is also well-defined for $\mu$ smooth with isolated atoms.

and that this distinguishes (in the smooth case) $N = 1$ from $N = 2$. See also remarks before (8.20). The case with isolated atoms is clear by continuity and removable singularities. It can also be treated by the combination principle, (4.18). Note also that the latter implies

**Lemma 8.9.** Given $\mu$ smooth with isolated atoms, the weight to multiplier relation for $f(x) = \text{Cen}_\mu(x)$ at fixed-points is the same as in (2.1).

In particular $df(x)$ at fixed-points is independent of the smooth part of $\mu$, and it is conformal.

**Proof.** (cf. (8.7) First $f$ determines a connection $D_f$ on $S^2$, which as in (4.18), is well-defined and smooth away from fixed-points. The curvature form (or measure) of $D_f$ (constructed using monodromies of $D_f$ as measures of open) provides a probability measure $\mu_f$, which, under the hypotheses above on fixed-points of $f$, includes atoms at the fixed-points. In fact one can separate the smooth and atomic parts of the measure by first decomposing the map using the combination principle

\begin{equation}
  f(x) = \text{Cen}(P(x), x), P(x) = \{f_a(x), f_s(x)\}
\end{equation}

as in (8.4) into (i) a part $f_a(x)$ (the atomic part) with the same fixed-points as $f$, and in view of (8.9) weights determined by multipliers as in (2.1) up to a normalizing factor (total mass one) and (ii) $f_s(x)$ (the smooth part) being the solution to (8.1), which will have no fixed-points. In fact $f_s(x)$ can be constructed directly from $f_a(x)$ and $f(x)$ using the combination principle (with a negative weight on $f_a(x)$}
to kill the atomic part-continuity has to be checked using (8.8.8.9) and removable
singularities applied where the fixed-points were “removed”.

By the linear relation of measures to connections in (4.18) underlying the combi-
nation principle, it thus suffices to establish (8.7) in the fixed-point free case, \( f = f_s \)
and assume \( \mu_f \) smooth. Also (4.18,4.21), already show that for holomorphic maps
\( \text{Cen}(\mu_f, x) = f \).

That this holds generally essentially follows from the correspondences in section
4; recall first that smooth maps \( f \) with no fixed-points correspond 1-1 to smooth
vector fields, \( v_f \), (4.1), or smooth connections \( D_f \) (4.18) (and these are equivalent
as noted after (4.21)). Thus it remains, for example, to show that smooth con-
nections correspond 1-1 to smooth (complex-valued) curvature forms (it is the curva-
ture form—not the function which one is prescribing, otherwise the PDEs involved
would be nonlinear) satisfying the Gauss-Bonnet normalization, but we have seen
that these connections correspond faithfully to 1-forms, and the correspondence
becomes a standard application of the complex-one-dimensional d-bar theory. □

We will now also sketch the relation to Laplace’s equation as in section 4. First,
we fix an auxiliary round metric on \( S^2 \), \( \text{vol} \) its volume form, as in section 4 and
5, and restricting to the subclass defined by \( \text{curl}(v_f) \equiv 0 \) there is a 1-1 cor-
respondence of smooth vector fields, \( v_f \), to smooth energy functions \( \mathcal{E}_f \) (mod con-
stant summands), and to \( \text{div}(v_f) = \Delta \mathcal{E}_f \), which is in 1-1 correspondence with the
(curvature-form) \( = \mu_f = h_f \text{vol} = -e \Delta \mathcal{E}_f \text{vol} + \kappa \text{vol} \) with \( \kappa \) the curvature of the
round metric as in (4.21), and we restrict to the subclass of curvature forms with
values in \( \mathbb{R} \), as in (4.21), satisfying Gauss-Bonnet. Here we use a density function,
\( \mu_f = h_f \text{vol} \). This provides a proof for the special case of maps \( f(x) = \text{Cen}(x, \mu) \)
determined by real measures \( \mu \).

The proof also shows that the theorem is not only analogous to existence and
uniqueness theory for Laplace’s equation on the 2-sphere, but it is formally equiva-
 lent, via adding a round metric in the background. In fact the theorem gives
a conformal (metric free) version where the flat affine structures play the role of
Green’s functions. Note that we implicitly use the uniqueness theory for \( \Delta \mathcal{E} \equiv \frac{1}{2} - \mu \)
to prove that for a given measure there is at most one map. Compare to the tech-
nique used in (2.1) where we could easily reduce to finite dimensional spaces.

This leaves some questions involved in weakening the condition on conform-
ality at fixed-points. Note that a smooth map \( f \) the linearization at a fixed-point
is not conformal will have an associated measure whose smooth part is not abso-
lutely convergent, hence some work is necessary to have a well-defined inversion
\( \text{Cen} \) associating maps to such measures. These measures do have other interesting
properties (integrals vanishing on fundamental domains of the linearization) and
could be worth considering.

It is not at all straightforward (not even rectifiable) to prove the correspondence
above for smooth measures using discrete approximations; the latter have too many
fixed-points, and too large a degree to converge smoothly! It would be interesting
to know if there is any geometrically well motivated notion of convergence of maps
that corresponds to convergence of measures (i.e. finitely supported, converging to
smooth) in this context? This is related in spirit to bubbling phenomena.

In higher dimensions, maps of \( S^N \) determine connections, hence curvature and
monodromy at fixed-points. But the inversion that would recover a map, from cur-
vature and monodromy, is less evident, owing essentially to the noncommutativity
of the Lie algebra associated to the connections. For \( N = 2 \) we have seen that one doesn't need any elliptic PDE theory for (stating the) solution for the connection given the curvature form; affine center-of-mass and stereographic projection suffices. On the other hand the \( Sym^2 \Lambda^2 \)-structure of curvature tensors suggests a possible generalization of the use of complex weights to \( N > 2 \); the weights can be assigned to every pair of oriented 2-planes, with appropriate Bianchi identities in case of sufficient regularity. While we do not pursue this here, we now consider a further development of \( \mu_f \) in the 2-D case that suggests an easier way to recover the measure from the map.

