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by

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Abstract

We consider the scalar wave equation in the Kerr geometry for Cauchy data which is smooth and compactly supported outside the event horizon. We derive an integral representation which expresses the solution as a superposition of solutions of the radial and angular ODEs which arise in the separation of variables. In particular, we prove completeness of the solutions of the separated ODEs.

This integral representation is a suitable starting point for a detailed analysis of the long-time dynamics of scalar waves in the Kerr geometry.

Contents

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1 Introduction

In a recent paper [6], the long-term behavior of Dirac spinor fields in the Kerr-Newman geometry, which describes a charged rotating black hole in equilibrium, was investigated. It was shown that solutions of the Dirac equation for Cauchy data in L^2 outside the event horizon and bounded near the event horizon, decay in L^∞_{loc} as $t \to \infty$. In this paper, we turn our attention to the scalar wave equation in the Kerr geometry. Our main result is to derive an integral representation for the propagator, similar to the one obtained for the Dirac equation in [6]. In our next paper [7], we will use this integral representation to analyze the long-time dynamics and the decay of solutions in L^∞_{loc}.

The analysis of the wave equation is quite different from that for the Dirac equation. The main difficulty is that, in contrast to the Dirac equation, there is no conserved density for the scalar wave equation which is *positive* everywhere outside the event horizon. This is due to the fact that the charge density, which was positive for the Dirac equation, is not positive for the wave equation. The other conserved density, the energy density, is non-positive either: it is in general negative inside the *ergosphere*, a region outside the event horizon in which the Killing vector corresponding to time translations becomes space-like. For these reasons, it is not possible to introduce a positive scalar product which is conserved in time. In more technical terms, we are faced with the difficulty that it is impossible to represent the Hamiltonian (i.e. the operator generating time translations) as a selfadjoint operator on a Hilbert space.

We remark that the existence of the ergosphere is a direct consequence of the fact that the Kerr black hole has angular momentum [4]. Thus the ergosphere vanishes in the spherically symmetric limit. This simplifies the analysis considerably; in particular c.f. [8] for the Schwarzschild case.

A number of important contributions have been made to the rigorous study of the scalar wave equation in black hole geometries. In the spherically symmetric case, Kay and Wald [11] proved, using energy estimates together with a reflection argument, that all solutions of the wave equation in the Schwarzschild geometry are bounded in L^∞. More recently, Klainerman, Machedon, and Stalker [12] proved decay in L^∞_{loc} of spherically symmetric solutions. These papers are the only general stability results for the scalar wave equation in a black hole geometry, and use the spherical symmetry of the Schwarzschild metric in an essential way. Whiting [16] proved the absence of exponentially growing modes for the Teukolsky equation with general spin $s = 0, \frac{1}{2}, 1, \ldots$ (the case $s = 0$ gives the scalar wave equation). Whiting's approach is based on interesting differential and integral transforms, which for a fixed angular momentum mode and fixed energy, convert the reduced ODEs into an equation admitting a positive conserved energy. Beyer [2] studied the wave and Klein-Gordon equations in the Kerr metric, using an approach based on C^0 semigroup theory. He proved that for each angular momentum mode, the Cauchy problem is well-posed, and he also obtained a stability result for the Klein-Gordon equation, provided that the mass parameter in this equation is sufficiently large. Finally, Nicolas [15] constructs a global solution for a non-linear Klein-Gordon equation in Kerr.

Since the Hamiltonian cannot be represented as a selfadjoint operator on a Hilbert space, we are forced to employ methods which are quite different from those which we used in [6]. More precisely, the conserved energy gives rise to an *indefinite* scalar product, with respect to which the Hamiltonian is selfadjoint. By considering the system in finite volume with Dirichlet boundary conditions, we can arrange that the scalar product is positive on the complement of a finite-dimensional subspace. This allows us to use the general theory of Pontrjagin spaces [3, 13]. In particular, the Hamiltonian is essentially selfadjoint, and has a spectral decomposition involving a finite set of complex spectral points, which appear in complex conjugate pairs, together with a discrete spectrum of real eigenvalues. We write the projectors onto the invariant subspaces as contour integrals of the resolvent. In order to obtain estimates for the resolvent, it is useful to consider the Hamiltonian as a non-selfadjoint operator on a Hilbert space. This procedure also works in the original infinite volume setting, and we derive operator estimates which compare the resolvent in finite volume to that in infinite volume. Using these estimates, we can represent the spectral projector corresponding to the non-real spectrum as integrals over contours which are not closed and lie inside a region of of the form $|\mathrm{Im}\,\omega| < c(1+|\mathrm{Re}\,\omega|)^{-1}$ around the real axis. At this point, we make use of the fact that the scalar wave equation in the Kerr geometry is separable into ordinary differential equations for the radial and angular parts [4]. For the angular equation, we rely on the results of [9], where a spectral representation is obtained for the angular operator, and estimates for the eigenvalues and spectral projectors are derived. For the radial equation, we here derive rigorous estimates which are based on the semi-classical WKB approximation. Using these estimates, we can express the resolvent in terms of solutions of the ODEs. Using furthermore Whiting's result that the ODEs admit no normalizable solutions for complex ω, we can deform the contours onto the real line. This finally gives an integral representation for the propagator in terms of the solutions of the ODEs with ω real.

To be more precise, recall that in Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $r > 0$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$, the Kerr metric takes the form [4]

$$\begin{aligned} ds^2 &= g_{jk}\, dx^j x^k \\ &= \frac{\Delta}{U}\,(dt \;-\; a\,\sin^2\vartheta\, d\varphi)^2 \;-\; U\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) \;-\; \frac{\sin^2\vartheta}{U}\,(a\,dt \;-\; (r^2+a^2)\,d\varphi)^2 \end{aligned} \tag{1.1}$$

with

$$U(r,\vartheta) \;=\; r^2 + a^2\,\cos^2\vartheta\;, \qquad \Delta(r) \;=\; r^2 - 2Mr + a^2\;,$$

where M and aM denote the mass and the angular momentum of the black hole, respectively. We shall restrict attention to the case $M^2 \geq a^2$, because otherwise there is a naked singularity. In the *non-extreme case* $M^2 > a^2$, the function Δ has two distinct zeros,

$$r_0 \;=\; M \;-\; \sqrt{M^2 - a^2} \qquad \text{and} \qquad r_1 \;=\; M \;+\; \sqrt{M^2-a^2}\;,$$

corresponding to the Cauchy and the event horizon, respectively. In the *extreme case* $M^2 = a^2$, the Cauchy and event horizons coincide,

$$r_0 \;=\; r_1 \;=\; M\;.$$

We shall consider only the region $r > r_1$ outside the event horizon, and thus $\Delta > 0$.

In order to determine the ergosphere, we consider the norm of the Killing vector $\xi = \frac{\partial}{\partial t}$,

$$g_{ij}\,\xi^i\xi^j \;=\; g_{tt} \;=\; \frac{\Delta - a^2\,\sin^2\vartheta}{U} \;=\; \frac{r^2 - 2Mr + a^2\cos^2\vartheta}{U}\;. \tag{1.2}$$

This shows that ξ is space-like in the open region of space-time where

$$r^2 - 2Mr + a^2\,\cos^2\vartheta \;<\; 0\;, \tag{1.3}$$

the so-called *ergosphere*. It is a bounded region of space outside the event horizon, and intersects the event horizon at the poles $\vartheta = 0, \pi$.

The scalar wave equation in the Kerr geometry is

$$\Box\,\Phi := g^{ij}\nabla_i\nabla_j\,\Phi = \frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,g^{ij}\,\frac{\partial}{\partial x^j}\right)\Phi \;=\; 0\,, \tag{1.4}$$

where g denotes the determinant of the metric g_{ij}. In Boyer-Lindquist coordinates this becomes

$$\left[-\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} + \frac{1}{\Delta}\left((r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\varphi}\right)^2 - \frac{\partial}{\partial\cos\vartheta}\sin^2\vartheta\frac{\partial}{\partial\cos\vartheta}\right.$$
$$\left.-\frac{1}{\sin^2\vartheta}\left(a\sin^2\vartheta\frac{\partial}{\partial t} + \frac{\partial}{\partial\varphi}\right)^2\right]\Phi \;=\; 0\,. \tag{1.5}$$

In what follows, we denote the square bracket in this equation by $\Box$ (although strictly speaking, it is a scalar function times the wave operator in (1.4)). We now state our main result.

Theorem 1.1 *Consider the Cauchy problem*

$$\Box\,\Phi \;=\; 0\,, \qquad (\Phi, i\partial_t\Phi)(0,x) \;=\; \Psi_0(x) \tag{1.6}$$

for initial data $\Psi_0 \in C^\infty_0((r_1,\infty)\times S^2)^2$ *which is smooth and compactly supported outside the event horizon. Then there exists a unique global solution* $\Psi(t) = (\Phi(t), i\partial_t\Phi(t))$ *which can be represented as follows,*

$$\Psi(t,r,\vartheta,\varphi)$$
$$= \;-\frac{1}{2\pi i}\sum_{k\in\mathbf{Z}} e^{-ik\varphi}\sum_{n\in\mathbb{N}}\lim_{\varepsilon\searrow 0}\left(\int_{C_\varepsilon} - \int_{\overline{C_\varepsilon}}\right) d\omega\; e^{-i\omega t}\;(Q_{k,n}(\omega)\;S_\infty(\omega)\;\Psi^k_0)(r,\vartheta)\,. \tag{1.7}$$

Here the sums and integrals converge in L^2_{loc}.

The notation and objects used in the statement of this theorem will be introduced later on, but we now give a brief outline: The parameters k and ω are the angular momentum in z-direction and the energy, respectively; they appear in the separation of variables (2.6). The function Ψ^k_0 is the kth angular Fourier component of Ψ_0, i.e. $\Psi^k_0(r,\vartheta) = (2\pi)^{-1}\int_0^{2\pi} e^{ik\varphi}\,\Psi_0(r,\vartheta,\varphi)\,d\varphi$. We consider ω in the lower complex half plane $\{\mathrm{Im}\,\omega < 0\}$, and C_ε is a contour which joins the points $\omega = -\infty$ with $\omega = \infty$ and stays in an ε-neighborhood of the real line. A typical example is

$$C_\varepsilon \;=\; \{x - i\varepsilon\, e^{-x^2} \;:\; x\in\mathbb{R}\}.$$

$\overline{C_\varepsilon}$ is the complex conjugated contour. Thus the integrals in (1.7) can be thought of as a "contour integral around the real axis" (see Figure 1), in analogy to the well-known Cauchy integral formula for matrices

$$e^{-iAt} \;=\; -\frac{1}{2\pi i}\oint_C e^{-i\omega t}\,(A-\omega)^{-1}\,d\omega\;,$$

where A is a finite-dimensional matrix and C a contour which encloses the whole spectrum of A. For given ω and k, the wave operator is a sum of a radial operator $\mathcal{R}_{\omega,k}$ and an

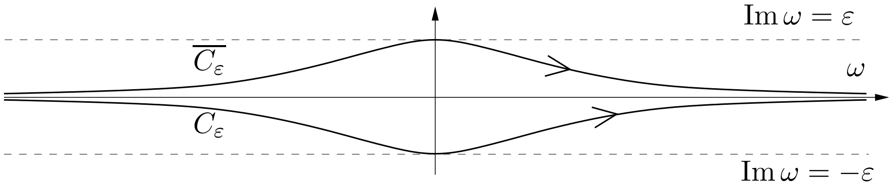

Figure 1: Integration contours

angular operator $\mathcal{A}_{\omega,k}$. As shown in [9], the angular operator has for ω near the real line a purely discrete spectrum consisting of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ (see Lemma 2.1). The spectral projectors onto the corresponding eigenspaces, which are one-dimensional, are denoted by $Q_{k,n}(\omega)$. Furthermore, we write the wave equation in the Hamiltonian form (2.19, 2.20) and let $S_\infty(\omega) = (H-\omega)^{-1}$ be the resolvent. The operator product $Q_{k,n}\,S_\infty$ can be expressed in terms of solutions of the reduced ordinary differential equations (see Proposition 5.4 with $g = s$ and s as in Lemma 5.1). Relying on Whiting's mode stability [16], we shall see below that the integrand in (1.7) is well-defined and holomorphic in the lower half plane, and thus the value of the integrals is indeed independent of the choice of C_ε. If the integrand were continuous up to the real axis, we could in (1.7) take the limit $\varepsilon \searrow 0$ to obtain an integral along the real line. However, we do not know whether the integrand in (1.7) is continuous on the real axis. Thus the integral in (1.7) can be regarded as an integral over the real axis, with an "$i\varepsilon$-regularization procedure" of possible singularities (if for instance the integrand had a simple pole at $\omega_0 \in \mathbb{R}$, this would give rise to a δ-contribution at ω_0).

We point out that the global existence and uniqueness of solutions of the Cauchy problem can be obtained more generally in globally hyperbolic space-times (see e.g. [14]). The main point of Theorem 1.1 is that we give an explicit decomposition of the propagator as a superposition of solutions of the ODEs which arise in the separation of variables. In particular, Theorem 1.1 shows completeness in the sense that the solutions of the coupled ODEs for real ω form a basis of the solution space. The explicit form of (1.7) is useful for the study of the dynamics, because the time-dependence of Ψ is related by a simple Fourier transform to the ω-dependence of the integrand in (1.7), which can in turn be analyzed by getting suitable ODE estimates [7].

We finally remark that the case of the wave equation for a scalar field minimally coupled to an electromagnetic field,

$$g^{jk}(\nabla_j - ieA_j)(\nabla_k - ieA_k)\,\Phi = 0, \tag{1.8}$$

could be treated by similar methods in the non-extreme Kerr-Newman geometry, where now the metric is given by (1.1) with

$$U(r,\vartheta) \;=\; r^2 + a^2\,\cos^2\vartheta\;, \qquad \Delta(r) \;=\; r^2 - 2Mr + a^2 + q^2\;, \tag{1.9}$$

and the electromagnetic potential is

$$A_j\,dx^j \;=\; -\frac{q\,r}{U}\,(dt \;-\; a\,\sin^2\vartheta\,d\varphi)\;, \tag{1.10}$$

where q denotes the charge of the black hole, and the parameters M, a, q satisfy the inequality $M^2 > a^2 + q^2$.

2 Preliminaries

In this section we briefly recall the variational formulation of the wave equation and the separation of variables. Furthermore, we bring the equation into a first-order Hamiltonian form. Finally, we introduce and discuss scalar products which are needed for the construction of the propagator.

The wave equation (1.5) is the Euler-Lagrange equation corresponding to the action

$$S = \int_{-\infty}^{\infty} dt \int_{r_1}^{\infty} dr \int_{-1}^{1} d(\cos\vartheta) \int_0^{\pi} d\varphi\, \mathcal{L}(\Phi, \nabla\Phi)\;, \tag{2.1}$$

where the Lagrangian $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L} \;&=\; -\Delta|\partial_r\Phi|^2 + \frac{1}{\Delta}\left|((r^2+a^2)\partial_t + a\partial_\varphi)\Phi\right|^2 \\ &\quad\; -\sin^2\vartheta\,|\partial_{\cos\vartheta}\varphi|^2 - \frac{1}{\sin^2\vartheta}\left|(a\sin^2\vartheta\partial_t + \partial_\varphi)\Phi\right|^2 . \end{aligned} \tag{2.2}$$

According to Noether's theorem, symmetries of the Lagrangian give rise to conserved quantities. The symmetry under local gauge transformations yields that the vector field

$$J_k \;=\; -\mathrm{Im}\,(\overline{\Phi}\,\nabla_k\Phi)\;, \tag{2.3}$$

called the (electromagnetic) current, is divergence free, and integrating the normal component of this current over the hypersurface $t = \text{const}$ yields the conserved charge Q. More precisely,

$$Q[\Phi] \;=\; \int_{r_1}^{\infty} dr \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi}\, \mathcal{Q}\,,$$

where $\mathcal{Q}$ is the charge density

$$\begin{aligned} \mathcal{Q} \;&=\; i\frac{\partial\mathcal{L}}{\partial\Phi_t}\,\Phi \\ &=\; \mathrm{Re}\left\{ \frac{(r^2+a^2)^2}{\Delta}\,\overline{\Phi}\left(i\partial_t\Phi + \frac{a\, i\partial_\varphi\Phi}{r^2+a^2}\right) - a^2\sin^2\vartheta\,\overline{\Phi}\left(i\partial_t\Phi + \frac{i\partial_\varphi\Phi}{a\sin^2\vartheta}\right)\right\}. \end{aligned}$$

Moreover, since the Kerr metric is stationary, the Lagrangian is invariant under time translations. The corresponding conserved quantity is the energy E,

$$E[\Phi] \;=\; \int_{r_1}^{\infty} dr \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi}\, \mathcal{E}, \tag{2.4}$$

where $\mathcal{E}$ is the energy density

$$\begin{aligned} \mathcal{E} \;=\; \frac{\partial\mathcal{L}}{\partial\Phi_t}\,\Phi_t - \mathcal{L} \;&=\; \left(\frac{(r^2+a^2)^2}{\Delta} - a^2\,\sin^2\vartheta\right)|\partial_t\Phi|^2 + \Delta\,|\partial_r\Phi|^2 \\ &\quad\; + \sin^2\vartheta\,|\partial_{\cos\vartheta}\Phi|^2 + \left(\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta}\right)|\partial_\varphi\Phi|^2 . \end{aligned} \tag{2.5}$$

One sees that all the terms in the energy density are positive, except for the coefficient of $|\partial_\varphi\Phi|^2$, which is positive if and only if $r^2 - 2Mr + a^2\,\cos^2\vartheta > 0$, i.e. outside the ergosphere. As a consequence, E is in general not positive.

