Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Application of The Method of Moving Planes to Conformally Invariant Equations

by

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Preprint no.: 96 2003

APPLICATION OF THE METHOD OF MOVING PLANES TO CONFORMALLY INVARIANT EQUATIONS

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1. Introduction

The main theme of this paper is the application of the method of moving planes to conformally invariant fully nonlinear elliptic equations. Throughout this paper, we assume that (M, g) is a smooth, compact locally conformally flat Riemannian manifold of dimension $n \geq 3$ and $[g]$ denotes the conformal class of g. The Schouten tensor of the metric g is defined by

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}g\right),$$

where Ric_g and R_g are the Ricci tensor and scalar curvature of g respectively. We want to investigate the following conformally invariant equation:

$$f(\lambda(S_{\hat{g}})) = 1. \tag{1.1}$$

where $\hat{g} \in [g]$, $S_{\hat{g}}$ is the Schouten tensor of $\hat{g}$, $\lambda(S_{\hat{g}})$ is the set of the eigenvalues of $S_{\hat{g}}$ with respect to $\hat{g}$, and f is a certain function on symmetric matrices we will specify. If we write $\hat{g} = u^{\frac{4}{n-2}} g$ for some positive smooth function u, the Schouten tensor $S_{\hat{g}}$ can be computed as

$$S_{\hat{g}} = -\frac{2}{n-2}u^{-1}\nabla_g^2 u + \frac{2n}{(n-2)^2}u^{-2}\nabla_g u \otimes \nabla_g u - \frac{2}{(n-2)^2}u^{-2}|\nabla_g u|^2 g + S_g. \tag{1.2}$$

Equation (1.1) is indeed a second order nonlinear differential equation on u.

The most important case is $f = \sigma_k$, where σ_k is the k-th elementary symmetric function,

$$\sigma_k(\lambda(S_{\hat{g}})) = 1, \tag{1.3}$$

for $\hat{g} \in [g]$. We note that if $k = 1$, equation (1.3) is the Yamabe equation. When $k > 1$, it is fully nonlinear. By the fundamental work of Caffarelli-Nirenberg-Spruck [1], equation (1.3) is elliptic in certain cone Γ_k^+ defined by Garding [7]. We recall

$$\Gamma_k^+ = \{\lambda \in \mathbf{R}^n \,|\, \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}.$$

Let

$$\mathcal{C}_k = \{\hat{g} \in [g] \,|\, \lambda(S_{\hat{g}})(x) \in \Gamma_k^+, \forall x \in M.\}$$

We say $\hat{g} \in [g]$ is admissible to equation (1.3) if $\hat{g} \in \mathcal{C}_k$.

The study of equation (1.3) was initiated in [24] by Viaclovsky. There have been many activities recently (e.g., see [24, 25, 26, 2, 3, 11, 12, 19, 15, 16]). Except [2], all the existence

Research of the first author was supported in part by NSERC Grant OGP-0046732.

results obtained so far for equation (1.3) are under the assumption of $\mathcal{C}_k \neq \emptyset$. In the case $n = 4, k = 2$, Chang-Gursky-Yang [2] proved that $\mathcal{C}_2 \neq \emptyset$ if the Yamabe constant Y_1 and $\int_M \sigma_2(g^{-1}S_g)dvol(g)$ (which is a conformal invariant when $n = 4$) are positive. Here we generalize their result to higher dimensions for locally conformally flat manifolds.

Recall the Yamabe constant of $[g]$ can be defined as

$$Y_1([g]) = \inf_{g\in[g]} (vol(g))^{-\frac{n-2}{n}} \int_M \sigma_1(g)dvol(g).$$

We define a sequence of conformal invariants for $2 \le l \le n/2$ by letting

$$Y_l = \begin{cases} \inf\limits_{g\in\mathcal{C}_{l-1}} (vol(g))^{-\frac{n-2l}{n}} \displaystyle\int_M \sigma_l(g)dvol(g) & \text{if } \mathcal{C}_{l-1} \neq \emptyset, \\ -\infty, & \text{if } \mathcal{C}_{l-1} = \emptyset. \end{cases}$$

We note that if $l \ge n/2$ and $\mathcal{C}_l \neq \emptyset$, then (M, g) is conformally equivalent to a spherical space form by Theorem 1 in a recent paper jointly with Viaclovsky [10]. Also in view of Theorem 1 in [13], only the case $l \le n/2$ is of interest to us as long as Y_l is concerned on the locally conformally flat manifolds.

