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BLOW-UP OF POSITIVE SOLUTIONS OF A SEMILINEAR PARABOLIC EQUATION WITH A GRADIENT TERM

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Abstract. We study the blow-up behavior (in time and space) of positive solutions of a semilinear parabolic equation with a gradient term. Our main result is a sharp estimate for the spatial blow-up profile of radially decreasing solutions on a ball. This result illustrates the influence of the gradient term on the profile.
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1 Introduction

In this paper we will study the initial-boundary value problem

$$u_t = \Delta u - |\nabla u|^q + u^p \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$
$$u(x,t) = 0 \qquad \text{if } (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$
$$u(x,0) = u_0(x) \qquad \text{if } x \in \Omega, \tag{1.3}$$

where $p, q > 1$, Ω is a bounded domain in $\mathbb{R}^n$, with C^2 boundary, and $u_0 \geq 0$.

This problem was introduced in [3] and it was studied later in [5, 7–11, 14], for instance. The main issue in those works was to determine for which p and q blow-up in finite time (in the L^∞-norm) may occur. It turns out (see [14]) that it occurs if and only if $p > q$. Equation (1.1) in $\mathbb{R}^n$ was considered in [1, 14, 15] from a similar point of view. In this case, blow-up in finite time is also known to occur when $p > q$ (see [14]) but unbounded solutions always exist (see [15]).

The main aim of this paper is to show that if $\Omega = B_R = \{x \in \mathbb{R}^n : |x| < R\}$ and $u_0 = u_0(r)$, $u_0'(r) \leq 0$, then the estimate

$$u(r,t) \leq Cr^{-\alpha}, \quad (r,t) \in (0,R] \times [0,T) \tag{1.4}$$

holds for any $\alpha > 2/(p-1)$ if $q \in (1, 2p/(p+1))$ and for any $\alpha > q/(p-q)$ if $q \in [2p/(p+1), p)$. Let

$$\alpha_0 := \inf\{\alpha > 0 : (1.4) \text{ holds for some } C = C(\alpha) > 0\}. \tag{1.5}$$

By comparison with

$$u_t = \Delta u + u^p, \qquad p > 1, \tag{1.6}$$

and using results from [2], for instance, one can see that $\alpha_0 \geq 2/(p-1)$. For $n = 1$ and some suitably chosen initial data u_0 we shall prove that $\alpha_0 = q/(p-q)$ if $q \in (2p/(p+1), p)$. This means that the behavior of solutions at blow-up is different for equations (1.6) and (1.1) with $q \in (2p/(p+1), p)$.

For convex Ω we show that the set of blow-up points is a compact subset of Ω if $q \in (1, 2p/(p+1))$.

As far as we know, the only previous studies of the behavior of solutions of (1.1) near blow-up were performed in [4, 12, 13].

In [13], the special case $q = 2p/(p+1)$ was considered and selfsimilar solutions of (1.1) in $\mathbb{R}^n$ were studied.

In [4], the blow-up rate (in t) of solutions of (1.1) in $\mathbb{R}^n$ with the initial condition

$$u(\cdot, 0) = u_0 \geq 0, \qquad u_0, |\nabla u_0| \in L^\infty(\mathbb{R}^n),$$

was established for

$$q < \frac{2p}{p+1}, \qquad p \leq 1 + \frac{2}{n}.$$

It turns out that in this case the rate is the same as for (1.6). This means

$$u(x,t) \leq C(T-t)^{-\frac{1}{p-1}}, \tag{1.7}$$

here $T < \infty$ is the blow-up time.

It was also proved in [4] that

$$|\nabla u(x,t)| \leq C(T-t)^{-\frac{p+1}{2(p-1)}}.$$

In [12], the estimate (1.7) was established for solutions of (1.1-3) under the assumptions that Ω is a ball in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$, $(n-2)p < (n+2)$, $q < 2p/(p+1)$, u_0 is radially symmetric and nonincreasing and $\Delta u_0 - |\nabla u_0|^q + u_0^p \geq 0$.

In this paper we show that (1.7) holds for solutions of (1.1-3) if Ω is convex and bounded, $q < 2p/(p+1)$ and either $\Delta u_0 - |\nabla u_0|^q + u_0^p \geq 0$ or $p \leq 1 + 2/n$.

