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by

Ursula Ludwig

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Ursula Ludwig*

Max-Planck-Institut für Mathematik in den Naturwissenschaften
Inselstraße 22-26
04103 Leipzig, Germany

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Abstract Our objective is to develop a stratified Morse theory with tangential conditions. We define a continuous strata-wise smooth Morse function on an abstract stratified space by using control conditions and radiality assumptions on the gradient vector field. For critical points of a Morse function one can show that the local unstable set is a manifold and the local stable set is itself an abstract stratified space. We also give a normal form for the gradient dynamics in the neighborhood of critical points. For a stratified Morse pair which satisfies the generic Morse-Smale condition we can build the Morse-Witten complex and show that its homology is equivalent to the singular homology of the space.

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*email:uludwig@mis.mpg.de

1 Introduction

In this article we want to develop a Morse theory on a compact abstract stratified space with tangential conditions. Our theory differs in many respects from the stratified Morse theory developed by Goresky/MacPherson [GM88].
King [Kin90] remarked that in view of a deformation lemma the following definition of a critical point for a function on a stratified space makes sense: Suppose that a neighborhood $U(0)$ of the critical point 0 is homeomorphic to a product $V \times W$, and furthermore that in these coordinates the function f can be written as $f(u, v) = g(u) + h(v)$ where $g : V \to \mathbb{R}$ and $h : W \to \mathbb{R}$ have locally cone-like fibers, then the Morse data of f is the product of the Morse data for g and h. He showed that in a neighborhood of a critical point a Morse function in the theory of Goresky/MacPherson has such a normal form. But their theory cannot treat the normal form $f(r, l, x) = r^2 + f_1(x)$. Here (r, l, x) are coordinates for $cone(L) \times \mathbb{R}^m \simeq U(0)$ corresponding to the local cone structure of the abstract stratified space, which is a consequence of Thom's First Isotopy Lemma. In some sense we will attack this kind of critical points in our theory.
A critical point in our theory will be called nondegenerate if it is a nondegenerate critical point both in the tangential directions and the normal directions of the stratum S. The nondegeneracy in the normal directions is expressed with the help of radial vector fields. Radial vector fields on real analytic Whitney stratified spaces $X \subset \mathbb{R}^N$ have been constructed by Schwartz [Sch91] (see also [Bra00] for an overview). For these vector fields the Poincaré-Hopf index theorem has been generalized, example 6.2.1 in [Sch91] shows that the theorem does not hold without the radiality condition. As there are connections between Hopf index theorem and Morse theory, it seems reasonable to impose a radial condition as a kind of nondegeneracy condition in the critical points of a stratified Morse theory. Actually we will show elsewhere that there are more reasons for studying them. We refer to [KT94] for further generalization of the Poincaré-Hopf index theorem on stratified spaces.
In contrast to the theory developed by Goresky/MacPherson, the gradient vector field in our theory is a stratified vector field satisfying control conditions like the ones introduced by Mather [Mat70]. These vector fields are (locally) integrable and so we can study the induced flow and make use of notions and theorems from the theory of dynamical systems, like

stable/unstable set, trajectory spaces. In particular, our theory can be generalized to stratified flows and Conley theory.

A goal of Morse theory is to deduce global information on the topology of the space from the knowledge of local data (critical points) of the Morse function. Also in this respect the difference between the two theories is revealed. Goresky/MacPherson introduced their theory in order to show properties of the intersection homology of the space. We will study the Morse-Witten complex for a Morse pair and show that its homology is equivalent to the singular homology of the space (for coefficient field $\mathbb{Z}_2$). Since our definition of a critical point is a degenerate situation for the theory of Goresky/MacPherson, one can see from our results that there are good reasons for their theory not to treat this situation.

For a compact manifold, the Morse complex for a Morse function has first been studied by Thom, Smale and Milnor [Mil65]. Witten [Wit82], motivated by methods arriving in supersymmetric quantum mechanics, studied a certain deformation of the complex of differential forms on a manifold, where only the critical points of a Morse function play a role. Notice that in the case of a stratified Morse function there cannot be a duality between the homological and the cohomological version of the Morse-Witten complex since Poincaré duality does not hold for singular homology on a singular space. The homology of the stratified space is just dual to the cohomology of the differential complex of controlled forms [Pfl01].

We were motivated to study stratified Morse theory with tangential conditions by an infinite dimensional example: the Plateau problem for surfaces of genus $g > 0$. Jost/Struwe [JS90] showed that the gradient vector field of the Dirichlet energy (with respect to the Weil-Petersson metric on the Mumford-Deligne compactification of the moduli space) is tangential to the stratification (by genus). Thus in isolated critical points of the functional the conditions needed in the stratified Morse theory of Goresky/MacPherson [GM88] are not fulfilled. A stratified Morse-Conley theory for this example would relate critical points of the Dirichlet-functional, i.e. minimal surfaces, to the homology of the moduli space of curves. Even though we cannot yet treat this example with the theory developed here, we hope that our approach gives hints in this direction.

The tangential conditions on the gradient vector field imposed in our theory also arrive naturally in other important examples. The G-equivariance of a vector field on a G manifold M – where G is a compact Lie group – translates into tangential conditions on the quotient M/G [dRdS88]. This quotient is known to be an abstract stratified space (cf. e.g. [Pfl01, Ver84].

2 Preliminaries

Abstract stratified spaces Let X be a topological space, Hausdorff, locally compact, paracompact and with countable basis of the topology. A stratification $\mathcal{S}$ of the topological space X is a locally finite family of disjoint locally closed subsets $S \subset X$ called pieces, such that $X = \bigcup_{S \in \mathcal{S}} S$. Moreover the strata S are smooth manifolds without boundary in the induced topology.
A tubular system for a stratified space is a family of tuples

$$\{(T_S, \pi_S : T_S \to S, \rho_S : T_S \to \mathbb{R}_{\geq 0})\}_{S \in \mathcal{S}},$$

where T_S is an open neighborhood of S, $\pi_S : T_S \to S$ is a continuous retraction and $\rho : T_S \to \mathbb{R}_{\geq 0}$ is continuous and such that $\rho_S^{-1}(0) = S$.

Definition 2.1. *(see [Ver84]) An abstract stratified space is a topological space X with a stratification $\mathcal{S}$ and a tubular system $\{(T_S, \pi_S : T_S \to S, \rho_S : T_S \to \mathbb{R})\}_{S \in \mathcal{S}}$ satisfying the following conditions:*
(1) For each pair of strata (S, R), $T_S \cap R \neq \emptyset$ implies that $S \leq R$, i.e. $S \subset \overline{R}$.
(2) For each pair of strata (S, R) with $S < R$ the map

$$(\pi_S, \rho_S) : R \cap T_S \to S \times \mathbb{R}_{>0}$$

is smooth and submersive.
(3) The tubular system is controlled, i.e. for each pair of strata (S, R) with $S \subset \overline{R}$ and all $x \in T_S \cap T_R$ the following conditions are satisfied:

($C\pi$) $\pi_S \pi_R(x) = \pi_S(x)$,

($C\rho$) $\rho_S \pi_R(x) = \rho_S(x)$.

A consequence of Thoms First Isotopy Lemma [Ver84] is that these spaces have locally a cone-like structure, i.e. let $p \in S$ be a point in a stratum S then there is a neighborhood of p in X homeomorphic (by a strata preserving, strata-wise smooth homeomorphism) to $B_\epsilon^{\dim S}(p) \times cone(L)$. Here $L \simeq (\pi_S, \rho_S)^{-1}(p, \epsilon)$ is the normal link in p (see also [GM88] for the notion of the normal link for a Whitney-stratified subspace $X \subset \mathbb{R}^N$). The normal link is independent of the choice of $p \in S$ and $\epsilon > 0$ small enough. By taking the obvious restrictions one can easily see that L is itself an abstract stratified space of smaller depth, where the depth of X is defined as follows:

$$\mathrm{depth}(X) = \max\{m \mid \text{ there exist a chain of strata } X_0 < X_1 < ... < X_m\}.$$

Example 2.2. *(1) The most prominent stratifications are the so-called Whitney stratifications of an analytic subset of* $\mathbb{R}^N$ *(see e.g.[GM88] for the definition). Whitney stratified spaces are an example of abstract stratified spaces (cf. e.g. [Tho69, Pfl01]).*

(2) Let M be a smooth manifold with a compact Lie group G acting on M. If the action of the group is not free, the orbit space M/G is in general not a manifold, but it is still an abstract stratified space (cf. e.g. [Pfl01, Ver84]).

Controlled stratified vector fields A (smooth) stratified vector field on an abstract stratified space X is given by a family $\xi = \{\xi_S : S \to TS\}_{S \in \mathcal{S}}$ of (smooth) vector fields on each stratum. We will say that ξ is a stratified C^k vector field ($k \in \mathbb{N}$) if the restriction ξ_S is C^k for all $S \in \mathcal{S}$.

Definition 2.3. *(see [Ver84]) A stratified vector field on X is called controlled (with respect to a tubular system) if for all pairs of strata (S, R) with $S < R$ and all $x \in T_S \cap R$ the following conditions are satisfied:*

($C\pi$) $d\pi_S \xi_R(x) = \xi_S(\pi_S(x))$,

($C\rho$) $d\rho_S \xi_R(x) = 0$.

There is no canonical notion of continuity for stratified vector fields, but using the control conditions ($C\pi$) and ($C\rho$) we can still obtain a (local) continuous flow (see [Ver84]). Moreover the flow Φ induced by a ($C\pi$)-controlled vector field is itself controlled.
Indeed we can relax the control condition ($C\rho$) into the condition

$$(C\rho') \mid d\rho_S \xi_R(x) \mid < A\rho_S(x) \text{ for all pair of strata } S < R \text{ and all } x \in T_S \cap R$$

introduced by [Loo78] in order to still have local integrability of the vector field (see [dPW95]).
The condition

$$(d\rho\xi < 0) \quad d\rho_S \xi_R(x) < 0 \text{ for all pairs of strata } S < R \text{ and all } x \in T_S \cap R$$

implies that the induced flow goes only from larger into smaller strata.