Recalling that the liouville symplectic form \( S \) on \( S^2 \times S^2 - \mathcal{D} \) in section 4.0.5 is \( SO^+(3,1) \) invariant, it is natural to try to recover \( \mu_f \) as \( S \) evaluated on the graph \( GF \subset S^2 \times S^2 - \mathcal{D} \), up to fixed-points, where the intersection with \( \mathcal{D} \) provides atomic contributions. In fact this is almost correct, the point though is that \( \mu_f \) is a complex measure, so we extend to a complex valued symplectic form \( \omega = \omega_C(X, Y) = S(X, Y) + \sqrt{-1}S(X, JY) \) using the complex structure \( J \) on \( S^2 \). This is the ("anti-holomorphic") complex conjugate of the holomorphic extension \( \sigma \) discussed in 4.0.5. Thus given a map \( f \) as above.

**Theorem 8.10.** \( \mu_f = c\omega_C(GF) \) on the smooth part (up to fixed-points, \( c \) is some normalizing constant factor) and this is also the curvature of the canonical connection associated to \( f \).

**Proof.** By conformal invariance of both \( \omega \) and \( \mu_f \) it suffices to check this for \( f, x \) normalized, \( st \ x = n \), \( (north) \) and \( f(n) = An = s \), by fixing an auxiliary round metric as in section 4.5. Thus the restriction \( S_{GF} = C \times \text{tr}(d(Af)) \) for some constant \( C \). On the other hand, recall that in section 5.1 we saw that (at least for maps \( f(x) = \text{Cen}_\mu(x) \) determined by real measures \( \mu \) \) \( d(Af) = I - 4\nabla^2\mathcal{E}, \) so \( \text{tr}(d(Af)) = 2 - 4\Delta\mathcal{E} \), and, in (4.21), that the (curvature-form) \( = \mu_f = h_f\text{vol} = -c\Delta\mathcal{E}\text{vol} + \text{vol}, \) so \( \text{tr}(d(Af)) = a + bh_f \), for some constants \( a, b. \) We also remarked in section 5.1 that \( \text{tr}(d(Af)) = 0 \) in the holomorphic case, and in (4.21), that \( h_f = 0 \) in the holomorphic case, so \( a = 0. \) But this now gives \( S_{GF} = Ch_f \) as desired, (in the case \( \mu_f = h_f\text{vol} \) real). The proof can be systematically translated to extend to the general case, replacing \( \nabla\mathcal{E} \) by the connection-1-form \( \eta, \Delta\mathcal{E} \) by \( \mathcal{D}\eta, \) etc.

In fact all one really needs, as in section 5, is to remark the existence of an affine relation between \( d(Af) \) and \( \nabla\eta \) (the latter being the same as \( \nabla^2\mathcal{E} \) in the special case). By linearity, it now suffices to find the contraction of \( d(Af) \) that corresponds to \( h_f = \sqrt{-1}. \) But symmetry and uniqueness reduce it to the 2 possibilities, \( \pm\text{tr}(d(Af)(X, JY)) = \pm S(X, JY). \) Finally, one uses the vanishing of \( h_f \) associated to any complex-linear \( d(Af) \), as in (4.15), to fix the choice of sign to get the ("anti-holomorphic") \( \omega \) as claimed. \( \square \)

**Corollary 8.11.** Holomorphic maps are defined by \( \omega_C|_{GF} = 0 \) i.e. pointwise vanishing of the pullback form, i.e. their graphs are Lagrangian for the (anti-holomorphic) complexified Liouville form.

8.3. Schwarz-Christoffel type uniformization.

**Definition 8.12.** A subvariety \( V \) of a manifold is said to be invariant for a connection \( D \) if \( \forall X, Y \in TV, D_X Y \in TV \); we also say \( V \) is totally geodesic or has the \( D \)-convexity property, or \( \text{CP}(D) \), or just \( \text{CP} \).

If \( p \in \mathcal{V} \subset S^{N+k} \) then \( S^N \) is totally geodesic for \( A_p \), and since
Lemma 8.13. CP is preserved under averaging of connections. (Averaging must use real weights W, not necessarily positive though.)

we have shown.

Proposition 8.14. Given any divisor PW in $S^{N} \subset S^{N+k}$, with real weights W, $S^{N}$ has the $D_{PW}$-convexity property. ($D_{PW}$ as in (4.18))

One can immediately conclude that for $N = k = 1$, $D_{PW}$ gives a Schwarz-Christoffel type uniformization of the polygonal region whose angles correspond, (4.18), to the weights of $W$; $D_{PW}$ is flat on each complementary component of $S^{1}$ and each arc of $S^{1}$ complementary to $P$ is (totally) geodesic for $D_{PW}$.

The higher dimensional versions similarly give polyhedra with totally geodesic faces, though the regions and faces are not flat except when their dimension is $N = 1, 2$.

Schwarz-Christoffel uniformization gives a conformally intrinsic center of mass constructions for divisors $PW$ in $S^{1} \subset S^{2}$; the flat convex structure on the disc allows one to assign the $D_{PW}$ affine center of mass to $PW$. This doesn’t generalize easily to higher dimension. We call these hyperbolic centers of mass. The Douady-Earle center of $PW$ is another such hyperbolic center, and we do not know if the two are equal (probably not) or somehow related?