Theorem 1.1. *Let (M, g) be a compact locally conformal flat n-dimensional manifold and $k \le n/2$. Assume that $Y_k([g]) > 0$, then $\mathcal{C}_k \neq \emptyset$ and equation (1.3) has an admissible solution $g \in \mathcal{C}_k$. If $Y_k([g_0]) = 0$, then either there is $g \in \mathcal{C}_k$ such that $\sigma_k(g) = 1$, or there is $g \in C^{1,1}$ in $\bar{\mathcal{C}}_k$ such that $\sigma_k(g) = 0$.*

Theorem 1.1 has the following consequence.

Corollary 1.1. *Let (M, g) be an n-dimensional compact, oriented and connected locally conformally flat manifold and $n = 2m$. If $\mathcal{C}_{m-1} \neq \emptyset$ and*

$$\int_M \sigma_m(g)dvol(g) > 0, \tag{1.4}$$

then (M, g_0) is conformally equivalent to $\mathbf{S}^{2m}$.

When $n = 4$, Corollary 1.1 was proved in [14]. A similar result was obtained for $n = 6$ in [14] under a weaker condition. Note that $\int_M \sigma_m(g)dvol(g)$ is a topological invariant. The product metric g of $\mathbf{S}^{n-1} \times \mathbf{S}$ is in $\mathcal{C}_{m-1}$ with $\int \sigma_m(g)dvol(g) = 0$.

Theorem 1.1 is proved through the establishment of certain global a priori estimates using the method of moving planes and the fundamental result on Schoen-Yau [23] on developing maps for locally conformally flat manifolds. The key point of this paper is that the method of moving planes is particularly appropriate for conformally invariant equations. This leads us to consider a general equation (1.1). Let us first specify conditions on f so that (1.1) is elliptic. Let Γ be an open symmetric convex cone in $\mathbf{R}^n$, that is, for $\lambda \in \Gamma$ and any permutation σ, $\sigma \cdot \lambda = (\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)}) \in \Gamma$. It is clear that $(1, 1, \cdots, 1) \in \Gamma$. Set $\tilde{\Gamma} = \{S \mid S \text{ is a symmetric matrix whose eigenvalues } (\lambda_1, \cdots, \lambda_n) \in \Gamma\}$. We assume

$$\tilde{\Gamma} \text{ is convex and } \Gamma \subseteq \Gamma_1^+. \tag{1.5}$$

where $\Gamma_1^+ = \{\lambda \mid \sum_{j=1}^n \lambda_j > 0\}$.

Since the regularity of f is not an issue here, throughout this paper, we assume that f is a smooth function defined in $\Gamma \subseteq \Gamma_1^+$, and satisfies

$$\frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \quad \text{for} \quad i = 1, 2, \ldots, n \quad \text{and} \quad \lambda \in \Gamma. \tag{1.6}$$

It is easy to check condition (1.6) implies that f is elliptic in $\tilde{\Gamma}$. A metric $\hat{g}$ is called *admissible* if $\hat{g}^{-1} S_{\hat{g}} \in \tilde{\Gamma}$ for every point in M. This is equivalent to say that $\lambda(S_{\hat{g}}) \in \Gamma$ for every point in M. We further assume a concavity condition on f:

$$f \quad \text{is concave in } \Gamma. \tag{1.7}$$

From a result in section 3 in [1], f is concave in Γ implies f is concave in $\tilde{\Gamma}$. If there is no confusion, we will also simply write Γ for $\tilde{\Gamma}$ in the rest of the paper. Since we are concerned with equation (1.1), it is necessary that there is $\gamma \in \Gamma$ such that $f(\gamma) = 1$. The symmetry and the concavity of f imply $f(t, \cdots, t) \geq 1$ for some $t > 0$. Therefore, we assume

$$f(t_0, \cdots, t_0) = 1, \qquad \text{for some} \quad t_0 > 0. \tag{1.8}$$

Our next result is concerned with a Harnack type inequality.