The wave equation can be separated into ordinary differential equations by the usual multiplicative ansatz

$$\Phi(t,r,\vartheta,\varphi) = e^{-i\omega t - ik\varphi}\, R(r)\, \Theta(\vartheta), \tag{2.6}$$

where ω is a quantum number which could be real or complex and which corresponds to the "energy", and k is an integer quantum number corresponding to the projection of angular momentum onto the axis of symmetry of the black hole. Substituting (2.6) into (1.5), we see that

$$\Box\Phi \;=\; (\mathcal{R}_{\omega,k} + \mathcal{A}_{\omega,k})\Phi, \tag{2.7}$$

where $\mathcal{R}_{\omega,k}$ and $\mathcal{A}_{\omega,k}$ are the radial and angular operators

$$\mathcal{R}_{\omega,k} \;=\; -\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} - \frac{1}{\Delta}((r^2+a^2)\omega + ak)^2 \tag{2.8}$$

$$\mathcal{A}_{\omega,k} \;=\; -\frac{\partial}{\partial\cos\vartheta}\,\sin^2\vartheta\,\frac{\partial}{\partial\cos\vartheta} + \frac{1}{\sin^2\vartheta}(a\omega\sin^2\vartheta + k)^2. \tag{2.9}$$

We can therefore separate the variables r and ϑ to obtain for fixed ω and k the system of ODEs

$$\mathcal{R}_{\omega,k}\, R_\lambda \;=\; -\lambda\, R_\lambda, \qquad \mathcal{A}_{\omega,k}\,\Theta_\lambda \;=\; \lambda\,\Theta_\lambda\,, \tag{2.10}$$

where the separation constant λ is an eigenvalue of the angular operator $\mathcal{A}_{\omega,k}$ and can thus be regarded as an angular quantum number. In the spherically symmetric case (i.e. $a=0$), λ goes over to the usual eigenvalues $\lambda = l(l+1)$ of total angular momentum. Since the k-modes are obtained simply by expanding the φ-dependence in a Fourier series, we can in what follows restrict attention to one fixed k-mode and omit the index k. We point out that for the λ-modes the situation is more difficult because λ as well as the corresponding angular eigenfunction $\Theta_\lambda(\vartheta)$ will in general depend on ω. As a consequence, it is at this point not clear how to decompose the initial data into λ-modes.

Our analysis is based on a few properties of the angular operator, which we now state. For real ω, the angular operator $\mathcal{A}_\omega$ clearly is formally selfadjoint on $L^2(S^2)$. However, this is not sufficient for our purpose, because we need to consider the case that ω is complex. In this case, $\mathcal{A}_\omega$ is a non-selfadjoint operator. Nevertheless, we have the following spectral decomposition, which is proved in [9].

Lemma 2.1 (angular spectral decomposition) *For any given $c>0$, we define the open set $U \subset \mathbb{C}$ by the condition*

$$|\mathrm{Im}\,\omega| \;<\; \frac{c}{1+|\mathrm{Re}\,\omega|}\,. \tag{2.11}$$

Then there is an integer N and a family of operators $Q_n(\omega)$ defined for $n \in \mathbb{N}\cup\{0\}$ and $\omega \in U$ with the following properties:

(i) *The Q_n are holomorphic in ω.*

(ii) *Q_0 is a projector onto an N-dimensional invariant subspace of $\mathcal{A}_\omega$. For $n>0$, the Q_n are projectors onto one-dimensional eigenspaces of $\mathcal{A}_\omega$ with corresponding eigenvalues $\lambda_n(\omega)$. These eigenvalues satisfy a bound of the form*

$$|\lambda_n(\omega)| \;\leq\; C(n)\,(1+|\omega|) \tag{2.12}$$

for suitable constants $C(n)$. Furthermore, there is a parameter $\varepsilon>0$ such that for all $n\in\mathbb{N}$ and $\omega\in U$,

$$|\lambda_n(\omega)| \;\geq\; n\,\varepsilon\,. \tag{2.13}$$

(iii) *The Q_n are complete, i.e.*

$$\sum_{n=0}^{\infty} Q_n \;=\; \mathbb{1}$$

with strong convergence of the series in $L^2(S^2)$.

(iv) *The Q_n are uniformly bounded in $L^2(S^2)$, i.e. for all $n \in \mathbb{N}_0$,*

$$\|Q_n\| \;\leq\; c_1 \tag{2.14}$$

with c_1 independent of ω and n.

If c is sufficiently small, $c < \varepsilon$, or the real part of ω is sufficiently large, $|\mathrm{Re}\,\omega| > C(c)$, one can choose $N = 1$, i.e. $\mathcal{A}_\omega$ is diagonalizable with non-degenerate eigenvalues.

The proof of this lemma is outlined as follows. If $|\mathrm{Im}\,\omega|$ is sufficiently small, the imaginary part of the potential can be treated as a slightly non-selfadjoint perturbation (see [10, Chapter 5, §4.5]), giving a spectral decomposition into one-dimensional eigenspaces. On the other hand, for any fixed $\omega \in \mathbb{C}$, an asymptotic analysis of the resolvent $(\mathcal{A}_\omega - \lambda)^{-1}$ for large λ (see e.g. [5, Chapter 12]) yields a spectral decomposition into invariant subspaces which for large $|\lambda|$ are one-dimensional eigenspaces. Thus the difficult point in the above lemma is to show that N and the constant c_1 can be chosen uniformly in $\omega \in U$. To this end, one must show that for real ω, the eigenvalue gaps of the selfadjoint operator $\mathcal{A}_\omega$ become large as $|\omega| \to \infty$. These gap estimates are worked out in [9] by analyzing the solutions of the corresponding complex Riccati equation.

We also write the reduced wave equation in the form

$$\begin{aligned}\Bigg[&-\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} - \frac{1}{\Delta}\left((r^2+a^2)\omega + ak\right)^2 \\ &\quad -\frac{\partial}{\partial\cos\vartheta}\sin^2\vartheta\frac{\partial}{\partial\cos\vartheta} + \frac{1}{\sin^2\vartheta}(a\omega\sin^2\vartheta + k)^2\Bigg]\,\Phi \;=\; 0. \end{aligned}\tag{2.15}$$

Under the separation, the above expressions for the charge and energy densities become

$$\mathcal{Q} \;=\; |\Phi|^2 \left\{ \frac{(r^2+a^2)^2}{\Delta}\left(\mathrm{Re}\,\omega + \frac{ak}{r^2+a^2}\right) - a^2\sin^2\vartheta\left(\mathrm{Re}\,\omega + \frac{k}{a\sin^2\vartheta}\right)\right\} \tag{2.16}$$

$$\begin{aligned}\mathcal{E} \;=\;& |\Phi|^2 \left\{ \frac{(r^2+a^2)^2}{\Delta}\left(|\omega|^2 - \frac{a^2k^2}{(r^2+a^2)^2}\right) - a^2\sin^2\vartheta\left(|\omega|^2 - \frac{k^2}{a^2\sin^4\vartheta}\right)\right\} \\ &+\Delta\,|\partial_r\Phi|^2 + \sin^2\vartheta\,|\partial_{\cos\vartheta}\Phi|^2\,. \end{aligned}\tag{2.17}$$

It is convenient to reformulate the wave equation in first-order Hamiltonian form. Letting

$$\Psi = \begin{pmatrix} \Phi \\ i\partial_t\Phi \end{pmatrix}, \tag{2.18}$$

the wave equation (1.5) takes the form

$$i\,\partial_t\Psi = H\,\Psi, \tag{2.19}$$

where H is the Hamiltonian

$$H = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}. \tag{2.20}$$

Here α and β are the differential operators

$$\begin{aligned} \alpha &= \left(\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2 \vartheta \right)^{-1} \left[-\partial_r \Delta\, \partial_r - \partial_{\cos\vartheta} \sin^2 \vartheta \partial_{\cos\theta} + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\vartheta} \right) \partial_\varphi^2 \right] \\ \beta &= -2a \left(\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2 \vartheta \right)^{-1} \left(\frac{r^2+a^2}{\Delta} - 1 \right) i\partial_\varphi \,. \end{aligned}$$

It is a subtle point to find a scalar product $<.,.>$ which is well-suited to the analysis of the wave equation. It is desirable to choose the scalar product such that the Hamiltonian is Hermitian (i.e. formally selfadjoint) with respect to it. Since H is the infinitesimal generator of time translations, H is Hermitian w.r. to $<.,.>$ if and only if the inner product $<\Psi, \Psi>$ is time independent for all solutions $\Psi = (\Phi, i\partial_t \Phi)$ of the wave equation. This can for example be achieved by imposing that $<\Psi, \Psi>$ should be equal to the energy E corresponding to Ψ. This leads us to introduce a scalar product by polarizing the formula for the energy, (2.4, 2.5). We thus obtain the so-called *energy scalar product*

$$\begin{aligned} <\Psi, \Psi'> = &\int_{r_1}^\infty dr \int_{-1}^1 d(\cos\vartheta) \left\{ \left(\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2 \vartheta \right) \overline{\partial_t \Phi}\, \partial_t \Phi' \right. \\ &\left. + \Delta\, \overline{\partial_r \Phi}\, \partial_r \Phi' + \sin^2\vartheta\, \overline{\partial_{\cos\vartheta} \Phi}\, \partial_{\cos\vartheta} \Phi' + \left(\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta} \right) \overline{\partial_\varphi \Phi}\, \partial_\varphi \Phi' \right\} , \qquad (2.21) \end{aligned}$$

where again $\Psi = (\Phi, i\partial_t\Phi)$ and $\Psi' = (\Phi', i\partial_t \Phi')$. If Ψ' is a solution of the reduced system of ODEs (2.10), Ψ' can be written as $\Psi' = (\Phi_{\omega,\lambda}, \omega \Phi_{\omega,\lambda})$ with $\Phi_{\omega,\lambda}(r,\vartheta) = R_\lambda(r) \Theta_\lambda(\vartheta)$. Integrating by parts and dropping the boundary terms (which is certainly admissible when we consider the system in finite volume or when Ψ has compact support), we can substitute the radial and angular equations into (2.21) to obtain

$$\begin{aligned} <\Psi, \Psi_{\omega,\lambda}> = \;&\omega \int_{r_1}^\infty dr \int_{-1}^1 d(cos\vartheta) \\ &\times \left[\left(\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\vartheta \right) \overline{(i\partial_t + \overline{\omega})\Phi}\; \Phi_{\omega,\lambda} + 2ak \left(\frac{r^2+a^2}{\Delta} - 1 \right) \overline{\Phi}\, \Phi_{\omega,\lambda} \right] . \quad (2.22) \end{aligned}$$

In the special case $\Psi = \Psi'$, this reduces to

$$\begin{aligned} <\Psi_{\omega,\lambda}, \Psi_{\omega,\lambda}> = \;&2\omega \int_{r_1}^\infty dr \int_{-1}^1 d(cos\vartheta)\, |\Phi_{\omega,\lambda}|^2 \\ &\times \left[\left(\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\vartheta \right) \mathrm{Re}\,\omega \;+\; ak \left(\frac{r^2+a^2}{\Delta} - 1 \right) \right] . \qquad (2.23) \end{aligned}$$

By construction, the Hamiltonian is Hermitian with respect to the energy scalar product. However, the energy scalar product is in general not positive definite. This is obvious in (2.21) because the factor $(\sin^{-2}\vartheta - a^2/\Delta)$ is negative inside the ergosphere. Likewise, the integrand in (2.23) can be negative because the factor ak in the second term in the brackets can have any sign.

Apart from the energy, also the charge Q gives rise to a conserved scalar product. It is a natural idea to try to obtain a positive scalar product by taking a suitable linear combination of these two scalar products. Unfortunately, comparing (2.17) and (2.16) one sees that it is impossible to form a non-trivial linear combination of $\mathcal{Q}$ and $\mathcal{E}$ which is manifestly positive everywhere. One might argue that a suitable linear combination might nevertheless be positive because the positive term $\Delta|\partial_r\Phi|^2 + \sin^2\vartheta|\partial_{\cos\vartheta}\Phi|^2$ might compensate the negative terms. However, comparing (2.23) with (2.16), one sees that there is a simple relation between the energy scalar product and the charge,

$$<\Psi_{\omega,\lambda}, \Psi_{\omega,\lambda}> \;=\; 2\omega\, Q[\Psi_{\omega,\lambda}]\;,$$

making it again impossible to form a linear combination such that the integrand of the corresponding scalar product is everywhere positive. Stephen Anco showed that it is indeed impossible to introduce a conserved density for the wave equation which gives rise to a positive definite scalar product [1]. We conclude that if we want to consider H as a selfadjoint operator, the underlying scalar product will necessarily be indefinite.

But we can clearly consider H as a non-selfadjoint operator on a Hilbert space, and this point of view will indeed be useful for the estimates of Section 4. Our method for constructing a positive scalar product is to simply replace the negative term $-a^2/\Delta$ in (2.21) by a positive term. More precisely, we introduce the scalar product $(.,.)$ by

$$\begin{aligned}(\Psi, \Psi') \;=\; &\int_{r_1}^\infty dr \int_{-1}^1 d(\cos\vartheta) \left\{ \left(\frac{(r^2+a^2)^2}{\Delta} - a^2\,\sin^2\vartheta \right) \overline{\partial_t\Phi}\,\partial_t\Phi' \right. \\ &\left. +\Delta\,\overline{\partial_r\Phi}\,\partial_r\Phi' + \sin^2\vartheta\,\overline{\partial_{\cos\vartheta}\Phi}\,\partial_{\cos\vartheta}\Phi' + \frac{1}{\sin^2\vartheta}\,\overline{\partial_\varphi\Phi}\,\partial_\varphi\Phi' + \frac{(r^2+a^2)^2}{\Delta}\,\overline{\Phi}\,\Phi' \right\}. \end{aligned} \tag{2.24}$$

We denote the corresponding Hilbert space by $\mathcal{H}$ and the norm by $\|.\|$. This norm dominates the energy scalar product in the sense that there is a constant $c_1 > 0$ depending only on the geometry such that the "Schwarz-type" inequality

$$|<\Psi, \Psi'>| \;\leq\; c_1\, \|\Psi\|\, \|\Psi'\| \tag{2.25}$$

holds for all $\Psi, \Psi' \in \mathcal{H}$.

We finally bring the Hamiltonian and the above inner products into a more convenient form. First, we introduce the "tortoise variable" u by

$$\frac{du}{dr} \;=\; \frac{r^2+a^2}{\Delta}\;, \qquad \frac{\partial}{\partial r} \;=\; \frac{r^2+a^2}{\Delta}\,\frac{\partial}{\partial u}\;. \tag{2.26}$$

The variable u ranges over $(-\infty, \infty)$ as r ranges over (r_1, ∞). Furthermore, we introduce the functions

$$\rho \;=\; r^2+a^2 \;-\; a^2\,\sin^2\vartheta\,\frac{\Delta}{r^2+a^2} \tag{2.27}$$

$$\beta \;=\; -\frac{2ak}{\rho}\left(1 - \frac{\Delta}{r^2+a^2}\right) \tag{2.28}$$

$$\delta \;=\; \frac{1}{\rho}\left(r^2+a^2 \;+\; \frac{a^2k^2}{r^2+a^2}\right) \tag{2.29}$$

as well as the operator

$$A \;=\; \frac{1}{\rho}\left[-\frac{\partial}{\partial u}(r^2+a^2)\frac{\partial}{\partial u} - \frac{\Delta}{r^2+a^2}\,\Delta_{S^2} - \frac{a^2k^2}{r^2+a^2} \right], \tag{2.30}$$

where Δ_{S^2} denotes the Laplacian on the 2-sphere (recall that the parameter k is fixed throughout). Then, after integrating by parts, our inner products can on $C^2(\mathbb{R}\times S^2)^2$ be written as

$$<\Psi_1, \Psi_2> \;=\; \int_{-\infty}^{\infty} du \int_{-1}^{1} d\cos\vartheta\, \rho\, \langle \Psi_1, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \Psi_2 \rangle_{\mathbb{C}^2} \tag{2.31}$$

$$(\Psi_1, \Psi_2) \;=\; \int_{-\infty}^{\infty} du \int_{-1}^{1} d\cos\vartheta\, \rho\, \langle \Psi_1, \begin{pmatrix} A+\delta & 0 \\ 0 & 1 \end{pmatrix} \Psi_2 \rangle_{\mathbb{C}^2}\,, \tag{2.32}$$

and the Hamiltonian takes the form

$$H \;=\; \begin{pmatrix} 0 & 1 \\ A & \beta \end{pmatrix}. \tag{2.33}$$

The functions ρ, β, and δ satisfy for a suitable constant $c > 0$ the bounds

$$\frac{1}{c} \;\leq\; \frac{\rho}{r^2 + a^2} \;\leq\; c\,, \qquad |\beta|, |\delta| \;\leq\; c\,.$$

We abbreviate the integration measure in (2.31) and (2.32) by

$$d\mu \;:=\; \rho\, du\, d\cos\vartheta\,. \tag{2.34}$$

3 Spectral Properties of the Hamiltonian in a Finite Box

We saw in the preceding section that the energy inner product (2.21), with respect to which the Hamiltonian is formally selfadjoint, is in general indefinite. This fact remains true even when as in [6] we consider the system in a "finite box," i.e. when the range of the radial variable r is restricted to a bounded interval $r \in [r_L, r_R]$ with $r_1 < r_L < r_R < \infty$. Accordingly, in order to derive a spectral representation for the propagator corresponding to the wave equation (1.5), we will need to consider the spectral theory of operators on indefinite inner product spaces. Since there is an extensive literature on this topic, we here only recall the basic facts needed for our analysis, referring the reader to [3, 13] for details.

A *Krein space* is a complex vector space $\mathcal{K}$ endowed with a non-degenerate inner product $<.\,,\,.>$ and an orthogonal direct sum decomposition

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-, \tag{3.1}$$

such that $(\mathcal{K}_+, <.,.>)$ and $(\mathcal{K}_-, - <.,.>)$ are both Hilbert spaces. A selfadjoint operator A on a Krein space $\mathcal{K}$ is said to be *definitizable* if there exists a non-constant real polynomial p of degree k such that

$$<p\,(A)\,x\;,\; x> \geq 0 \tag{3.2}$$

for all $x \in \mathcal{D}(A^k)$. Definitizable operators have a spectral decomposition, which is similar to the spectral theorem in Hilbert spaces, except that there is in general an additional finite point spectrum in the complex plane (see [3, p. 180], [13, Thm 3.2, p. 34] and Lemma 3.3 below). An important special case of a Krein space is when $\mathcal{K}$ is positive except on a finite-dimensional subspace, i.e.

$$\kappa := \dim \mathcal{K}_- \;< \infty. \tag{3.3}$$

In this case the Krein space is called a *Pontrjagin space* of index κ. Classical results of Pontrjagin (see [3, Thms 7.2 and 7.3, p. 200] and [13, p. 11-12]) yield that any selfadjoint operator A on a Pontrjagin space is definitizable, and that it has a κ-dimensional negative subspace which is A-invariant.