8.4. Polyhedral volume and hyperbolic center of mass. Note that the connection provided in theorem 4.18 is conformally intrinsic, but the metric in proposition 4.21 depends a priori on the choice of a metric on the sphere, $S^{2}$ and these are parametrized by $H^{3}$. In fact it is easy to see that the metric is determined up to a scale factor, $v(h)$ in $\mathbb{R}^{+}$, (which stands for volume), $h \in SO^{+}(3, 1)/SO(3) = H^{3}$. More generally we can define $v(Z)$ in $\mathbb{R}^{+}$ on configurations in $S^{2}$, assuming some metric, so that $v(h) = v(hZ)$, and show it is non-trivial;

Lemma 8.15. $v(h) \to \infty$ as $h \to \infty$ in $H^{3}$.

Proof. To see this, note that $v(h)$ is finite iff $\forall i, w_{i} < 1/2$ and use continuity of $v$ on configurations $Z$ in $S^{2}$ into $\mathbb{R}$ as $h \to \infty$, $Z$ tends to a configuration with 1 or 2 points, hence $\forall i, w_{i} < 1/2$ fails in the limit. \hfill \□

Theorem 8.16. $v$ along an orbit is well-defined and convex on $H^{3}$, and the local minimum of $v$ exists and is unique.

So this also defines a hyperbolic center of mass. The proof uses:

Lemma 8.17. Given fixed positive weights $v(Z)$ is strictly plurisubharmonic on $SDiv$, the nondegenerate configurations on $S^{2}$.

Again one wonders if this has any nice relation to the Douady-Earle center? Note the relation of the proof here to that of (4.28), and in particular the method there for extending to higher dimension.

Proof. $v(Z)$ is an $S^{2}$-integral of $e^{\mathcal{E}}$ as in proposition 4.21. Fixing a point $x$ at which we evaluate $\mathcal{E}(x; P)$ and varying the configuration, $\mathcal{E}$ is strictly plurisubharmonic in $P$. Thus $v$ is strictly plurisubharmonic in $P$. $v$ is well-defined along an orbit as a function of $SO^{+}(3, 1)/SO(3) = H^{3}$ using isometry invariance of $v$. Now strict plurisubharmonicity and the vanishing of the gradient and hessian along $SO(3)$, which is totally real in configuration space, implies positivity along $H^{3}$, hence convexity. \hfill \□

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8.5. A sketchy form of a Projective Paley-Wiener formalism. Our goal here is to introduce a decomposition, of self maps $g$ of $S^1$, using the combination principle (4.18): $g(t) = \text{Cen}(\{g_+(t), g_-(t)\}, t)$, where $g_{\pm}$ are themselves maps of $S^1$ to $S^2$ which extend to holomorphic fixed-point free maps $g_{\pm} : H_{\pm} \to \mathbb{CP}^1$, of each hemisphere. Perhaps the most natural viewpoint though, is to consider the connection $D^g$ on $S^1$ associated to $g$ as in (4.18). Then the decomposition here provides holomorphic connections $D^g_{\pm}$ on the hemispheres $H_{\pm}$, st $D^g = D^g_+ + D^g_-$. 

The term Paley-Wiener usually refers to decomposition of real or complex functions on $S^1$ to summands which extend holomorphically (or harmonically) to either hemisphere $H_{\pm}$ of $S^2$. In fact this gives one way of decomposing self maps $g$ of a metric space $S^1$; consider the first derivative as a function and apply Paley-Wiener. The construction here is different in that it is projective intrinsic and requires no metric. The basic idea, which we sketch here, is to associate a divisor in $S^2$ as we have been doing throughout the paper, to a map $g$ of $S^1$. Here we must extend $g$ from $S^1$ to $S^2$ appropriately, and then split the divisor into its hemispherical pieces.

Given a sufficiently regular self map $g$ of $S^1$, $g$ is associated to an energy function $E_g$ in section 4; by this we intend (4.11), over $S^1 = \mathbb{RP}^1$, rather than $\mathbb{CP}^1$. One can extend approximations $E_i$ of $E_g$ to a nhld (in $\mathbb{CP}^1$) st $\Delta E_i = c$, but there are well-known global approximation methods that allow us to assume without loss of generality that $E_i$ is defined globally with simple logarithmic poles, which by the same, reversed, $E \leftrightarrow g$ correspondence, give approximations of $g$ by self maps of $\mathbb{CP}^1$, $g_i$, that extend globally as holomorphic maps. Furthermore, reflection symmetry of $g$ implies existence of global reflection symmetric approximations.

We note that the associated divisors $PW_i$ are reflection symmetric, where reflection symmetry of weights means that weights are complex-conjugate under reflection; we write $PW_i = PW_i^+ + PW_i^-$ to express the decomposition of $PW$ into pieces in each hemisphere (one can divide weights in half for $p_i \in S^1$). Finally, each part $PW_\pm$ defines a map $g_\pm = \text{Cen}(PW_\pm, x)$ which is well-defined in the limit and holomorphic on the hemisphere opposite the support of its divisor, and $g_i$ is the combination of $g_\pm^i$ in the sense of (4.18), since this corresponds to the sum of divisors.

In the case of positive real weights we get in addition that $g_\pm^i$ maps each hemisphere to itself, but we cannot expect this in general.

To finish we should pass back to the limit $g_i \to g$, and take limits of the associated $PW_i$. This leads to the question of the nature of the limiting divisor; whether it is a measure or perhaps a distribution, and how this depends on regularity hypotheses for the original circle map $g$? We do not discuss these questions here, which is why we referred above to a “formalism”. Using a Paley-Wiener decomposition on the level of connections or one-forms, as suggested above, should provide some ready made regularity statements.

