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A representation formula for the inverse harmonic mean curvature flow

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A REPRESENTATION FORMULA FOR THE INVERSE HARMONIC MEAN CURVATURE FLOW

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ABSTRACT. Let M_t be a smooth family of embedded, strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ evolving by the inverse harmonic mean curvature flow

$$\frac{d}{dt}F = \mathcal{H}^{-1}\nu.$$

Surprisingly, we can determine the explicit solution of this nonlinear parabolic equation with some Fourier analysis. More precisely, there exists a representation formula for the evolving hypersurfaces M_t that can be expressed in terms of the heat kernel on S^n and the initial support function.

1. Introduction

Let M_0 be a smooth, closed, strictly convex hypersurface in euclidean space $\mathbb{R}^{n+1}$ and suppose that M_0 is given by a smooth embedding $F_0 : S^n \to \mathbb{R}^{n+1}$ of the unit n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. We consider the initial value problem for the *inverse harmonic mean curvature flow*

$$(*) \qquad \begin{aligned} \frac{d}{dt}F(x,t) &= \mathcal{H}^{-1}(x,t)\nu(x,t), \\ F(\cdot,0) &= F_0, \end{aligned}$$

where

$$\mathcal{H} := \frac{1}{\frac{1}{\kappa_1} + \cdots + \frac{1}{\kappa_n}}$$

is the harmonic mean curvature of the hypersurface M_t parameterized by $F_t := F(\cdot,t) : S^n \to \mathbb{R}^{n+1}$, $\kappa_1, \ldots, \kappa_n$ denote the principal curvatures of M_t and $\nu(\cdot,t)$ is the outer unit normal vectorfield along M_t.

There are numerous important works on this flow. One should for example consult Andrews [3], [4], Chow-Liou-Tsai [8], Gerhardt [10]

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and Urbas [13]. It has been shown in Urbas [13] that (∗) admits a smooth solution for $t \in [0, \infty)$ and that the solutions converge to infinity as $t \to \infty$. Moreover, the hypersurfaces stay strictly convex and embedded and after a time dependent homothetic rescaling the rescaled hypersurfaces converge smoothly to a round sphere (see also Gerhardt [10] for an extension to starshaped hypersurfaces). In Chow-Liou-Tsai [8] the authors considered hypersurfaces driven by functions of the inverse harmonic mean curvature and also proved that convexity is preserved for a wide class of such flows, including (∗). Andrews [3], [4] treated both inward and outward directed flows.

For a geometric evolution equation it is in general not possible to determine the explicit solution. If T denotes the first time where a singularity occurs, one rather studies the blow-up behaviour of such flows as $t \to T$. Under certain conditions for the initial hypersurface it is often possible to classify the singularities, at least after a suitable rescaling procedure. E.g. under the assumption that the initial hypersurface is convex one was able to prove for a wide class of such flows (inward and outward directed) that a homothetically rescaled flow smoothly converges to a round sphere as $t \to T$.

If a convex hypersurface is evolving under the nonlinear parabolic equation (∗) given by the inverse harmonic mean curvature flow, it is therefore astonishing that it is possible to obtain the explicit solution. We state the main theorem

Theorem 1.1. *Let M_0 be a smooth, closed, strictly convex hypersurface in euclidean space $\mathbb{R}^{n+1}$ and suppose that M_0 is given by a smooth embedding $F_0 : S^n \to \mathbb{R}^{n+1}$ of the unit n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. The inverse harmonic mean curvature flow*

$$\frac{d}{dt} F(x,t) = \mathcal{H}^{-1}(x,t)\nu(x,t),$$
$$F(\cdot, 0) = F_0,$$

admits a smooth, strictly convex solution for $t \in [0, \infty)$. The hypersurfaces $M_t := F(S^n, t) \subset \mathbb{R}^{n+1}$ can be parameterized by their inverse Gauss maps $\mathcal{Y}_t : S^n \to M_t$ in the following way

$$\mathcal{Y}_t(x) = D\bar{S}(x,t), \quad \textit{for all } (x,t) \in S^n \times [0,\infty)$$

where $\bar{S}(\cdot, t) : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is the homogeneous extension of degree one of the support function $S(\cdot, t) : S^n \to \mathbb{R}$ of M_t defined by

$$\bar{S}(\lambda x, t) := \lambda S(x,t), \quad \textit{for all } (x,t) \in S^n \times [0,\infty), \textit{and all } \lambda > 0.$$