Most of our proofs were inspired by classical arguments from [6] where the nonlinearity did not depend on the gradient of the solution. Because of the gradient dependence, some auxiliary functions have to be chosen differently. To derive differential inequalities for them, we employ terms which were not needed in [6]. Another novelty in the proof of the main result (Theorem 2.2) is the use of a bootstrap argument.

The paper is organized as follows. In Section 2 we derive (1.4) in the radial case, in Section 3 we show that blow-up takes place away from the boundary if Ω is convex and $q < 2p/(p+1)$ and in Section 4 we study the blow-up rate in t.

2 The Radially Symmetric Case

In what follows we shall consider the more general equation

$$u_t = \Delta u - h(|\nabla u|) + f(u) \qquad \text{in } \Omega \times (0,T), \tag{2.1}$$

where

$$f \in C^1([0,\infty)),\ f(u) > 0 \quad \text{for } u > 0, \tag{2.2}$$

$$h \in C^1([0,\infty)),\ h(v) > 0,\ h'(v) \geq 0 \quad \text{for } v > 0, \tag{2.3}$$

and

$$\tilde{h}(v) := vh'(v) - h(v) \leq Kv^q, \quad \text{for } v > 0 \text{ and some } 0 \leq K < \infty,\ q > 1. \tag{2.4}$$

Consider (2.1), (1.2-3) in the case

$$\Omega = B_R = \{x \in \mathbb{R}^n : |x| < R\} \tag{2.5}$$

and assume that

$$u_0 = u_0(r), \quad \text{and} \quad u_0'(r) \leq 0 \ \text{ if } 0 < r < R. \tag{2.6}$$

Then the solution is radial and $u_r < 0$ in $(0,R) \times (0,T)$. Problem (2.1), (1.2-3) becomes

$$u_t = u_{rr} + \frac{n-1}{r}u_r - h(-u_r) + f(u) \qquad \text{in } (0,R) \times (0,T), \tag{2.7}$$

$$u_r(0,\cdot) = 0, \quad u(R,\cdot) = 0 \qquad \text{in } [0,T), \tag{2.8}$$

$$u(\cdot,0) = u_0 \qquad \text{in } [0,R]. \tag{2.9}$$

Theorem 2.1 *Let the assumptions* (2.2-6) *be satisfied. Assume further that there exist functions $F \in C^2([0,\infty))$ and $c_\varepsilon \in C^2([0,R])$, $\varepsilon > 0$, such that*

$$c_\varepsilon(0) = F(0) = 0, \quad c_\varepsilon', F' \geq 0, \quad F'' \geq 0,$$

$$f'F - fF' - 2c_\varepsilon' F'F + c_\varepsilon^2 F'' F^2 - 2^{q-1}K c_\varepsilon^q F^q F' + \left(\frac{c_\varepsilon''}{c_\varepsilon} + \frac{n-1}{r}\frac{c_\varepsilon'}{c_\varepsilon} - \frac{n-1}{r^2}\right) F \geq 0, \qquad u > 0,\ 0 < r < R, \tag{2.10}$$

$$\frac{c_\varepsilon(r)}{r} \to 0 \quad \textit{uniformly on } [0,R] \textit{ as } \varepsilon \to 0, \tag{2.11}$$

and

$$G(s) := \int_s^\infty \frac{du}{F(u)} < \infty \qquad \textit{for } s > 0.$$

If the solution u blows up in finite time T then the point $r = 0$ is the only blow-up point, and there is $\tilde{\varepsilon} > 0$ such that

$$u(r,t) \leq G^{-1}\left(\int_0^r c_{\tilde{\varepsilon}}(\rho)\, d\rho\right), \quad (r,t) \in (0,R] \times (0,T).$$

Proof. Set $w = u_r$. Differentiating (2.7) with respect to r we get

$$w_t - \frac{n-1}{r} w_r - w_{rr} = \frac{1-n}{r^2} w + f'(u)w + h'(-u_r)u_{rr}. \tag{2.12}$$