Theorem 1.2. *Suppose that f satisfies (1.6), (1.7) and (1.8). Then there exists a constant $C > 0$ such that for any admissible solution $u^{\frac{4}{n-2}}|dx|^2$ of (1.1) in a open ball B_{3R}, we have*

$$\max_{B_R} u(x) \cdot \min_{B_{2R}} u(x) \leq \frac{C}{R^{n-2}}. \tag{1.9}$$

As an application, the following global regularity and existence for equation (1.1) on a general compact locally conformally flat manifold (M, g) will be proved via fundamental work of Schoen-Yau on developing maps in [23]. Here, we need an additional condition:

$$\overline{\lim}_{t \to +\infty} f(t\gamma) > 1, \text{ for all } \gamma \in \Gamma \text{ and } \overline{\lim}_{\lambda \to p} f(\lambda) < 1, \text{ for all } p \in \partial\Gamma. \tag{1.10}$$

We note that (1.10) implies (1.8).

Theorem 1.3. *Let (M, g) be an n-dimensional smooth compact locally conformally flat manifold with g admissible. Suppose that f satisfies (1.6), (1.7) and (1.10), and (M, g) is not conformally equivalent to the standard n-sphere. Then there exists a positive constant $C > 0$, such that*

$$\|u\|_{C^3} + \|u^{-1}\|_{C^3} \leq C. \tag{1.11}$$

Furthermore, there is a smooth admissible solution $u^{\frac{4}{n-2}} g$ satisfying equation (1.1).

Such type of inequality in Theorem 1.2 was first discovered by Schoen for the Yamabe problem. A different proof was given by Chen and Lin [4]. We follow the argument of [4, 5] by employing the method of moving planes here. The inequality was proved for $f = \sigma_k$ by Li-Li in [19]. They tried to generalize it under some cumbersome conditions, see Remark 2.1. Recently, Li-Li [20] announced similar results of Theorem 1.2 and Theorem 1.3.

The paper is organized as follows. In section 2, we prove the Harnack type inequality via the method of moving planes. The method of moving planes indeed are our main theme in this paper. A global gradient estimate (Proposition 3.1) for locally conformally flat manifolds not conformal to $\mathbf{S}^n$ is also obtained by this method in section 3. Here, we make use of the fundamental work of Schoen-Yau [23] on the developing maps on locally conformally flat manifolds. Theorem 1.1 and Theorem 1.3 will be proved in section 3.

Acknowledgement. Part of the work has been done while the third author was visiting the National Center for Theoretical Science (CTS) in Taiwan. He would like to thank CTS for the invitation and kind hospitality.

After this paper was completed, we received the preprint [17] of Gursky-Viaclovsky. They treated a fully nonlinear equation in 4-manifolds and obtained various results. Among them, they gave a more direct proof of the main result of [2].

2. A Harnack type inequality

Theorem 1.2 will be proved by contradiction. Before going to the proof, we want to give a sketch of our idea first. Suppose that the inequality does not hold. Then there exists a sequence of blowup solutions for equation (1.1). We then rescale the solutions. The main step is to give C^1 estimates for these rescaled solutions. Actually, the C^1-norm of the rescaled solution will be proved to be uniformly small, and then the C^2 estimates or higher-order derivatives follows by the concave assumption accordingly. Therefore, the rescaled solutions converges to a constant in $C^{2,\alpha}$ and that will yield a contradiction to assumptions (1.6) and (1.8).

Obviously, the crucial step is the C^1 estimate of those rescaled solutions. Here, the method of moving planes will be employed to obtain a local gradient estimates. As in previous works, we first extend our rescaled solutions to the whole space $\mathbf{R}^n$, and obtain a viscosity super-solution. Then, we apply the Kelvin transformation twice on those extended super-solutions. Finally the local gradient estimates follow from the application of the method of moving planes.