For our purposes we need vector fields which are "nearly ($C\pi$)-controlled"; we call a stratified vector field ($C\pi'$)-controlled if it satisfies

$$(C\pi') \quad d\pi_S \xi_R(x) = \xi_S(\pi_S(x)) + \rho_S^2(x)\chi(\pi_S(x)) \text{ for all } x \in T_S \cap R,$$

where χ is a stratified bounded vector field.

Radial vector fields The construction of radial vector fields on a Whitney-stratified set $X \subset \mathbb{R}^N$ in [Sch91] uses two kinds of tubular systems, geodesic ones and parametrized ones. We want to give the following easier definition which still keeps track of the main ideas:

Definition 2.4. *Let X be an abstract stratified space with a controlled tubular system $\{(T_S, \pi_S : T_S \to S, \rho_S : T_S \to \mathbb{R})\}_{S \in \mathcal{S}}$. A vector field ξ_{rad} is called radial with respect to the stratum S and the tubular system if it satisfies*
(i) $\xi_{rad|S} = 0$ and
(ii) $\xi_{rad|T_S - S}$ is the controlled lift of the vector field $-t\frac{\partial}{\partial t} \in \Gamma(\mathbb{R})$ along the controlled submersion $d\rho : T_S - S \to \mathbb{R}$.

Remark 2.5. *From the definition of an abstract stratified space it follows that $\rho_S : T_S - S \to \mathbb{R}$ is a stratified controlled submersion (i.e. it satisfies the control condition ($C\pi$) and the restriction to each stratum is submersive). Thus by [Ver84] (2.4) there exists a controlled vector field ξ_{rad} on X which is a lift of $-t\frac{\partial}{\partial t}$ satisfying*

$$d\rho \xi_{rad} = -t\frac{\partial}{\partial t}.$$

Note that, because we study the negative gradient vector field, for us a radial vector field is pointing inward on ∂T_S^δ and not outward like in [Sch91].

A radial extension ζ_{rad} of $\zeta_S \in \Gamma(TS)$ is the sum of a parallel extension and the radial vector field defined above. It has the same singularities as ζ_S.

Definition 2.6. *Let $\zeta_S : S \to TS$ be a smooth vector field on a stratum S. Let $\{(T_S, \pi_S, \rho_S)\}_{S \in \mathcal{S}}$ be a controlled tubular system on X.*

(1) Let $\zeta_{||}$ be a stratified controlled lift of ζ_S on the open neighborhood T_S (with respect to the given tubular system). A parallel extension of the vector field ζ_S is a stratified vector field χ such that

$$\chi(x) = \zeta_{||}(x) + \rho_S^2(x)\zeta_{pert}(x)$$

for all $x \in T_S$, where ζ_{pert} is a stratified bounded vector field on T_S.

(2) *A radial extension ζ_{rad} of ζ_S on T_S is given by*

$$\zeta_{rad}(x) = \chi(x) + \xi_{rad}(x) \text{ for all } x \in T_S$$

where ξ_{rad} denotes the radial vector field in Def. 2.4 and χ is the parallel extension of ζ_S in part (1).

Riemannian metric on an abstract stratified space For Whitney stratified spaces $X \subset \mathbb{R}^N$ one can find the definition of a Riemannian metric in [Pfl01], it is just the pullback of the metric on the ambient space. We will define here the notion of a Riemannian metric on an abstract stratified space. Note that the definition below contains fewer smoothness assumptions than the definition in [Pfl01].

Definition 2.7. *Let X be an abstract stratified space with a stratification $\mathcal{S}$ and a controlled tubular system (π, ρ). A Riemannian metric on X compatible with the given tubular system is a family*

$$\{g^S \in \Gamma(Sym(TX_S^* \otimes TX_S^*))\}_{S \in \mathcal{S}}$$

of smooth Riemannian metrics satisfying the following conditions:

(1) *Let S, R be a pair of strata S, R with $S < R$, let $v_{||}$ resp. $w_{||}$ be parallel extensions (with respect to (π, ρ)) of smooth vector fields $v : S \to TS$ resp. $w : S \to TS$. For each sequence $x_k \in R$ with $\lim x_k = x \in S$ the following equality holds:*

$$\lim_{x_k \to x} g^R(x_k)(v_{||}, w_{||}) = g^S(x)(v, w).$$

(2) *The distance*

$$d(p, q) = \inf_{\gamma} \int_p^q \sum_{j \in J} \sqrt{g(\gamma_j(t))(\dot{\gamma}_j(t), \dot{\gamma}_j(t))} dt$$

yields a metric on X and induces the original topology on X. (γ runs over all piecewise C^1-curves in X with endpoints p, q and J is a decomposition of the differentiability set into its connected components.)

(3) *Let S, R be a pair of strata with $S \subset \overline{R}$, then for all points $x \in R$ the distance from the lower stratum $d(x, S)$ is of order $O(\rho_S)$.*

By a Riemannian metric on X we mean a Riemannian metric compatible with an arbitrary controlled tubular system. If we fix the tubular system, we will speak about a compatible Riemannian metric.
Let us show the existence of a compatible Riemannian metric on a compact abstract stratified space with fixed tubular system:

Proposition 2.8. *Let X be a compact abstract stratified space with controlled tubular system (π, ρ). Then there exists a Riemannian metric compatible with (π, ρ).*

Proof: We show the assumption by induction on the depth of the stratification.
Let us first show the case of two strata $\{X_1, X_2\}$, i.e. of depth 1. Let $\{U_i\}_{i \in I}$ be a finite open covering of X. Because of the existence of a controlled partition of unity ([Ver84]) it is enough to construct a metric on each open set. If U_i is included in the larger stratum X_2, then $U_i \simeq \mathbb{R}^{\dim X_2}$ and we take the Euclidean metric on $\mathbb{R}^{\dim X_2}$. If U_i is the neighborhood of a point $p \in X_1$, there exists a stratified diffeomorphism $\mathbb{R}^{\dim X_1} \times cone(L(p))$ where the cone coordinate is given by $t = \rho_1$ ($t = 0$ corresponds to the cone point). The normal link is a compact manifold and thus there exists a metric g_L on L. The required Riemannian metric on U_i satisfying (1)-(3) of Definition 2.7 is given by the warped metric $g = g_{X_1} + t^2 g_L + dt \otimes dt$.
Let us assume that we can construct a Riemannian metric on stratified spaces of depth $< k$. Let X be a compact abstract stratified space of depth k. Let $U(p)$ be an open neighborhood of $p \in S$. It can be written as $U(p) \simeq cone(L(p)) \times \mathbb{R}^{\dim S}$, where the normal link is an abstract stratified space of depth $< k$ and thus there exists a Riemannian metric g_L on L compatible with the induced tubular system. We can now define a metric $g_S + t^2 g_L + dt \otimes dt$ on $U(p)$, and with a controlled partition of unity we obtain a metric on X satisfying the conditions (1)-(3) of Definition 2.7.

□

Note that in contrast to the case of a manifold there can exist different Riemannian metrics on a stratified space compatible with the same tubular system, which are, however, not Lipschitz equivalent. For instance for a stratified space with two strata the two warped metrics $g_i = g_{X_1} + g_i(\rho) g_L + d\rho \otimes d\rho$ ($i = 1, 2$) with different coefficients $g_i(\rho) = O(\rho^i)$ are not Lipschitz equivalent.

3 Definition of a stratified Morse pair

From now on we will assume that X is compact. We denote by $C^n_{strat}(X,\mathbb{R})$ the space of continuous strata-wise C^n-smooth functions on the abstract stratified space X. Let (f,g) be a pair consisting of a function $f \in C^n_{strat}(X,\mathbb{R}), (n \geq 2)$, and a Riemannian metric on X. A point $0 \in S$ in the stratum S of X is called a critical point of (f,g) if it is a critical point for the restriction $f_{|S}$.

A critical point will be called nondegenerate if it is a nondegenerate critical point both in the tangential directions and the normal directions of the stratum S. The nondegeneracy in the normal directions will be expressed in terms of a radiality condition on the gradient vector field. The gradient vector field of a function $f \in C^n_{strat}(X,\mathbb{R})$ is the stratified vector field

$$\{\nabla_{g^S} f_{|S} \mid g^S(x)(\nabla f(x), v) = d(f_{|S})(x)(v) \text{ für alle } x \in S, v \in T_x S\}_{S \in \mathcal{S}}.$$

Definition 3.1. *A critical point $0 \in S$ is a nondegenerate critical point for the pair (f,g) if the following conditions are satisfied:*
(i) 0 is a nondegenerate critical point of the restriction $f_{|S}$.
(ii) There exists a controlled tubular system $\{(T_S, \pi_S, \rho_S)\}_{S \in \mathcal{S}}$ on X such that the stratified vector field $-\nabla_g f$ is a radial extension of $-\nabla_g f_{|S}$.

It is easily seen that nondegenerate critical points are isolated.

We can now give the definition of a Morse function. We require tangential conditions for the gradient vector field. This tangency will be expressed with the help of control conditions.

Definition 3.2. *A pair (f,g) is called a stratified Morse pair if there exists a controlled tubular system (π,ρ) on X such that the stratified vector field $\xi := -\nabla_g f$ satisfies the control conditions $(C\pi'), (C\rho')$ and $d\rho\xi < 0$ and all critical points are nondegenerate.*

Proposition 3.3. *Let (f,g) be a Morse pair. The flow induced by the negative gradient flow $\xi = -\nabla_g f$ exists for all time, in particular flow lines do not leave a stratum in finite time. Moreover in infinite (positive) time the flow lines can only go from a larger into a smaller stratum.*

Proof: The control condition $(C\rho')$ makes sure that the flow does not leave a stratum in finite time (see [dPW95]). The two control conditions $(C\pi')$ and $(C\rho')$ yield the continuity of the flow (in analogy to [dPW95]). The last statement follows from the condition $d\rho\xi < 0$.