Here, D is the gradient in $\mathbb{R}^{n+1}$ and the support function $S(\cdot, t)$ is given by the formula

$$S(x,t) = e^{nt} \int_{S^n} H(x,y,t) S(y,0) d\sigma(y), \tag{1.1}$$

where $H(x, y, t)$ is the heat kernel and $d\sigma$ the standard volume element on S^n. $S(\cdot, 0)$ denotes the support function of the initial hypersurface M_0.

Remark 1.2. *The following theorem about the heat kernel is well-known (cf. Berger-Gauduchon-Mazet* [5]*)*

Theorem: Let M be a compact Riemannian manifold, $\{f_i\}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions with corresponding eigenvalues λ_i (i.e. $\Delta f_i = -\lambda_i f_i$), then

$$H(x,y,t) = \sum e^{-\lambda_i t} f_i(x) f_i(y).$$

Moreover, the eigenfunctions f_k on the unit n-sphere are the spherical harmonics $Y_{n,k}$ which are restrictions to S^n of the homogeneous harmonic polynomials of degree k in $\mathbb{R}^{n+1}$. They can be expressed in terms of the Legendre polynomials (see Müller [12] *for more details on spherical harmonics).*

Example 1.3. Let us briefly discuss the one-dimensional situation. If $n = 1$, then $\mathcal{H}^{-1} = \frac{1}{k}$, where k denotes the curvature of the evolving convex curves γ_t. In this case, the flow

$$\frac{d}{dt}\gamma_t = \frac{1}{k}\,\nu \tag{$*'$}$$

can also be viewed as the one-dimensional version of the inverse mean curvature flow

$$\frac{d}{dt} F = \frac{1}{H}\,\nu$$

which is important in General Relativity (see Huisken-Ilmanen [11] for details). The eigenvalues λ_k of the Laplacian on $S^1 \cong [0, 2\pi)$ are $\lambda_k = k^2$, $k \in \mathbb{N}$ with multiplicity 2. Moreover, the functions $\frac{1}{\sqrt{\pi}} \cos(kx)$, $\frac{1}{\sqrt{\pi}} \sin(kx)$ form an orthonormal basis of $L^2(S^1)$. For the heat kernel on S^1 we get

$$H(x,y,t) = \frac{1}{\pi} \sum_{k \in \mathbb{N}} e^{-k^2 t} \big(\cos(kx)\cos(ky) + \sin(kx)\sin(ky)\big).$$

According to Theorem 1.1, the support function $S(\cdot, t)$ of γ_t is given by the formula

$$S(x,t) = \sum_{k\in\mathbb{N}} e^{(1-k^2)t}\big(c_k \cos(kx) + s_k \sin(kx)\big), \tag{1.2}$$

where the constants c_k, s_k are defined by

$$c_k := \frac{1}{\pi}\int_0^{2\pi} \cos(ky)S(y,0)dy, \quad s_k := \frac{1}{\pi}\int_0^{2\pi} \sin(ky)S(y,0)dy$$

and $S(\cdot, 0)$ denotes the support function of the initial curve γ_0. If $\bar{S}$ denotes the extension of S to $\mathbb{R}^2 \setminus \{0\}$ as above, then

$$D\bar{S}(x,t) = S(x,t)\begin{pmatrix}\cos x\\ \sin x\end{pmatrix} + S'(x,t)\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix}, \quad \text{for all } x \in [0, 2\pi),$$

where we have set

$$S'(x,t) := \frac{\partial}{\partial x}S(x,t).$$

Consequently

$$\begin{aligned}\mathcal{Y}(x,t) &= \sum_{k\in\mathbb{N}} e^{(1-k^2)t}\cos(kx)\begin{pmatrix}c_k\cos x - ks_k\sin x\\ c_k\sin x + ks_k\cos x\end{pmatrix}\\ &+ \sum_{k\in\mathbb{N}} e^{(1-k^2)t}\sin(kx)\begin{pmatrix}s_k\cos x + kc_k\sin x\\ s_k\sin x - kc_k\cos x\end{pmatrix}\end{aligned}$$

is the parameterization of γ_t by the inverse Gauss map.