It seems a new idea to obtain the local gradient estimates via the method of moving planes for the fully nonlinear elliptic equation. For geometric fully nonlinear elliptic equation with the concave assumption, the local gradient estimate is generally the crucial step to obtain the a priori bound for solutions. Here, our proof relies on the conformal invariance of the equation. This leads us to suspect that for conformally invariant fully nonlinear elliptic equation, the concave assumption alone should be enough for the a priori bound. This is partially confirmed in our proof of Theorem 1.2 here. We shall study this problem for general manifolds later.

Since we use Kelvin transformations repeatedly in our proof, we shall keep our notations as clean as possible.

Suppose u is a C^2 function. Recall that the Schouten tensor $S(u)$ related to the metric $u^{\frac{4}{n-2}}|dx|^2$ is the matrix whose (i,j)-th component is defined by

$$S_{ij}(x) = u^{-\frac{4}{n-2}}\left(-\frac{2}{n-2}u^{-1}u_{x_ix_j} + \frac{2n}{(n-2)^2}u^{-2}u_iu_j - \frac{2}{(n-2)^2}u^{-2}|\bigtriangledown u|^2\delta_{ij}\right).$$

Let $\lambda(S(u))(x) = (\lambda_1, \ldots, \lambda_n)$ denote the eigenvalues of $(S_{ij}(x))$. Assume that u satisfies

$$\left\{\begin{array}{l} f(\lambda(S(u)))(x) = 1 \\ \lambda(S(u))(x) \in \Gamma \quad \text{for} \quad x \in B_{3R}(0), \end{array}\right. \tag{2.1}$$

where $B_r(p)$ is the open ball with center p and radius $r > 0$. Sometimes, we denote $B_r(p)$ by $B(p,r)$.

Proof of Theorem 1.2. By scaling invariance of the equation, we may assume $R = 1$. Inequality (1.9) will be proved by contradiction. Suppose it does not hold. Then there exists a sequence of solutions u_i such that

$$\max_{B_1} u_i \cdot \min_{B_2} u_i \geq i. \tag{2.2}$$

Let

$$M_i = \max_{\bar{B}_1} u_i = u_i(\bar{x}_i),$$

and $x_i \in B_1$ with $\bar{B}(x_i, r_i) \subset \bar{B}_1$ and $|x_i - \bar{x}_i| = r_i$, where $r_i = M_i^{-\frac{2}{n-2}}$. By (2.2), $M_i \to +\infty$ as $i \to +\infty$. Set

$$\left\{\begin{array}{l} v_i(y) = M_i^{-1} u_i(x_i + M_i^{-\frac{2}{n-2}} y) \\ \bar{x}_i = x_i + M_i^{\frac{-2}{n-2}} \bar{y}_i. \end{array}\right. \tag{2.3}$$

Then $|\bar{y}_i| = 1$ and v_i satisfies

$$\left\{\begin{array}{l} f(\lambda(S(v_i))(x)) = 1, \\ \lambda(S(v_i))(x) \in \Gamma, \end{array}\right. \tag{2.4}$$

for $|x| < M_i^{\frac{2}{n-2}}$.

For simplicity, we let $L_i = M_i^{\frac{2}{n-2}}$ and choose $l_i \to +\infty$ as $i \to +\infty$ such that

$$l_i^2 < L_i, \tag{2.5}$$

and

$$l_i^{n-2} < i. \tag{2.6}$$

We extend v_i to $\mathbf{R}^n$ via the Kelvin transformation, i.e., $\tilde{v}_i(y)$ is defined by

$$\tilde{v}_i(y) = \left(\frac{l_i}{|y|}\right)^{n-2} v_i\left(\frac{l_i^2 y}{|y|^2}\right) \quad \text{for} \quad |y| \geq l_i.$$

Then $\tilde{v}_i(y)$ also satisfies (2.4) for $|y| \geq l_i$, since equation (2.4) is conformally invariant.