We now explain how the abstract theory applies to the wave equation in the Kerr geometry. In order to have a spectral theorem, the Hamiltonian must be definitizable. There is no reason why H should be definitizable on the whole space $(r_1, \infty) \times S^2$, and this leads us to consider the wave equation in "finite volume" $[r_L, r_R] \times S^2$ with Dirichlet boundary conditions. Thus setting $\Psi = (\phi, i\phi_t)$ and regarding the two components as independent functions, we consider the vector space $\mathcal{P}_{r_L, r_R} = (H^{1,2} \oplus L^2)([r_L, r_R] \times S^2)$ with boundary conditions

$$\Psi_1(r_L) \;=\; 0 \;=\; \Psi_1(r_R)\,. \tag{3.4}$$

We endow this vector space with the inner product associated to the energy; i.e. in analogy to (2.21),

$$\begin{aligned} <\Psi, \Psi'> = & \int_{r_L}^{r_R} dr \int_{-1}^{1} d(\cos\vartheta) \left\{ \left(\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\vartheta \right) \overline{\Psi_2}\, \Psi_2' \right. \\ & \left. + \Delta\, \overline{\partial_r \Psi_1}\, \partial_r \Psi_1' + \sin^2\vartheta\, \overline{\partial_{\cos\vartheta} \Psi_1}\, \partial_{\cos\vartheta} \Psi_1' + \left(\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta} \right) \overline{\partial_\varphi \Psi_1}\, \partial_\varphi \Phi_1' \right\} . \end{aligned} \tag{3.5}$$

Lemma 3.1 *For every $r_R > r_1$ there is a countable set $E \subset (r_1, r_R)$ such that for all $r_L \in (r_1, r_R) \setminus E$, the inner product space $\mathcal{P}_{r_L, r_R}$ is a Pontrjagin space.*

Proof. Since (3.5) involves no terms which mix the first component of Ψ with the second component, $\mathcal{P}_{r_L, r_R}$ clearly has an orthogonal direct sum decomposition $\mathcal{P}_{r_L, r_R} = V_1 \oplus V_2$ with $V_{1/2} = \{\Psi \in \mathcal{P}_{r_L, r_R} \;:\; \Psi_{2/1} \equiv 0\}$. Furthermore, it is obvious that the space $(V_2, <.,.>)$ has a positive scalar product and that the corresponding norm is equivalent to the L^2-norm. Hence it remains to consider V_1, i.e. the space $H^{1,2}([r_L, r_R] \times S^2)$ with Dirichlet boundary conditions and the inner product

$$\begin{aligned} <\phi, \phi'> \;=\; & \int_{r_L}^{r_R} dr \int_{-1}^{1} d(\cos\vartheta) \\ & \times \left\{ \Delta\, \overline{\partial_r \phi}\, \partial_r \phi' + \sin^2\vartheta\, \overline{\partial_{\cos\vartheta} \phi}\, \partial_{\cos\vartheta} \phi' + \left(\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta} \right) k^2\, \overline{\phi} \phi' \right\} . \end{aligned} \tag{3.6}$$

Transforming to the variable u, (2.26), and using the representation (2.31), one sees that on the subspace $C^2([u_L, u_R] \times S^2)$ the inner product (3.6) can be written as

$$<\phi\;,\;\phi'> = \;(\phi, A\phi')_{L^2([u_L, u_R] \times S^2, d\mu)} \tag{3.7}$$

with A according to (2.30). Here we set $u_L = u(r_L)$, $u_R = u(r_R)$, and $d\mu$ is the measure (2.34). A is a Schrödinger operator with smooth potential on a compact domain. Thus it is selfadjoint in the Hilbert space $\mathcal{H} = L^2([u_L, u_R] \times S^2, d\mu)$ with suitable domain of definition $\mathcal{D}(A) \subset H^{1,2}$. It has a purely discrete spectrum which is bounded from below and has no limit points. The corresponding eigenspaces are finite-dimensional, and the eigenfunctions are smooth.

Let us analyze the kernel of A. Separating and using that the Laplacian on S^2 has eigenvalues $-l(l+1)$, $l \in \mathbb{N}_0$, A has a non-trivial kernel if and only if for some $l \in \mathbb{N}_0$, the solution of the ODE

$$\left[-\frac{\partial}{\partial u} (r^2+a^2) \frac{\partial}{\partial u} + \frac{\Delta}{r^2+a^2}\, l(l+1) - \frac{a^2 k^2}{r^2+a^2} \right] \phi(u) = 0 \tag{3.8}$$

with boundary conditions $\phi(u_R) = 0$ and $\phi'(u_R) = 1$ vanishes at $u = u_L$. Since this ϕ has at most a countable number of zeros on $(-\infty, u_R]$ (note that $\phi(u) = 0$ implies $\phi'(u) \neq 0$ because otherwise ϕ would be trivial), ϕ vanishes at u_L only if $u_L \in E_l$ with E_l countable. We conclude that there is a countable set $E = \cup_l E_l$ such that the kernel of A is trivial unless $u_L \in E$.

Assume that $u_L \notin E$. Then A has no kernel, and so we can decompose $\mathcal{H}$ into the positive and negative spectral subspaces, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Clearly, $\mathcal{H}_-$ is finite-dimensional. Since its vectors are smooth functions, we can consider $\mathcal{H}_-$ as a subspace of $\mathcal{P}_{r_L,r_R}$, and according to (3.7) it is a negative subspace. Its orthogonal complement in $\mathcal{P}_{r_L,r_R}$ is contained in $\mathcal{H}_+$ and is therefore positive. We conclude that $\mathcal{P}_{r_L,r_R}$ is positive except on a finite-dimensional subspace.

It remains to show that the topology induced by $<.\,,\,.>$ is equivalent to the $H^{1,2}$-topology. Since on finite-dimensional spaces all norms are equivalent, it suffices to consider for any $\lambda_0 > 0$ the spectral subspace for $\lambda \geq \lambda_0$, denoted by $\mathcal{H}_{\lambda_0}$. We choose λ_0 such that

$$1 - \lambda_0 \;\leq\; V_0 := \min_{[r_L,r_R]} \left(-\frac{a^2 k^2}{r^2 + a^2} \right) < 0 \;.$$

Then for every $\Psi \in C^2 \cap \mathcal{H}_{\lambda_0}$,

$$\begin{aligned}
<\Psi, \Psi> &= \langle \Psi, A\Psi \rangle_{L^2(d\mu)} \overset{(*)}{\leq} c\, \|\Psi\|^2_{H^{1,2}} \\
<\Psi, \Psi> &\geq \frac{1}{2} \langle \Psi, A\Psi \rangle_{L^2(d\mu)} + \frac{\lambda_0}{2} \|\Psi\|^2_{L^2(d\mu)} \\
&\overset{(*)}{\geq} \frac{1}{2c}\, \|\Psi\|^2_{H^{1,2}} + \frac{V_0 - 1}{2}\, \|\Psi\|^2_{L^2(d\mu)} + \frac{\lambda_0}{2}\, \|\Psi\|^2_{L^2(d\mu)} \;\geq\; \frac{1}{2c}\, \|\Psi\|^2_{H^{1,2}} \;,
\end{aligned}$$

where in $(*)$ we used that the coefficients of the ODE (3.8) are bounded from above and below and that the zero order term is bounded from below by V_0. ■

We always choose r_L and r_R such that $\mathcal{P}_{r_L,r_R}$ is a Pontrjagin space and that our initial data is supported in $[r_L, r_R] \times S^2$.

We now consider the Hamiltonian (2.20) on the Pontrjagin space $\mathcal{P}_{r_L,r_R}$ with domain $\mathcal{D} = C^\infty([r_L, r_R] \times S^2)^2 \subset \mathcal{P}_{r_L,r_R}$. For clarity, we denote this operator by H_{r_L,r_R}.

Lemma 3.2 *H_{r_L,r_R} is essentially selfadjoint on $\mathcal{P}_{r_L,r_R}$.*

Proof. On the domain of H_{r_L,r_R}, the scalar product can be written in analogy to (2.31) as

$$<\Psi, \Psi'> = \;(\Psi, S\Psi')_{L^2([u_L,u_R]\times S^2, d\mu}\;,$$

where the operator S acts on the two components of Ψ as the matrix

$$S = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.9}$$

where A is again given by (2.30) and $d\mu$ is the measure (2.34). According to the definition of essential selfadjointness, H is essentially selfadjoint in $\mathcal{P}_{r_L,r_R}$ if and only if the operator SH is essentially selfadjoint on $L^2([r_L, r_R] \times S^2)^2$. Multiplying S with H gives the operator

$$S\,H \;=\; \begin{pmatrix} 0 & A \\ A & \beta \end{pmatrix}.$$

This operator is essentially selfadjoint according to standard elliptic theory. ∎

We now prove a basic lemma on the structure of the spectrum of the Hamiltonian H_{r_L,r_R}.

Lemma 3.3 *The spectrum of H_{r_L,r_R} is purely discrete. It consists of finitely many complex spectral points appearing as complex conjugate pairs, and of an infinite sequence of real eigenvalues with no accumulation points.*

Proof. Since the operator H_{r_L,r_R} is essentially selfadjoint, there exists a negative definite subspace L_- of $\mathcal{P}_{r_L,r_R}$ of dimension κ which is H_{r_L,r_R}-invariant (see [13, p. 11]). Let p_0 denote the minimal polynomial of H_{r_L,r_R} on L_-, i.e.

$$p_0(H_{r_L,r_R})\, L_- \;=\; 0$$

with $\deg p_0 \leq \kappa$ minimal. Furthermore, we let p be the real polynomial of degree $\leq 2\kappa$ defined by $p = p_0\,\overline{p_0}$. We claim that $\operatorname{im} p\,(H_{r_L,r_R})$ is a positive semi-definite subspace. Indeed, we have for all $x \in \mathcal{P}_{r_L,r_R}$,

$$<\overline{p_0}\,(H_{r_L,r_R})\,x, L_-> = <x, p_0\,(H_{r_L,r_R})L_-> = \;0, \tag{3.10}$$

so that

$$\operatorname{im} p\,(H_{r_L,r_R}) \;\subset\; \operatorname{im} \overline{p}_0\,(H_{r_L,r_R}) \;\subset\; (L_-)^{\perp} \;\subset\; (\mathcal{P}_{r_L,r_R})_+, \tag{3.11}$$

as claimed.

Next, from the ellipticity of H_{r_L,r_R}, it follows that

$$\dim \ker\,(p\,(H_{r_L,r_R}) < \infty, \tag{3.12}$$

and from [13, Prp. 2.1], we know that for each eigenvalue ξ of H_{r_L,r_R}, the corresponding Jordan chain has finite length bounded by $2\kappa + 1$. It follows that $p^{2\kappa+1}(H_{r_L,r_R})$ has a finite-dimensional kernel and no Jordan chains. This implies that

$$\operatorname{im} p^{2\kappa+1}\,(H_{r_L,r_R}) \cap \ker p^{2\kappa+1}\,(H_{r_L,r_R}) \;=\; \{0\}\,. \tag{3.13}$$

Furthermore, since the operator $p^{2k+1}(H_{r_L,r_R})$ is selfadjoint, its image and kernel are clearly orthogonal.

The image of $p^{2k+1}(H_{r_L,r_R})$ is contained in $\operatorname{im} p\,(H_{r_L,r_R})$ and is therefore positive semi-definite. We shall now show that the space $\operatorname{im} p^{2\kappa+1}\,(H_{r_L,r_R})$ is actually positive definite. To this end, we let N be its null space,

$$N := \{x \in \operatorname{im} p^{2\kappa+1}\,(H_{r_L,r_R}),\; <x\;,\; x>= 0\}. \tag{3.14}$$

For all $x \in N$ and $y \in \mathcal{D}(H_{r_L,r_R})$, we have

$$<x\;,\; p^{2\kappa+1}\,(H_{r_L,r_R})\,y>= 0, \tag{3.15}$$

which is equivalent to

$$<p^{2\kappa+1}\,(H_{r_L,r_R})\,x\;,\; y>= 0, \tag{3.16}$$

because p is real. Since the scalar product is non-degenerate, this implies that

$$p^{2\kappa+1}\,(H_{r_L,r_R})\,x = 0\,. \tag{3.17}$$

But we have just shown that $\ker p^{2\kappa+1}(H_{r_L,r_R})$ and $\mathrm{im}\, p^{2\kappa+1}(H_{r_L,r_R})$ have trivial intersection. It follows that $x = 0$ and therefore that $\mathrm{im}\, p^{2\kappa+1}(H_{r_L,r_R})$ is positive definite, as claimed.

Restricting H_{r_L,r_R} to $\mathrm{im}\, p^{2\kappa+1}(H_{r_L,r_R})$, we have a self-adjoint operator on a Hilbert space. Thus the spectral theorem in Hilbert space applies, and the ellipticity of H_{r_L,r_R} yields that the spectrum is purely discrete. On the finite-dimensional orthogonal complement $\ker(p^{2\kappa+1}(H_{u_a,u_b}))$ we bring H_{r_L,r_R} into the Jordan canonical form. ∎

4 Resolvent Estimates

In this section we consider the Hamiltonian H as a non-selfadjoint operator on the Hilbert space $\mathcal{H}$ with the scalar product $(.,.)$ according to (2.24). We work either in infinite volume with domain of definition $\mathcal{D}(H) = C_0^\infty((r_1,\infty)\times S^2)^2$ or in the finite box $r \in [r_L, r_R]$ with Dirichlet boundary conditions and $\mathcal{D}(H) = C^\infty((r_L,r_R)\times S^2)^2$. Some estimates will hold in the same way in finite and infinite volume. Whenever this is not the case, we distinguish between finite and infinite volume with the subscripts ${}_{r_L,r_R}$ and ${}_\infty$, respectively. We always consider a fixed k-mode.

The next lemma shows that the operator $H - \omega$ is invertible if either $|\mathrm{Im}\,\omega|$ is large or $|\mathrm{Im}\,\omega| \neq 0$ and $|\mathrm{Re}\,\omega|$ is large. The second case is more subtle, and we prove it using a spectral decomposition of the elliptic operator A which generates the energy scalar product. This lemma will be very useful in Section 7, because it will make it possible to move the contour integrals so close to the real axis that the angular estimates of Lemma 2.1 apply.

Lemma 4.1 *There are constants $c, K > 0$ such that for all $\Psi \in \mathcal{D}(H)$ and $\omega \in \mathbb{C}$,*

$$\|(H-\omega)\Psi\| \;\geq\; \frac{1}{c}\left(|\mathrm{Im}\,\omega| - \frac{K}{1+|\mathrm{Re}\,\omega|}\right)\|\Psi\|\,.$$

Proof. For every unit vector $\Psi \in \mathcal{D}(H)$,

$$\begin{aligned}\|(H-\omega)\Psi\| \;&\geq\; |(\Psi,(H-\omega)\Psi)| \;\geq\; |\mathrm{Im}\,(\Psi,(H-\omega)\Psi)| \\ &\geq\; |\mathrm{Im}\,\omega| \;-\; \frac{1}{2}\,|(\Psi,(H-H^*)\Psi)|\,.\end{aligned} \tag{4.1}$$

It is useful to work again in the variable u and the representation (2.32) of the scalar product $(.,.)$ on $C^2(\mathbb{R}\times S^2)^2$. We introduce on $C^2(\mathbb{R}\times S^2)^2$ the operator H_+ by

$$H_+ \;=\; \begin{pmatrix} 0 & 1 \\ A+\delta & 0 \end{pmatrix}.$$

Comparing with (2.32) one sees that H_+ is formally selfadjoint w.r. to the scalar product $(.,.)$. Furthermore, one sees from (2.33) that H_+ differs from H only by a bounded operator,

$$\|H - H_+\| \;=\; \left\|\begin{pmatrix} 0 & 0 \\ -\delta & 0 \end{pmatrix}\right\| \;\leq\; c\,.$$

Thus on $C^2(\mathbb{R} \times S^2)$,

$$\begin{aligned} \|H - H^*\| &= \|(H - H_+) - (H - H_+)^*\| \\ &\leq \|H - H_+\| + \|(H - H_+)^*\| \;=\; 2\,\|H - H_+\| \;\leq\; 2c\,, \end{aligned}$$

and substituting this bound into (4.1), we conclude that

$$\|(H - \omega)\Psi\| \;\geq\; (|\mathrm{Im}\,\omega| - c)\,\|\Psi\|\,.$$

In view of this inequality it remains to consider the case where $|\mathrm{Re}\,\omega|$ is large.