Example 1.4. We give an explicit example. Let $a \in [0, 1)$ be a number and assume that the initial support function is given by

$$S(y,0) = 1 + a\sin^2(y) = \frac{2+a}{2} - \frac{a}{2}\cos(2y).$$

It then easily follows that

$$s_k = 0 \quad \text{for all } k \in \mathbb{N}$$

$$c_0 = 2 + a,\ c_2 = -\frac{a}{2} \text{ and } c_k = 0 \quad \text{for all } k \in \mathbb{N} \setminus \{0, 2\}.$$

By formula (1.2) the support function of the evolving curves γ_t is

$$S(x,t) = (2+a)e^t - \frac{a}{2}e^{-3t}\cos(2x)$$

and the inverse Gauss maps are

$$\mathcal{Y}(x,t) = \Big((2+a)e^t - \frac{a}{2}e^{-3t}\cos(2x)\Big)\begin{pmatrix}\cos x\\ \sin x\end{pmatrix} + ae^{-3t}\sin(2x)\begin{pmatrix}-\sin x\\ \cos x\end{pmatrix}.$$

If we consider the rescaled curves $\tilde{\gamma}_t := e^{-t}\gamma_t$, then the support functions $\tilde{S}$ and inverse Gauss maps $\tilde{\mathcal{Y}}(x,t)$ of $\tilde{\gamma}_t$ are

$$\tilde{S}(x,t) = 2 + a - \frac{a}{2}\, e^{-4t} \cos(2x),$$

$$\tilde{\mathcal{Y}}(x,t) = \big(2 + a - \frac{a}{2} e^{-4t} \cos(2x)\big) \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} + a e^{-4t} \sin(2x) \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}.$$

In particular, if $t \to \infty$, then the support functions $\tilde{S}(x,t)$ tend to the constant $a+2$ which means that the curves converge uniformly to the circle of radius $a+2$ centered at the origin. Figure 1 shows the flow for $a = -\frac{3}{4}$ at different time steps, Figure 2 depicts the rescaled solution and Figure 3 shows the curves in a single coordinate plane.

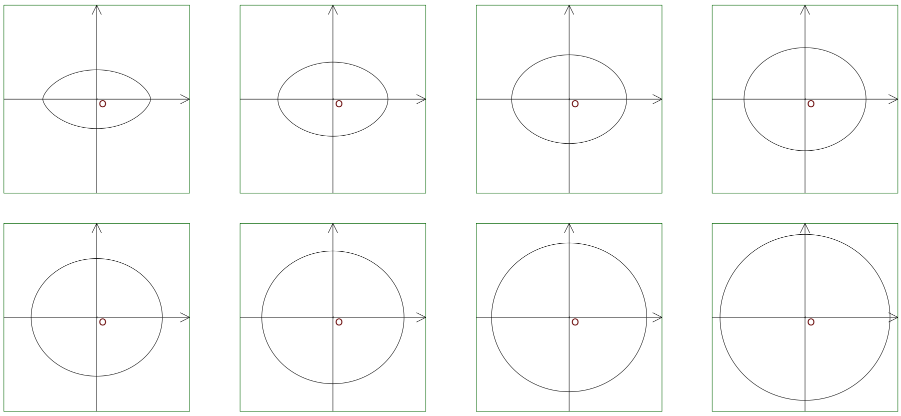

FIGURE 1. *The flow* $\frac{d}{dt}\gamma_t = \frac{1}{k}\nu$ *for the curve* γ_0 *with support function* $S(x) = 1 - \frac{3}{4}\sin^2(x)$ *at the different time steps* $t = \frac{j}{10}, j \in \{0,1,2,3,4,5,6,7\}$.