We wish to obtain an estimate from below for $-w = -u_r$ near $r = 0$. As in [6], we introduce the function

$$J = w + c_\varepsilon(r)F(u) \tag{2.13}$$

Our aim is to show that $J \le 0$ in $[0,R] \times [0,T)$. Using (2.12-13) and writing F instead of $F(u)$ we compute the equation for J:

$$\begin{aligned} J_t - \frac{n-1}{r} J_r - J_{rr} =& \frac{1-n}{r^2} w + f'(u)w + h'(-u_r)u_{rr} + c_\varepsilon F'[f(u) - h(-u_r)] \\ & - 2wc'_\varepsilon F' + F\left[\frac{1-n}{r} c'_\varepsilon - c''_\varepsilon\right] - c_\varepsilon w^2 F''. \end{aligned}$$

Using the relations $u_r = w = J - c_\varepsilon F$, $w^2 = c_\varepsilon^2 F^2 + (J - 2c_\varepsilon F)J$ and

$$u_{rr} = J_r - c'_\varepsilon F - c_\varepsilon F' u_r$$

we obtain

$$\begin{aligned} J_t &- \left(\frac{n-1}{r} + h'(-u_r)\right) J_r - J_{rr} - b_0 J \\ &= c_\varepsilon \Big\{ F\Big(-f' - \frac{c'_\varepsilon}{c_\varepsilon} h'(-u_r) + \frac{n-1}{r^2} - \frac{c''_\varepsilon}{c_\varepsilon} - \frac{n-1}{r}\frac{c'_\varepsilon}{c_\varepsilon}\Big) \\ &\quad + F'[f - u_r h'(-u_r) - h(-u_r)] + 2c'_\varepsilon F' F - c_\varepsilon^2 F'' F^2 \Big\}, \end{aligned} \tag{2.14}$$

where

$$b_0 = f' - (n-1)r^{-2} - 2c'_\varepsilon F' - c_\varepsilon F''(J - 2c_\varepsilon F).$$

From (2.4) it follows that

$$-u_r h'(-u_r) - h(-u_r) = \tilde{h}(-w) = \tilde{h}(c_\varepsilon F - J) \le 2^{q-1} K (c_\varepsilon^q F^q + |J|^q). \tag{2.15}$$

By (2.10), (2.14), (2.15) and $h' c'_\varepsilon / c_\varepsilon \ge 0$, we obtain that

$$J_t - \left(\frac{n-1}{r} J_r + h'(-u_r)\right) J_r - J_{rr} - bJ \le 0 \quad \text{in } (0,R) \times (0,T),$$

where $b = b_0 + 2^{q-1} K c_\varepsilon F' |J|^{q-2} J$. It follows that J cannot attain a positive maximum in $(0,R) \times (0,t]$ for any $t < T$.

If we choose $\tilde{\varepsilon} > 0$ such that

$$c_{\tilde{\varepsilon}}(r) \le -\frac{u'_0(r)}{\max\limits_{[0,R]} F(u_0)}$$

then $J(\cdot, 0) \le 0$ in $[0,R]$. Obviously, $J(0,\cdot) = 0$ and $J(R,\cdot) < 0$ in $(0,T)$. Hence, $J \le 0$ in $[0,R] \times [0,T)$. Integrating this inequality we obtain the conclusion. ☐

Next we consider the particular case $f(u) = u^p$, $p > 1$.

Theorem 2.2 *Let the assumptions* (2.3-6) *be satisfied. If* $f(u) = u^p$, $p > 1$, *and* u *blows up in finite time then the point* $r = 0$ *is the only blow-up point. Moreover,* (1.4) *holds for any* $\alpha > 2/(p-1)$ *if* $q \in (1, 2p/(p+1))$ *and for any* $\alpha > q/(p-q)$ *if* $q \in [2p/(p+1), p)$.

Proof. Choosing $c_\varepsilon(r) = \varepsilon r^{1+\delta}$ the assumption (2.11) is satisfied and (2.10) becomes

$$\begin{aligned} f'F - fF' - 2\varepsilon(1+\delta)r^\delta F'F + \varepsilon^2 r^{2+2\delta} F''F^2 \\ -2^{q-1}K\varepsilon^q r^{q+\delta q} F^q F' + \delta(n+\delta)r^{-2}F \geq 0. \end{aligned} \tag{2.16}$$

We shall show (2.16) for $F(u) = u^\gamma$ with some suitable $\gamma > 1$. With this choice of F, the inequality (2.16) takes the form

$$\begin{aligned} (p-\gamma)u^{p+\gamma-1} + (\varepsilon r^{1+\delta})^2 \gamma(\gamma-1)u^{3\gamma-2} + \delta(n+\delta)r^{-2}u^\gamma \\ \geq 2\varepsilon\gamma(1+\delta)r^\delta u^{2\gamma-1} + 2^{q-1}K\gamma(\varepsilon r^{1+\delta})^q u^{\gamma q+\gamma-1}. \end{aligned}$$

In the proof of this inequality, we shall need the following lemma.