$\square$

Before presenting our results let us first discuss the case of a manifold $M \subset \mathbb{R}^N$ with convex boundary in view of the two stratified Morse theories.

Example 3.4. *The manifold M with convex boundary can be seen as a stratified space with two strata, where the boundary ∂M is the smaller stratum and the interior $\operatorname{int} M$ is the larger stratum.*
In [Bra74] there are given two possible normal forms for a Morse function in a critical point $0 \in \partial M$. One can find local coordinates $\{x_1, ..., x_n\}$ around the critical point 0 such that $\partial M = \{x_n = 0\}$ and the function can be written as:

$$f_{GM}(x) = \Sigma_{i=1}^{n-1} \pm x_i^2 \pm x_n,$$

or

$$f(x) = \Sigma_{i=1}^{n-1} \pm x_i^2 \pm x_n^2.$$

In the first case we see that the gradient vector field $\nabla_g f_{GM} = \pm\frac{\partial}{\partial x_n}$ (with respect to the Euclidean metric in these coordinates) is transversal to the boundary. (Note that in this case $df_{GM}(0) \neq 0$ but $d(f_{GM|\partial M})(0) = 0$.) This normal form corresponds to critical points in the sense of Goresky/MacPherson. The deformation lemma in this case is proven by moving the critical point into the interior and then applying smooth theory. Note that by passing a critical point with normal form $f_{GM}(x) = \Sigma_{i=1}^{n-1} \pm x_i^2 - x_n$ the topological type of the sublevel set does not change.
In the second case the negative gradient vector field $-\nabla_g f$ (where g is the Euclidean metric) is tangential to the boundary. Moreover the vector field satisfies the control condition $(C\pi')$ and $(C\rho')$ with respect to the standard tubular neighborhood of the boundary. Note that we can apply the homeomorphic coordinate change $u_i = x_i, (i = 1, ...n-1); u_n = x_n^2$ and thus get again into the first case. (In the general case of an abstract stratified space the two situations are not even homeomorphic). In order to apply our theory, we restrict ourselves to the normal form $f(x) = \Sigma_{i=1}^{n-1} \pm x_i^2 + x_n^2$ because we require additionally that the flow does not leave the boundary in (infinite) positive time. The other critical points do not yield any contribution to the CW-complex anyway.
Note that in most cases where Morse theory has been studied on manifolds with boundary the condition of transversality of the gradient field to the boundary is required. This has the advantage of being a generic condition.

But also Morse theory with tangential conditions has been studied, see for example [LP90].

4 Existence of a stratified Morse pair

Proposition 4.1. *Let X be a compact abstract stratified space with a tubular system (π, ρ). We can construct a Morse pair (f, g) on X such that g is a compatible Riemannian metric on X and the negative gradient vector field $\xi := -\nabla_g f$ satisfies the control condition $(C\pi')$, $(C\rho')$ and $d\rho\xi < 0$ (with respect to (π, ρ)).*

Proof: Let g be the Riemannian metric on X compatible with the given tubular system (π, ρ) constructed in Prop. 2.8. We will moreover construct a function $f \in C^\infty_{strat}(X, \mathbb{R})$, such that (f, g) is a stratified Morse pair.
Because the stratification is locally finite and because of condition (2) of an abstract stratification, two strata of the same dimension are not comparable and thus (by taking the union of all strata of the same dimension) we can assume without loss that the stratification $\mathcal{S} = \{X_i\}_{i=1,...,n}$ on X is such that $\dim X_1 < ... < \dim X_n$. <u>The case of two strata</u>: Let X be a stratified space with two strata $\{X_1, X_2\}$ with $X_1 \subset \overline{X_2} = X$. Let $f_1 : X_1 \to \mathbb{R}$ be a smooth Morse function on X_1 (the existence of a Morse function on X_1 is clear since X_1 is a compact smooth stratum). We can extend f_1 to the tubular neighborhood T_{X_1} of X_1 by $f(x) = f_1 \circ \pi_1(x) + \rho_1^2(x)$ $(x \in T_{X_1})$. It is easily seen that $-\nabla_g f$ satisfies the control conditions $(C\pi)$,$(C\rho')$ and $d\rho\xi < 0$ with respect to the given tubular neighborhood. Moreover the stratified vector field $\xi = -\nabla_g f$ is a radial lift of $\xi_1 = -\nabla_{g_1} f_1 \in \Gamma(TX_1)$. By using a partition of unity one can construct a Morse function on X where the critical points in a tubular neighborhood of X_1 lie already in X_1 and are all nondegenerate in the sense of Def. 3.1.
<u>General case</u>: Let X be a stratified space with n strata. We will construct a stratified Morse pair by induction on the number of strata.
Let us assume that we have already constructed a Morse function

$$f : \cup_{1=1}^{k-1} X_i \to \mathbb{R}$$

such that the stratified vector field $\{\xi_i\}_{i=1,...,k-1}$ satisfies the required control conditions and that moreover in the neighborhood of critical points $p \in X_i, (i = 1, ..., k-2)$ of the pair (f, g) the gradient vector field $\{\xi_l\}_{l=i,...,k-1}$

is a radial extension of ξ_i. We want to extend the Morse function on the stratum X_k.
From the case of two strata applied to the pair of strata (X_i, X_k) we know already that we can extend the function $f_{|X_i} : X_i \to \mathbb{R}$ to a continuous strata-wise smooth function

$$f_i : T_{X_i} \cap X_k \to \mathbb{R}$$

through $f_i = f_{|X_i} \circ \pi_i(x) + \rho_i^2(x)$. The vector field

$$\{\xi_i : X_i \to TX_i, \xi_{ik} : X_k \cap T_{X_i} \to T(X_k \cap T_{X_i})\}$$

is a radial extension of ξ_i on $\{X_i, X_k\}$ and satisfies the control conditions $(C\pi)$, $(C\rho')$ and $d\rho\xi < 0$. We must choose the tubes T_{X_i} small enough such that all critical points of the already defined Morse function $f : \cup_{i=1}^{k-1} X_i \to \mathbb{R}$ which lie in $T_{X_i} \cap X_{k-1}$ already lie in X_i. One can check by using the induction assumption and eventually shrinking the neighborhoods T_{X_i} of X_i that the vector fields $\{\{\xi_i\}_{i<j}, \xi_{jk} : T_{X_j} \cap X_k \to T(T_{X_j} \cap X_k)\}$ satisfy the control conditions $(C\pi')$ and $(C\rho')$.
We chose the following covering of $\cup_1^k X_i$:
$V_1 = T_{U_1} \cap X_k$,
$V_2 = T_{U_2} \cap X_k - \overline{\mathrm{U}_\epsilon(X_1)}$,
...
$V_{k-1} = T_{U_{k-1}} \cap X_k - \overline{\mathrm{U}_\epsilon(\cup_{i=1,\ldots,k-2} X_i)}$,
$V_k = X_k - \overline{\mathrm{U}_\epsilon(\cup_{i=1,\ldots,k-1} X_i)}$
and a subordinate controlled partition of unity $\{\varphi_i\}$. We can now extend the Morse function to the stratum X_k by $f(x) = \Sigma\varphi_i(x) f_i(x)$. The gradient vector field

$$\{\xi_i\}_{i<k} \cup \{\xi_k = \Sigma\big(\varphi_i \nabla_g f_i + f_i \nabla_g \varphi_i\big)\}$$

satisfies the control condition $(C\pi')$. The first summand $\Sigma\varphi_i \nabla f_i$ is a linear combination of $(C\pi')$-controlled vector fields and thus itself $(C\pi')$-controlled. For the second summand we have

$$d\pi_j(f_i \nabla\varphi_i)(x) = f_i(\pi_j(x)) \nabla\varphi(\pi_j(x)) \text{ for } i < j$$

and

$$d\pi_j(f_j \nabla\varphi_j)(x) = \Big(f_j(\pi_j(x)) + \rho_j^2(x)\Big) \nabla_g \varphi_j(\pi_j(x)).$$

The expressions $d\pi_j(f_i \nabla\varphi_i)(x)$ for $i > j$ do not yield any contribution.

Similarly we can check the control conditions $(C\rho')$ and $d\rho\xi < 0$.
By the choice of the covering $\{V_i\}$ and the construction we can easily deduce the radiality of the vector field in the neighborhood of critical points.

$\square$

Remark 4.2. *Note that the question of the existence of a stratified Morse function for a Whitney-stratified space $X \subset \mathbb{R}^N$ is not meaningful for our theory. We cannot always construct a Morse function $f : X \to \mathbb{R}$ being a restriction of a smooth function $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ and satisfying the conditions of Def. 3.2.*

5 Normal form in a critical point

An important result in smooth Morse theory is the so-called Morse lemma [Mil65], which provides a normal form for a Morse function in the neighborhood of a critical point (see also example 3.4 for a Morse lemma on manifolds with boundary). We do not have a similar statement for a stratified Morse pair, however in this section we will obtain a "normal form" for the induced dynamics of the negative gradient flow. We first analyze the stable and unstable set of a critical point 0 for a Morse pair, establishing the stratified analogue of the Stable/Unstable Manifold Theorem. Then we generalize the Hartman-Grobman Theorem to the stratified case.