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2. Support functions

Let M be a smooth, closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$. We shall recall some facts about the support function of convex hypersurfaces (for more results see Bonnesen-Fenchel [6]). Since M is strictly convex, the Gauss map is invertible. Thus we may assume that M is parameterized by the inverse Gauss map $\mathcal{Y} : S^n \to M \subset \mathbb{R}^{n+1}$. This

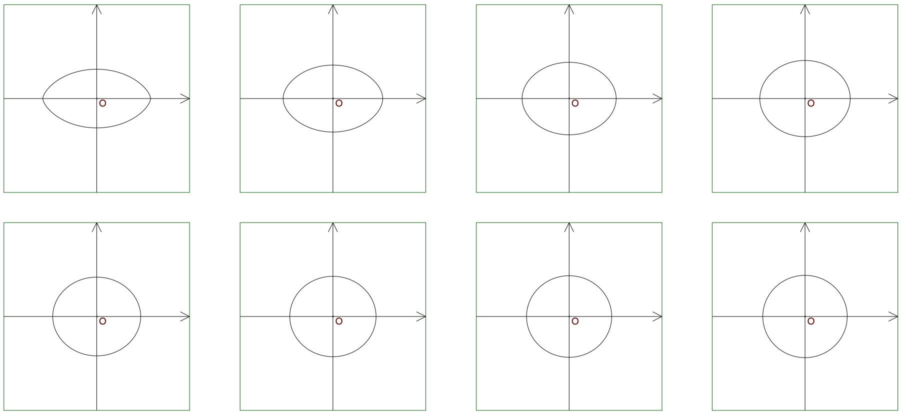

FIGURE 2. *The rescaled curves $\tilde{\gamma}_t = e^{-t}\gamma_t$ with γ_t as in Figure 1*

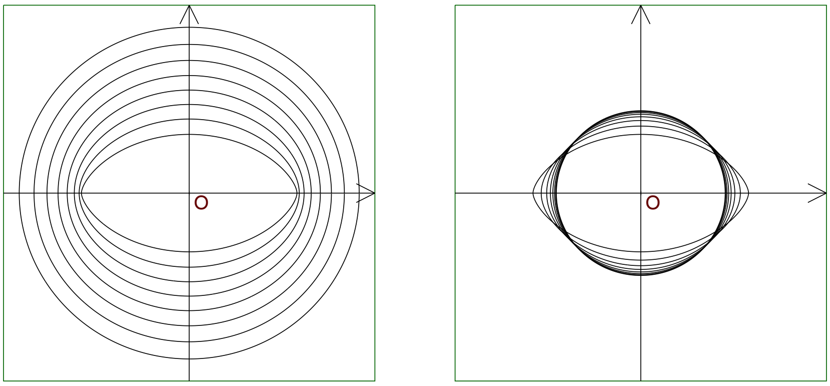

FIGURE 3. *The curves in Figure 1 resp. Figure 2 in a single coordinate plane.*

means that $\nu(x) = x$. Without loss of generality, we may assume that M encloses the origin. The support function S of M is defined by

$$S(x) := \langle x, \mathcal{Y}(x)\rangle \quad \text{for all } \; x \in S^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^{n+1}$. One can extend S to a homogeneous function $\bar{S}$ on $\mathbb{R}^{n+1} \setminus \{0\}$ of degree one by

$$\bar{S}(\lambda x) := \lambda S(x) \quad \text{for all } x \in S^n \text{ and } \lambda > 0.$$

It then follows

$$D\bar{S}(x) = \mathcal{Y}(x) \quad \text{for all } x \in S^n,$$

where $D\bar{S}$ is the gradient of $\bar{S}$ in $\mathbb{R}^{n+1}$. Let $\sigma = \sigma_{ij} dx^i \otimes dx^j$ denote the standard metric on S^n and ∇ its induced Levi-Civita connection. We want to compute the Hessian $\nabla^2 S$ of S. We have

$$\nabla_i S = \nabla_i \langle \mathcal{Y}, x\rangle = \langle \mathcal{Y}, \nabla_i x\rangle$$

because $\nu(x) = x$ and $\langle \nabla_i \mathcal{Y}, \nu\rangle = 0$. Taking another covariant derivative we obtain

$$\nabla_i \nabla_j S = \langle \nabla_i \mathcal{Y}, \nabla_j x\rangle + \langle \mathcal{Y}, \nabla_i \nabla_j x\rangle.$$