Lemma 2.3 *Let* n *be a positive integer,* $R \in (0, \infty)$, $K \in (0, \infty)$ *and* $p \in (1, \infty)$.

1. If $1 < \gamma < p$ *and* $0 < \delta < \infty$ *then for* $\varepsilon > 0$ *small enough*

$$\frac{1}{2}\Big((p-\gamma)u^{p+\gamma-1} + \delta(n+\delta)r^{-2}u^\gamma\Big) \geq 2\varepsilon\gamma(1+\delta)r^\delta u^{2\gamma-1} \tag{2.17}$$

holds for every $r \in (0, R]$ *and* $u \geq 0$.

2. If $1 < q < 2p/(p+1)$, $\gamma \in (p/q, p)$ *and* $\delta \in [-1, \infty)$ *then for* $\varepsilon > 0$ *small enough*

$$\begin{aligned} \frac{1}{2}\Big((p-\gamma)u^{p+\gamma-1} + (\varepsilon r^{1+\delta})^2\gamma(\gamma-1)u^{3\gamma-2}\Big) \\ \geq 2^{q-1}K\gamma(\varepsilon r^{1+\delta})^q u^{\gamma q+\gamma-1} \end{aligned} \tag{2.18}$$

holds for every $r \in (0, R]$ *and* $u \geq 0$.

3. If $1 < q < p$, $\gamma = p/q$ *and* $\delta \in [-1, \infty)$ *then for* $\varepsilon > 0$ *small enough*

$$\frac{1}{2}(p-\gamma)u^{p+\gamma-1} \geq 2^{q-1}K\gamma(\varepsilon r^{1+\delta})^q u^{\gamma q+\gamma-1}. \tag{2.19}$$

holds for every $r \in (0, R]$ *and* $u \geq 0$.

Proof. The inequalities (2.17) and (2.18) are consequences of the Hölder inequality

$$\frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta} \geq ab, \qquad a, b \geq 0,\ \alpha, \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

where we choose $\alpha = (p-1)/(\gamma-1)$ in the case of (2.17) and $\alpha = (2\gamma-1-p)/(2\gamma-1-\gamma q)$ in the case of (2.18) (the choice of a, b is straightforward; we set $a^\alpha/\alpha = (p-\gamma)u^{p+\gamma-1}$ and $b^\beta/\beta = \delta(n+\delta)r^{-2}u^\gamma$ in (2.17), for example).

In the inequality (2.19), it is sufficient to choose

$$\varepsilon \le \left(\frac{p-\gamma}{2^q K\gamma}\right)^{\frac{1}{q}} R^{-1-\delta}. \quad \Box$$

Now we continue with the proof of Theorem 2.2. Consider first the case $q < 2p/(p+1)$. In this case we choose $F(u) = u^\gamma$, $1 < \gamma < p$ and Lemma 2.3, parts 1 and 2 yield that (2.16) holds. Theorem 2.1 implies that

$$u(r,t) \le Cr^{-\frac{2+\delta}{\gamma-1}}.$$

(Obviously, $(2+\delta)/(\gamma-1)$ $(> 2/(p-1))$ can be made arbitrarily close to $2/(p-1)$.)

Consider next the case $q \in [2p/(p+1), p)$. Now we choose $F(u) = u^\gamma$, $\gamma = p/q$ and Lemma 2.3, parts 1 and 3 yield that (2.16) holds. Theorem 2.1 implies that

$$u(r,t) \le Cr^{-\alpha}, \qquad \alpha = \alpha(\delta,\gamma) = \frac{2+\delta}{\gamma-1}.$$