8.6. Schwarz-Pick for $\text{Cen}$. The following is standard;

**Lemma 8.18.** Given $S^N \subset S^{N+1}$ there is a canonical reflection map, $\rho : S^N :\subset$, (self-map), is a conformal involution of $S^{N+1}$ fixing $S^N$, and equivariant for $\text{Aut}(S^N) \subset \text{Aut}(S^{N+1})$, [1, 7].

e.g. use canonical boundary measures associated to the $H^{N+1}$ structures. Applying the canonical reflection $\rho$ to the maps described in the last point of (8.1)
provides self maps $\rho \text{Cen}_{P \text{W}*}$ of $H^{N+1}$ and (continuing with notation as in (8.1), $P \subset S^N$)

**Lemma 8.19.** If $\forall i, 0 < w_i < 1/2$, then there is a unique fixed-point in $H^{N+1}$ of $\rho \text{Cen}_{P \text{W}*}$ and it is the Donady-Earle center of $P \text{W}$.

This follows immediately from the existence and uniqueness of the Donady-Earle center, [7], especially using the characterization in terms of measures on the unit sphere of $T_2 H$, which is exactly the equilibrium condition involving the dipole moment, in section 4.3. We have given a self-contained proof, using $E$, in section 4.3.

It is natural to ask then if the Donady-Earle center is an attracting fixed-point for $\rho \text{Cen}_{P \text{W}*}$, and whether there is an extension of the Schwarz lemma which would prove this? Our goal here is to show this is so, and to discuss some of its consequences—or further questions that this suggests.

Note also that if $\forall i, w_i < 1/2$, then the $H^{N+1}$-boundary fixed-points of $\rho \text{Cen}_{P \text{W}*}(x)$ are repelling in $S^{N+1} \subset H^{N+1}$.

This affine construction makes sense for other symmetric spaces $H$ of noncompact type; reflection of $x \in H$ can be generalized; it gives a sphere or other space (at $\infty$), rather than a point, but this still determines an affine structure $A_{x*}$ on $H$ (by Harish-Chandra embedding theory, [19]) thus a divisor $P \text{W}$ on $\partial H$ determines a well-defined map denoted $\text{Cen}_{P \text{W}}(x)$ for any $H$ of noncompact type, but we do not know if the extension of the Schwarz lemma is valid in such generality—we only consider real hyperbolic space here.

We restrict ourselves to the real hyperbolic space, and measures with finite support. But in general, any measure can be handled using finite approximations, (using $x \in H^{N+1} P \subset \partial H^{N+1}$ since $\text{Cen}$ is continuous in this larger class. Two extreme cases of the latter are worth noting:

(1) The case of standard spherical measure, $\mu$: for the $H^2$ case this determines a constant map, it suffices to approximate by measures with support on the roots of unity, and to note that $z^n \to 0$ as $n \to \infty$. One should use this to see that for any smooth $\mu$, the image of $\text{Cen}_{\mu}(x)$ is bounded in $H^2$.

For the $H^{N+1}$ case equivariance shows that $\text{Cen}_{\mu}(x) = \lambda_N(|x|) x$ for some function $\lambda$. But for $N \geq 2$, one can define the associated map $\text{Cen}(x, \mu) : S^N \to S^N$, see (8.8), and by continuity of extensions this implies that $\text{Cen}_{\mu}(x)$ is proper! A simple calculation shows that the critical dimension is $\frac{N}{2}$, so we would guess that the property of unboundedness holds even for very singular boundary measures in the case $N \geq 2$. It is also interesting to relate this to existence of the affine connection on $S^N$ associated to singular $\mu$, see (8.7) for the smooth case.

(2) At the opposite extreme, the possibility exists that (with $N = 1$) for very singular measures one can hope to construct inner functions using finite approximations, and the fixed-point property of the support of $\mu$ in the finite case.

**Theorem 8.20.** ("Schwarz lemma") Suppose $P \subset S^N$ and $\forall i, 0 < w_i$, then $f = \rho \text{Cen}_{P \text{W}} : H^{N+1} \subset$ is (weakly) distance decreasing with respect to the hyperbolic metric.

(1) In fact the map is strongly contracting (but not uniformly) unless the support $P$ only spans a totally geodesic subspace, but even in this case a
reflection property in the transverse directions (see below) ensures the existence and uniqueness of the fixed-points desired. (**the span being the smallest totally geodesic subspace containing (possibly in the ideal boundary) the set.**)

(2) Observe that \( N = 1 \) involves the case, \( H^2 \to H^2 \), and for any \( P \) this is a standard fact of complex analysis; essentially the sharp form of the Schwarz lemma, but our statement, as it stands, only applies to the case of maps with positive real weights and with all fixed-points in the boundary, in the correspondence established in (2.1). We will overcome these limitations, to some extent, further below. On the other hand our proof is quite different from the usual (known?) proofs even in this case.