Using standard elliptic theory, the operator A with domain $\mathcal{D}(A) = C_0^\infty(\mathbb{R} \times S^2)$ is essentially self-adjoint on the Hilbert space $L^2(d\mu) := L^2(\mathbb{R} \times S^2, d\mu)$, with $d\mu$ according to (2.34). Clearly, A is bounded from below, $A \geq -c$, and thus $\sigma(A) \subset [-c, \infty)$. For given $\Lambda \gg 1$ we let P_0 and P_Λ be the spectral projector corresponding to the sets $[-c, \Lambda^2)$ and $[\Lambda^2, \infty)$, respectively. We decompose a vector $\Psi \in \mathcal{H}$ in the form $\Psi = \Psi_0 + \Psi_\Lambda$ with

$$\Psi_0 \;=\; \begin{pmatrix} P_0 & 0 \\ 0 & P_0 \end{pmatrix} \Psi\,, \qquad \Psi_\Lambda \;=\; \begin{pmatrix} P_\Lambda & 0 \\ 0 & P_\Lambda \end{pmatrix} \Psi\,.$$

This decomposition is orthogonal w.r. to the energy scalar product,

$$<\!\Psi_\Lambda, \Psi_0\!> \,=\, \langle \Psi_\Lambda, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \Psi_0 \rangle_{L^2(d\mu)} \;=\; 0\,.$$

However, our decomposition is *not* orthogonal w.r. to the scalar product $(.,.)$, because

$$(\Psi_\Lambda, \Psi_0) \;=\; \langle \Psi_\Lambda, \begin{pmatrix} A+\delta & 0 \\ 0 & 1 \end{pmatrix} \Psi_0 \rangle_{L^2(d\mu)} \;=\; \langle \Psi_\Lambda, \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} \Psi_0 \rangle_{L^2(d\mu)}\,.$$

But at least we obtain the following inequality,

$$|(\Psi_\Lambda, \Psi_0)| \;\leq\; c\,\|\Psi_0\|\,\|\Psi_\Lambda^1\|_{L^2(d\mu)}\,, \tag{4.2}$$

where Ψ_Λ^1 denotes the first component of Ψ_Λ. Using that

$$\|\Psi_\Lambda^1\|^2_{L^2(d\mu)} \;=\; \langle \Psi_\Lambda^1, A^{-1}\Psi_\Lambda^1 \rangle_{L^2(d\mu)} \;\leq\; \frac{1}{\Lambda^2}\,\|\Psi_\Lambda\|^2\,,$$

we can also write (4.2) in the more convenient form

$$|(\Psi_\Lambda, \Psi_0)| \;\leq\; \frac{c}{\Lambda}\,\|\Psi_0\|\,\|\Psi_\Lambda\|\,.$$

Choosing Λ sufficiently large, we obtain

$$\|\Psi\|^2 \;=\; \|\Psi_\Lambda\|^2 \;+\; 2\,\mathrm{Re}\,(\Psi_\Lambda, \Psi_0) \;+\; \|\Psi_0\|^2 \;\leq\; 4\,(\Psi_\Lambda\| + \|\Psi_0\|)^2$$

and thus

$$\|\Psi\| \;\leq\; 2\,(\|\Psi_\Lambda\| + \|\Psi_0\|)\,. \tag{4.3}$$

Furthermore, we can arrange by choosing Λ sufficiently large that

$$<\!\Psi_\Lambda, \Psi_\Lambda\!> \,=\, \langle \Psi_\Lambda, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \Psi_\Lambda \rangle_{L^2(d\mu)} \;\geq\; \frac{1}{2}\,\langle \Psi_\Lambda, \begin{pmatrix} A+\delta & 0 \\ 0 & 1 \end{pmatrix} \Psi_\Lambda \rangle_{L^2(d\mu)} \;=\; \frac{1}{2}\,\|\Psi_\Lambda\|^2.$$

Next we estimate the inner products $<\Psi_\Lambda, H\Psi_0>$, $(\Psi_0, H\Psi_\Lambda)$ and $(\Psi_0, H\Psi_0)$. The calculations

$$\begin{aligned}
<\Psi_\Lambda, H\Psi_0> &= \langle \Psi_\Lambda, \begin{pmatrix} 0 & A \\ A & \beta \end{pmatrix} \Psi_0 \rangle_{L^2(d\mu)} = \langle \Psi_\Lambda, \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \Psi_0 \rangle_{L^2(d\mu)} \\
(\Psi_0, H\Psi_\Lambda) &= \langle \Psi_0, \begin{pmatrix} 0 & A+\delta \\ A & \beta \end{pmatrix} \Psi_\Lambda \rangle_{L^2(d\mu)} = \langle \Psi_0, \begin{pmatrix} 0 & \delta \\ 0 & \beta \end{pmatrix} \Psi_\Lambda \rangle_{L^2(d\mu)} \\
|(\Psi_0, H\Psi_0)| &= \left| \langle \Psi_0, \begin{pmatrix} 0 & A+\delta \\ A & \beta \end{pmatrix} \Psi_0 \rangle_{L^2(d\mu)} \right| \\
&\le c\, \|\Psi_0\|_{L^2(d\mu)} + 2\, \|A\Psi_0^1\|_{L^2(d\mu)}\, \|\Psi_0^2\|_{L^2(d\mu)} \\
\|A\Psi_0^1\|^2_{L^2(d\mu)} &= \langle \Psi_0, \begin{pmatrix} A^2 & 0 \\ 0 & 0 \end{pmatrix} \Psi_0 \rangle_{L^2(d\mu)} \\
&\le \Lambda^2\, \langle \Psi_0, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \Psi_0 \rangle_{L^2(d\mu)} = \Lambda^2\, \|\Psi_0\|^2
\end{aligned}$$

give us the bounds

$$\begin{aligned}
|<\Psi_\Lambda, H\Psi_0>| &\le c\, \|\Psi_\Lambda\|\, \|\Psi_0\| \\
|(\Psi_0, H\Psi_\Lambda)| &\le c\, \|\Psi_0\|\, \|\Psi_\Lambda\| \\
|(\Psi_0, H\Psi_0)| &\le (c + 2\Lambda)\, \|\Psi_0\|^2 .
\end{aligned}$$

Using the above inequalities, we can estimate the inner product $<\Psi_\Lambda, (H-\omega)\Psi>$ by

$$\begin{aligned}
|<\Psi_\Lambda, (H-\omega)\Psi>| &\ge |<\Psi_\Lambda, (H-\omega)\Psi_\Lambda>| - |<\Psi_\Lambda, (H-\omega)\Psi_0>| \\
&\ge \frac{|\mathrm{Im}\,\omega|}{2}\, \|\Psi_\Lambda\|^2 - c\, \|\Psi_0\|\, \|\Psi_\Lambda\| .
\end{aligned}$$

Applying the Schwarz inequality $|<\Psi_\Lambda, (H-\omega)\Psi>| \le c_1\, \|\Psi_\Lambda\|\, \|(H-\omega)\Psi\|$ and dividing by $\|\Psi_\Lambda\|$, we obtain (possibly after increasing c) that

$$\|(H-\omega)\Psi\| \ \ge\ \frac{|\mathrm{Im}\,\omega|}{c}\, \|\Psi_\Lambda\| \ -\ \|\Psi_0\| . \tag{4.4}$$

Next we estimate the inner product $(\Psi_0, (H-\omega)\Psi)$,

$$\begin{aligned}
|(\Psi_0, (H-\omega)\Psi)| &\ge |(\Psi_0, (H-\omega)\Psi_0)| - |(\Psi_0, (H-\omega)\Psi_\Lambda)| \\
&\ge (|\omega| - c - 2\Lambda)\, \|\Psi_0\|^2 - c \left(1 + \frac{|\omega|}{\Lambda}\right) \|\Psi_0\|\, \|\Psi_\Lambda\| .
\end{aligned}$$

We apply the Schwarz inequality $(\Psi_0, (H-\omega)\Psi) \le \|\Psi_0\|\, \|(H-\omega)\Psi\|$ and divide by $\|\Psi_0\|$,

$$\|(H-\omega)\Psi\| \ \ge\ (|\omega| - c - 2\Lambda)\, \|\Psi_0\| - c\left(1 + \frac{|\omega|}{\Lambda}\right) \|\Psi_\Lambda\| . \tag{4.5}$$

Choosing $\Lambda = (|\omega| - c)/4$ and increasing c, the inequalities (4.4) and (4.5) give for sufficiently large $|\omega|$ the bounds

$$\begin{aligned}
\|(H-\omega)\Psi\| &\ge \frac{|\mathrm{Im}\,\omega|}{c}\, \|\Psi_\Lambda\| \ -\ \|\Psi_0\| \\
\|(H-\omega)\Psi\| &\ge \frac{|\omega|}{2}\, \|\Psi_0\| - c\, \|\Psi_\Lambda\| .
\end{aligned}$$

Multiplying the second inequality by $4/|\omega|$ and adding the first inequality, we conclude that

$$2\,\|(H-\omega)\Psi\| \;\geq\; \left(\frac{|\mathrm{Im}\,\omega|}{c} - \frac{4c}{|\omega|} \right) \|\Psi_\Lambda\| + \|\Psi_0\|\,.$$

The result now follows from (4.3). ■

The last lemma allows us to introduce the resolvent. Namely, we let

$$\Omega \;=\; \left\{ \omega \in \mathbb{C} \;:\; |\mathrm{Im}\,\omega| \;\geq\; \frac{2K}{1+|\mathrm{Re}\,\omega|} \right\} \tag{4.6}$$

with K as in Lemma 4.1.

Corollary 4.2 *If $\omega \in \Omega$, the operator $H-\omega$ is invertible. The corresponding* **resolvent**

$$S(\omega) \;:=\; (H-\omega)^{-1}$$

satisfies the bound

$$\|S(\omega)\| \;\leq\; \frac{c}{|\mathrm{Im}\,\omega|} \tag{4.7}$$

with c independent of $\omega \in \Omega$.

Using (4.6) in (4.7), we immediately get the bound

$$\|S(\omega)\| \;\leq\; c\,(1+|\mathrm{Re}\,\omega|)\,. \tag{4.8}$$

Since $S(\omega)$ is a bounded operator, its domain of definition can clearly be chosen to be the whole Hilbert space. We shall assume until the end of this section that $\omega \in \Omega$.

The next lemma gives detailed estimates for the difference of the resolvents S_{r_L,r_R} and S_∞ in finite and infinite volume, respectively. By $Q_\lambda(\omega)$ we denote a given projector onto an invariant subspace of the angular operator $\mathcal{A}_\omega$ corresponding to the spectral parameter λ of dimension at most N (see Lemma 2.1 for details).

Lemma 4.3 *For every $\Psi \in C^\infty_0((r_L,r_R)\times S^2)^2$ and every $p \in \mathbb{N}$, there is a constant $C = C(\Psi,p)$ (independent of ω) such that*

$$|\langle\Psi, [S_{r_L,r_R}(\omega) - S_\infty(\omega)]\,\Psi\rangle| \;\leq\; \frac{C}{1+|\omega|^p}\,\frac{1}{|Im\,\omega|-c}\,. \tag{4.9}$$

Furthermore, for every $\Psi \in C^\infty_0((r_L,r_R)\times S^2)^2$ and every $p \in \mathbb{N}$ and $q \geq N$, there is a constant $C = C(\Psi,p,q)$ (independent of ω and λ) such that

$$|\langle\Psi, Q_\lambda\,[S_{r_L,r_R}(\omega) - S_\infty(\omega)]\,\Psi\rangle| \;\leq\; \frac{C}{(1+|\omega|^p)(1+|\lambda|^q)}\,\frac{1}{|Im\,\omega|-c}\,\|Q_\lambda\|\,. \tag{4.10}$$

Proof. By definition of the resolvent, $(H-\omega)\,S(\omega)\Psi = \Psi$. This relation holds both in finite and in infinite volume, and thus

$$((H-\omega)\,[S_{r_L,r_R}(\omega) - S_\infty(\omega)]\,\Psi)\,(r,\vartheta) \;=\; 0 \qquad \text{if } r_L \leq r \leq r_R.$$

Iterating this identity and using the fact that H and S commute, we see that on $[r_L,r_R]\times S^2$,

$$\omega^{p+1}\,[S_{r_L,r_R}(\omega) - S_\infty(\omega)]\,\Psi \;=\; [S_{r_L,r_R}(\omega) - S_\infty(\omega)]\,H^{p+1}\,\Psi\,. \tag{4.11}$$

Combining this identity with the Schwarz-type inequality (2.25), we obtain

$$\begin{aligned}
|<\Psi, [S_{r_L,r_R}(\omega) - S_\infty(\omega)]\,\Psi>| &\leq c_1\,\|S_{r_L,r_R}(\omega) - S_\infty(\omega)\|\,\|\Psi\|^2 \\
|\omega^{p+1}|\,|<\Psi, [S_{r_L,r_R}(\omega) - S_\infty(\omega)]\,\Psi>| &\leq \left|<\Psi, [S_{r_L,r_R}(\omega) - S_\infty(\omega)]\,H^{p+1}\Psi>\right| \\
&\leq c_1\,\|S_{r_L,r_R}(\omega) - S_\infty(\omega)\|\,\|\Psi\|\,\|H^{p+1}\Psi\|\,.
\end{aligned}$$

Since Ψ is smooth and has compact support, $H^{p+1}\Psi$ also has these properties. The estimate (4.8) gives (4.9).

In order to prove (4.10), we first combine (4.11) with (2.25) to obtain

$$\begin{aligned}
&(1+|\omega|^{p+1})\,|<\Psi, Q_\lambda(S_{r_L,r_R} - S_\infty)\Psi>| \\
&\quad\leq\; c_1\,\|Q_\lambda\|\,\|S_{r_L,r_R} - S_\infty\|\,\|\Psi\|\,\big(\|\Psi\| + \|H^{p+1}\Psi\|\big)\,. \qquad (4.12)
\end{aligned}$$

Since q is at least as large as the dimension of the invariant subspace corresponding to λ, $(\mathcal{A}_\omega - \lambda)^q Q_\lambda = 0$. Therefore, for every $\Psi' \in C^\infty_0([r_L, r_R] \times S^2)^2$,

$$0 \;=\; <\Psi, (\mathcal{A}_\omega - \lambda)^q Q_\lambda \Psi'> \;=\; <(\mathcal{A}^*_\omega - \overline{\lambda})^q \Psi, Q_\lambda \Psi'>\;\;.$$

Expanding the power $(\mathcal{A}^*_\omega - \overline{\lambda})^q$ and using (2.25), we obtain

$$|\lambda|^q\,\left|<\Psi, Q_\lambda\Psi'>\right| \;\leq\; \sum_{l=1}^q c_l\,|\lambda|^{q-l}\,\left\|(\mathcal{A}^*_\omega)^l\Psi\right\|\,\left\|\chi_{[r_L,r_R]}\,Q_\lambda\Psi'\right\|$$

with combinatorial factors c_l (here $\chi_{[r_L,r_R]}$ is the operator of multiplication by the characteristic function). Since the angular operator $\mathcal{A}^*_\omega$ is according to (2.9) a polynomial in ω of degree two, the function $(\mathcal{A}^*_\omega)^l\Psi$ is also polynomial in ω, i.e.

$$(\mathcal{A}^*_\omega)^l\Psi \;=\; \sum_{p=0}^{2l} \omega^p\,\Psi_p\,,$$

where the functions Ψ_p are composed of Ψ and its angular derivatives, as well as the coefficient functions of $\mathcal{A}^*_\omega$. This gives the estimate

$$\|(\mathcal{A}^*_\omega)^l\Psi\| \;\leq\; \sum_{p=0}^{2l} |\omega|^p\,\|\Psi_p\| \;\leq\; c\,(1+|\omega|^{2l})$$

with a constant c which depends only on Ψ and l. We thus obtain

$$|\lambda|^q\,\left|<\Psi, Q_\lambda\Psi'>\right| \;\leq\; \sum_{l=1}^q c_l(\Psi)\,|\lambda|^{q-l}\,(1+|\omega|^{2l})\,\left\|\chi_{[r_L,r_R]}\,Q_\lambda\Psi'\right\|.$$

Young's inequality allows us to compensate the lower powers of λ,

$$|\lambda|^q\,\left|<\Psi, Q_\lambda\Psi'>\right| \;\leq\; c(q,\Psi)\,(1+|\omega|^{2q})\,\left\|\chi_{[r_L,r_R]}\,Q_\lambda\Psi'\right\|.$$

We now choose Ψ' equal to the left side of (4.11) with $p=0$ and $p=r$ and take the sum of the resulting inequalities. Applying again the Schwarz inequality, we obtain

$$|\lambda|^q(1+|\omega|^r)\,|<\Psi, Q_\lambda(S_{r_L,r_R} - S_\infty)\Psi>| \;\leq\; c(1+|\omega|^{2q})\|Q_\lambda\|\|S_{r_L,r_R} - S_\infty\|\,(\|\Psi\| + \|H^r\Psi\|)\,.$$

By choosing r sufficiently large, we can compensate the factor $(1+|\omega|^{2q})$ on the right. More precisely,

$$|\lambda|^q(1+|\omega|^{p+1})\,|<\Psi, Q_\lambda(S_{r_L,r_R} - S_\infty)\Psi>| \;\leq\; c'\,\|Q_\lambda\|\,\|S_{r_L,r_R} - S_\infty\|\,\big(\|\Psi\| + \|H^{p+2q+1}\Psi\|\big)\,.$$

Adding this inequality to (4.12) and substituting the estimate (4.8) gives (4.10). ∎

5 Separation of the Resolvent

In this section we fix $\omega \notin \sigma(H)$, so that the resolvent $S = (H-\omega)^{-1}$ exists. As in the previous section, we assume that Q_λ is a given projector onto a finite-dimensional invariant subspace of the angular operator $\mathcal{A}_\omega$ corresponding to the spectral parameter λ. Our goal is to represent the operator product $Q_\lambda S$ in terms of the solutions of the radial ODE.

According to (2.10) and (2.8), the radial ODE is

$$\left[-\frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} - \frac{(r^2+a^2)^2}{\Delta} \left(\omega + \frac{ak}{r^2+a^2} \right)^2 + \lambda \right] R(r) \;=\; 0\,, \tag{5.1}$$

where λ is the separation constant. We can assume that $k \geq 0$ because otherwise we reverse the sign of ω. We again work in the "tortoise variable" u, (2.26), and set

$$\phi(r) \;=\; \sqrt{r^2+a^2}\, R(r)\,. \tag{5.2}$$

Then equation (5.1) can be written as

$$\left[\frac{1}{r^2+a^2} \frac{\partial}{\partial u} (r^2+a^2) \frac{\partial}{\partial u} + \left(\omega + \frac{ak}{r^2+a^2} \right)^2 - \frac{\lambda\, \Delta}{(r^2+a^2)^2} \right] \frac{\phi}{\sqrt{r^2+a^2}} \;=\; 0\,. \tag{5.3}$$

Using that

$$(r^2+a^2)\, \frac{\partial}{\partial u}\, (r^2+a^2)^{-\frac{1}{2}} \;=\; -\frac{1}{2}\, (r^2+a^2)^{-\frac{1}{2}}\, \frac{\partial}{\partial u} (r^2+a^2) \;=\; -\frac{\partial}{\partial u}\, (r^2+a^2)^{\frac{1}{2}}\,,$$

(5.3) simplifies to the Schrödinger-type equation

$$\left(-\frac{\partial^2}{\partial u^2} + V(u) \right) \phi(u) \;=\; 0 \tag{5.4}$$

with the potential

$$V(u) \;=\; -\left(\omega + \frac{ak}{r^2+a^2} \right)^2 \;+\; \frac{\lambda\, \Delta}{(r^2+a^2)^2} \;+\; \frac{1}{\sqrt{r^2+a^2}}\, \partial_u^2 \sqrt{r^2+a^2}\,. \tag{5.5}$$

We let ϕ_1 and ϕ_2 be two solutions of (5.4) which are compatible with the boundary conditions. More precisely, in finite volume we satisfy the Dirichlet boundary conditions $\phi_1(u_L) = 0$ and $\phi_2(u_R) = 0$ (again with $u_L = u(r_L)$ and $u_R = u(r_R)$). Likewise, in infinite volume we only consider the case $\mathrm{Im}\,\omega < 0$ and let ϕ_1 and ϕ_2 be the fundamental solutions which decay exponentially at $u = -\infty$ and $u = +\infty$, respectively (the existence of these fundamental solution will be established in Corollary 6.4). If the solutions ϕ_1 and ϕ_2 were linearly dependent, they would give rise to a vector in the kernel of $H-\omega$, in contradiction to our assumption $\omega \notin \sigma(H)$. Thus the Wronskian

$$w(\phi_1, \phi_2) \;:=\; \phi_1'(u)\, \phi_2(u) - \phi_1(u)\, \phi_2'(u) \tag{5.6}$$

is non-zero (note that w is by definition independent of u).