The inequality (2.19) is equivalent to

$$u^{\gamma q-p} \le \frac{p-\gamma}{K\gamma}(2\varepsilon r^{1+\delta})^{-q}$$

and, due to the above estimate on u, it holds also if γ is replaced by $\tilde\gamma \in (\gamma, p)$ such that $(\tilde\gamma q - p)\alpha < (1+\delta)q$, or, equivalently,

$$\tilde\gamma < \frac{p}{q} + \frac{1+\delta}{2+\delta}(\gamma-1).$$

If δ is close to zero, this reduces to

$$\tilde\gamma < \frac{p}{q} + \frac{\gamma-1}{2}.$$

Clearly,

$$\gamma < \frac{p}{q} + \frac{\gamma-1}{2} \quad \text{if} \quad \gamma < \frac{2p}{q} - 1 \ \ (\le p),$$

and

$$\alpha(\delta,\gamma) \to \frac{q}{p-q} \quad \text{as} \quad \delta \to 0,\ \gamma \to \left(\frac{2p}{q}-1\right).$$

Consequently, an obvious bootstrap argument implies the assertion. $\Box$

Theorem 2.4 *Consider Problem* (1.1-3) *with* $n = 1$, $\Omega = (-R,R)$ *and assume that* $u_0(x) = u_0(-x) \ge 0$, $u_0'' - |u_0'|^q + u_0^p \ge 0$ *for* $|x| \le R$, $u_0' \le 0$ *for* $x \in [0,R]$. *Let the solution* u *blow up in finite time. If* $q \in (2p/(p+1), p)$ *and* α_0 *is as in* (1.5) *then*

$$\alpha_0 \ge \frac{q}{p-q}.$$

Proof. Given $\varepsilon > 0$, there is $C_1 > 0$ such that

$$u(x,t) \le C_1 x^{-(\alpha_0+\varepsilon)}. \tag{2.20}$$

By Theorem 2.2 and the definition of α_0, we obtain the existence of a sequence $\{(x_k,t_k)\}$ in $(0,R)\times(0,T)$ such that $t_k \uparrow T$, $x_k \downarrow 0$,

$$M_k := x_k^{\alpha_0-\varepsilon} u(x_k,t_k) \to \infty. \tag{2.21}$$

The estimate (2.20) implies

$$M_k \le C_1 x_k^{-2\varepsilon}. \tag{2.22}$$

Without loss of generality we may assume

$$|u_x(x_k,t_k)| \le M_k(\alpha_0-\varepsilon)x_k^{-(\alpha_0-\varepsilon+1)}. \tag{2.23}$$

(Otherwise we set

$$\tilde{x}_k := \inf\{0 < x \le x_k : |u_x(x,t_k)| > M_k(\alpha_0-\varepsilon)x^{-(\alpha_0-\varepsilon+1)}\}.$$

Then (2.23) holds for $\tilde{x}_k$ and

$$u(\tilde{x}_k,t_k) = u(x_k,t_k) - \int_{\tilde{x}_k}^{x_k} u_x(x,t_k)\,dx \ge M_k \tilde{x}_k^{\alpha_0-\varepsilon}.)$$

Set $y_k := x_k^{1-\delta}$, $\delta := 1-(\alpha_0-\varepsilon)/(\alpha_0+\varepsilon)$, then (2.20) and (2.21) yield

$$u(y_k,t_k) \le C_1 y_k^{-(\alpha_0+\varepsilon)} = C_1 x_k^{-(\alpha_0-\varepsilon)} < \frac{M_k}{2} x_k^{-(\alpha_0-\varepsilon)} = \frac{1}{2}u(x_k,t_k). \tag{2.24}$$

Since $u_t \ge 0$ and $u_x \le 0$ in $(0,R)\times(0,T)$, we have

$$\begin{aligned} \int_{x_k}^{y_k} |u_x(x,t_k)|^q\,dx &\le \int_{x_k}^{y_k} (u_{xx}(x,t_k) + u^p(x,t_k))\,dx \\ &\le -u_x(x_k,t_k) + \int_{x_k}^{y_k} u^p(x,t_k)\,dx. \end{aligned} \tag{2.25}$$