For the Unstable Manifold Theorem we do not need yet the condition of the gradient vector field being a radial extension:

Theorem 5.1. *Let (f, g) be a pair consisting of a function $f \in C^n_{strat}(X, \mathbb{R})$ ($n \geq 2$) and a Riemannian metric on X. Let $0 \in S$ be a nondegenerate critical point of the restriction $f_{|S}$ of index k. Moreover the negative gradient vector field $\xi := -\nabla_g f$ should satisfy the control conditions $(C\pi'), (C\rho')$ and $d\rho\xi < 0$. Then the local unstable set*

$$W^u_{loc}(0) := \{y \in X \mid \lim_{t \to -\infty} \Phi(y, t) = 0\} \cap U_\epsilon(0)$$

is a k-dimensional C^{n-1} submanifold of S.

Proof: Because of the control condition $(C\rho')$ and $d\rho\xi < 0$, the induced flow cannot leave the stratum S. Then the unstable set $W^u_{loc}(0)$ is completely contained in the stratum S, and the proposition follows by applying the Unstable Manifold Theorem of the smooth theory (compare e.g. [KH95]).

Note that the control condition $(C\pi')$ is needed in order to provide the existence of the flow (cf. [Ver84]). $\square$

We now prepare the analogue of the "Stable Manifold Theorem". Here the radiality assumption is crucial.
Let $0 \in S$ be the critical point of the Morse pair (f, g). Denote by m the dimension of the stratum S. Let G be the time-1-map of the induced flow for the negative gradient vector field $-\nabla_g f$ and $A : \mathbb{R}^m \to \mathbb{R}^m$ the linearisation of $G_{|S}$. Denote by $\{x_1, ...x_{n-k}, y_1, ..., y_k\}$ local coordinates of S in a neighborhood of 0, where x resp. y denote the stable resp. unstable directions of A.
The next lemma states the existence of local cone coordinates where the gradient vector field $-\nabla_g f$ has almost standard form:

Lemma 5.2. *Let $0 \in S$ be a critical point of the Morse pair (f, g). Then there exists a neighborhood $U(0)$ of the critical point as well as a stratified diffeomorphism $U(0) \simeq cone(L) \times \mathbb{R}^m$ so that the stratified gradient vector field $\xi := -\nabla_g f$ can be written in these coordinates $(r, l, x, y) \in cone(L) \times \mathbb{R}^m$ as*

$$\xi(r, l, x, y) = \xi_S(0, 0, x, y) - r\frac{\partial}{\partial r} + O(r^2).$$

Proof: It follows from Def. 3.2 of a stratified Morse pair that ξ is the radial extension of $-\nabla_g f_{|S}$ with respect to a controlled tubular system (π, ρ) on X. Then $(\pi_S, \rho_S) : T_S - S \to S \times (0, \epsilon)$ is a controlled submersion and because of Thom's First Isotopy Lemma (see e.g.[Ver84]) locally trivial. This yields local coordinates $U(0) \simeq cone(L) \times \mathbb{R}^m$ as in the statement. The expression of ξ in these coordinates follows from the control condition $(C\pi')$ and the fact that the radial vector field with respect to the tubular system (π, ρ) and the stratum S is $\xi_{rad} = -r\frac{\partial}{\partial r}$.

$\square$

Lemma 5.3. *Let 0 be a critical point of the Morse pair (f, g). Then the time-1-map of the negative gradient flow $\xi = -\nabla_g f$ can be expressed in the local coordinates of Lemma 5.2 in the following way:*

$$\begin{aligned}
G^r(r, l, x, y) &= ar, \\
G^l(r, l, x, y) &= l, \\
G^x(r, l, x, y) &= A^s x + \alpha_s(r, l, x, y), \\
G^y(r, l, x, y) &= A^u y + \alpha_u(r, l, x, y),
\end{aligned}$$

where $\alpha_u : cone(L) \times \mathbb{R}^m \to \mathbb{R}^k, \alpha_s : cone(L) \times \mathbb{R}^m \to \mathbb{R}^{m-k}$ are Lipschitz continuous with Lipschitz constant δ with respect to the warped metric in prop. 2.8. Moreover the restriction of $\alpha_{s,u}$ to each stratum is C^1, and $\alpha_{s,u}(0) = 0$. A^s is a linear contraction, A^u is a linear expansion, and $a = \frac{1}{e}$.

The assumption follows from Lemma 5.2 and the following criterium for Lipschitz continuity, which one can check easily using the warped metric:

Lemma 5.4. *Let X be a stratified cone, $X = cone(L)$, with a stratification $\{X_2 = cone(L) - \{0\}, 0\}$. Let g be a warped metric $g = dr^2 + g(r)g_L$, where $g(r)$ is decreasing with $\lim_{r\to 0} g(r) = 0$. Let*

$$G = (G^r, G^l) : X_2 \to X_2$$

be a C^1-diffeomorphism on X_2, such that G^r is a contraction with $\lim_{r\to 0} G^r(r, l) = 0$.
Then the following statements hold:
(1) There is a continuous continuation of G on X by $G(0) = 0$.
(2) If in addition there is an r_0 such that for $r < r_0$ the following expressions are uniformly bounded

$$\begin{array}{lcl} G^r_r & := & \frac{\partial G^r}{\partial r}, \\ \frac{1}{g(r)} G^r_l & := & \frac{1}{g(r)} \frac{\partial G^r}{\partial l}, \\ g(r) G^l_r & := & g(r) \frac{\partial G^l}{\partial r}, \\ G^l_l & := & \frac{\partial G^l}{\partial l}, \end{array}$$

then G is Lipschitz in a neighborhood of 0.

Let us now embed $cone(L) \times S \subset \mathbb{R}^N \times \mathbb{R}^m$ by a metric (strata-wise) embedding (this is possible as one can easily see by an induction on the depth of the stratification). Now we use Kirszbrauns Lemma [Kir34] to extend the dynamics on a neighborhood of $0 \in \mathbb{R}^{N+m}$. We apply the lemma to α_s resp. α_u and obtain Lipschitz continuous functions $\widetilde{\alpha_s} : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^{m-k}$ resp. $\widetilde{\alpha_u} : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^k$ with the same Lipschitz constant δ as before. We can now deduce properties of the stable set for the critical points of a stratified Morse pair by using the Stable Manifold Theorem of the smooth theory for the extended dynamics:

Theorem 5.5. *Let $0 \in S$ be a critical point of a stratified Morse pair (f, g). Then the local stable set*

$$W^s_{loc}(0) := \{y \in X \mid \lim_{t\to\infty} \Phi(y, t) \to 0\} \cap U_\epsilon(0)$$

is an abstract stratified space. The restriction of $W^s(0)$ on each stratum is C^{n-1}.

Proof: As we indicated above, we can use Kirszbrauns lemma to extend the time-1-map G to a neighborhood of 0 in $\mathbb{R}^{N+m}$. Thus G is the restriction of a map $\widetilde{G} : \mathbb{R}^N \times \mathbb{R}^{m-k} \times \mathbb{R}^k \to \mathbb{R}^N \times \mathbb{R}^{m-k} \times \mathbb{R}^k$ with

$$\begin{array}{rcll} \widetilde{G}^z(z,x,y) & = & az, & \\ \widetilde{G}^x(z,x,y) & = & A^s x + \widetilde{\alpha_s}(r,l,x,y), & \quad (*) \\ \widetilde{G}^y(z,x,y) & = & A^u y + \widetilde{\alpha_u}(r,l,x,y), & \end{array}$$

where $\widetilde{\alpha_{s,u}}$ are Lipschitz continuous functions with Lipschitz constant δ and $\widetilde{\alpha_{s,u}}(0) = 0$. For an adapted norm we have $0 < \|A^s\| < \lambda < 1$ and $\|(A^u)^{-1}\| < \mu^{-1} < 1$. Moreover $a = \frac{1}{e} < 1$. We now apply the Stable Manifold Theorem of the smooth theory (see e.g. [KH95]) for the map $\widetilde{G}$ and the fixed point 0. The local stable set $W^s_{loc}(\widetilde{G}, 0, \mathbb{R}^N) = \{x \in \mathbb{R}^N \mid \lim_{n\to\infty} \widetilde{G}^n(x) = 0\} \cap U_\epsilon(0)$ is a Lipschitz graph. The stratified space X is $\tilde{G}$-invariant and thus the local stable set for the map $G : X \to X$ and the stratified fixed point 0 is given by the restriction $W^s_{loc}(G, 0, X) = W^s_{loc}(\widetilde{G}, 0, \mathbb{R}^N) \cap X$. The stable set is a transversal intersection of the Lipschitz graph $W^s_{loc}(\widetilde{G}, 0, \mathbb{R}^N)$ with the stratified space X and thus itself an abstract stratified space (cf. [KTL89]). The stratification is parametrized by the strata of the normal link L at 0. The strata-wise smoothness follows by using cone fields as in the proof of the classical statement in [KH95] (see the proof in [Lud03]).

$\square$

Until now we described the behavior of points in the stable and unstable set. Actually in a neighborhood of a critical point one can also describe the orbit structure of orbits outside of W^s_{loc} and W^u_{loc}. In analogy to the smooth case (see e.g. [KH95]) we can construct a flow invariant stable/unstable foliation in the neighborhood of a critical point. The unstable leaves are as smooth as the flow and are completely contained in a stratum, whereas the stable leaves are Lipschitz graphs (with respect to the metric constructed in Prop. 2.8) transversal to the stratum S of the critical point and thus abstract stratified spaces themselves (cf. [KTL89]). The restriction on each stratum is as smooth as the flow. Using this foliations we can especially establish a kind of normal form of the dynamics of the gradient flow in the neighborhood of a critical point: We can prove the stratified Grobman-Hartman theorem:

Proposition 5.6. *Let* 0 *be a critical point of a stratified Morse pair* (f, g)*. Then, in a neighborhood of the critical point, the stratified vector field* $\xi := -\nabla_g f$ *is topologically conjugate to the linearisation* $\xi_{lin} = D\xi_S(0) - r\frac{\partial}{\partial r}$ *(in the coordinates of Lemma 5.2). Thus there exist neighborhoods* $U(0)$ *and* $V(0)$ *of* 0 *as well as a stratified homeomorphism* $h : U(0) \to V(0)$ *such that the following diagram is commutative*

$$\begin{array}{ccc} U(0) & \xrightarrow{h} & V(0) \\ {\scriptstyle G}\Big\downarrow & & \Big\downarrow{\scriptstyle G_{lin}} \\ U(0) & \xrightarrow{h} & V(0), \end{array}$$

where G *is the time-1-map for* ξ *and* G_{lin} *is the time-1-map for the linearisation.*

6 Morse-Conley index for the critical point

One important generalization of Morse theory to flows on metric spaces is due to Conley. He introduced the notion of homological Conley index for a flow-invariant set I (cf. [Con78]). In the smooth case one can show that the Conley index of a fixed point for the negative gradient flow of a Morse function can be identified with the Morse index, i.e. with the number of negative eigenvalues of the Hessian of f.