(3) In higher dimensions \( N > 1 \), for \(|P| = 3\); we will exhibit a fibration at \( \text{Cen}_{PW} \) is handled using the \( H^2 \to H^2 \) case along fibers, and is isometric along the base. This thus describes the extremal case where weak contraction is in fact isometry. \( \mathbb{P} \) spans an \( H^2 = H^2_P \), which is invariant for \( f \), so \( H^2_P \subset S^2_P \) is \( \text{Cen}_{PW} \) invariant. Note that the isotropy \( G_P \) of \( P \) in \( O(N + 2, 1) \) is compact—it fixes the Douady-Earle center of \( P \) in \( H^{N+2} \), and \( \text{Cen}_{PW} \) is \( G_P \)-equivariant, so \( \forall g \in G_P, gS^2_P \) is \( \text{Cen}_{PW} \) invariant. The contraction property follows along each \( gS^2_P \cap H^{N+1} \) noting the cancellation of the conformal factors (in domain and range) which relate the hyperbolic metrics on the fiber to that of the ambient space. Also \( G_P \)-orbits, \( G_P x \) are orthogonal to \( x \in S^2_P \) by symmetry, clearly \( G_P x \cap S^2_P = x \), hence conformal invariance implies \( G_P x \) is orthogonal to \( gS^2_P \), \( \forall g \in G_P \). These orbits are spheres, as one can see that \( P \subset S^1 \) and \( G_P x \) correspond to fibers associated to an embedding \( SO(2) \times SO(N) \to SO(N+2) \). Since \( S^N = \partial H^{N+1} \) is fibered by \( gS^2_P \)'s, \( G_P x \) is orthogonal to \( \partial H^{N+1} \) so \( G_P x \cap H^{N+1} \) is totally geodesic (wrt the hyperbolic metric) in \( H^{N+1} \). Thus the fibration, with fibers \( gS^2_P, \ g \in G_P \), induces mutual isometries of the orbits \( G_P x \cap H^{N+1}, \ x \in H^{N+1} \) which are thus isometric copies of the base space of this fibration. Transverse to the fibers the map \( \rho \text{Cen}_{PW} \) is now clearly isometric along the base, hence only weakly contracting. In fact it corresponds to a map of the base of the fibration, which is reflection through a point, \( s \), where \( s \) is the (class of the ) fiber \( H^2 \) spanned by \( P \). In particular, there is a unique fixed-point. By an obvious extension to other codimensions, we can assume without loss of generality that \( P \) spans \( H^{N+1} \), so \(|P| > N + 1\), this being useful for intuition but not strictly necessary in our proof.

(4) Note also that for \(|P| = 2\), \( \rho \text{Cen}_{PW} \) is an isometry and it follows that for any \(|P| \) finite, \( x \to p \in P \), nontangential approach entails asymptotically isometric \( \rho \text{Cen}_{PW} \).

(5) The proof here is augmented below to work also for measures with support in \( H^N \), in (8.21, 8.22) allowing support in the interior (of the complementary target space). It is not clear if the Douady-Earle or other proofs cover this generalization. One can weaken the hypotheses to allow negative weights in the (interior of) \( H^N \), and ask if \( \rho \text{Cen}_{PW} \) being a self map still suffices for the contraction property? (See also 8.6.2 below.)

(6) The idea of the proof is to use the correspondence of minima to attractors (8.6). Critical points are denoted \( \text{crit}(x') \), in terms of a point \( x' \in H^{N+2} \) used to parametrize the round metrics on \( S^{N+1} \). We show that for enough
of the associated energy functions interior critical points, \( \text{crit}(x') \) have to be local minima. It will be useful to now consider a couple of special situations where this is easy to establish; we consider the boundary behaviour of \( \text{crit}(x') \) as well as an obvious case where \( \text{crit}(x') \) has a local minimum in \( H^{N+1}_+ \).

(7) For the latter, choose \( P \) to be the vertices of a regular simplex inscribed in \( S^N \subset S^{N+1} \), with symmetric weights, \( |P| = N + 2 \). The energy \( E_P \) and its hessian, \( H \), are invariant by the symmetries, which fix the centroid \( x_c \in \text{crit}(E) \cap H^{N+1}_+ \), so \( H \) is either strictly positive or strictly negative, or zero. For \( N = 1 \) it has positive trace, (4.1), so it is strictly positive. For \( N = 1 \) with larger \( |P| \), any \( |P| > N + 1 \), one can perturb the simplex, replacing each vertex by a cluster, and the hessian perturbed slightly is strictly positive. For larger \( N \), it is easy to see that at \( x_c \), \( H \) has positive trace: \( x_c \) is at radial distance \( \pi/2 \) from each \( p \in P \) by symmetry, so the hessian of each \( E_p \) is positive in the radial direction and zero along the totally geodesic level set. In fact this works for energy \( E_P(x) \) of any \( P \subset S^N \), at \( x \) the pole in \( S^{N+1} \) (ie the center of the hemisphere).

(8) The boundary behaviour of \( \text{crit}(E) \); as we approach the case where \( P \) is clustered at one point \( Aq \) “opposite” to \( q \), \( E_{PW} \) is a small perturbation of the function \( E_{Aq} \) (case \( |P| = 1 \)), but \( E_{Aq} \) has a unique local minimum at \( q \) and this is the only critical point of \( E_{PW} \) in the limit as the size of perturbations vanishes. One can see directly then that the perturbation doesn’t change the critical point structure outside a nbhd of \( Aq \); in a nbhd of \( Aq \) it must create saddle points, but they clearly stay near \( Aq \) in \( S^{N+1} \). We claim these must stay in \( S^N \) for sufficiently small perturbations; it suffices to note that for \( p \) near \( Aq \) the \( \nabla E_p \) are all transverse to the concentric spheres \( S^N \) (perturbations of \( S^N \)) near \( Aq \). The version of this last point which is relevant below involves perturbing the choice of round metric rather than \( P \), ie \( x' \to \partial H^{N+2} \).

\textbf{Proof.} (of (8.20)). To simplify notation and facilitate the comparison to the usual Schwarz lemma for holomorphic maps in \( H^2 \subset \mathbb{C} \), we will consider \( \text{Cen}_{PW} : H^{N+1}_+ \to H^{N+1}_+ \), ie we don’t use the reflection map here. This isometric change doesn’t affect contraction properties.