Using first (2.23), (2.20) and then (2.22), we obtain

$$\begin{aligned} -u_x(x_k,t_k) &+ \int_{x_k}^{y_k} u^p(x,t_k)\,dx \\ &\le M_k(\alpha_0-\varepsilon)x_k^{-(\alpha_0-\varepsilon+1)} + C_1\int_{x_k}^{y_k} x^{-p(\alpha_0+\varepsilon)}\,dx \\ &\le C_1(\alpha_0-\varepsilon)x_k^{-(\alpha_0+\varepsilon+1)} + \frac{C_1}{p(\alpha_0+\varepsilon)-1}x_k^{1-p(\alpha_0+\varepsilon)}. \end{aligned} \tag{2.26}$$

As we mentioned in the introduction, comparison with (1.6) yields that $\alpha_0 \geq 2/(p-1)$, hence $p(\alpha_0+\varepsilon)-1 \geq \alpha_0+\varepsilon+1$. Using this and combining (2.25), (2.26) we have

$$\int_{x_k}^{y_k} |u_x(x,t_k)|^q\,dx \leq C_2 x_k^{1-p(\alpha_0+\varepsilon)}, \qquad C_2 := C_1\left(\alpha_0-\varepsilon+\frac{1}{p(\alpha_0+\varepsilon)-1}\right). \tag{2.27}$$

By (2.21) and (2.24) (together with the definition of M_k)

$$C_2^{1/q} x_k^{-\alpha_0+\varepsilon} \leq \frac{M_k}{2} x_k^{-\alpha_0+\varepsilon} = \frac{1}{2} u(x_k,t_k) \leq \int_{x_k}^{y_k} |u_x(x,t_k)|\,dx. \tag{2.28}$$

The Hölder inequality and (2.27) imply

$$\int_{x_k}^{y_k} |u_x(x,t_k)|\,dx \leq \left(\int_{x_k}^{y_k} |u_x(x,t_k)|^q\,dx\right)^{1/q} y_k^{1/q'} \leq C_2^{1/q} x_k^{-[p(\alpha_0+\varepsilon)-q(1-\delta)-\delta]/q}. \tag{2.29}$$

From (2.28) and (2.29) it follows that

$$\frac{p(\alpha_0+\varepsilon)-q(1-\delta)-\delta}{q} \geq \alpha_0-\varepsilon.$$

If $\varepsilon \to 0$ then also $\delta \to 0$, therefore $\alpha_0 \geq q/(p-q)$. □

3 The Convex Domain Case

In this section we show the following:

Theorem 3.1 *Let Ω be a convex bounded domain. If $f(u) = u^p$ with $p > 1$ and h satisfies* (2.4) *with some $q \in \big(1, 2p/(p+1)\big)$, then the set of blow-up points of any solution of* (2.1), (1.2-3) *is a compact subset of Ω.*

Proof. We may assume without loss of generality that

$$\frac{\partial u_0}{\partial \nu} < 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

otherwise we could work with the initial value $u(\cdot,\tau)$ for any small $\tau > 0$.

Take any point $y_0 \in \partial\Omega$. Now, let the new orthonormal coordinates be chosen in such a way that y_0 is the origin and $(1,0,\ldots,0)$ is the outward normal at y_0.

Let $\Omega_a^+ := \Omega \cap \{x \in \mathbb{R}^n : x_1 > a\}$, where $a < 0$. Using standard reflection principle we easily conclude from (3.1) that

$$u_{x_1} < 0 \quad \text{on } \Omega_a^+ \times (0,T) \tag{3.2}$$

provided $|a|$ is small enough. To obtain an estimate from below on $-u_{x_1}$ in $\Omega_a^+ \times (0,T)$, we introduce a function

$$J = u_{x_1} + c(x_1)F(u) \tag{3.3}$$

in $\Omega_a^+ \times (0,T)$, where c, F are nonnegative functions to be determined, and $c' \geq 0$, $F' \geq 0$, $F'' \geq 0$.

We compute that

$$\begin{aligned} J_t - \Delta J + h'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot \nabla J - \left(f' + \frac{c'}{|\nabla u|}h'(|\nabla u|)F - 2c'F'\right) J \\ = c\Big\{F'f - f'F + [|\nabla u|h'(|\nabla u|) - h(|\nabla u|)]F' - \frac{c'}{|\nabla u|}h'(|\nabla u|)F^2 \\ - \frac{c''}{c}F + 2c'F'F - F''|\nabla u|^2\Big\}. \end{aligned} \tag{3.4}$$

If F and c satisfy

$$f'F - fF' - 2c'F'F - K|\nabla u|^q F' + \frac{c''}{c}F + |\nabla u|^2 F'' \geq 0, \tag{3.5}$$

then the right-hand side in (3.4) is nonpositive. Therefore, J cannot attain a positive maximum in $\Omega_a^+ \times (0,t]$ for any $t < T$.