The Gauss-Weingarten equations imply

$$\nabla_i \nabla_j x = -\tau_{ij} x,$$

where τ_{ij} is the second fundamental form of S^n and because $\tau_{ij} = \sigma_{ij}$ we have

$$\nabla_i \nabla_j x = -\sigma_{ij} x.$$

On the other hand

$$\langle \nabla_i \mathcal{Y}, \nabla_i x\rangle = \langle \nabla_i \mathcal{Y}, \nabla_i \nu\rangle = h_{ij}$$

is the second fundamental form of M, so that we derive

$$\nabla_i \nabla_j S = h_{ij} - \sigma_{ij} S. \tag{2.1}$$

Moreover, the Weingarten equation gives

$$\nabla_i \nu = h_{ij} g^{jk} \nabla_k \mathcal{Y}.$$

Then

$$\sigma_{ij} = \langle \nabla_i x, \nabla_j x\rangle = \langle \nabla_i \nu, \nabla_j \nu\rangle = \langle h_{ik} g^{kl} \nabla_l \mathcal{Y}, h_{js} g^{st} \nabla_t \mathcal{Y}\rangle = h_{ik} h_{jl} g^{kl}$$

so that

$$\sigma_{ij} = h_{ik} h_{jl} g^{kl}, \tag{2.2}$$

where g^{kl} is the inverse of the induced metric g_{ij} on M. From (2.1) and (2.2) we immediately obtain

$$\Delta S = \sigma^{ij} \nabla_i \nabla_j S = \mathcal{H}^{-1} - nS. \tag{2.3}$$

Next we will compute the evolution equation of the support function S. To this end let us assume that M_t is a smooth family of closed, strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ parameterized by a smooth embedding $F_t : S^n \to M_t \subset \mathbb{R}^{n+1}$ such that

$$\frac{d}{dt} F_t(x) = f(x,t)\nu(x,t),$$

where $f(x,t)$ is a smooth speed function. It is then possible to find a uniquely determined diffeomorphism $\Psi_t : S^n \to S^n$ such that the embedding

$$\mathcal{Y}_t : S^n \to M_t, \quad \mathcal{Y}_t(x) := F_t(\Psi_t(x))$$

is the inverse Gauss map. Thus we obtain

$$\begin{aligned}\frac{d}{dt}S_t &= \frac{d}{dt}\langle \mathcal{Y}_t(x), x\rangle \\ &= \frac{d}{dt}\langle F_t(\Psi_t(x)), x\rangle \\ &= \left\langle \frac{\partial}{\partial t}F_t(\Psi_t(x)) + DF_t\left(\frac{\partial \Psi}{\partial t}\right), x\right\rangle \\ &= \left\langle \frac{\partial}{\partial t}F_t(\Psi_t(x)), x\right\rangle \\ &= \langle f(\Psi_t(x), t)\nu(\Psi_t(x), t), x\rangle \\ &= f.\end{aligned}$$

In particular, if f is given by the inverse of the harmonic mean curvature, then (2.3) implies

Lemma 2.1. *If M_t is a smooth family of closed, strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ evolving by the inverse harmonic mean curvature flow $(*)$, then the support function satisfies the linear equation*

$$\frac{d}{dt}S = \Delta S + nS,$$

where Δ is the Laplacian w.r.t. the standard metric on S^n.

Corollary 2.2. *If M_t is a smooth family of closed, strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ evolving by the inverse harmonic mean curvature flow $(*)$, then the support function $S(\cdot, t)$ of M_t is given by*

$$S(x,t) = e^{nt}\int_{S^n} H(x,y,t)S(y,0)d\sigma(y),$$

where $H(x,y,t)$ is the heat kernel on S^n and $d\sigma$ the standard volume element on S^n.