Let $0 \in S$ be a critical point of a stratified Morse pair (f, g) on an abstract stratified space. Then by definition the restriction $f_{|S}$ to the smooth stratum S is a Morse function in the usual sense and we can calculate the Morse index of 0 for $f_{|S}$. Since the flow cannot leave the stratum S in positive time, it is intuitively clear that this is a good definition of an index in 0. We will rigorously prove this in prop. 6.2.

Lemma 6.1. *Let* ξ *be a stratified vector field satisfying the control conditions* $(C\rho')$ *and* $d\rho\xi < 0$ *and inducing a continuous stratified flow* Φ*. Let* I *be a flow invariant set completely contained in a stratum* S*. Assume that there exists a continuous projection* $\pi' : Tub(S) \to S$ *commuting with the flow. Let* (N_1, N_0) *be an index pair for* I *in* S *(for the restricted flow). Then* $(N_1^X, N_0^X) := (\pi')^{-1}(N_1, N_0) \cap \{\rho_S(x) \leq \rho_0\}$ *is an index pair for* I *in* X*. The two pairs* (N_1, N_0) *and* (N_1^X, N_0^X) *are homotopic.*

Proof: Since π' is continuous and N_1^X and N_0^X are closed and contained in a compact set they are themselves compact. <u>We show $Inv(\overline{N_1^X - N_0^X}) = \{I\}$:</u>

The inclusion $\supset$ is evident. Let now $x \notin I$ be a point in $Inv(\overline{N_1^X - N_0^X})$. Because of the control condition $(C\rho')$, there exists a $t(x)$ such that $\rho(\Phi(x,t)) > \rho_0$ for $t < t(x)$. This is a contradiction to $x \in Inv(\overline{N_1^X - N_0^X})$.
We show that N_0^X is an exit set: Let $y \in N_1^X$ with $\Phi(y,t) \notin N_1^X$ for a time t. Two cases can arise. 1st case: $\pi'(\Phi(y,t)) \notin N_1$. Since $\pi'(y) \in N_1$ and N_0 is an exit set for N_1, there exists a time t_1 such that $\Phi(\pi'(x), t_1) = \pi'(\Phi(x,t_1)) \in N_0$. Then $\Phi(y,t_1) \in N_0^X$ and the assumption follows. 2nd case: $\pi'(\Phi(y,t_1)) \in N_1$, i.e. the inequality $\rho(\Phi(x,t_1)) > \rho_0$ must hold. This is a contradiction to $d\rho\xi < 0$.
We show the positive invariance of N_0^X: Let $y \in N_0^X$ and $\Phi(y,t) \in N_1^X$. Then $\pi'(\Phi(y,t)) \in N_1$ and since N_0 is positive invariant, $\pi'(\Phi(y,t)) \in N_0$ follows. Thus we have $\Phi(y,t) \in N_0^X$.
The last part of the statement follows from the existence of a stratified diffeomorphism $(N_1^X, N_0^X) \simeq (N_1, N_0) \times cone(L)$ and the contractibility of $cone(L)$. □

Proposition 6.2. *The index*

$$\mu(p) := dimW^u(p)$$

of a critical point $p \in S$ for a stratified Morse pair (f,g) is well-defined.

Proof: We can apply Lemma 6.1 to the continuous projection $\pi' := \pi^u : U(p) \to W^u(p)$ given through the local stable foliation in a neighborhood of the critical point. All assumptions of Lemma 6.1 are then verified for the invariant set $I = \{p\}$ and the negative gradient flow and according to the lemma each index pair for 0 is homotopic to an index pair in the smooth stratum S. Thus the Conley index for p can be identified with the Morse index for $f_{|S}$.

□

7 Morse homology for a stratified Morse pair

In this section we will build the Morse-Witten complex for a stratified Morse pair (f,g) satisfying the Morse-Smale condition. The homology of this complex is equivalent to the singular homology of the stratified space. As we saw already in example 3.4 of a manifold with boundary, not all critical points gave a contribution to the cell complex. If we want each critical point to contribute to the Morse-Witten complex, the following condition on the negative gradient flow is essential:

(*) The negative gradient flow Φ does not leave a stratum in finite time and moreover in infinite positive time the flow can go at most from a larger to a smaller stratum.

This property is a consequence of the control condition $d\rho(-\nabla_g f) < 0$. Remember that by definition of a Morse pair (f, g) the negative gradient vector field $\xi = -\nabla_g f$ satisfies the control conditions $(C\pi'), (C\rho')$ and $d\rho\xi < 0$ with respect to a controlled tubular system (π, ρ). For the whole section we will make the additional assumption that the Riemannian metric g is compatible with (π, ρ).

Global stable/unstable set Before we start studying the trajectory spaces we want to indicate some more properties of the gradient flow. There are some minor changes to the manifold case: for example the level sets as well as the (global) stable sets of critical points are, in general, abstract stratified spaces.

Proposition 7.1. *On each stratum the gradient $\nabla_g f$ is orthogonal on regular level sets $f^{-1}(c)$. For each regular value c the level set $f^{-1}(c)$ is an abstract stratified space.*

Proof: (1) The first assumption follows strata-wise like in the smooth theory. In order to show that each regular level set is an abstract stratification we will verify the Whitney conditions (A) and (B) (with respect to the compatible metric g) for the compact set $f^{-1}(c)$ and its stratification (induced from the stratification $\mathcal{S}$ of X) $\{S \cap f^{-1}(c)\}_{S\in\mathcal{S}}$.
We first verify Whitney (A). Let (S, R) be a pair of strata in X with $S \subset \overline{R}$. Let $\{x_n\} \in R \cap f^{-1}(c)$ be a sequence with $\lim x_n = x \in S \cap f^{-1}(c)$. We must show that $\lim_{x_n\to x} T_{x_n}(R\cap f^{-1}(c)) \supset T_x(S\cap f^{-1}(c))$. The pair (S, R) is locally trivial $R \simeq S\times cone(L_S)$ and we can choose local coordinates $\{x, l, r\}$ adapted to the tubular system. We denote the metric resp. its inverse in these coordinates by (g_{ij}) resp. $(a_{ij}) = (g_{ij})^{-1}$.
Each $v_n \in T_{x_n}(R \cap f^{-1}(c))$ satisfies the equality $df_{x_n}(v_n) = <\nabla_g f(x_n), v_n>_g = 0$ and thus

$$\begin{aligned} <\nabla_g f(x_n), v_n> &= \sum_{i,k} g_{ik} \sum_j \left(a_{ij}\frac{\partial f}{\partial x_j}\right) v_n^k = \sum \delta_{jk}\frac{\partial f}{\partial x_j} v_n^k \\ &= \sum_k \frac{\partial f}{\partial x_k} v_n^k = 0 \end{aligned} \qquad (T)$$

In the chosen coordinates the control conditions $(C\pi')$ and $(C\rho')$ can be rewritten as

$$\lim_{r\to 0}\frac{\partial f}{\partial x_1}(x,r,l)=\frac{\partial f}{\partial x_1}(x,0)$$

and

$$a_{13}\frac{\partial f}{\partial x_1}+a_{23}\frac{\partial f}{\partial x_2}+a_{33}\frac{\partial f}{\partial x_3}=O(r).$$

Since the metric is compatible with the tubular neighborhood, the distance $\int\sqrt{g_{33}}dt$ from the lower strata S is of order $O(r)$ and thus $\frac{\partial f}{\partial x_3}=O(r)$.
Let now $v_n=(v_n^1,0,v_n^3)\in T_{x_n}(R\cap f^{-1}(c))$. By inserting the control conditions and the compatibility of the metric (see Def. 2.7) into (T) we obtain:

$$\frac{\partial f}{\partial x_1}(x_n)v_n^1+\frac{\partial f}{\partial x_3}(x_n)v_n^3=0.$$

Then $\lim(v_n^1,0,v_n^3)=(v,0,0)\in T_xS$ satisfies

$$\frac{\partial f}{\partial x_1}(x)v=0$$

and thus lies in $T_x(S\cap X^c)$.
The Whitney condition (B) follows from the following argument: Let λ be the limit of secants in $x\in S\cap f^{-1}(c)$. Since the pair (S,R) satisfies the Whitney condition (B) one can write $\lambda=w_1\oplus w_2$ with $w_1\in T_x\pi^{-1}(x)$ and $w_2\in T_xS$ [Tro79]. But from the above calculations it follows that $w_2\in T_x(S\cap f^{-1}(c))$.

□

For deducing from the local theorem 5.5 that the global stable set is also a stratified space, we need to understand what happens with a transversal section under the negative gradient flow:

Lemma 7.2. *Let $D\subset X$ be an abstract stratified subspace of X which is transversal to the stratum S, i.e. there exists a tubular system (π,ρ) such that $D\cap T_S^\delta=\pi_S^{-1}(S\cap D)$. Let ξ be a stratified vector field satisfying the control conditions $(C\pi'),(C\rho')$ and $d\rho\xi<0$. Let Φ be the induced flow. Then for all times $t\in\mathbb{R}$ also $\Phi(D,t)$ is an abstract stratified set transversal to S.*

Proof: One checks the lemma by using the control conditions for the gradient vector field.