For $|y| = L_i$, we have $|\frac{l_i^2 y}{|y|^2}| \leq 1$ and then, $\tilde{v}_i(y) \leq \left(\frac{l_i}{L_i}\right)^{n-2}$. On the other hand, by (2.6), $v_i(y)$ satisfies,

$$v_i(y) \quad \geq \quad M_i^{-1} \inf_{B_2(0)} u_i \geq \frac{i}{M_i^2} = \frac{i}{L_i^{n-2}} > \left(\frac{l_i}{L_i}\right)^{n-2}.$$

Therefore,

$$v_i(y) > \tilde{v}_i(y) \quad \text{for} \quad |y| = L_i. \tag{2.7}$$

Set

$$\tilde{\tilde{v}}_i(y) = \begin{cases} v_i(y) & |y| \le l_i, \\ \min(v_i(y), \tilde{v}_i(y)) & l_i \le |y| \le L_i, \\ \tilde{v}_i(y) & |y| \ge L_i. \end{cases} \tag{2.8}$$

By (2.7), $\tilde{\tilde{v}}_i$ is a continuous function defined in the whole space $\mathbf{R}^n$ and one may try to prove that $\tilde{\tilde{v}}_i$ is a viscosity super-solution. But, we will not pursue this fact in our proof. We will rather keep both v_i and $\tilde{v}_i(y)$ as solutions of (2.4) in the regions $\{y \,|\, |y| < L_i\}$ and $\{y \,|\, |y| \ge l_i\}$ respectively.

In what follows, we want to prove the first derivatives of v_i are uniformly small in the ball $B(\bar{y}_i, \frac{1}{2})$. In fact, we will prove for each $j = 1, 2, \ldots, n$ and $\delta > 0$,

$$\left|\frac{\partial v_i(y)}{\partial y_j}\right| \le \delta \left\{ 1 + \sum_{k \ne j} \sup_{y \in B(\bar{y}_i, \frac{1}{2})} \left|\frac{\partial v_i}{\partial y_k}(y)\right| \right\} \tag{2.9}$$

for all $i \ge i_0 = i_0(\delta)$, and $|y - \bar{y}_i| \le \frac{1}{2}$. Without loss of generality, we may assume $j = 1$, and $\bar{y}_i = (-1, 0, \ldots, 0)$. To obtain (2.9), we apply the Kelvin transformation twice on $\tilde{\tilde{v}}_i$. In the rest of the proof, in order to keep the simplicity, we will abuse some notations if there is no confusion. For any small δ, we first make the inversion T_1 with respect to the ball $B_1(e_\delta)$ and denote the Kelvin transformation of $\tilde{\tilde{v}}_i$ by u_i, that is,

$$u_i(x) = |x - e_\delta|^{2-n} \tilde{\tilde{v}}_i \left(\frac{x - e_\delta}{|x - e_\delta|^2} + e_\delta \right), \tag{2.10}$$

where $e_\delta = (\delta^2, 0, \ldots, 0)$. From now on, u_i will be the one defined in (2.10). So $u_i(x)$ satisfies (2.4) except the small ball $\{x \mid |x - e_\delta| < 2l_i^{-1}\}$. We choose i large so that the small ball is contained in the ball $B(e_\delta, \frac{1}{2}\delta^2)$. We also denote Y and $\tilde{Y}$ as the image of $\{y \mid l_i \le |y| \le L_i\}$ and $\{y \mid |y| \ge L_i\}$ under the inversion T_1. Next, we denote T_2 to be the inversion $x \to \frac{x}{|x|^2}$, and $u_i^*(y)$ to be the corresponding Kelvin transform, that is,

$$u_i^*(y) = |y|^{2-n} \left| \frac{y}{|y|^2} - e_\delta \right|^{2-n} \tilde{\tilde{v}}_i \left(\frac{\frac{y}{|y|^2} - e_\delta}{|\frac{y}{|y|^2} - e_\delta|^2} + e_\delta \right). \tag{2.11}$$