Given \( x \in H^{N+1}_+ \), as in (8.1), \( \text{Cen}_{PW}(x) \in H^{N+1}_+ \subset S^{N+1} \). Now \( S^{N+1} \) bounds a copy of \( H^{N+2} \) and \( S^N = \partial H^{N+1} \) bounds a totally geodesic copy of \( H^{N+1}_+ \subset H^{N+2} \), and the geodesic in \( H^{N+2} \) from \( x \) to \( \text{Cen}_{PW}(x) \) crosses this \( H^{N+1}_+ \) at exactly one point \( x' = n(x) \), the normalizing projection. We can assume without loss of generality that \( x' = 0 \in H^{N+2} \), so that \( x \) and \( \text{Cen}_{PW}(x) \) are antipodal (with respect to the round metric determined by \( x' \)). The antipodal property implies \( E_{PW} \) is critical at \( x \), and we claim that \( E_{PW} \) has a local minimum at \( x \).

But first, taking into account equality of the conformal factors of the hyperbolic metrics at \( x \) and \( \text{Cen}_{PW}(x) \) with respect to the round metric, (by symmetry; both are antipodal invariant), proposition 8.6 implies the distance decreasing property at \( x \), a local minimum. It remains to verify the claim.

To confirm the claim now, we will need some explicit formulae for derivatives of \( E_p(x) = -\log(|p - x|) \), (we do not need to worry about the positive constant normalizing factors here); by explicit calculation, for the single point \( p \)
where \( p, x \) are points on the sphere \(|p|^2 = |x|^2 = 1\) as well as vectors in \( \mathbb{R}^{N+2} \), we use inner products \((p, x)\) multiplying vectors as usual, and \( \nabla \mathcal{E}_p(x) \in T_x S^{N+1} \) is the usual projection. We record also

\[
\nabla^2 \mathcal{E}_p(x) = (1 - (p, x))^{-2}((p, v)^2 + (p, x)^2 - (p, x)),
\]

\(|v|^2 = 1, v \perp x\) and one can check easily it has constant trace for \( N = 1\).

Now the critical point condition is

\[
0 = \nabla \mathcal{E}_{PW}(x) = -\sum w_i \frac{(p_i, x) - p_i}{1 - (p_i, x)} = \xi(x, PW) - (\xi(x, PW), x),
\]

where we write

\[
\xi(x, PW) = \sum \frac{w_i}{1 - (p_i, x)} p_i.
\]

But \( P \subset S^N \subset \mathbb{R}^{N+1} \implies \xi(x, PW) \subset \mathbb{R}^{N+1} \), and \( x \in \mathbb{R}^{N+2} - \mathbb{R}^{N+1} \) so \((8.4)\) implies \((\xi(x, PW), x) = 0\), and moreover, \( \xi(x, PW) = 0\).

Now it is easy to see that

\[
\nabla^2 \mathcal{E}_{PW}(x) = \sum w_i (1 - (p_i, x))^{-2}((p_i, v)^2 + (p_i, x)^2 - (p_i, x)) = \frac{1}{1 - (p_i, x)^2} - (\xi(x, PW), x) \geq 0,
\]

finishing the proof. \(\Box\)

An obvious candidate for generalizing \((8.20)\) is to allow divisors \( PW \) with support in \( H^{N+1}_\pm \) rather than \( S^N \), and just as well, in \( H^{N+1}_+ \). We can easily adapt the proof above to cover this case: we pick up in the preceding proof from \((8.5)\), and rather than \( \xi(x, PW) = 0 \), we will show \((\xi(x, PW), x) \leq 0\) which clearly suffices. But the \( p_i \) all have positive components in the \( e_{N+2} \) direction, where zero components in the \( e_{N+2} \) direction cuts out \( S^N \), and \( x \) has negative component in the \( e_{N+2} \) direction. Thus \((8.4)\) implies \((\xi(x, PW), x) \leq 0\).

Finally we can apply \((8.20)\) to \( \mathcal{E}_\mu \) for any positive measure, regarding \( \mu \) as a limit of divisors with finite support in \( H^{N+1}_\pm \), and using continuity with respect to \( H_- \) and \( \text{Div} \) (noting disjointness) to get the decreasing property in the limit.

**Theorem 8.21.** Suppose \( P \subset H^{N+1}_+ \) and \( \forall i, 0 < w_i \), then \( \text{Cen}_{PW} : H^{-N+1}_- \to H^{N+1}_+ \) is (weakly) distance decreasing with respect to the hyperbolic metric. The same holds for \( \text{Cen}_{\mu} \) for any (nontrivial positive probability) measure \( \mu \) with support in \( H^{N+1}_+ \).

As a consequence, and using remark 3 following \((8.20)\), we can extend the Donahue-Earle center.

**Corollary 8.22.** Given \( P \subset H^{N+1}_+ \) and weights \( \forall i, 0 < w_i < 1/2 \), or a measure \( \mu \) as above but without atoms of mass \( \geq 1/2 \), there is a unique fixed point in \( H^{N+1}_+ \) of \( \rho \text{Cen}_{PW} \) and it agrees with the Donahue-Earle center where the latter is well defined.
8.6.1. unique critical point for $\mathcal{E}_{PW}$ in $H^{N+1}_0$. (This subsection is just a technical follow-up to things touched upon in the proof.) Note that there is a natural candidate for the left-inverse of $x' = n(x)$, from the proof above, namely the map $\text{crit}(x')$ that assigns to each $x' \in H^{N+1}_0$ the critical points $x_i$ of $\mathcal{E}_{PW}$ associated to this normalization. This is a priori multivalued, but it is natural to conjecture

**Conjecture 8.23.** If $P$ spans $H^{N+1}$ then $\mathcal{E}_{PW}$ has at most one critical point for $\mathcal{E}_{PW}$ in $H^{N+1}_0$ (assuming positive weights).