□

Proposition 7.3. *Let $p \in S$ be a critical point of a stratified Morse pair (f, g). Then the following holds:*
(1) The global unstable set

$$W^u(p) = \{x \in X \mid d(\Phi(x,t), p) \overset{t\to-\infty}{\longrightarrow} 0\} = \cup_{t\geq 0}\Phi(W^u_{loc}(p), t)$$

is an embedded C^{n-1}-submanifold of S.
(2) The global stable set

$$W^s(p) = \{x \in X \mid d(\Phi(x,t), p) \overset{t\to\infty}{\longrightarrow} 0\} = \cup_{t\leq 0}\Phi(W^s_{loc}(p), t)$$

is an abstract stratified set. For all strata R with $S \leq R$ (i.e. $S \subset \overline{R}$) the restriction $W^s(p) \cap R$ is an embedded C^{n-1}-submanifold of R.

Proof:
Part (1) follows from the smooth theory since the flow does not leave a stratum in positive time. Part (2) follows from the Stable Set Theorem 5.5 and the above lemma.

Transversality and the Morse-Smale condition The trajectory space between two critical points $p \in R$ and $q \in S$ is defined as

$$\mathcal{M}_{pq}(f,g) = \{x(t), t \in \mathbb{R} \mid \dot{x}(t) = -\nabla_g f(x(t)), \lim_{t\to-\infty} x(t) = p, \lim_{t\to\infty} x(t) = q\}.$$

It is a flow invariant set and equals the intersection of the stable set of q and the unstable set of p

$$\mathcal{M}_{pq}(f,g) := W^u(p) \cap W^s(q).$$

We already saw that the unstable manifold of a point $p \in R$ is itself always completely contained in the stratum R. Thus also the trajectory space $\mathcal{M}_{pq}$ must be contained in R, and it can be nonempty only if the point q lies in a smaller stratum $S \leq R$. Thus with the assumption $(*)$ on the gradient flow for a stratified Morse pair the Morse-Smale condition makes sense also for the stratified case.

Definition 7.4. *The stratified Morse pair (f, g) is said to satisfy the Morse-Smale condition if for all critical points $p \in R$ and $q \in S$ ($S \leq R$) the intersection $W^u(p) \cap W^s(q)$ is transversal, i.e. for all points $x \in W^u(p) \cap W^s(q) \subset R$ we have*

$$T_xW^u(p) + T_xW^s(q) = T_xR.$$

It follows from the implicit function theorem that for a Morse pair which satisfies the Morse-Smale condition the trajectory space $\mathcal{M}_{pq}$ is a submanifold of R of dimension $\mu(p,q) := \mu(p) - \mu(q)$. Equally by using Prop. 7.1 the unparametrized trajectory space

$$\widetilde{\mathcal{M}_{pq}} := \mathcal{M}_{pq} \cap f^{-1}(a),$$

where $a \in (f(q), f(p))$ is a regular value, is a submanifold of dimension $\mu(p,q) - 1$.

In the smooth theory the Morse-Smale condition is a generic condition (cf.[Sch93]). We will show that also in the stratified case the Morse-Smale condition is in an appropriate sense generic. We show this result in two steps. First we approximate (strata-wise in the strong C^1-topology) the gradient vector field $\xi := -\nabla_g f$ by a stratified Morse-Smale vector field $\widetilde{\xi}$. Then we show that this perturbed vector field is also the gradient vector field of a stratified Morse pair.

Proposition 7.5. *Let (f,g) be a stratified Morse pair on the compact stratified space X. Then the stratified vector field $\xi = -\nabla_g f$ can be approximated in the C^1-topology by a stratified vector field $\tilde{\xi}$ satisfying the following conditions:*

(1) The singularities of $\widetilde{\xi}$ coincide with the singularities of ξ and $\xi = \widetilde{\xi}$ in a neighborhood of each singularity.

(2) If ξ satisfies the control conditions $(C\pi'), (C\rho')$ and $d\rho\xi < 0$ with respect to a tubular system (π, ρ) then also $\widetilde{\xi}$ satisfies the control conditions with respect to the same tubular system.

(3) For all fixed points of $\widetilde{\xi}$ the unstable set is a C^1-manifold lying in the stratum which contains the point. The stable set is an abstract stratified space and the restriction to each stratum is C^1. For each pair (p_i, p_j) of fixed points the intersection $W^u(p_j, \tilde{\xi}) \cap W^s(p_i, \tilde{\xi})$ lies in the stratum which contains p_j and is transversal.

Proof: We can approximate f by a Morse function which has distinct values at distinct critical points. Thus we can assume that f has critical points ordered by $p_i < p_j \Leftrightarrow f(p_i) < f(p_j)$. We will first indicate the perturbation needed to achieve the transversality in (3) for a pair of singularities (p_i, p_j) lying respectively in the strata S and R. As by the assumption (*) flow lines can at most leave a larger stratum in positive infinite time, the intersection $W^u(p_j) \cap W^s(p_i)$ can only be non-void if $S \leq R$. We want to indicate how to obtain a perturbed vector field ξ' out of ξ such that the conditions (1) and (2) of the statement are satisfied, all stable sets are C^1-manifolds, all unstable ones are abstract stratified spaces and the intersection $W^u(p_j) \cap W^s(p_i)$ lies in R and is transversal.

Choose $\epsilon > 0$ small enough for $[f(p_j) - 3\epsilon, f(p_j)]$ not containing any other critical value. Let $\overline{p_j} := f(p_j)$. We first do the perturbation of the vector field in the stratum R, i.e. we perturb ξ_R into ξ'_R. The perturbation is done in $R \cap f^{-1}[\overline{p_j} - 3\epsilon, \overline{p_j} - \epsilon]$ by applying the smooth theory on the vector field ξ_R (see [Sma61]). There, the transversality of the intersection is obtained exploring Sard's theorem.

We need to indicate the perturbation in $R' \cap f^{-1}[\overline{p_j} - 3\epsilon, \overline{p_j} - \epsilon]$, where R' is running over all strata larger than R. We know that ξ satisfies the control condition $(C\pi')$. We can write $\xi_{R'} = \xi^{||}_{R'} + \xi^{\perp}_{R'}$, where $\xi^{||}_{R'}$ designs the parallel extension of ξ_R like in resp. Definition 2.6. (The decomposition is orthogonal with respect to the compatible metric g.) In a small tubular neighborhood V of R we can define the perturbed vector field to be $\xi'_{R'} = \xi^{'||}_{R'} + \xi^{\perp}_{R'}$, where $\xi^{'||}_{R'}$ designs a parallel lift of ξ'_R with respect to (π, ρ). Outside of V we define $\xi' = \xi$ and we use a (controlled) partition of unity to obtain the perturbation ξ' on X. By construction, ξ' satisfies (1) and the control conditions in (2). The local stable resp. unstable sets for ξ' are the same as the ones for ξ, so they are C^1-manifolds or abstract stratified spaces resp.. With the control conditions we can deduce the statement for the global stable/unstable sets like in Prop.7.3.

The statement of the proposition now follows from an inductive application of the above construction. □

Proposition 7.6. *Let $\widetilde{\xi}$ be the stratified vector field obtained in the previous proposition. There exists a Morse pair $(\tilde{f}, \tilde{g})$ which satisfies the Morse-Smale condition such that the vector field $\widetilde{\xi}$ is a gradient field:*

$$\widetilde{\xi} = -\nabla_{\tilde{g}} \tilde{f}.$$

Moreover $\tilde{f}$ can be chosen being self-indexing, i.e. for each critical point p of index k we have $\tilde{f}(p) = k$.

Proof: The proof goes along the lines of the classical proof in [Sma61]. First one constructs a function $\tilde{f}$ such that $\tilde{f}(p) = k$ for all critical points of index k and the vector field $\tilde{\xi}$ is transversal to the level sets of the function. Then one constructs a metric $\tilde{g}$ on the stratified space X compatible with (π, ρ)

and such that $\tilde{\xi} = -\nabla_{\tilde{g}} \tilde{f}$. Because of the existence of a controlled partition of unity, it is enough to construct the metric on open neighborhoods. In a neighborhood of a critical point $\widetilde{\xi} = \xi = -\nabla_g f$, and so we can put $(f + c, g) = (\widetilde{f}, \widetilde{g})$, where the constant c is such that $f(p) + c = \tilde{f}(p) = k$. Let U be an open set which does not contain singularities. The level set $f^{-1}(a) \cap U$ is an abstract stratification (Lemma 7.1).

Let us assume first that U is a stratified space with two strata $\{X_2, X_1\}$, thus $f^{-1}(a)$ has also two strata. We have a framing $E_1(y), ...E_{n-1}(y)$ on $U \cap f^{-1}(a)$ where $\lim_{y \to x \in X_1} E_i(y) = E_i(x)$ for $i = 1, ..., m-1$. The vectors $E_1, ..., E_{n-2}$ are tangential to $\{f = a\} \cap \{\rho = \rho(y)\}$ and E_{n-1} is the vector in the radial direction.

As by the former construction the vector field $\widetilde{\xi}$ is transversal to the level sets of $\widetilde{f}$, the vectors $\{\widetilde{\xi}(y), E_1(y), ..., E_{n-1}(y)\}, y \in X_2$ form a basis for $T_y X_2$. In this basis we can now define the metric as

$$\widetilde{g}_{ij}(y) = \begin{pmatrix} -d\tilde{f}(\tilde{\xi}) & d\tilde{f}(E_i) & \rho^2(y)d\tilde{f}(E_i) & d\tilde{f}(E_{n-1}) \\ d\tilde{f}(E_i) & (\delta_{ij})_{i,j<m} & 0 & 0 \\ \rho^2(y)d\tilde{f}(E_i) & 0 & \rho^2(y)(\delta_{ij})_{m \le i,j<n-1} & 0 \\ d\tilde{f}(E_{n-1}) & 0 & 0 & 1 \end{pmatrix}.$$

We complete the proof by induction on the depth of the stratification of U.