Next we show that (3.5) is satisfied for

$$c(x_1) = (x_1 - a)^2 \tag{3.6}$$

with $|a|$ small enough and some suitably chosen F. Recall that $f(u) = u^p$, $1 < p$. Choosing $F(u) = u^\gamma$, $1 < \gamma < p$, it is sufficient to prove

$$\begin{aligned} (p-\gamma)u^{p+\gamma-1} - 4\gamma|a|u^{2\gamma-1} - K\gamma|\nabla u|^q u^{\gamma-1} \\ + 2|a|^{-2}u^\gamma + \gamma(\gamma-1)|\nabla u|^2 u^{\gamma-2} \geq 0. \end{aligned} \tag{3.7}$$

Similarly as in Lemma 2.3, we can prove that

$$\frac{1}{2}[(p-\gamma)u^{p+\gamma-1} + 2|a|^{-2}u^\gamma] \geq 4\gamma|a|u^{2\gamma-1}, \tag{3.8}$$

for every $u \geq 0$ provided $|a|$ is small enough. Moreover, the Hölder inequality

$$\frac{A^\alpha}{\alpha} + \frac{B^\beta}{\beta} + \frac{C^\nu}{\nu} \geq ABC, \qquad A, B, C \geq 0,\ \alpha, \beta, \nu > 1,\ \ \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\nu} = 1,$$

with the choice

$$\alpha = \frac{p-1}{q-1}, \qquad \beta = \frac{2(p-1)}{2p-(p+1)q}, \qquad \nu = \frac{2}{q},$$

$$\frac{A^\alpha}{\alpha} = (p-\gamma)u^{p+\gamma-1}, \quad \frac{B^\beta}{\beta} = 2|a|^{-2}u^\gamma \quad \text{and} \quad \frac{C^\nu}{\nu} = \gamma(\gamma-1)v^2 u^{\gamma-2}$$

implies that the following lemma holds.

Lemma 3.2 *If $1 < q < 2p/(p+1)$, $1 < \gamma < p$ and $K \in (0,\infty)$ then for $|a|$ small enough*

$$\frac{1}{2}\Big[(p-\gamma)u^{p+\gamma-1} + 2|a|^{-2}u^\gamma + \gamma(\gamma-1)v^2u^{\gamma-2}\Big] \ge K\gamma v^q u^{\gamma-1} \qquad (3.9)$$

for every nonnegative u and v.

Inequality (3.7) follows immediately from (3.8) and (3.9).

Next, observe that $J < 0$ on $\{x : x_1 = a\}$ by (3.2), and $J < 0$ on $\{t = 0\}$ by (3.1). The maximum principle yields also $J < 0$ on $\Gamma \times (0,T)$, where $\Gamma = \partial\Omega \cap \{x : x_1 > a\}$. Hence,

$$J < 0 \quad \text{in } \Omega_a^+ \times (0,T).$$

Consequently,

$$-u_{x_1} \ge (x_1 - a)^2 F(u)$$

at any (x,t) such that $x = (x_1, 0)$, $a \le x_1 < 0$. Integrating with respect to x_1 and denoting $G(s) = \int_s^\infty \frac{du}{F(u)}$, we get

$$G(u((x_1,0),t)) \ge \int_a^{x_1} c(\rho)\,d\rho = \frac{1}{3}(x_1 - a)^3$$

and therefore

$$u((x_1,0),t) \le G^{-1}\Big(\frac{1}{3}(x_1-a)^3\Big).$$

Thus, u is uniformly bounded on

$$\Big\{(x_1,0) : x_1 \in \Big[\frac{a}{2}, 0\Big]\Big\} \times (0,T).$$

The above proof shows that a can be chosen independently of the initial point $y_0 \in \partial\Omega$. Hence, by varying y_0 along $\partial\Omega$ we conclude that there is a neighborhood Ω' of $\partial\Omega$ (in Ω) such that u is uniformly bounded in $\Omega' \times (0,T)$. □

Using the scaling argument from [4] it is not difficult to see that Theorem 3.1 implies the following:

Corollary 3.3 *Let Ω be a convex bounded domain. If $f(u) = u^p$, $1 < p \le 1 + 2/n$ and h satisfies* (2.4) *with some $q \in (1, 2p/(p+1))$, then any solution u of* (2.1), (1.2-3) *which blows up at $t = T$ satisfies*

$$u(x,t) \le C(T-t)^{-\frac{1}{p-1}}.$$

4 Blow-Up Rate

Theorem 4.1 *Let Ω be a convex bounded domain and let $u_0 \in C^2(\overline{\Omega})$ be such that*

$$\Delta u_0 + f(u_0) - h(|\nabla u_0|) \geq 0 \quad \text{in } \Omega.$$

If $f(u) = u^p$ with $p > 1$ and h satisfies (2.4) with some $q \in (1, 2p/(p+1))$, then any solution u of (2.1), (1.2-3) which blows up at $t = T$ satisfies

$$u(x,t) \leq C(T-t)^{-\frac{1}{p-1}}.$$

Proof. For any $\eta > 0$ small enough, set

$$\Omega^\eta = \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > \eta\}.$$

We shall now derive a lower bound on u_t away from the parabolic boundary of $\Omega \times (0,T)$.

As in [6] we introduce the function

$$J = u_t - \delta F(u),$$

where $\delta > 0$ and F is a nonnegative function to be determined, $F' \geq 0$, $F'' \geq 0$. Since $u_t = \delta F + J$,

$$\begin{aligned}
J_t - \Delta J &= u_{tt} - \Delta u_t - \delta F'[u_t - \Delta u] + \delta F''|\nabla u|^2 \\
&= f'[J + \delta F] - \frac{h'(|\nabla u|)}{|\nabla u|}\nabla u \cdot (\nabla J + \delta F' \nabla u) \\
&\quad - \delta F'[f - h(|\nabla u|)] + \delta F''|\nabla u|^2 \\
&= f'J - \frac{h'(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla J + \delta f'F - \delta F'[h'(|\nabla u|)|\nabla u| - h(|\nabla u|)] \\
&\quad - \delta F'f + \delta F''|\nabla u|^2.
\end{aligned}$$

Our aim is to have

$$f'F - F'f + F''|\nabla u|^2 - F'[h'(|\nabla u|)|\nabla u| - h(|\nabla u|)] \geq 0\,. \tag{4.1}$$

It will then follow that J cannot attain a negative minimum in $\Omega \times (0,t]$ for any $t < T$.

By Theorem 3.1, the set of blow-up points is a compact subset of Ω. Therefore, if $\eta > 0$ is small enough then

$$F(u) \leq C_0 < \infty \quad \text{if } x \in \partial\Omega^\eta,\ 0 < t < T.$$

Applying the maximum principle to u_t we also have $u_t \geq c > 0$ on the parabolic boundary of $\Omega^\eta \times (\eta, T)$. It follows that $J > 0$ on the parabolic boundary of $\Omega^\eta \times (\eta, T)$ provided δ is chosen sufficiently small and, consequently, $J > 0$ in $\Omega^\eta \times (\eta, T)$. Hence

$$\frac{u_t}{F(u)} \geq \delta \quad \text{in } \Omega^\eta \times (\eta, T). \tag{4.2}$$

Let $G(s) = \int_s^\infty \frac{du}{F(u)}$. Then (4.2) implies that

$$-\frac{dG(u)}{dt} = \frac{u_t}{F(u)} \geq \delta$$

or, by integration,

$$G(u(x,t)) \geq G(u(x,t)) - G(u(x,T)) \geq \delta(T-t).$$

Therefore also

$$u(x,t) \leq G^{-1}(\delta(T-t)), \quad (x,t) \in \Omega^\eta \times (\eta, T). \tag{4.3}$$

This gives an upper bound on the blow-up rate as $t \uparrow T$.

Since $f(u) = u^p$ with $p > 1$, we can choose

$$F(u) = u^p + K_\varepsilon u^{p-\varepsilon} + C_\varepsilon$$

with $\varepsilon > 0$ small enough and some $K_\varepsilon, C_\varepsilon > 0$ and (4.1) will be satisfied. For this choice of F, inequality (4.3) takes the form as claimed. □

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6 References

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