We begin by constructing the "Green's function" corresponding to (5.4).

Lemma 5.1 *The function*

$$s(u,u') \;:=\; \frac{1}{w(\phi_1,\phi_2)} \times \left\{ \begin{array}{ll} \phi_1(u)\,\phi_2(u') & \text{if } u \leq u' \\ \phi_2(u)\,\phi_1(u') & \text{if } u > u' \end{array} \right. \tag{5.7}$$

satisfies the distributional equation

$$\left(-\frac{\partial^2}{\partial u^2} + V(u) \right) s(u,u') \;=\; \delta(u-u') \,.$$

Proof. By definition of the distributional derivative,

$$\int_{-\infty}^{\infty} \eta(u)\,(-\partial_u^2 + V) s(u,u')\,du \;=\; \int_{-\infty}^{\infty} \big((-\partial_u^2+V)\eta(u)\big)\, s(u,u')\,du$$

for every test function $\eta \in C_0^\infty(\mathbb{R})$. It is obvious from its definition that the function $s(.,u')$ is smooth except at the point $u=u'$, where its first derivative has a discontinuity. Thus after splitting up the integral, we can integrate by parts twice to obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \big((-\partial_u^2+V)\eta(u)\big)\, s(u,u')\,du \\ &\quad= \int_{-\infty}^{u'} \eta(u)\,(-\partial_u^2+V)s(u,u')\,du \;+\; \lim_{u \nearrow u'} \eta(u)\,\partial_u s(u,u') \\ &\qquad + \int_{u'}^{\infty} \eta(u)\,(-\partial_u^2+V)s(u,u')\,du \;-\; \lim_{u \searrow u'} \eta(u)\,\partial_u s(u,u')\,. \end{aligned}$$

Since for $u \neq u'$, s is a solution of (5.4), the obtained integrals vanish. Computing the limits with (5.7), we get

$$\begin{aligned} &\int_{-\infty}^{\infty} \big((-\partial_u^2+V)\eta(u)\big)\, s(u,u')\,du \;=\; \left(\lim_{u \nearrow u'} - \lim_{u \searrow u'}\right) \eta(u)\,\partial_u s(u,u') \\ &\quad= \frac{1}{w(\phi_1,\phi_2)}\,\eta(u')\,\big(\phi_1'(u')\,\phi_2(u') - \phi_2'(u')\,\phi_1(u')\big) \;=\; \eta(u')\,, \end{aligned}$$

where in the last step we used the definition of the Wronskian (5.6). ∎

In what follows we also regard $s(u,u')$ as the integral kernel of a corresponding operator s, i.e.

$$(s\phi)(u) \;:=\; \int du'\; s(u,u')\,\phi(u')\,du'\,.$$

If Q_λ projects onto an eigenspace of $\mathcal{A}_\omega$, we see from (2.10), (2.7), and (5.2) that

$$\Box \left((r^2+a^2)^{-\frac{1}{2}}\, Q_\lambda(\vartheta,\vartheta')\;\, s(u,u') \right) \;=\; (r^2+a^2)^{-\frac{1}{2}}\, Q_\lambda(\vartheta,\vartheta')\,\delta(u-u')\,. \tag{5.8}$$

Loosely speaking, this relation means that the operator product $Q_\lambda\, s$ is an angular mode of the Green's function of the wave equation. Unfortunately, Q_λ might project onto an invariant subspace of $\mathcal{A}_\omega$ which is *not* an eigenspace. In this case, the angular operator has on the invariant subspace the "Jordan decomposition"

$$\mathcal{A}_\omega\, Q_\lambda \;=\; (\lambda + \mathcal{N})\, Q_\lambda \tag{5.9}$$

with $\mathcal{N} = \mathcal{N}(\omega,\lambda)$ a nilpotent operator. Lemma 5.3 extends (5.8) to this more general case. In preparation, we need to consider powers of the operator s.

Lemma 5.2 *For every $l \in \mathbb{N}_0$, the operator s^l is well-defined. Its kernel $(s^l)(u,u')$ has regularity C^{2l-2}.*

Proof. Writing out the operator products with the integral kernel, one sees that the operator s^l is obtained from s by iterated convolutions,

$$s^{p+1}(u,u') \;=\; \int s(u,u'')\, s^p(u'',u')\, du\,. \tag{5.10}$$

In the finite box, these convolution integrals are all finite because $s(u,u')$ is continuous and the integration range is compact. In infinite volume, the function $s(u,u')$ decays exponentially as $u,u' \to \pm\infty$ (see Corollary 6.4), and so the integrals in (5.10) are again finite. Hence s^l is well-defined.

Let us analyze the regularity of the integral kernel of s^l. By definition, $s(u,u')$ is continuous, and (5.10) immediately shows that the same is true for $s^p(u,u')$. Differentiating through (5.10) and applying Lemma 5.1, one sees that s^p satisfies for $p > 1$ the distributional equation

$$\left(-\frac{\partial^2}{\partial u^2} + V(u)\right)(s^p)(u,u') \;=\; (s^{p-1})(u-u')\,.$$

This shows that incrementing p indeed increases the order of differentiability by two. ∎

Lemma 5.3 *For given $\lambda \in \sigma(\mathcal{A}_\omega)$ we let g be the operator*

$$g \;=\; \sum_{l=0}^{\infty} (-\mathcal{N})^l\, s^{l+1}, \tag{5.11}$$

where $\mathcal{N}$ is the nilpotent matrix in the Jordan decomposition (5.9). Then

$$\Box\left((r^2+a^2)^{-\frac{1}{2}}\, Q_\lambda(\vartheta,\vartheta')\;\, g(u,u')\right) \;=\; (r^2+a^2)^{-\frac{1}{2}}\, Q_\lambda(\vartheta,\vartheta')\, \delta(u-u')\,. \tag{5.12}$$

Note that if Q_λ projects onto an eigenspace, $\mathcal{N}$ vanishes and thus $g = s$. Furthermore, since $\mathcal{N}$ is nilpotent, the series in (5.11) is actually a finite sum. Thus in view of Lemma 5.2, (5.11) is indeed well-defined.

Proof of Lemma 5.3. Denoting the radial operator with integral kernel $\delta(u-u')$ by $\mathbb{1}_u$, we can write the result of Lemma 5.1 in the compact form $(-\partial_u^2 + V)s = \mathbb{1}_u$. Hence on the invariant subspace, we can do a Neumann series calculation,

$$(-\partial_u^2 + V)\, g \;=\; \sum_{l=0}^{\infty} (-\mathcal{N})^l\, (-\partial_u^2 + V) s^{l+1} \;=\; \sum_{k=0}^{\infty} (-\mathcal{N})^l\, s^l \;=\; \mathbb{1}_u - \mathcal{N}\, g\,,$$

to obtain that $(-\partial_u^2 + V + \mathcal{N})\, g = \mathbb{1}_u$. According to (2.10), (2.7), and (5.2), this is equivalent to (5.12). ∎

We come to the separation of the resolvent. In order to explain the difficulty, we point out that H and Q_λ do not in general commute, and thus

$$(H-\omega)\, Q_\lambda \;\neq\; Q_\lambda\, (H-\omega) \quad \text{and} \quad Q_\lambda\, S \;\neq\; S\, Q_\lambda\,.$$

Therefore, one must be very careful with the orders of multiplication; in particular, it is not possible to simplify the operator product $(H-\omega)Q_\lambda S$. However, we know from the separation of variables that for every solution Ψ of the equation $(H-\omega)\Psi = 0$, its projection $Q_\lambda(\omega)\Psi$ is again a solution. In other words, H and Q_λ do commute on the kernel of $(H-\omega)$. This fact will be exploited in the proof of the following proposition.

Proposition 5.4 *For $\omega \not\in \sigma(H)$ we let Q_λ be a spectral projector of the angular operator $\mathcal{A}_\omega$. Then the resolvent of H has the representation*

$$Q_\lambda\, S(\omega) \;=\; Q_\lambda\, T(\omega, \lambda)\;,$$

where T is the operator with integral kernel

$$\begin{aligned} T(u,\vartheta; u', \vartheta') \;&=\; \delta(\cos\vartheta - \cos\vartheta')\, \delta(u-u') \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &+\delta(\cos\vartheta - \cos\vartheta')\, (r^2+a^2)^{-\frac{1}{2}}\, g(u,u') \begin{pmatrix} \tau(u',\vartheta') & \sigma(u',\vartheta') \\ \omega\,\tau(u',\vartheta') & \omega\,\sigma(u',\vartheta') \end{pmatrix} . \end{aligned} \tag{5.13}$$

Here g is the Green's function (5.11) (which depends on ω and λ), and τ, σ are the functions

$$\begin{aligned} \tau \;&=\; (r^2+a^2)^{-\frac{3}{2}} \left((r^2+a^2)^2 - \Delta\, a^2 \sin^2\vartheta \right) \\ \sigma \;&=\; 2ak\, (r^2+a^2)^{-\frac{3}{2}} \left((r^2+a^2) - \Delta \right) + \omega\tau\,. \end{aligned}$$

Proof. Let us compute the operator product $(H-\omega)Q_\lambda T$. We first consider the first summand in (5.13), which we denote by T_1. In this case, the operator product is particularly simple because the second column in the matrix (2.20) involves no u- or ϑ-derivatives. We obtain

$$((H-\omega)\, Q_\lambda\, T_1)(u,\vartheta; u',\vartheta') \;=\; Q_\lambda(\vartheta,\vartheta')\, \delta(u,u') \begin{pmatrix} 1 & 0 \\ \beta(u,\vartheta)-\omega & 0 \end{pmatrix} . \tag{5.14}$$

Next we consider the second summand in (5.13), which we denote by T_2. Fixing u', ϑ' and considering T_2 as a function of u, ϑ, we see from Lemma 5.3 that each column of $Q_\lambda T_2$ is for $u < u'$ a vector of the form $\Psi = (\Phi, \omega\Phi)$ with Φ a solution of the separated wave equation (2.7). The same is true for $u > u'$. Hence for $u \neq u'$, $Q_\lambda T_2$ is composed of eigenfunctions of the Hamiltonian,

$$((H-\omega)\, Q_\lambda\, T_2)(u,\vartheta; u',\vartheta') \;=\; 0 \qquad \text{if } u \neq u'.$$

It remains to compute the distributional contribution to $(H-\omega)Q_\lambda T_2$ at $u = u'$. Since T_2 is continuous at $u = u'$, we only get a contribution when both radial derivatives act on the factor g. According to Lemma 5.2, the higher powers of s are in C^2, and thus we may replace g by s. Applying (2.20), (2.26), and Lemma 5.1, we obtain

$$((H-\omega)\, Q_\lambda\, T_2)(u,\vartheta; u',\vartheta') \;=\; \frac{1}{\sigma(u,\vartheta)}\, Q_\lambda(\vartheta,\vartheta')\, \delta(u-u') \begin{pmatrix} 0 & 0 \\ \tau(u',\vartheta') & \sigma(u',\vartheta') \end{pmatrix} . \tag{5.15}$$

We add (5.14) to (5.15) and carry out the sum over $\lambda \in \sigma(\mathcal{A}_\omega)$. Since the spectral projectors Q_λ are complete (see Lemma 2.1 **(iii)**), $\sum_\lambda Q_\lambda(\vartheta,\vartheta')$ gives a contribution only for $\vartheta = \vartheta'$. We thus obtain a multiplication operator,

$$\sum_{\lambda \in \sigma(\mathcal{A}_\omega)} (H-\omega)\, Q_\lambda\, T(\lambda) \;=\; \mathbb{1} + \left((\beta - \omega) + \frac{\tau}{\sigma} \right) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Using the explicit form of the functions τ, σ, and β, one sees that the second term vanishes. Thus

$$\sum_{\lambda \in \sigma(\mathcal{A}_\omega)} (H - \omega)\, Q_\lambda\, T(\lambda) \;=\; \mathbb{1}\,.$$

Multiplying from the left by $Q_{\lambda'} S$ and using the orthogonality of the angular spectral projectors gives the result. ■

6 WKB Estimates

In this section we shall derive estimates for the radial ODE (5.4) in the regime

$$|\mathrm{Re}\,\omega| \gg 1 \qquad \text{and} \qquad |\mathrm{Im}\,\omega| \leq c\,. \tag{6.1}$$

In this "high-energy regime", the semi-classical WKB-solution should be a good approximation. In order to quantify this statement rigorously, we shall make an ansatz for ϕ which involves the WKB wave function and estimate the error.

Our first lemma gives control of the sign of $\mathrm{Re}\,\sqrt{V}$.

Lemma 6.1 *There is a constant C such that for all ω with*

$$\mathrm{Im}\,\omega \;\neq\; 0\,, \qquad |\mathrm{Re}\,\omega| \;>\; C$$

and all $\lambda \in \sigma(\mathcal{A}_\omega)$, the function $Re\,\sqrt{V}$ has no zeros.

Proof. At a zero of $\mathrm{Re}\,\sqrt{V}$, the function V is real and non-positive. Thus it suffices to show that the imaginary part of V has no zeros.

We first estimate the imaginary part of the angular spectrum. For any $\lambda \in \sigma(\mathcal{A}_\omega)$ we let Φ_λ be a corresponding eigenvector. Then

$$\mathrm{Im}\,(\lambda)\, \langle \Phi, \Phi\rangle_{L^2} \;=\; \frac{1}{2i}\left(\langle \Phi, \mathcal{A}_\omega \Phi\rangle_{L^2} - \langle \mathcal{A}_\omega \Phi, \Phi\rangle_{L^2}\right) \;=\; \frac{1}{2i}\langle \Phi, (\mathcal{A}_\omega - \mathcal{A}_\omega^*)\Phi\rangle_{L^2}\,,$$

where $\langle .,.\rangle L^2$ is the L^2-scalar product on S^2. Hence, according to (2.9),

$$\begin{aligned} |\mathrm{Im}\,\lambda| \;&\leq\; \frac{1}{2}\, \|\mathcal{A}_\omega - \mathcal{A}_\omega^*\| \;=\; \sup_{S^2} \left| \mathrm{Im}\left(\frac{1}{\sin^2\vartheta} (a\omega \sin^2\vartheta + k)^2 \right) \right| \\ &\leq\; 2a^2\, |\mathrm{Re}\,\omega|\, |\mathrm{Im}\,\omega| \;+\; |2ak\, \mathrm{Im}\,\omega|\;. \end{aligned} \tag{6.2}$$

The imaginary part of (5.5) is computed to be

$$\mathrm{Im}\,V \;=\; -2\left(\mathrm{Re}\,\omega + \frac{ak}{r^2+a^2}\right) \mathrm{Im}\,\omega \;+\; \frac{\Delta}{(r^2+a^2)^2}\, \mathrm{Im}\,\lambda \tag{6.3}$$

Using (6.2), the second summand is estimated by

$$\left| \frac{\Delta}{(r^2+a^2)^2}\, \mathrm{Im}\,\lambda \right| \;\leq\; 2\, \frac{a^2 \Delta}{(r^2+a^2)^2} \left(|\mathrm{Re}\,\omega| + \frac{|k|}{a} \right) |\mathrm{Im}\,\omega|\,.$$

The factor $a^2\Delta(r^2+a^2)^{-2}$ vanishes on the event horizon and at infinity and is always smaller than one. Thus there is a constant c with $a^2\Delta(r^2+a^2)^{-2} \leq c < 1$. This shows that after choosing $|\mathrm{Re}\,\omega|$ sufficiently large, the first summand in (6.3) dominates the second, and so $\mathrm{Im}\,V$ has no zeros. ■

In what follows, we assume that the assumptions of the above lemma are satisfied. We choose the sign convention for the square root such that

$$\mathrm{Re}\,\sqrt{V(u)},\ \mathrm{Re}\,V(u)^{\frac{1}{4}} \;\geq\; 0 \qquad \text{for all } u \in \mathbb{R}. \tag{6.4}$$

Furthermore, we shall restrict attention to ω in the range

$$-c \;<\; \mathrm{Im}\,\omega \;<\; 0\,, \qquad |\mathrm{Re}\,\omega| \;>\; C\,, \tag{6.5}$$

where c is any fixed constant and C will be chosen depending on the particular application. The next lemma, which we will need in Section 7, estimates $\sqrt{V}$ inside the "finite box" $[u_L, u_R]$ uniformly in ω for large $\mathrm{Re}\,\omega$.