We also denote Z and $\tilde{Z}$ to be the image of Y and $\tilde{Y}$ under T_2 respectively. Clearly, Z and $\tilde{Z}$ lie in a small ball with center $(\frac{1}{\delta^2}, 0, \ldots, 0)$. Note that the composition $T_2 \circ T_1(y) \to y$ in C^2 for $\bar{B}(\bar{y}_i, \frac{1}{2})$ as $\delta \to 0$. Hence

$$\frac{\partial}{\partial y_1} \left(\frac{\frac{y}{|y|^2} - e_\delta}{|\frac{y}{|y|^2} - e_\delta|^2} + e_\delta \right) = (1, 0, \cdots, 0) + O(\delta^2), \tag{2.12}$$

and

$$\frac{\partial}{\partial y_1}\left(|y|^{2-n}\left|\frac{y}{|y|^2}-e_\delta\right|^{2-n}\right)=O(\delta^2) \tag{2.13}$$

for $y\in B(\bar{y}_i,\frac{1}{2})$. Both (2.12) and (2.13) can be computed by straightforward way.

Now we fix i and δ and apply the method of moving planes to u_i^*. For any $\lambda\in\mathbf{R}$ we set $\Sigma_\lambda=\{y\mid y_1>\lambda\}$ and y^λ to denote the reflection of $y\in\Sigma_\lambda$ with respect to the hyperplane $y_1=\lambda$. $u_i^*(y)$ has a harmonic expansion at ∞. We list here for the convenience of reference (see [8]).

$$\begin{aligned} u_i^* &= \frac{1}{|x|^{n-2}}(a_0+\frac{a_jx_j}{|x|^2}+\frac{a_{jk}x_jx_k}{|x|^4})+o(\frac{1}{|x|^n}),\\ (u_i^*)_j &= \frac{-a_0(n-2)x_j}{|x|^n}+\frac{a_j}{|x|^n}-\frac{n(\sum_l a_lx_l)x_j}{|x|^{n+2}}+\frac{2\sum_l a_{jl}x_l}{|x|^{n+2}}\\ &\quad -\frac{(n+2)(\sum a_{lk}x_lx_k)x_j}{|x|^{n+4}}+o(\frac{1}{|x|^{n+1}}).\end{aligned} \tag{2.14}$$

Set $y=x-x_0$, where $x_0=(-\frac{a_1}{(n-2)a_0},\cdots,-\frac{a_n}{(n-2)a_0})$. Then (2.14) is reduced to

$$u_i^*(x)=\frac{a_0}{|y|^{n-2}}+\frac{\tilde{a}_{jk}y_jy_k}{|y|^n}+o(|y|^{-n}),\quad (u_i^*)_{x_j}(x)=\frac{-a_0(n-2)y_j}{|y|^n}+O(|y|^{-n-1}).$$

As a consequence of the previous expansion, we have (for the proof see [8])

Lemma 2.1. *For any $\lambda<\frac{a_1}{(n-2)a_0}$, there exists $R=R(\lambda)$ depending only on* $\min(1+|a_1|,\lambda)$ *such that for $x=(x_1,y')$ and $y=(y_1,y')$ satisfying*

$$x_1<y_1, x_1+y_1\le 2\lambda, |y|\ge R$$

we have

$$u_i^*(x)<u_i^*(y).$$

Before we start the process of moving planes by using Lemma 2.1, we note that a_0,a_j and R in the Lemma could be large, because it also depends on i and δ. By our construction, $u_i^*(y)$ is a positive C^2 function except at $Z\cup\tilde{Z}$. But $u_i^*(y)$ is a super-harmonic function in the distribution sense. Therefore, for any small neighborhood N of $Z\cup\tilde{Z}$,

$$u_i^*(y)\ge\inf_{\partial N}u_i^*\ge c_0=c_0(i,\delta)>0 \tag{2.15}$$

for $y\in\bar{N}$. Thus, by Lemma 2.1, λ can be chosen negatively large so that

$$u_i^*(y^\lambda)<u_i^*(y)\quad\text{for}\quad y\in\Sigma_\lambda. \tag{2.16}$$

As usual, we set

$$\lambda_0=\sup\{\lambda\mid u_i^*(y^{\lambda'})<u_i^*(y)\ \text{ for } y\in\Sigma_{\lambda'} \text{ and } \lambda'<\lambda\}.$$