We do not expect there to be a natural metric in which the functions are convex though. In the degenerate case where $P$ spans a subspace, we expect a unique linear slice of the sphere to correspond to the critical set. We also remark that there is an interesting relation between degeneration of the lesion at a critical point and reversal of orientation of the map $n$ which we will not be able to discuss here. The conjecture is easily motivated by a continuity method argument; as long as every critical point is a local minimum no bifurcation is possible. One can use real analyticity of $\mathcal{E}$ as well. We give a linear-algebra argument below in a special case.

The functions $\mathcal{E}_{PW}$ are thus conjectured to be perfect in the sense of Morse theory in these hemispheres (our proof of (8.20) has even ruled out degenerate local minima). If the geometry of $\mathcal{E}_{PW}$ reflects a global topological property we will say that it is topologically adapted (in fact we would prefer to say “tau”, but the term is already taken). It would be interesting to know how well $\mathcal{E}_{PW}$ is topologically adapted for other classes of divisors as well. The motivation comes from complex analysis, where holomorphic functions have well known perfect intersection theoretic properties. One is always interested in finding other classes of functions or maps with analogous behaviour.

We now reduce to a simplified setting: noting that (8.5) is linear in $W$, as is $\nabla^2_x \mathcal{E}_{PW}(x)$ it would have sufficed to check positivity of the latter at extreme points of the compact convex space of $\{W : \xi(x, PW) = 0\}$, and by standard convexity theory this reduces to weights with at most $N + 2$ nonzero entries. We can also write $\xi(x, PW) = \sum w_i \tilde{p}_i$, with $\tilde{p}_i = \frac{1}{1 - (p_i, x)} p_i$. By fixing a moebius map $M$ sending $x \mapsto \infty, Ax \mapsto 0$, and letting $\tilde{p}_i = M p_i$, the latter lie in a fixed sphere $\tilde{S}$ and a similar formula can be derived from $\text{Cen}_{PW}(x) = Ax$.

So it is of interest to consider the case $|P| = N + 2$, and fixing a generic such $P$, $x$ now determines $W$ uniquely by $\xi(x, PW) = 0$.

We will now show that $W$ determines $x \in H^{N+1}_0$ uniquely assuming $|P| = N + 2$. But for generic $P$ the $N + 1$ by $N + 2$ matrix of $p_i$ is of rank $N + 1$, so it has a one dimensional kernel, say $0 = \sum w_i(1) p_i$. Clearly $\xi = 0 \Rightarrow \lambda w_i(1) = \lambda w_i(P) \Rightarrow w_i = \lambda w_i(1)(1 - (p_i, x))$. Rewriting $(p_i, x) = P x$ as a matrix product, we need to check that the image in $\mathbb{R}^{N+2}$ of the unit ball $B^{N+1}_1$ by $e - P x, e = (1, \ldots, 1)$ intersects any line $\ell$ through 0 at most once (no 2 $x$’s give proportional $e - P x$).

(or the image projects 1-1 into the sphere or simplex). If the contrary were true then the affine linear space $V = \text{range of } e - P x$ on $E^{N+1}$ (alias $\mathbb{R}^{N+1}$) contains $\ell \ni 0$ which implies that the range of $P y$ on $y \in E^N$ contains $P y_0 = e$ for some $y_0$.

Since $w_i, 1 - (p, x) > 0$, $\sum a_i p_i = 0$ has positive coefficients, showing that 0 is in the convex hull of $p_i$. (Note also that if 0 is not in the convex hull of $p_i$, this implies there is no critical point in $H^{N+1}_0$ and it is clear in this case that the local minimum should be in the boundary, this is an intuitively satisfying
analysis of the case where \( P \) lies in a hemisphere.\) So \( 0 < \sum a_i = (\sum a_i p_i, g_0) = 0 \) and by the contradiction, \( W \) determines \( x \) uniquely for generic \( P \).

**Proposition 8.24.** If \( P \) spans \( H^{N+1} \) and \( |P| = N + 2 \) then \( \mathcal{E}_{PW} \) has at most one critical point for \( \mathcal{E}_{PW} \) in \( H^{N+1} \).

Motivated by the continuity method suggested above, we discuss, in the case \( |P| = N + 2 \), the set of all solutions, \( \text{Crit}^0 \), in \( x, PW \), its connectedness properties, with attention to it’s interior \( \text{Crit}^0 \), and the relation of the boundary to the degenerate set (nonmaximal rank of \( \hat{P} \)).

We consider \( \text{Crit} \) fibered over \( x \in H^{N+1} \) and parametrize the fiber by \( \hat{P} \), i.e. configurations of \( \hat{p}_i \in S \) as above. \( W \) exists iff \( 0 \in \text{Hull}(\hat{p}_i) \) and the \( \hat{p}_i \) determine \( W \) uniquely if the the \( \hat{p}_i \) have maximal rank, this holds if \( 0 \in \text{Hull}^0(\hat{p}_i) \), the interior. One would like to view this as the interior of a solution-space, and we expect that modulo permutations of the indices \( i \in I \), there is a unique open connected set \( P_0 \) of configurations \( \hat{P} \), in the complement of the degenerate set (nonmaximal rank in \( \text{dim(hull)} \) whose closure is \( \text{Crit} \)).