Lemma 6.2 *For every angular momentum mode n and every $c, \varepsilon > 0$, there are constants C and c' such that for all $u \in [u_L, u_R]$ and all ω in the range (6.5),*

$$|\mathrm{Re}\,\sqrt{V}| \;\leq\; c' \tag{6.6}$$

$$|\mathrm{Im}\,\sqrt{V(\omega)} - \mathrm{Im}\,\sqrt{V(\mathrm{Re}\,\omega)}| \;\leq\; \varepsilon\,. \tag{6.7}$$

Proof. We set $\omega_0 = \mathrm{Re}\,\omega$, $\lambda = \lambda(\omega_0)$ and introduce for a parameter $\tau \in [0,1]$ the potential

$$\begin{aligned} V_\tau &= V(\omega_0) + \tau W \quad \text{with} \\ W &= -2i\,\mathrm{Im}\,\omega \left(\mathrm{Re}\,\omega + \frac{ak}{r^2+a^2} \right) + (\mathrm{Im}\,\omega)^2 + (\lambda - \lambda_0)\,\frac{\Delta}{(r^2+a^2)^2}\,. \end{aligned}$$

Then $V_0 = V(\omega_0)$ and $V_1 = V(\omega)$. The mean value theorem yields that

$$\left| \mathrm{Re}\,\sqrt{V(\omega)} - \mathrm{Re}\,\sqrt{V(\omega_0)} \right| \;\leq\; \sup_{\tau\in[0.1]} \mathrm{Re}\left(\frac{W}{2\sqrt{V_\tau}} \right) \tag{6.8}$$

$$\left| \mathrm{Im}\,\sqrt{V(\omega)} - \mathrm{Im}\,\sqrt{V(\omega_0)} \right| \;\leq\; \sup_{\tau\in[0.1]} \mathrm{Im}\left(\frac{W}{2\sqrt{V_\tau}} \right). \tag{6.9}$$

By choosing C sufficiently large, we can clearly arrange that $V(\omega_0) < 0$, and thus $\mathrm{Re}\,\sqrt{V(\omega_0)} = 0$. Furthermore, one sees immediately from the explicit formulas for V, W together with the estimate for the angular eigenvalue (2.12) that

$$\begin{aligned} \mathrm{Re}\,W &= \mathcal{O}(|\mathrm{Re}\,\omega|^0)\,, \qquad \mathrm{Im}\,W \;=\; \mathcal{O}(|\mathrm{Re}\,\omega|^1) \\ \sqrt{V_\tau} - i\mathrm{Re}\,\omega &= \mathcal{O}(|\mathrm{Re}\,\omega|^0)\,. \end{aligned}$$

Using this in (6.8) and (6.9) gives the claim. ■

We introduce the WKB solutions $\acute{\alpha}$ and $\grave{\alpha}$ by

$$\acute{\alpha}(u) \;=\; \acute{c}\,V^{-\frac{1}{4}}\,\exp\left(\int_0^u \sqrt{V} \right), \quad \grave{\alpha}(u) \;=\; \grave{c}\,V^{-\frac{1}{4}}\,\exp\left(-\int_0^u \sqrt{V} \right), \tag{6.10}$$

where $\acute{c}$ and $\grave{c}$ are some normalization constants. A straightforward calculation shows that these functions satisfy the Schrödinger equation

$$\alpha'' \;=\; \tilde{V}\alpha \qquad \text{with} \qquad \tilde{V} \;=\; V - \frac{1}{4}\,\frac{V''}{V}\,. \tag{6.11}$$

We can hope that $\acute{\alpha}$ and $\grave{\alpha}$ are approximate solutions of the radial equation (5.4). In order to estimate the error, we first write (5.4) as a first order system,

$$\Psi' = \begin{pmatrix} 0 & 1 \\ V & 0 \end{pmatrix} \Psi \qquad \text{with} \qquad \Psi = \begin{pmatrix} \phi \\ \phi' \end{pmatrix}. \tag{6.12}$$

Next we make for Ψ the ansatz

$$\Psi = A\Phi \qquad \text{with} \qquad A = \begin{pmatrix} \acute{\alpha} & \grave{\alpha} \\ \acute{\alpha}' & \grave{\alpha}' \end{pmatrix} \tag{6.13}$$

and Φ a 2-component complex function. A is the fundamental matrix of the ODE (6.11) and thus

$$A' = \begin{pmatrix} 0 & 1 \\ \tilde{V} & 0 \end{pmatrix} A\,. \tag{6.14}$$

Differentiating through the ansatz for (6.13) and using (6.12, 6.14), we obtain that

$$A\,\Phi' = \begin{pmatrix} 0 & 0 \\ V - \tilde{V} & 0 \end{pmatrix} A\,\Phi\,. \tag{6.15}$$

The determinant of A is a Wronskian and thus constant. A short computation using (6.10) shows that $\det A = -2\acute{c}\grave{c}$. Hence we can easily compute the inverse of A by Cramer's rule,

$$A^{-1} = -\frac{1}{2\acute{c}\grave{c}} \begin{pmatrix} \grave{\alpha}' & -\grave{\alpha} \\ -\acute{\alpha}' & \acute{\alpha} \end{pmatrix},$$

and multiplying (6.15) by A^{-1} gives

$$\Phi' = -\frac{1}{2\acute{c}\grave{c}}\,(V - \tilde{V}) \begin{pmatrix} -\grave{\alpha}\acute{\alpha} & -\grave{\alpha}^2 \\ \acute{\alpha}^2 & \acute{\alpha}\grave{\alpha} \end{pmatrix} \Phi\,.$$

Finally, we put in the explicit formulas (6.11) and (6.10) to obtain the equation

$$\Phi' = W \begin{pmatrix} -1 & -f^{-1} \\ f & 1 \end{pmatrix} \Phi\,, \tag{6.16}$$

where W and f are the functions

$$W \;=\; \frac{1}{8\acute{c}\grave{c}}\,\frac{V''}{V^{\frac{3}{2}}}\,, \qquad f \;=\; \exp\left(2\int_0^u \sqrt{V}\right). \tag{6.17}$$

We shall now derive an estimate for the solutions of the ODE (6.16). The main difficulty is that when the function f is very large or close to zero, the matrix in (6.16) has large norm, making it impossible to use simple Gronwall estimates. Instead, we can use that according to (6.4), the function

$$|f| \;=\; \exp\left(2\int_0^u \mathrm{Re}\sqrt{V}\right)$$

is monotone.

Theorem 6.3 *Assume that the potential V in the Schrödinger equation (5.4) satisfies the conditions (6.4) and that the function W defined by (6.17) is in $L^1(\mathbb{R})$. Then there is a solution Φ of the system of ODEs (6.16) with boundary conditions*

$$\lim_{u\to -\infty} \Phi(u) \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This solution satisfies for all $u \in \mathbb{R}$ the bounds

$$\begin{aligned} \left| \Phi_1(u) - \exp\left(-\int_{-\infty}^{u} W \right) \right| \;&\leq\; e^{4\|W\|_1}\, \|W\|_1 \\ |\Phi_2(u)| \;&\leq\; e^{4\|W\|_1}\, \|W\|_1\, |f(u)|\,. \end{aligned}$$

Proof. We set $\rho = |f|$ and introduce new functions a and b by

$$a = \Phi_1 \qquad \text{and} \qquad b = \frac{\Phi_2}{\rho}\,. \tag{6.18}$$

According to (6.16), they satisfy the following ODEs,

$$\begin{aligned} a' \;&=\; -Wa - W\frac{\rho}{f} b \\ b' + \frac{\rho'}{\rho} b \;&=\; W\frac{f}{\rho} a + Wb\,. \end{aligned}$$

This gives rise to the following differential inequalities,

$$\begin{aligned} |a|' \;&=\; \frac{d}{du}\sqrt{\overline{a}a} \;=\; \frac{1}{|a|}\,\mathrm{Re}\left(\overline{a}\, a'\right) \;\leq\; -|a|\,\mathrm{Re}\, W \;+\; |b| \left| W\,\frac{\rho}{f} \right| \\ |b|' \;&=\; \frac{1}{|b|}\,\mathrm{Re}\left(\overline{b}\, b'\right) \;\leq\; -|b|\,\frac{\rho'}{\rho} \;+\; |a| \left| W\,\frac{f}{\rho} \right| \;+\; |b|\,\mathrm{Re}\, W\,. \end{aligned}$$

Using that ρ is monotone and that $|f| = \rho$, we obtain the simple inequality

$$(|a| + |b|)' \;\leq\; 2|W|\,(|a| + |b|)\,.$$

Integrating this inequality from v to u, $-\infty < v < u < \infty$, gives the "Gronwall estimate"

$$(|a| + |b|)(u) \;\leq\; (|a| + |b|)(v)\, \exp\left(2 \int_v^u |W| \right) \;\leq\; (|a| + |b|)(v)\, e^{2\|W\|_1}\,. \tag{6.19}$$

We now let $\Phi^{(v)}$ be the solution of (6.16) with boundary conditions $\Phi^{(v)}(v) = (1, 0)$. In order to estimate $\Phi^{(v)}$, we rewrite (6.16) as

$$\begin{aligned} \left(e^{\int_v^u W}\, \Phi_1^{(v)}(u) \right)' \;&=\; -W\, e^{\int_v^u W}\, \frac{\Phi_2^{(v)}}{f} \\ \left(e^{-\int_v^u W}\, \Phi_2^{(v)}(u) \right)' \;&=\; W\, e^{-\int_v^u W}\, f\, \Phi_1^{(v)}\,. \end{aligned}$$

We integrate and use (6.19) to obtain the inequalities

$$\left| e^{\int_v^u W}\, \Phi_1^{(v)}(u) - 1 \right| \;\leq\; e^{3\|W\|_1} \int_v^u |W| \tag{6.20}$$

$$\left| e^{-\int_v^u W}\, \Phi_2^{(v)}(u) \right| \;\leq\; e^{3\|W\|_1} \int_v^u \rho\, W \;\leq\; e^{3\|W\|_1}\, \rho(u) \int_v^u |W|\,, \tag{6.21}$$

where in the last step we used the monotonicity of ρ.

The inequalities (6.20) and (6.21) yield that for every $\varepsilon > 0$, there is a $\tilde{u}$ such that for all $v, v' < \tilde{u}$, the exponential on the left side of (6.20) and (6.21) are arbitrarily close to one, and the integrals on the right can be made arbitrarily small. Thus $|(\Phi^{(v)} - \Phi^{(v')})(\tilde{u})| < \varepsilon$. Due to the factor $\rho(u)$ on the right of (6.21), we even know that $|(b^{(v)} - b^{(v')})(\tilde{u})| < \varepsilon$ (with b according to (6.18)). Since (6.16) is linear, $\Phi^{(v)} - \Phi^{(v')}$ is also a solution. Applying (6.19) for this solution and choosing $v = \tilde{u}$, we obtain that for all $u > \tilde{u}$, $|(\Phi^{(v)} - \Phi^{(v')})(u)| < c\varepsilon$ with a constant c being independent of $\tilde{u}$. This shows that $\Phi^{(v)}(u)$ converges as $v \to -\infty$, and that the above estimates are still true for $v = -\infty$.

The theorem now follows from (6.20) and (6.21) if we set $v = -\infty$ and pull out a factor of $e^{\int_{-\infty}^u W}$ and $e^{-\int_{-\infty}^u W}$, respectively. ∎

The above theorem has two immediate consequences: First, it yields the existence of solutions $\acute{\phi}$ and $\grave{\phi}$ which decay exponentially at minus and plus infinity, respectively. Second, it gives very good control of the global behavior of these solutions if $|\mathrm{Re}\,\omega|$ is large.

Corollary 6.4 *For every angular momentum mode n and every ω with $\mathrm{Im}\,\omega < 0$, there are solutions $\acute{\phi}$ and $\grave{\phi}$ of the Schrödinger equation (5.4) which satisfy the boundary conditions*

$$\lim_{u\to-\infty} \left| e^{-i\omega u}\, \acute{\phi}(u) \right| \;=\; 1 \;=\; \lim_{u\to\infty} \left| e^{i\omega u}\, \grave{\phi}(u) \right| .$$

Proof. It suffices to construct $\acute{\phi}$, because $\grave{\phi}$ is obtained in exactly the same way if one considers the ODEs backwards in u (i.e. after transforming the radial variable according to $u \to -u$). We choose Φ as in Theorem 6.3 and let $\acute{\phi} = \phi$ be the corresponding solution of the Schrödinger equation given by (6.13) and (6.12). Note that the corollary only makes a statement on the asymptotic behavior of $\acute{\phi}$ as $u \to -\infty$, and thus the behavior of $\acute{\phi}$ on any interval $[u_0, \infty)$, $u_0 < 0$ is irrelevant. Thus we may freely modify the potential V on any such interval. In particular, we can change the potential V on $[u_0, \infty)$ such that it is constant for large u. For any $\varepsilon > 0$, we choose u_0 so small and modify $V_{|[u_0,\infty)}$ such that $\mathrm{Re}\sqrt{V} \geq 0$ and $\|W\|_1 < \varepsilon/3$ (this is possible because V'' decays for large $|u|$ at least at the rate $\sim |u|^{-3}$). Then Theorem 6.3 applies, and we obtain that

$$|\Phi_1 - 1| \;\leq\; \varepsilon\,, \qquad |\Phi_2| \;\leq\; \varepsilon\, |f|\,.$$

Using these bounds in (6.13), one sees that $|\phi/\acute{\alpha} - 1| < \varepsilon$ and thus, after choosing the normalization constants $\acute{c}$ and $\grave{c}$ in (6.10) appropriately,

$$\limsup_{u\to-\infty} |e^{-i\omega u}\, \acute{\phi}| \;\leq\; 1+\varepsilon \quad \text{and} \quad \liminf_{u\to-\infty} |e^{-i\omega u}\, \acute{\phi}| \;\geq\; 1-\varepsilon\,.$$

Since ε is arbitrary, the result follows. ∎

Proposition 6.5 *For every n and $c, \varepsilon > 0$, there is a constant $C > 0$ such that for all ω in the range (6.5), the solutions $\acute{\phi}$ and $\grave{\phi}$ of Corollary 6.4 are close to the (suitably normalized) WKB wave functions $\acute{\alpha}$ and $\grave{\alpha}$, (6.10), in the sense that for all $u \in \mathbb{R}$,*

$$\left| \frac{\acute{\phi}}{\acute{\alpha}} - 1 \right| + \left| \frac{\acute{\phi}'}{\acute{\alpha}'} - 1 \right| \;\leq\; \varepsilon \qquad \textit{and} \qquad \left| \frac{\grave{\phi}}{\grave{\alpha}} - 1 \right| + \left| \frac{\grave{\phi}'}{\grave{\alpha}'} - 1 \right| \;\leq\; \varepsilon\,.$$

The reason why we need to choose C large is that the functions $V'/|V|^{3/2}$ and W must be sufficiently small. More specifically, one can choose C such that

$$\|W\|_1 \;\leq\; \frac{\varepsilon}{3} \qquad \text{and} \qquad \frac{|V'|}{|V|^{\frac{3}{2}}} \;\leq\; \frac{1}{3}\,.$$

Proof of Proposition 6.5. Using (2.12) in (5.5), one sees that in the strip $-2c < \mathrm{Im}\,\omega < 0$, the potential V satisfies the bound

$$|V(u)+\omega^2| \;\leq\; c_1(1+|\omega|)\,.$$

On the other hand, differentiating (5.5) and using (2.26) and (2.12), one sees that in this strip,

$$|V'(u)| + |V''(u)| \;\leq\; (1+|\omega|)\, g(u)\,,$$

where g is a function which decays for large $|u|$ at least at the rate $\sim u^{-2}$. Putting these estimates for V and V'' into (6.17), one sees that by choosing C sufficiently large, we can arrange that for all ω in the range (6.5), $\|W\|_1 \leq \varepsilon/3$. Theorem 6.3 yields that

$$|\Phi_1 - 1| \;\leq\; \varepsilon\,, \qquad |\Phi_2| \;\leq\; \varepsilon\,|f|\,. \tag{6.22}$$

Dividing the first row in (6.13) by $\acute{\alpha}$, we obtain the identity

$$\frac{\acute{\phi}}{\acute{\alpha}} \;=\; \Phi_1 + \frac{\grave{\alpha}}{\acute{\alpha}}\,\Phi_2\,,$$

and using (6.22) gives

$$\left|\frac{\acute{\phi}}{\acute{\alpha}} - 1\right| \;\leq\; \varepsilon\left(1 + |f|\left|\frac{\grave{\alpha}}{\acute{\alpha}}\right|\right).$$

From the second row in (6.13) we obtain similarly,

$$\left|\frac{\acute{\phi}'}{\acute{\alpha}'} - 1\right| \;\leq\; \varepsilon\left(1 + |f|\left|\frac{\grave{\alpha}'}{\acute{\alpha}'}\right|\right).$$

Finally, we apply the elementary estimates for the WKB wave functions

$$\left|\frac{\grave{\alpha}}{\acute{\alpha}}\right| \;=\; \frac{1}{|f|}\,, \qquad \left|\frac{\grave{\alpha}'}{\acute{\alpha}'}\right| \;=\; \frac{1}{|f|}\left|\frac{V'\,V^{-\frac{5}{4}} + V^{\frac{1}{4}}}{V'\,V^{-\frac{5}{4}} - V^{\frac{1}{4}}}\right| \;\leq\; \frac{2}{|f|}\,,$$

where in the last step we applied the above bounds for V' and V and possibly increased C.

The solution $\grave{\phi}$ is obtained similarly if one considers the Schrödinger equation (5.4) backwards in u and repeats the above arguments. ∎

The next two propositions give estimates for composite expressions.