Proof. The function $\tilde{S}(x,t) := e^{-nt}S(x,t)$ satisfies the heat equation

$$\frac{d}{dt}\tilde{S} = \Delta\tilde{S} \tag{2.4}$$

and then

$$\tilde{S}(x,t) = \int_{S^n} H(x,y,t)\tilde{S}(y,0)d\sigma(y).$$

But since $\tilde{S}(y,0) = S(y,0)$ we obtain the result. □

Corollary 2.3. *Let M_0 be a smooth, closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ and let M_t be the corresponding smooth family of hypersurfaces evolving by their inverse harmonic mean curvature. Then* *the rescaled hypersurfaces* $\tilde{M}_t := e^{-nt} M_t$ *converge smoothly to a round sphere centered at the origin as* $t \to \infty$.

Proof. If $S(\cdot, t)$ and $\tilde{S}(\cdot, t)$ are the support functions of M_t resp. $\tilde{M}_t$, then

$$\tilde{S}(x, t) = e^{-nt} S(x, t).$$

In addition, by equation (2.4) $\tilde{S}$ solves the heat equation on S^n and therefore smoothly converges to a constant as $t \to \infty$. It is clear that a smooth convergence of the support function implies a smooth convergence of the corresponding hypersurfaces as well. On the other hand, the support function is constant if and only if the hypersurface is a round sphere centered at the origin. □

Proof of the main theorem. It is well-known that a solution of (∗) exists for $t \in [0, \infty)$ and that the hypersurfaces M_t stay convex and embedded during the flow (cf. Urbas [13]). It is also well-known the the rescaled hypersurfaces $\tilde{M}_t := e^{-nt} M_t$ converge smoothly to a round sphere centered at the origin. It remains to prove the precise formula for the support function and the inverse of the Gauss maps. This has been shown in Corollary 2.2 and the equation for the inverse of the Gauss maps $\mathcal{Y}$ follows from $D\bar{S}_{|S^n} = \mathcal{Y}$. *q.e.d.*

References

1. K. Anada: Contraction of surfaces by harmonic mean curvature flows and nonuniqueness of their self similar solutions. Calc. Var. Partial Differential Equations **12** (2001), 109–116.
2. K. Anada; M. Tsutsumi: Stability of solutions of nonlinear parabolic equations for harmonic mean curvature flows. Nonlinear Anal. **51** (2002), Ser. A, 305–319.
3. B. Andrews: Evolving convex hypersurfaces. Martin, Gaven (ed.) et al., Proceedings of the miniconference on analysis and applications, held at the University of Queensland, Brisbane, Australia, September 20-23, 1993. Canberra: Australian National University, Centre for Mathematics and its Applications. Proc. Cent. Math. Appl. Aust. Natl. Univ. **33** (1994), 1–24.
4. B. Andrews: Contraction of convex hypersurfaces in Riemannian spaces. J. Diff. Geom. **39** (1994), 407–431.
5. M. Berger; P. Gauduchon; E. Mazet: Le spectre d'une variéte riemannienne. Lecture Notes in Mathematics **194**, Springer-Verlag, (1971).
6. T. Bonnesen; W. Fenchel: Theorie der konvexen Körper (German). Springer-Verlag, (1934).
7. I. Chavel: Eigenvalues in Riemannian geometry. Including a chapter by Burton Randol. With an appendix by Jozef Dodziuk. Pure and Applied Mathematics, **115**, Academic Press, Inc., Orlando, FL, (1984).

8. B. Chow; L.-P. Liou; D.-H. Tsai: On the nonlinear parabolic equation $\partial_t u = F(\Delta u + nu)$ on S^n. Communications in Analysis and Geometry **4** (1996), 415–434.
9. B. Chow; D.-H. Tsai: Expansion of convex hypersurfaces by nonhomogeneous functions of curvature. Asian J. Math. **1** (1997), 769–784.
10. C. Gerhardt: Flow of nonconvex hypersurfaces into spheres. J. Diff. Geom. **32** (1990), 299–314.
11. G. Huisken; T. Ilmanen: The inverse mean curvature flow and the Riemannian Penrose inequality. J. Diff. Geom. **59** (2001), 353–437.
12. C. Müller: Spherical harmonics. Lecture Notes in Mathematics **17**, Springer-Verlag, (1966).
13. J.I.E. Urbas: An expansion of convex hypersurfaces. J. Diff. Geom. **33** (1991), 91–125; Correction to, ibid. **35** (1992), 763–765.

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