(C_*, ∂_*) **is a complex** We denote by C_k, the k-th chain group, the $\mathbb{Z}_2$-vector field

$$C_k = \oplus_{x \in \mathrm{Crit}(f,g), \mu(x)=k} \mathbb{Z}_2 x,$$

where $\mathrm{Crit}(f, g)$ is the set of critical points of the Morse pair. Since X is compact and critical points are isolated, the sum appearing in the definition of the chain group is finite. The boundary operator is defined as follows:

$$\begin{array}{rccl} \partial_{k+1}: & C_{k+1} & \to & C_k, \\ & p & \to & \Sigma n(p,q)q, \end{array}$$

where $n(p, q)$ is the number modulo 2 of trajectories between the critical points p and q, i.e. the cardinality of $\widetilde{\mathcal{M}_{pq}}$.

Our aim here is to show that the boundary operator is well-defined, i.e that $n(p, q) < \infty$ and that $\partial \circ \partial = 0$. These two results will be shown by analyzing the asymptotic structure of the space of gradient lines. We show two compacticity results. The space of gradient lines is compact up to broken trajectories. The converse result is the gluing operation. Note that in the functional analytic proof of these compacticity results in the smooth case [Sch93], at this point the interplay of $H^{1,2}$- and C^∞_{loc}-convergence comes into play. Here however we just study the topological boundary of the set of trajectories for the negative gradient flow of a stratified Morse pair. In our proofs we exploit only the normal form (stratified Grobman-Hartman, Prop.5.6) in the neighborhood of a critical point. This approach was worked out in [Web93] for the case of a manifold.

For a point $y_i \in X$ we will denote by $y_i(\overline{\mathbb{R}})$ the set of points lying on the trajectory of the negative gradient flow (including the endpoints).

Definition 7.7. *Let $\{x_n\}$ be a sequence of trajectories in the trajectory space $\widetilde{\mathcal{M}_{pq}}$. We say that the sequence converges to a broken trajectory of order k if there exist critical points $p_0 := p, p_1, ..., p_k := q$ and trajectories $y_i \in \widetilde{\mathcal{M}_{p_i,p_{i+1}}}, i = 0, ..., k-1$ such that for all $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ and for all $n > N_0$ the trajectories x_n lie in an ϵ neighborhood of the broken trajectory, i.e.*

$$x_n(t) \in U_\epsilon(\cup_i y_i(\overline{\mathbb{R}})) \text{ for all } t \in \mathbb{R}.$$

Proposition 7.8. *The trajectory space $\widetilde{\mathcal{M}_{pq}}$ is compact up to k-order broken trajectories, where $k \leq \mu(p) - \mu(q) - 1$.*

For the proof of Prop.7.8 the following three lemmas are needed. The next lemma explains how to continue a broken trajectory.

Lemma 7.9. *Let $r \in S$ be a critical point of a stratified Morse pair (f, g). Let $x_i^s \in R, (S < R)$ be a sequence of points in a neighborhood $U(r)$ with $\lim x_i^s = x^s \in W^s(r)$. Then there exists a point $x^u \in W^u(r)$ with $x^u \in \overline{\cup_i x_i^s(\overline{\mathbb{R}})}$.*

Proof: Note that the statement of the lemma holds also in the case $S = R$. If the critical point as well as the sequence $\{x_i^s\}_{i\in I}$ lie in the stratum S, we can apply the smooth theory due to the assumption (*) on the gradient flow. Let $r \in S$, $x_i^s \in R$ where $S < R$. Since the statement is a pure topological one, it is enough to show the statement for a topologically conjugate situation. With Lemma 5.3 and Prop. 5.6 there exists a commutative diagram

$$\begin{array}{ccc} U(r) & \xrightarrow{h} & cone(L)\times\mathbb{R}^m \subset (\mathbb{R}^N\times\mathbb{R}^{m-k})\times\mathbb{R}^k \\ G\Big\downarrow & & \Big\downarrow(A^s,A^u) \\ U(r) & \xrightarrow{h} & cone(L)\times\mathbb{R}^m \subset (\mathbb{R}^N\times\mathbb{R}^{m-k})\times\mathbb{R}^k. \end{array}$$

where h is a stratified homeomorphism, G is the time-1-map for the negative gradient flow, A^s is a linear contraction and A^u is a linear expansion, and $cone(L)\times\mathbb{R}^m$ is invariant under the linear map A. The result follows by applying smooth theory (see e.g. [AB95], Theorem A.2) on the sequence $h(x_i^s)$. By the A-invariance and compactness of the cone, the point $h(x^u)$ obtained with the smooth theory is contained in $cone(L)\times\mathbb{R}^m$.

□

The following lemma will imply that the continued trajectory stays uniformly close to the broken trajectory through x^u:

Lemma 7.10. *Let ξ be a stratified vector field which satisfies the control condition $(C\pi')$ and $(C\rho')$ w.r.t. a tubular neighborhood. Moreover let ξ_S be Lipschitz continuous with Lipschitz constant K. For $x\in S$ and $y\in R$ with $S<R$ the following estimate holds:*

$$d(\Phi(x,t),\Phi(y,t))\le e^{Kt}d(y,x),$$

where d is the metric constructed in prop. 2.8.

Proof: The lemma is easily checked by using the control conditions for the vector field. □

The last lemma helps us to know where the broken trajectory stops:

Lemma 7.11. *Let $x\in\overline{\mathcal{M}_{pq}}$ and $f(x)=f(q)$ (resp.$f(x)=f(p)$) , then $x=q$ (resp. $x=p$).*

Proof: Since $x\in\overline{\mathcal{M}_{pq}}$, we have especially that $x\in\overline{W^u(q)}$. The result follows from the normal form of Section 5 because the only intersection point of the level set $f^{-1}(f(q))$ with $W^u(q)$ is the point q.

□

Proof of Prop. 7.8:

Let x_n be a sequence in $\mathcal{M}_{pq}$ having no convergent subsequence (in $\mathcal{M}_{pq}$). With the compactness of X we can find a subsequence which converges in X. The limit point x_0 lies in the adherence $\overline{\mathcal{M}_{pq}}$. Then all points of the

trajectory through x_0 are also in $\overline{\mathcal{M}_{pq}}$. The trajectory through x_0 connects two critical points p' and q' and moreover $f(p'), f(q') \in [f(p), f(q)]$. If $f(p') = f(p)$ resp. $f(q') = f(q)$ we are done because Lemma 7.11 yields $p' = p$ resp. $q' = q$. Otherwise we have to continue the broken trajectory.
We can analyze the situation in a neighborhood of the critical point $p' \in S$, and we can assume that $x_n \in U_\epsilon(p')$. Choose $T \in \mathbb{R}$, such that $x_0(T) \in U_\epsilon(q')$. Then there exists a constant K and $N \in \mathbb{N}$ such that for $N > n$ $x_n \in U_{e^{-KT}\epsilon}(p')$. With Lemma 7.10 we can deduce that $x_n(T) \in U_\epsilon(q')$. If $q \neq q'$, we can apply Lemma 7.9 for the sequence $x_n(T)$ and know how to continue the trajectory downward. The procedure stops after finitely many steps because there are only finitely many critical points.
Thus we obtain critical points $p_0, ..., p_k$ as well as a broken trajectory $\{y_i\}_{i=0,...,k-1}$ connecting them. There must be $\mu(p_0) > \mu(p_1) > ... > \mu(p_k)$ and thus the broken trajectory is of order $k < \mu(p,q)$. Note that these points can eventually lie in different strata but because of (*) there must be $S(p_0) \geq ... \geq S(p_k)$.

□

The converse result of Prop. 7.8 is the so-called gluing.

Proposition 7.12. *Let X_1, X_2, X_3 be strata with $X_1 \subset \overline{X_2} \subset \overline{X_3}$. Let $q \in X_1$, $r \in X_2$, $p \in X_3$ be critical points of the stratified Morse pair (f, g) which satisfies the Morse-Smale condition. Let the relative indices be $\mu(p,r) = 1$ and $\mu(r,q) = 1$. Then there exists a $\rho_0 > 0$ and a diffeomorphism*

$$\mathcal{G} : \widetilde{\mathcal{M}_{pr}} \times \widetilde{\mathcal{M}_{rq}} \times (\rho_0, \infty) \to \widetilde{\mathcal{M}_{pq}}.$$

Moreover if x_n is a sequence of gradient lines in $\widetilde{\mathcal{M}_{pq}}$, which converges to a broken trajectory in $\widetilde{\mathcal{M}_{pr}} \times \widetilde{\mathcal{M}_{rq}}$ in the sense of Definition 7.7, then there is a N_0 such that $x_n \in \mathrm{im}(\mathcal{G})$ for all $n > N_0$. The compactification of $\widetilde{\mathcal{M}_{pq}}$ in the sense of Definition 7.7 corresponds to the limit $\lim \rho = \infty$.