Proposition 6.6 *Under the assumptions of Proposition 6.5,*

$$\left|\frac{w(\acute{\phi},\grave{\phi})}{w(\acute{\alpha},\grave{\alpha})} - 1\right| \;\leq\; 4\varepsilon\,.$$

Proof. Rewriting the Wronskian as

$$w(\acute{\phi}, \grave{\phi}) \;=\; \acute{\phi}'\, \grave{\phi} - \acute{\phi}\, \grave{\phi}' \;=\; \acute{\alpha}'\, \grave{\alpha}\, \frac{\acute{\phi}'}{\acute{\alpha}'}\, \frac{\grave{\phi}}{\grave{\alpha}} - \acute{\alpha}\, \grave{\alpha}'\, \frac{\acute{\phi}}{\acute{\alpha}}\, \frac{\grave{\phi}'}{\grave{\alpha}'}\,,$$

we can put in the estimate of Proposition 6.5 to obtain

$$\left| w(\acute{\phi}, \grave{\phi}) - w(\acute{\alpha}, \grave{\alpha}) \right| \;\leq\; 4\varepsilon \left(|\acute{\alpha}'\, \grave{\alpha}| + |\acute{\alpha}\, \grave{\alpha}'| \right). \tag{6.23}$$

Furthermore, a short explicit calculation using (6.10) shows that

$$w(\acute{\alpha}, \grave{\alpha}) \;=\; 2\sqrt{V}\, \acute{\alpha}\grave{\alpha} \tag{6.24}$$

$$(\acute{\alpha}\, \grave{\alpha})' \;=\; -\frac{1}{2}\, \frac{V'}{V}\, \acute{\alpha}\grave{\alpha} \;=\; -\frac{1}{4}\, V'\, V^{-\frac{3}{2}}\, w(\acute{\alpha}, \grave{\alpha}) \tag{6.25}$$

and thus

$$\begin{aligned} \acute{\alpha}'\, \grave{\alpha} &= \frac{1}{2} \left(w(\acute{\alpha}, \grave{\alpha}) + (\acute{\alpha}\, \grave{\alpha})' \right) \;=\; \frac{1}{2}\, w(\acute{\alpha}, \grave{\alpha}) \left(1 - \frac{1}{4}\, V'\, V^{-\frac{3}{2}} \right) \\ \acute{\alpha}\, \grave{\alpha}' &= -\frac{1}{2} \left(w(\acute{\alpha}, \grave{\alpha}) - (\acute{\alpha}\, \grave{\alpha})' \right) \;=\; -\frac{1}{2}\, w(\acute{\alpha}, \grave{\alpha}) \left(1 + \frac{1}{4}\, V'\, V^{-\frac{3}{2}} \right). \end{aligned}$$

Substituting these relations into (6.23) gives

$$\left| \frac{w(\acute{\phi}, \grave{\phi})}{w(\acute{\alpha}, \grave{\alpha})} - 1 \right| \;\leq\; 4\varepsilon \left(1 + \frac{1}{4} \left| V'\, V^{-\frac{3}{2}} \right| \right).$$

Here the left side only involves Wronskians and is thus independent of u. Hence we may on the right side take the limit $u \to \infty$. This gives the result. ∎

For $u_L < u_R$ we set

$$\left. \begin{aligned} \phi_{[u_L, u_R]} &= \acute{\phi}(u_L)\, \grave{\phi}(u_R) - \acute{\phi}(u_R)\, \grave{\phi}(u_L) \\ \alpha_{[u_L, u_R]} &= \acute{\alpha}(u_L)\, \grave{\alpha}(u_R) - \acute{\alpha}(u_R)\, \grave{\alpha}(u_L)\,. \end{aligned} \right\} \tag{6.26}$$

Proposition 6.7 *Under the assumptions of Proposition 6.5,*

$$\left| \frac{\phi_{[u_L, u_R]}}{\alpha_{[u_L, u_R]}} - 1 \right| \;\leq\; 8\varepsilon\, \exp\left(2 \int_{u_L}^{u_R} Re\, \sqrt{V} \right) \left| \sin\left(2 \int_{u_L}^{u_R} \mathrm{Im}\, \sqrt{V} \right) \right|^{-1}.$$

Proof. Rewriting $\phi_{[u_L, u_R]}$ as

$$\phi_{[u_L, u_R]} \;=\; \acute{\alpha}(u_L)\, \grave{\alpha}(u_R)\, \frac{\acute{\phi}(u_L)}{\acute{\alpha}(u_L)}\, \frac{\grave{\phi}(u_R)}{\grave{\alpha}(u_R)} \;-\; \acute{\alpha}(u_R)\, \grave{\alpha}(u_L)\, \frac{\acute{\phi}(u_R)}{\acute{\alpha}(u_R)}\, \frac{\grave{\phi}(u_L)}{\grave{\alpha}(u_L)}\,,$$

Proposition 6.5 yields that

$$\left| \phi_{[u_L, u_R]} - \alpha_{[u_L, u_R]} \right| \;\leq\; 4\varepsilon \left(|\acute{\alpha}(u_L)\, \grave{\alpha}(u_R)| + |\acute{\alpha}(u_R)\, \grave{\alpha}(u_L)| \right) \tag{6.27}$$

Furthermore, it is obvious from (6.10) that

$$\acute{\alpha}(u_L)\, \grave{\alpha}(u_R) \;=\; \acute{\alpha}(u_R)\, \grave{\alpha}(u_L)\, \exp\left(-2 \int_{u_L}^{u_R} \sqrt{V} \right)$$

and thus from (6.26),

$$\begin{aligned}
\alpha_{[u_L,u_R]} &= \acute{\alpha}(u_R)\,\grave{\alpha}(u_L)\left(\exp\left(-2\int_{u_L}^{u_R}\sqrt{V}\right)-1\right) \\
&= \acute{\alpha}(u_L)\,\grave{\alpha}(u_R)\left(1-\exp\left(2\int_{u_L}^{u_R}\sqrt{V}\right)\right). \qquad (6.28)
\end{aligned}$$

Dividing (6.27) by $\alpha_{[u_L,u_R]}$ and putting in the last identities, we obtain

$$\left|\frac{\phi_{[u_L,u_R]}}{\alpha_{[u_L,u_R]}}-1\right| \;\leq\; 4\varepsilon\,\frac{1+\left|\exp\left(-2\int_{u_L}^{u_R}\sqrt{V}\right)\right|}{\left|\exp\left(-2\int_{u_L}^{u_R}\sqrt{V}\right)-1\right|} \;\leq\; \frac{8\varepsilon}{\left|\exp\left(-2\int_{u_L}^{u_R}\sqrt{V}\right)-1\right|},$$

where in the last step we used that $\mathrm{Re}\sqrt{V}\geq 0$. We finally estimate the obtained denominator from above,

$$\begin{aligned}
\left|\exp\left(-2\int_{u_L}^{u_R}\sqrt{V}\right)-1\right| &\geq \left|\mathrm{Im}\exp\left(-2\int_{u_L}^{u_R}\sqrt{V}\right)\right| \\
&= \exp\left(-2\int_{u_L}^{u_R}\mathrm{Re}\sqrt{V}\right)\left|\sin\left(2\int_{u_L}^{u_R}\mathrm{Im}\sqrt{V}\right)\right|.
\end{aligned}$$

■

7 Contour Deformations

In this section we shall use contour integral methods to prove the main theorem. Recall that in Section 3, we showed that the Hamiltonian H_{u_L,u_R} in finite volume is a selfadjoint operator on the Pontrjagin space $\mathcal{P}_{u_L,u_R}$. It has a purely discrete spectrum, and for each $\omega\in\sigma(H_{u_L,u_R})$, the projector E_ω onto the corresponding invariant subspace can be expressed as the contour integral

$$E_\omega \;=\; -\frac{1}{2\pi i}\oint_{|\omega'-\omega|<\varepsilon} S_{u_L,u_R}(\omega')\,d\omega'\,,$$

where ε is to be chosen so small that $B_\varepsilon(\omega)$ contains no other points of the spectrum. The theory of Pontrjagin spaces also yields that $\sigma(H_{u_L,u_R})$ will in general involve a finite number of non-real spectral points, which lie symmetrically around the real axis. We let $E_\mathbb{C}$ be the projector onto the invariant subspace corresponding to all non-real spectral points,

$$E_\mathbb{C} \;:=\; \sum_{\omega\in\sigma(H_{u_L,u_R})\setminus\mathbb{R}} E_\omega\,. \qquad (7.1)$$

Our first lemma represents $E_\mathbb{C}$ as a Cauchy integral over an unbounded contour. More precisely, we choose a contour C_{u_L,u_R} in the lower half plane which joins the points $+\infty$ with $-\infty$ and encloses the spectrum in the lower half plane from above. Furthermore, if $\mathrm{Re}\,\omega$ is outside the finite interval $[\omega_-,\omega_+]$, ω should be in the open set Ω (see (4.6)) and should approach the real axis as $|\mathrm{Im}\,\omega|\sim -|\mathrm{Re}\,\omega|^{-1}$ (see Figure 2).

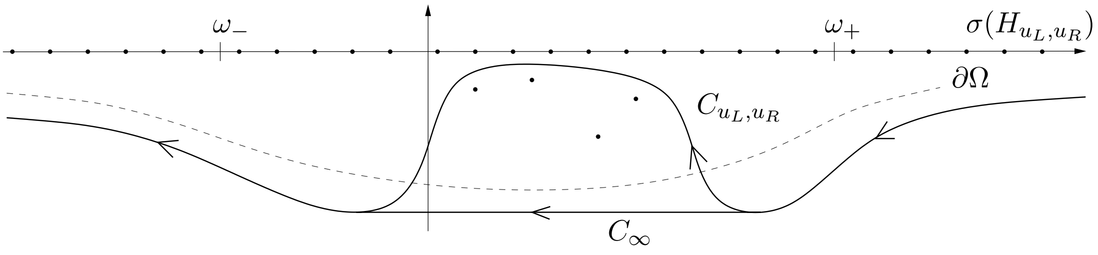

Figure 2: Unbounded contour representation of $E_{\mathbb{C}}$

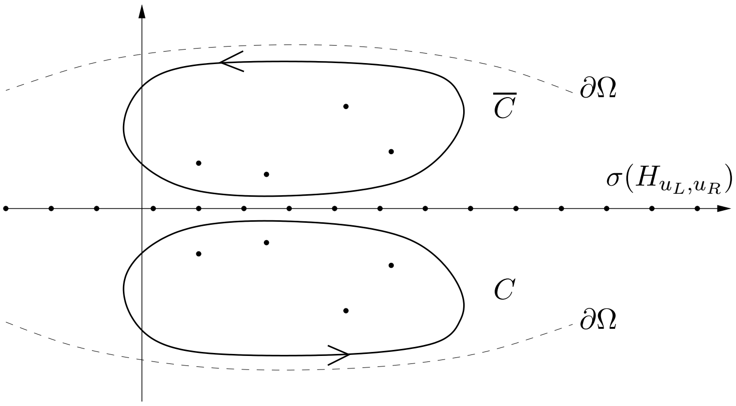

Figure 3: Closed contour representation of $E_{\mathbb{C}}$

Lemma 7.1 *The spectral projector corresponding to the non-real spectrum (7.1) has the representation*

$$E_{\mathbb{C}} \;=\; \frac{1}{\pi} \lim_{L\to\infty} \mathrm{Im} \int_{C_{u_L,u_R}} \frac{L^3}{(L+i\omega)^3}\, S_{u_L,u_R}(\omega)\, d\omega\,. \tag{7.2}$$

Proof. The Cauchy integral formula yields that

$$E_{\mathbb{C}} \;=\; -\frac{1}{2\pi i}\oint_C S_{u_L,u_R}(\omega)\, d\omega \;+\; \frac{1}{2\pi i}\oint_{\overline{C}} S_{u_L,u_R}(\omega)\, d\omega \;=\; \frac{1}{\pi}\,\mathrm{Im} \oint_C S_{u_L,u_R}(\omega)\, d\omega\,,$$

where C is a closed contour which encloses the spectrum in the lower half plane (see Figure 3). The dominated convergence theorem allows us to insert a factor $L^3/(L+i\omega)^3$,

$$E_{\mathbb{C}} \;=\; \frac{1}{\pi} \lim_{L\to\infty} \mathrm{Im} \oint_C \frac{L^3}{(L+i\omega)^3}\, S_{u_L,u_R}(\omega)\, d\omega\,.$$

The function $L^3/(L+i\omega)^3$ has no poles in the lower half plane and decays cubically for large $|\omega|$. Furthermore, according to (4.8), the resolvent grows at most linearly for large $|\omega|$. This allows us to deform the contour in such a way that C is closed in the lower half plane on larger and larger circles $|\omega| = R$. In the limit $R \to \infty$ the contribution along the circle tends to zero. Thus we end up with the integral along the contour C_{u_L,u_R}. ∎

Our next goal is to get rid of the "convergence generating factor" $L^3/(L+i\omega)^3$ in (7.2). We shall use the fact that when we take the difference $S_{u_L,u_R} - S_\infty$ and evaluate it with a test function, the resulting expression has much better decay properties at infinity (see Lemma 4.3). We choose a contour C_∞ which coincides with C_{u_L,u_R} if $\mathrm{Re}\,\omega \notin [\omega_-, \omega_+]$ and always stays inside Ω (see Figure 2).

Lemma 7.2 *For every $\Psi \in C_0^\infty((u_L, u_R) \times S^2)^2$,*

$$<\Psi, E_{\mathbb{C}}\Psi> = \frac{1}{\pi}\,\mathrm{Im}\left(\int_{C_{u_L,u_R}} <\Psi, S_{u_L,u_R}\Psi>\ d\omega \;-\; \int_{C_\infty} <\Psi, S_\infty \Psi>\ d\omega \right). \tag{7.3}$$

Furthermore,

$$\begin{aligned} <\Psi, E_{\mathbb{C}}\Psi> &= \sum_{n\in\mathbb{N}} I_n \qquad \textit{with} && (7.4)\\ I_n &= -\frac{1}{2\pi i}\left(\int_{C_{u_L,u_R}} <\Psi, Q_n\, S_{u_L,u_R}\Psi>\ d\omega \;-\; \int_{C_\infty} <\Psi, Q_n S_\infty \Psi>\ d\omega \right) \\ &\quad + \frac{1}{2\pi i}\left(\int_{\overline{C_{u_L,u_R}}} <\Psi, Q_n\, S_{u_L,u_R}\Psi>\ d\omega \;-\; \int_{\overline{C_\infty}} <\Psi, Q_n S_\infty \Psi>\ d\omega \right) && (7.5) \end{aligned}$$

The series in (7.4) converges absolutely.

We point out that the above integrals are merely a convenient notation and are to be given a rigorous meaning as follows. We formally rewrite the integrals in (7.3) (and similarly in (7.5)) as

$$\int_{C_\infty} <\Psi, (S_{u_L,u_R} - S_\infty)\Psi>\ d\omega \;+\; \left(\int_{C_{u_L,u_R}} - \int_{C_\infty} \right) <\Psi, S_{u_L,u_R}\Psi>\ d\omega\,. \tag{7.6}$$

Now the first summand is well-defined according to Lemma 4.3. In the second summand, the integrals combine to an integral over a bounded contour, and this is clearly well-defined because the contour does not intersect the spectrum of H_{u_L,u_R}.

Note that in (7.5) we cannot combine the integrals over C_{u_L,u_R} and $\overline{C_{u_L,u_R}}$ (and similarly over C_∞ and $\overline{C_\infty}$) to the imaginary part of one contour integral because Q_n in general does not commute with S_{u_L,u_R}, and so the integrands in (7.5) need not be real. For notational convenience, we abbreviate the second line in (7.5) by "$-$ccc" (for "complex conjugated contours").

Proof of Lemma 7.2. According to Corollary 4.2, the resolvent $S_\infty(\omega)$ is holomorphic for $\omega in \Omega$ and grows at most linearly in $|\omega|$. Thus for all $L > 0$,

$$\int_{C_\infty} \frac{L^3}{(L+i\omega)^3}\, S_\infty(\omega)\, d\omega \;=\; 0\,.$$

Combining this identity with (7.2), we obtain the representation

$$\begin{aligned} <\Psi, E_{\mathbb{C}}\Psi> = \frac{1}{\pi} \lim_{L\to\infty} & \\ \times\,\mathrm{Im}&\left(\int_{C_{u_L,u_R}} \frac{L^3}{(L+i\omega)^3}\ <\Psi, S_{u_L,u_R}\Psi>\ d\omega \;-\; \int_{C_\infty} \frac{L^3}{(L+i\omega)^3}\ <\Psi, S_\infty\Psi>\ d\omega \right). \end{aligned}$$

We now rewrite the integrals according to (7.6). If replace the contour C_{u_L,u_R} by C_∞, the integrands combine, and we obtain the expression

$$\frac{1}{\pi} \lim_{L\to\infty} \mathrm{Im} \int_{C_\infty} \frac{L^3}{(L+i\omega)^3} \;<\!\Psi, (S_{u_L,u_R} - S_\infty)\Psi\!>\; d\omega \;.$$

The estimate (4.9) allows us to apply Lebesgue's dominated converge theorem and to take the limit $L \to \infty$ inside the integrand. The error we made when replacing C_{u_L,u_R} by C_∞ is

$$\frac{1}{\pi} \lim_{L\to\infty} \mathrm{Im} \left\{ \left(\int_{C_{u_L,u_R}} - \int_{C_\infty} \right) \frac{L^3}{(L+i\omega)^3} \;<\!\Psi, S_{u_L,u_R}\Psi\!>\; d\omega \right\} .$$

Now the contour is bounded, and since the factor $<\!\Psi, S_{u_L,u_R}\Psi\!>$ is bounded, we can again apply Lebesgue's dominated convergence theorem to take the limit $L \to \infty$ inside the integrand. This gives (7.3).

Note that our contours were chosen such that the condition (2.11) is satisfied for a suitable constant $c > 0$, and so Lemma 2.1 applies. Using completeness of the $(Q_n)_{n\in\mathbb{N}}$ (see Lemma 2.1 **(iii)**), it immediately follows from (7.3) that

$$\begin{aligned} &<\!\Psi, E_{\mathbb{C}}\Psi\!> \\ &= \; -\frac{1}{2\pi i} \left(\int_{C_{u_L,u_R}} \sum_{n\in\mathbb{N}} <\!\Psi, Q_n S_{u_L,u_R}\Psi\!>\; d\omega \;-\; \int_{C_\infty} \sum_{n\in\mathbb{N}} <\!\Psi, Q_n S_\infty \Psi\!>\; d\omega \right) - \mathrm{ccc} \;. \end{aligned}$$

Again replacing the contour C_{u_L,u_R} by C_∞, we obtain the expression

$$-\frac{1}{2\pi i} \int_{C_\infty} \sum_{n\in\mathbb{N}} <\!\Psi, Q_n\,(S_{u_L,u_R} - S_\infty)\Psi\!>\; d\omega \;-\mathrm{ccc} \;.$$

According to (4.10), the summands decay faster than any polynomial in λ_n. Applying the angular estimates (2.13) and (2.14), we conclude that the sum over n converges absolutely, uniformly in $\omega \in C_\infty$. Thus the dominated convergence theorem allows us to commute summation and integration, and the series converges absolutely. It remains to consider the expression

$$-\frac{1}{2\pi i} \left(\int_{C_{u_L,u_R}} - \int_{C_\infty} \right) \sum_{n\in\mathbb{N}} <\!\Psi, Q_n S_{u_L,u_R}\Psi\!>\; d\omega \;-\mathrm{ccc} \;.$$

Now the contours are compact, and thus the absolute convergence of the n-series is uniform on the contour. Hence we can again apply Lebesgue's dominated convergence theorem to interchange the summation with the integration. ■

We shall now deform the contours C_{u_L,u_R} and C_∞ and analyze the resulting integrals. Our aim is to move the contours onto the real axis such that they reduce to an ω-integral over the real line. It is a major advantage of (7.4) that the series stands in front of the integrals, because this allows us to deform the contours in each summand I_n separately. Moreover, since our contour deformations will keep the values of the integrals unchanged, Lemma 7.2 guarantees that the series over n will converge absolutely. Thus we may in what follows restrict attention to fixed n.

For given n, we know from Section 3 that the function $<\Psi, Q_n S_{u_L,u_R} \Psi>$ is meromorphic, and all poles are points of $\sigma(H_{u_L,u_R})$. For the integrals over C_∞ in (7.5), we cannot use abstract arguments because we have hardly any information on the spectrum of H_∞ (we only know from Lemma 4.1 that the spectrum lies outside the set Ω, (4.6), but it may be continuous and complex). But from the separation of the resolvent we know that the operator $Q_\lambda S_\infty$ is well-defined and bounded unless the Wronskian $w(\acute{\phi}, \grave{\phi})$ vanishes (see Proposition 5.4 and (5.7)). If this Wronskian were zero and $\mathrm{Im}\, \omega < 0$, this would give rise to a solution ϕ of the reduced wave equation which decays exponentially as $u \to \pm\infty$. Such "unstable modes" were ruled out by Whiting [16]. We conclude that $<\Psi, Q_n S_\infty \Psi>$ is analytic in the whole lower half plane $\{\mathrm{Im}\, \omega < 0\}$.