8.6.2. the classical Schwarz lemma and divisors.** The comparison to the 2-D case, i.e. the classical Schwarz lemma, should be completed by (i) weakening the hypotheses to allow any weights in the (interior of) \( H_*^2 \), and extending the technique here to show that \( \rho \text{Cen}_{PW} \) being a self map still suffices for the contraction property, (recall as motivation the case of negative weights in \( H_*^2 \)) and (ii) analyzing which divisors/measures are associated to self-maps of \( H^2 \). The case of atomic measures with support \( P \subset S^1 \) on the boundary is easy; the fixed-points \( p \in P \) must have, for \( f : H^2 \to H_*^2 \), negative real multipliers, hence, by (2.1), \( 0 < w \leq 1 \). This is not at all clear for smooth support: note that the fixed-point property is lost in the smooth limit of atomic support, so a different approach is necessary. In fact we remarked that lebesgue measure on \( S^1 \) gives a constant map, so small complex perturbations still give self-maps of \( H^2 \). If the perturbation involves a smooth measure \( \mu \) on \( S^1 \), then one must be careful about uniformity, up to the boundary, of the resulting perturbation of \( \rho \text{Cen}_{\mu} \) over \( H^2 \). But one can perturb by a pair of atoms in \( H^2_* \) with small imaginary weights, and total weight real, to avoid the uniformity issue.

Then using the “\( S^2 \)-harmonic measure” construction, which we proceed to discuss, given any \( P \subset H^2 \), we produce \( \mu \) on \( S^1 \) with smooth support, which determines the same map on \( H^2_* \), and one then gets a perturbation by a nontrivial imaginary smooth measure \( \mu \) on \( S^1 \), which induces a self-map.

Suppose \( PW \) has support in \( \overline{H_*^2} \), then for each \( p_i \in P \), the restriction of \( \mathcal{E}_{p_i} \), to \( \overline{H_*^2} \) is by potential theory equal to \( \mathcal{E}_{\mu} \) where \( \mu = \mu_{p_i} \) is the “\( S^2 \)-harmonic measure” of \( p_i \) on \( S^1 \); this is just the jump of the normal derivative \( \partial_n \mathcal{E}_{p_i} \) at each \( x \in S^1 \), (or equivalently a distributional laplacian) where we construct \( \mathcal{E}_{p_i} \) by continuously extending \( \mathcal{E}_{p_i} \) from \( \overline{H_*^2} \) to \( H^2 \) as an \( S^2 \)-harmonic function. This can also be described using the standard euclidean harmonic measure of \( (p_i + \text{an additional mass}) \) where the additional mass corresponds to the forcing term in the definition of \( \mathcal{E} \) (coming from the curvature of the sphere). The main point is that

**Lemma 8.25.** \( \mu_{p_i} \) is positive on \( S^1 \),


despite this forcing term. This is best seen using \( \partial_n \mathcal{E}_{p_i} \); the maximum principle implies that \( \mathcal{E}_{p_i} |H_\circ \) is everywhere smaller than the \( (S^2) \)-harmonic function with the same boundary values, and this implies an inequality on the normal derivatives at
the boundary. But the reflection of this harmonic function is the extension used to define $S^2$-harmonic measure. Note that we use the isometry property of the reflection map here, this being a special feature of the hemisphere case.

Since an atom $PW$ in $H^2$ and its $S^2$-harmonic measure induce the same energy $E$ on $H^2$, the $\text{Cen}$ construction is not 1-1 from measures to maps in this context.

8.6.3. more questions: Q: It is natural to consider existence of a.e. boundary limits for $\rho\text{Cen}_{PW}$ as well as growth of $\rho\text{Cen}_\mu$ in relation to Hausdorff dimension, or entropy numbers, associated to the boundary measure.

Q: Are there notions of convexity associated to the distance decreasing property? It is tempting to use a convexity argument to get the unique local minimum in $H^{N+1}$, but we don’t see how.

Q: The Schwarz lemma is a powerful tool for studying holomorphic maps as dynamical systems. In the general context of maps constructed using $\text{Cen}$ form, this requires a Poincaré or Kobayashi metric type construction on the complement of the Julia set compatible with associated generalized Schwarz lemmas.

9. PROBLEMS

We summarize some but not all problems from the text, some not yet mentioned, and some relations to those already mentioned.

(1) (3.14) diagonalizing Hessians geometrically. Note the relation to discriminants implicit in 4.0.6.
(2) The conjectures in section 5.2.1 on degenerate attractors.
(3) General convergence of the maps in section 6. Classify all periodic-attractors for $\mathcal{F}$ and for the root-finding algorithm. Relate the combinatorics of triangulations to attractors for $\mathcal{F}$. This includes the case of collapsing attractors, 3.1, and the problems there.
(4) possible characterizations of $\mathcal{F}$, (3.8).
(5) (3.2).
(6) Morse-theory of $\mathcal{E}$ vs dynamics of $\mathcal{F}$: The energy functions we study provide strictly plurisubharmonic exhaustions with interesting equivariance properties and relations to the moment map. It should be interesting to compare the Morse-Smale complex of $\mathcal{E}$ to dynamics of $\mathcal{F}$. Does $Z^n$ (roots of unity) play a special role in the dynamics of $\mathcal{F}$, or $\nabla \mathcal{E}$? Are all local minima of $\mathcal{E}$ in the drainage basin of $Z^n$? Notice that this is the grad flow analogue of the question as to whether every basin of attraction for $\mathcal{F}$ has $Z^n$ in its closure? Is $Z^n$ the only critical point of index $n$, for $\mathcal{E}$?
(7) The holomorphic surgery analysis of $\text{Rat}$; section 2.1.1.
(8) The holomorphic Lefschetz fixed-point theorem in terms of weights, after (2.1) vs generalizations to the multivariable case including $\mathcal{F}$ and suspensions?
(9) (8.6.3) problems following up on the Schwarz lemma for $\text{Cen}_{PW}$.
(10) Regularity theory for section 8.5. The problem would be better motivated with some specific applications in mind.
(11) Deformation and rigidity theory for the $\text{Cen}_{PW}$ in higher dimension; topological conjugacy vs smooth(er) conjugacy. Qualitative descriptions of the branching behavior of $\text{Cen}_{PW}$.
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