Proof: Let $(u, v) \in \widetilde{\mathcal{M}_{pr}} \times \widetilde{\mathcal{M}_{rq}}$ be a broken trajectory. We will construct the gluing map by generalizing the geometric construction (see [Web93]) in the smooth case to the stratified case. Let $a \in (f(r), f(p))$ be a regular value, then

$$D(p) := f^{-1}(a) \cap W^u_{loc}(p) \subset X_3$$

is a transversal $\mu(r)$-sphere to the trajectory u. Using Lemma 7.10 we can choose a time $T(p)$ such that $\Phi(D(p), T(p)) \subset U(r)$, where $U(r)$ is a small neighborhood of the critical point r. Because of the assumption (*), $D^u := \Phi(D(p), T(p)) \subset W^u(p)$ is completely contained in the stratum X_3 and because of the Morse-Smale condition, D^u is transversal to $W^s_{loc}(r)$ (in X_3). Let $b \in (f(q), f(r))$ be a regular value. Then

$$D(q) := W^s(q) \cap f^{-1}(b)$$

is an abstract stratified space and there exists a $T(q)$ large enough such that

$$D^s := \Phi(-T(q), D(q)) \subset U(r).$$

From Lemma 7.2 we know that D^s is also an abstract stratified space. We can choose $U(r)$ small enough such that there exist local coordinates like in Lemma 5.2 such that

$$U(r) \simeq cone(L(r)) \times \mathbb{R}^{\dim X_2} \subset \mathbb{R}^N \times \mathbb{R}^{\dim X_2}.$$

In addition we can choose the coordinates such that

$$W^u_{loc}(r) \simeq \{0\} \times \{0\} \times \mathbb{R}^{\mu(r)} \cap B_\epsilon(r)$$

and

$$W^s_{loc}(r) \simeq cone(L(r)) \times \mathbb{R}^{\dim X_2 - \mu(r)} \times \{0\} \cap B_\epsilon(r).$$

In the neighborhood $U(r)$ the flow Φ can be seen as the restriction of a Lipschitz continuous flow $\widetilde{\Phi}$ in $\mathbb{R}^N \times \mathbb{R}^{\dim X_2}$ (more precisely $\widetilde{\Phi}$ is a small Lipschitz perturbation of a linear flow). The $\mu(r)$-sphere D^u is transversal to $W^s_{loc}(r)$ and is the graph $D^u = graph\varphi^u$ of a C^1-map

$$\varphi_u : \mathbb{R}^{\mu(r)} \to cone(L(r)) \times \mathbb{R}^{\dim X_2 - \mu(r)} \subset \mathbb{R}^N \times \mathbb{R}^{\dim X_2 - \mu(r)}.$$

Equally we can write D^s as graph of a Lipschitz continuous function

$$\varphi^s : cone(L(r)) \times \mathbb{R}^{dim X_2 - \mu(r)} \to \mathbb{R}^{\mu(r)}.$$

The restriction of D^s to each stratum is C^1. D^s is the restriction of a Lipschitz continuous graph $\widetilde{D^s} = \widetilde{\varphi^s}$ with $\widetilde{\varphi^s} : \mathbb{R}^N \times \mathbb{R}^{dim X_2 - \mu(r)} \to \mathbb{R}^{\mu(r)}$. By applying smooth theory we can show that there is a unique intersection point of D^u_t and $\widetilde{D^s_t}$. By the assumption $(*)$ on the gradient flow, D^u_t is completely included in X_3 and thus the intersection point lies actually in X_3, and $D^u_t \cap D^s_t = D^u_t \cap \widetilde{D^s_t}$ and thus yields a trajectory connecting the

points p and q. The restriction $D^s_t = \widetilde{D^s} \cap X_3$ is C^1 and thus the intersection is transversal and depends C^1 on the parameter t.
The gluing map $\mathcal{G}$ is defined as follows:

$$\mathcal{G}(u, v, t) = \pi(\text{trajectory through } D^u_t \cap D^s_t),$$

where $\pi : \mathcal{M}_{pq} \to \widetilde{\mathcal{M}_{pq}}$ is the canonical projection.

□

Note that, as they lie in one stratum, the trajectory spaces are very much like in the smooth case. The only difference is that in the case of an abstract stratification parts of the broken trajectories can lie in a smaller stratum. For index difference 1 we can deduce from Proposition 7.8 that the trajectory space $\widetilde{\mathcal{M}_{pq}}$ is compact and thus the boundary operator of the complex is well-defined. From the two compacticity results one can, like in the smooth case, deduce that broken trajectories for index difference two always come in pairs. Thus we have

Theorem 7.13. *(1) For $\mu(p) - \mu(q) = 1$ the trajectory space $\widetilde{\mathcal{M}_{pq}}$ is compact.*
(2) The boundary operator $\partial : C_{k+1} \to C_k$ is well-defined.
(3) (C_*, ∂_*) is a complex, i.e. $\partial \circ \partial = 0$.

The Morse homology for a stratified Morse pair (f, g) which satisfies the Morse-Smale condition is defined as

$$H_k(f, g, \mathbb{Z}_2) = \ker\partial_k / \mathrm{im}\partial_{k+1}.$$

Morse homology and singular homology For $c \in \mathbb{R}$ we will denote by X^c resp. S^c the sublevel sets

$$X^c = \{x \in X \mid f(x) \leq c\} \text{ resp. } S^c = \{x \in S \mid f(x) \leq c\}.$$

Proposition 7.14. *Let (f, g) be a stratified Morse pair.*
(1) Suppose that the interval $[a, b]$ does not contain critical values. Then the sublevel sets X^a and X^b are homotopy equivalent by a stratified homotopy.
(2) Suppose that there is exactly one critical value $c \in [a, b]$. Moreover let us assume that there is exactly one critical point $p \in f^{-1}(c)$. Then there exists a deformation retract of X^b to $X^a \cup e^k$ where e^k is a k-cell and k is the index of p.

Proof: (1) This deformation result is the same as in the theory of Goresky/MacPherson [GM88] . The proof of the proposition uses the First Isotopy Lemma of Thom.
(2) Denote by S the stratum in which the point p lies. In a small neighborhood of p the vector field $\xi = -\nabla_g f$ is a radial lift of ξ_S with respect to the tubular neighborhood (π, ρ). Thus we have a deformation retract $\pi : T_S \to S$. The restriction f_S is a smooth Morse function on S and thus we have a deformation retract $S^b \simeq S^a \cup e^k$ (see e.g. [Mil65]). By composing the two deformation retracts we obtain the deformation retract of the statement.

□

As a corollary of Prop. 7.14 we obtain the analogue of [Fra79] (theorem 2.3) for a stratified Morse pair. It states that the space X has the homotopy type of a CW-complex, where cells of the complex (denoted by $Y(p)$) are in one-to-one correspondence with critical points. We will study only self-indexing Morse functions because in this case the intersection $W^u(p) \cap f^{-1}(c)$ for $c \in (k-1, k)$ is a k-disc. In the general case this does not hold.
Also, we can extract more information on the CW-complex by looking at the trajectory manifolds $\widetilde{\mathcal{M}_{pq}} = W^u(p) \cap W^s(q) \cap f^{-1}(c)$ where $c \in (f(q), f(p))$. Recall that the Pontryagin manifold associated to a smooth map $F : M \to S^k$ is the (unique) framed cobordism class $(F^{-1}(p), F^* v)$, where y is a regular value of F and v is an ONB of $T_y S^k$. In addition one knows that two maps are homotopic iff the associated Pontryagin manifolds are cobordant.
For codimension 0 a framed cobordism class consists just of a set of points equipped with signs. In the following we want to forget the framing and thus count the points only $\bmod 2$. Let us denote by R the stratum of X in which the point p lies. Differing from the smooth case, the stratum $W^s(q) \cap R$ need not be connected in general and thus has no canonical orientation. This is why we use the coefficient field $\mathbb{Z}_2$.

Proposition 7.15. *Let p, q be critical points with $\mu(p) = k+1$ and $\mu(q) = k$ of a self-indexing Morse pair (f, g) satisfying the Morse-Smale condition. Let $c \in [k, k+1]$ and $\widetilde{\mathcal{M}_{pq}} = W^u(p) \cap W^s(q) \cap f^{-1}(c)$. The homotopy class of maps corresponding to $\widetilde{\mathcal{M}_{pq}}$ under the Pontryagin construction corresponds* $\bmod 2$ *to the homotopy class of the attaching map of the cell $Y(p)$ on the cell $Y(q)$.*

Proof: The attaching map of the cell $Y(p)$ on the cell $Y(q)$ is up to homotopy the composition of the two deformation retracts:

$$r : (S^k \simeq) W^u(p) \cap f^{-1}(c) \to f^{-1}(\leq c - \epsilon) \cup W^u(q)$$

and

$$h : f^{-1}(\leq c - \epsilon) \cup W^u(q) \to S^k.$$

Here $(S^k, *)$ denotes the compactification of $W^u(q) \simeq D^k$ and $f^{-1}(\leq c - \epsilon)$ is mapped to $*$ under h.
Because of the Morse-Smale condition, the point q is a regular point for $h \circ r$ and $(h \circ r)^{-1}(q) = \widetilde{\mathcal{M}_{pq}}$. The number mod 2 of points in $\widetilde{\mathcal{M}_{pq}}$ yields the degree mod 2 of the attaching map.

□

We can now deduce our main result:

Theorem 7.16. *Let X be a compact abstract stratified space. Let (f, g) be a stratified Morse pair on X satisfying the Morse-Smale condition. Then there is an isomorphism:*

$$H_*(f, g, \mathbb{Z}_2) \simeq H_{sing}(X, \mathbb{Z}_2).$$

Proof: As the degree of the attaching map is the only ingredient needed to calculate the homology of a CW-complex [Bre97], we can deduce the result for stratified Morse pairs with self-indexing f from the previous proposition. The general case can be reduced to the self-indexing case by the following: Let (f, g) be a stratified Morse pair, satisfying the Morse-Smale condition. By using the proof of Prop. 7.6 one can find a stratified Morse pair $(\widetilde{f}, \widetilde{g})$ such that $-\nabla_g f = -\nabla_{\widetilde{g}} \widetilde{f}$. Since the Morse complex is defined only using critical points and gradient trajectories, the two associated Morse complexes are equal and thus $H_*(\widetilde{f}, \widetilde{g}, \mathbb{Z}_2) = H_*(f, g, \mathbb{Z}_2)$.

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