Using the above analyticity properties of $<\Psi, Q_n S_{u_L,u_R} \Psi>$ and $<\Psi, Q_n S_\infty \Psi>$, we are free to deform the contours C_{u_L,u_R} and C_∞ in any compact set, provided that C_{u_L,u_R} never intersects $\sigma_n(H_{u_L,u_R})$. In particular, choosing ω_- and ω_+ real and outside of $\sigma(H_{u_L,u_R})$, we may deform the contours as shown in Figure 4. We let $E_{[\omega_-,\omega_+]}$ be the projector on all

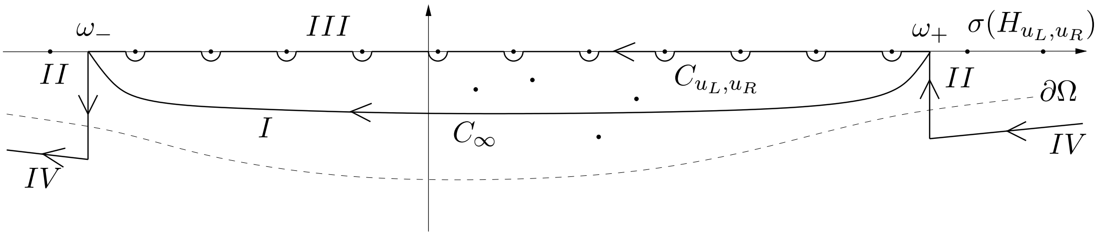

Figure 4: Contour deformation onto the real axis

invariant subspaces of H_{u_L,u_R} corresponding to real ω in the range $\omega_- \leq \omega \leq \omega_+$,

$$Q_n E_{[\omega_-,\omega_+]} \;=\; \sum_{\omega \in [\omega_-,\omega_+]} Q_n(\omega)\, E_\omega \,.$$

The next lemma shows that the integral over $C_{III} \cup \overline{C_{III}}$ equals $Q_n E_{[\omega_-,\omega_+]}$, whereas the integrals over the contours II and IV can be made arbitrarily small by choosing $|\omega_\pm|$ sufficiently large.

Lemma 7.3 *For every $\Psi \in C_0^\infty((u_L,u_R) \times S^2)^2$, $n \in \mathbb{N}$, and $\varepsilon > 0$ there are $\omega_-, \omega_+ \in \mathbb{R} \setminus \sigma(H_{u_L,u_R})$ such that*

$$\left| I_n + <\Psi, Q_n E_{[\omega_-,\omega_+]} \Psi> \;-\; \frac{1}{2\pi i} \left(\int_{C_I} - \int_{\overline{C_I}} \right) <\Psi, Q_n S_\infty \Psi> \right| \;\leq\; \varepsilon \,,$$

where I_n are the integrals (7.5) and C_I is any contour in the lower half plane which joins ω_- with ω_+ (see Figure 4).

Proof. Lemma 4.3 yields that by choosing ω_+ and $-\omega_-$ sufficiently large, we can make the contribution of the contour IV arbitrarily small. The integrals over C_{III} and $\overline{C_{III}}$ combine to contour integrals around the spectral points on the real axis,

$$-\frac{1}{2\pi i} \left(\int_{C_{III}} - \int_{\overline{C_{III}}} \right) <\Psi, Q_n\, S_{u_L,u_R} \Psi>\; d\omega \;=\; -\frac{1}{2\pi i} \sum_{\omega'} \oint_{|\omega-\omega'|=\delta} <\Psi, Q_n\, S_{u_L,u_R} \Psi>\; d\omega \,,$$

where the sum runs over all $\omega' \in \sigma(H_{u_L,u_R}) \cap [\omega_-, \omega_+]$, and δ must be chosen so small that each contour contains only one point of the spectrum. If we let $\delta \to 0$ and use that Q_n depends smoothly on ω, one sees that the integrals over the circles converge to $-2\pi i <\Psi, Q_n\, E_{\omega'} \Psi>$. We conclude that

$$-\frac{1}{2\pi i} \left(\int_{C_{III}} - \int_{\overline{C_{III}}} \right) <\Psi, Q_n\, S_{u_L,u_R} \Psi> \;d\omega \;=\; - <\Psi, Q_n\, E_{[\omega_-,\omega_+]} \Psi> \;.$$

It remains to show that by choosing $|\omega_\pm|$ sufficiently large, we can make the integral over the contour II arbitrarily small. According to Lemma 2.1, for sufficiently large $|\omega_\pm|$ the angular operator $\mathcal{A}_\omega$ is diagonalizable for all ω on the contour II. Thus we can assume that the nilpotent matrices $\mathcal{N}$ in the Jordan decomposition (5.9) all vanish. Hence we can separate the resolvents according to Proposition 5.4 to obtain

$$<\Psi, Q_n\, (S_{u_L,u_R} - S_\infty) \Psi> = \sum_{\lambda \in \Lambda_n} <\Psi, Q_\lambda\, \Delta T_\lambda \Psi> \,,$$

where ΔT_λ is the operator with integral kernel

$$\Delta T_\lambda(u; u', \vartheta') \;=\; (r^2+a^2)^{-\frac{1}{2}}\, (s_{u_L,u_R} - s_\infty)(u,u') \begin{pmatrix} \rho(u',\vartheta') & \sigma(u',\vartheta') \\ \omega\, \rho(u',\vartheta') & \omega\, \sigma(u',\vartheta') \end{pmatrix}.$$

Since the functions ρ and σ are smooth and the angular operators Q_λ are bounded (2.14), it suffices to show that for every $\varepsilon > 0$ and $g \in C^\infty_0((u_L, u_R))$, we can choose $\omega_\pm$ such that for all ω on the contour II,

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} du'\; g(u)\, g(u')\, (s_{[u_L,u_R]} - s_\infty)(u,u') \;\leq\; \varepsilon \,.$$

Let us derive a convenient formula for $s_{[u_L,u_R]} - s_\infty$. We let ϕ_1 and ϕ_2 be the two fundamental solutions which satisfy the Dirichlet boundary conditions $\phi_1(u_L) = 0 = \phi_2(u_R)$. Likewise, we let $\acute{\phi}$ and $\grave{\phi}$ be the two fundamental solutions in infinite volume as constructed in Corollary 6.4. Furthermore, assume that $u_L < u < u' < u_R$. Then, according to (5.7),

$$s_{u_L,u_R}(u,u') \;=\; \frac{1}{w(\phi_1,\phi_2)}\, \phi_1(u)\, \phi_2(u') \,, \qquad s_\infty(u,u') \;=\; \frac{1}{w(\acute{\phi},\grave{\phi})}\, \acute{\phi}(u)\, \grave{\phi}(u') \,.$$

Expressing ϕ_1 as a linear combination of $\acute{\phi}$ and ϕ_2,

$$\phi_1(u) \;=\; \acute{\phi}(u)\, \phi_2(u_L) - \acute{\phi}(u_L)\, \phi_2(u) \,,$$

and substituting into the above formula for s_{u_L,u_R}, we obtain

$$\begin{aligned} s_{u_L,u_R}(u,u') \;&=\; \frac{1}{\phi_2(u_L)\, w(\acute{\phi},\phi_2)} \Big(\acute{\phi}(u)\, \phi_2(u_L) - \acute{\phi}(u_L)\, \phi_2(u) \Big)\, \phi_2(u') \\ &=\; \frac{1}{w(\acute{\phi},\phi_2)}\, \acute{\phi}(u)\, \phi_2(u') \;-\; \frac{\acute{\phi}(u_L)}{\phi_2(u_L)}\, \frac{\phi_2(u)\, \phi_2(u')}{w(\acute{\phi},\phi_2)} \,. \end{aligned}$$

In the first summand, we can express ϕ_2 in terms of $\acute{\phi}$ and $\grave{\phi}$,

$$\phi_2(u) \;=\; \acute{\phi}(u)\, \grave{\phi}(u_R) - \acute{\phi}(u_R)\, \grave{\phi}(u) \,. \qquad (7.7)$$

This gives

$$\begin{aligned}\frac{1}{w(\acute{\phi},\phi_2)}\,\acute{\phi}(u)\,\phi_2(u') &= \frac{1}{-\acute{\phi}(u_R)\,w(\acute{\phi},\grave{\phi})}\,\acute{\phi}(u)\left(\acute{\phi}(u')\,\grave{\phi}(u_R)-\acute{\phi}(u_R)\,\grave{\phi}(u')\right)\\ &= -\frac{\grave{\phi}(u_R)}{\acute{\phi}(u_R)}\,\frac{\acute{\phi}(u)\,\acute{\phi}(u')}{w(\acute{\phi},\grave{\phi})}\;+\;s_\infty(u,u')\,.\end{aligned}$$

We conclude that

$$(s_{u_L,u_R}-s_\infty)(u,u') \;=\; -\frac{\acute{\phi}(u_L)}{\phi_2(u_L)}\,\frac{\phi_2(u)\,\phi_2(u')}{w(\acute{\phi},\phi_2)}\;-\;\frac{\grave{\phi}(u_R)}{\acute{\phi}(u_R)}\,\frac{\acute{\phi}(u)\,\acute{\phi}(u')}{w(\acute{\phi},\grave{\phi})}\,,$$

and because of its symmetry in u and u', this identity is also valid in the case $u'<u$. Using (7.7) and the notation (6.26), we get

$$(s_{u_L,u_R}-s_\infty)(u,u') \;=\; \frac{\acute{\phi}(u_L)\,\phi_2(u)\,\phi_2(u')}{\phi_{[u_L,u_R]}\,\acute{\phi}(u_R)\,w(\acute{\phi},\grave{\phi})}\;-\;\frac{\grave{\phi}(u_R)\,\acute{\phi}(u)\,\acute{\phi}(u')}{\acute{\phi}(u_R)\,w(\acute{\phi},\grave{\phi})}\,. \tag{7.8}$$

We choose $|\omega_\pm|$ such that

$$\int_{u_L}^{u_R}\mathrm{Im}\,\sqrt{V(\omega_\pm)}\;\in\;\frac{2\mathbb{Z}+1}{4}\,\pi\,.$$

According to the estimate (6.7) in Lemma 6.2, we can arrange that the function $\int_{u_L}^{u_R}\mathrm{Im}\,\sqrt{V}$ is nearly constant on the contour II, and thus

$$\left|\sin\left(2\int_{u_L}^{u_R}\mathrm{Im}\,\sqrt{V}\right)\right|\;\geq\;\frac{1}{2}\,. \tag{7.9}$$

Propositions 6.5, 6.6, and 6.7 allow us to estimate each term in (7.8) by the corresponding term in the WKB approximation. According to (7.9), the factor $|\sin(\ldots)|^{-1}$ which appears in Proposition 6.7 is bounded. Choosing ε sufficiently small, We thus obtain the estimate

$$\begin{aligned}&\left|(s_{u_L,u_R}-s_\infty)(u,u')\right|\\ &\quad\leq\;2\left|\frac{\acute{\alpha}(u_L)\,\alpha_2(u)\,\alpha_2(u')}{\alpha_{[u_L,u_R]}\,\acute{\alpha}(u_R)\,w(\acute{\alpha},\grave{\alpha})}\right|\,\exp\left(2\int_{u_L}^{u_R}\mathrm{Re}\sqrt{V}\right)+2\left|\frac{\grave{\alpha}(u_R)\,\acute{\alpha}(u)\,\acute{\alpha}(u')}{\acute{\alpha}(u_R)\,w(\acute{\alpha},\grave{\alpha})}\right|,\end{aligned} \tag{7.10}$$

where we introduced the function

$$\alpha_2(u)\;=\;|\acute{\alpha}(u)\,\grave{\alpha}(u_R)|+|\acute{\alpha}(u_R)\,\grave{\alpha}(u)|\,.$$

Using the explicit formulas (6.24, 6.28) together with (7.9), we get

$$|w(\acute{\alpha},\grave{\alpha})|\;\geq\;|\sqrt{V(u)}|\,|\acute{\alpha}(u)\,\grave{\alpha}(u)|\,,\qquad |\alpha_{[u_L,u_R]}|\;\geq\;\frac{1}{2}\,|\acute{\alpha}(u_R)\,\grave{\alpha}(u_R)|\,. \tag{7.11}$$

Substituting these bounds into (7.10), we get an estimate for $|s_{u_L,u_R}-s_\infty|$ in terms of expressions of the form

$$\frac{1}{\sqrt{|V(u)|}}\,\exp\left(2\int_{u_L}^{u_R}\mathrm{Re}\sqrt{V}\right)\left|\frac{\acute{\alpha}(u_1)}{\acute{\alpha}(v_1)}\right|\cdots\left|\frac{\grave{\alpha}(u_k)}{\grave{\alpha}(v_k)}\right|\cdots \tag{7.12}$$

with $u_i, v_i \in [u_L, u_R]$. The quotients of the WKB wave functions have according to (6.10) the explicit form

$$\left|\frac{\acute{\alpha}(u)}{\acute{\alpha}(v)}\right| \;=\; \left|\frac{V(u)}{V(v)}\right|^{-\frac{1}{4}} \exp\left(2\int_{u_L}^{u_R} \mathrm{Re}\sqrt{V}\right) , \tag{7.13}$$

and similarly for $\grave{\alpha}$. The inequality (6.6) shows that the exponentials in (7.12) and (7.13) are bounded uniformly in ω. Furthermore, it is obvious from (5.5) that $V(u)/V(v)$ is close to one if $|\omega|$ is large. We conclude that on the contour II,

$$|(s_{u_L,u_R} - s_\infty)(u,u')| \;\leq\; \frac{c}{\sqrt{|V(u)|}} ,$$

and this can be made arbitrarily small by choosing $|\omega_\pm|$ sufficiently large. ∎

We are now in the position to prove our main theorem.

Proof of Theorem 1.1. According to Lemma 7.3,

$$-\frac{1}{2\pi i} \lim_{\omega_\pm \to \pm\infty} \left(\int_{C_I} - \int_{\overline{C_I}}\right) <\Psi, Q_n S_\infty \Psi> \,d\omega \;=\; I_n + <\Psi, Q_n E_{\mathbb{R}} \Psi> ,$$

where $E_{\mathbb{R}}$ denotes the projector onto the invariant subspace corresponding to the real spectrum of H_{u_L,u_R}. Here the $\omega_\pm$ are to be chosen as in Lemma 7.3 and C_I is again any contour which joins ω_- with ω_+ in the lower half plane. Suppose that the contour C_ε intersects the lines $\mathrm{Re}\,\omega = \omega_\pm$ in the points $\omega_+ - i\delta_+$ and $\omega_- - i\delta_-$, respectively. Then we choose the contour C_I as follows,

$$C_I \;=\; (\omega_- - i[0,\delta_-]) \cup (C_\varepsilon \cap (\omega_-,\omega_+) + i\mathbb{R}) \cup (\omega_+ - i[0,\delta_+]) .$$

The first and last parts of the contour have lengths δ_- and δ_+, respectively, and these lengths clearly tend to zero as $\omega_\pm \to \pm\infty$. Furthermore, it is obvious from Propositions 6.5 and 6.6 as well as (7.11) and (7.13) that the integrand is uniformly bounded on these parts of the contour. Hence the contribution of these contours tends to zero as $\omega_\pm \to \pm\infty$. Thus

$$-\frac{1}{2\pi i}\left(\int_{C_\varepsilon} - \int_{\overline{C_\varepsilon}}\right) <\Psi, Q_n S_\infty \Psi> \,d\omega \;=\; I_n + <\Psi, Q_n E_{\mathbb{R}} \Psi> .$$

According to Lemma 7.2 and Lemma 2.1, the right side of this equation is absolutely summable in n and

$$\sum_n \left(I_n + <\Psi, Q_n E_{\mathbb{R}} \Psi>\right) \;= <\Psi, (E_{\mathbb{C}} + E_{\mathbb{R}})\, \Psi> .$$

Since the spectral projectors in the Pontrjagin space $\mathcal{H}_{u_L,u_R}$ are complete, $E_{\mathbb{C}} + E_{\mathbb{R}} = \mathbb{1}$. We conclude that

$$-\frac{1}{2\pi i} \sum_n \left(\int_{C_\varepsilon} - \int_{\overline{C_\varepsilon}}\right) <\Psi, Q_n S_\infty \Psi> \,d\omega \;= <\Psi, \Psi> .$$

Polarizing, we obtain for every $\Psi \in C_0^\infty((r_1,\infty) \times S^2)^2$ the simple identity

$$\Psi \;=\; -\frac{1}{2\pi i} \sum_n \left(\int_{C_\varepsilon} - \int_{\overline{C_\varepsilon}}\right) Q_n S_\infty \Psi \,d\omega . \tag{7.14}$$

The integral and sum converge in L^2_{loc}.

If we apply the Hamiltonian to the integrand in the above formula, we obtain according to Proposition 5.4

$$H\, Q_n S_\infty \Psi \;=\; (H-\omega)\, Q_n S_\infty \Psi \;+\; \omega\, Q_n S_\infty \Psi \;=\; \text{(holomorphic terms)} \;+\; \omega\, Q_n S_\infty \Psi\;.$$

The holomorphic terms are holomorphic in the whole neighborhood of the real axis enclosed by C_ε and $\overline{C_\varepsilon}$ (see Lemma 2.1 **(i)**), and therefore the contour integral over them drops out. We conclude that applying H reduces to multiplying the integrand by a factor ω. Iteration shows that the dynamics of Ψ is taken into account by a factor $e^{-i\omega t}$,

$$\Psi(t) \;=\; -\frac{1}{2\pi i} \sum_n \left(\int_{C_\varepsilon} - \int_{\overline{C_\varepsilon}} \right) e^{-i\omega t}\, Q_n S_\infty \Psi_0 \, d\omega\;.$$

Comparing this expansion with (7.14), one sees that the integrand in the last expansion is equal to the integrand in (7.14) if Ψ is replaced by $\Psi(t)$. Since $\Psi(t)$ is smooth and by causality has compact support, we conclude that the integral and sum again converge in L^2_{loc}. Finally, using that the contour integrals in this formula are all independent of ε, we may take the limit $\varepsilon \searrow 0$ of each of them. ■

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