We claim if δ is small enough, then

$$\lambda_0\ge\min\left(-\frac{1}{4},\frac{a_1}{(n-2)a_0}\right).$$

Clearly, by the continuity, we have

$$w_{\lambda_0}(y) := u_i^*(y) - u_i^*(y^{\lambda_0}) \geq 0 \quad \text{for} \quad y \in \Sigma_{\lambda_0}. \tag{2.17}$$

We claim

$$w_{\lambda_0}(y) > 0 \quad \text{for} \quad y \in \Sigma_{\lambda_0}. \tag{2.18}$$

Recall that $w_{\lambda_0}(y)$ is continuous in $\bar{\Sigma}_{\lambda_0}$ and is C^2 in $\Sigma_{\lambda_0}\backslash(Z\cup\tilde{Z})$. Now suppose $y_0 \in \Sigma_{\lambda_0}$ such that

$$w_{\lambda_0}(y_0) = 0. \tag{2.19}$$

If $y_0 \notin Z \cup \tilde{Z}$, by the strong maximum principle $w_{\lambda_0}(y) \equiv 0$ for $y \notin Z \cup \tilde{Z}$. Let $v_i^*(y)$ denote the double Kelvin transformation of $v_i(y)$ through the conformal mapping $T_2 \circ T_1$. Note that

$$v_i^*(y) = u_i^*(y) \quad \text{for} \quad y \in \mathbf{R}^n\backslash Z \cup \tilde{Z},$$

where $\mathbf{R}^n\backslash\tilde{Z}$ is *connected*. Since $w_{\lambda_0}(y) \equiv 0$ for $y \notin Z \cup \tilde{Z}$, by the unique continuation, we have

$$v_i^*(y^{\lambda_0}) = v_i^*(y) \quad \text{for} \quad y \in \Sigma_{\lambda_0}\backslash\tilde{Z}. \tag{2.20}$$

For $y \in Z$, by (2.20) and (2.17),

$$v_i^*(y^{\lambda_0}) = v_i^*(y) \geq u_i^*(y) \geq u_i^*(y^{\lambda_0}) = v_i^*(y^{\lambda_0}). \tag{2.21}$$

Thus, $v_i^*(y) = u_i^*(y)$ for $y \in Z$, which implies

$$v_i(y) \leq \tilde{v}_i(y) \quad \text{for} \quad l_i \leq |y| \leq L_i. \tag{2.22}$$

By (2.7), this is a contradiction. Thus, $y_0 \in Z \cup \tilde{Z}$.

If $y_0 \in Z$ and $v_i(y_0) \leq \tilde{v}_i(y_0)$, then $v_i(y_0) = v_i(y_0^{\lambda_0})$ and by (2.17), $v_i^*(y) \geq u_i^*(y) \geq u_i^*(y^{\lambda_0}) = v_i^*(y^{\lambda_0})$ for $y \in \Sigma_{\lambda_0}\backslash\tilde{Z}$. Thus, the strong maximum principle again yields

$$v_i^*(y) = v_i^*(y^{\lambda_0}) \quad \text{for} \quad y \in \Sigma_{\lambda_0}\backslash\tilde{Z}.$$

And it is reduced to the previous case. Thus, $v_i(y_0) > \tilde{v}_i(y_0)$. Set $\tilde{v}_i^*(y)$ be the corresponding double Kelvin transformation of $\tilde{v}_i$. Clearly, $\tilde{v}_i^*(y)$ is defined only on $Z \cup \tilde{Z}$. By (2.17), $\tilde{v}_i^*(y) \geq u_i^*(y^{\lambda_0})$ for $y \in \tilde{Z}$ and the equality holds at y_0, which implies

$$\tilde{v}_i^*(y) = u_i^*(y^{\lambda_0}) \quad \text{in} \quad \tilde{Z}. \tag{2.23}$$

Therefore

$$\tilde{v}_i(y) \leq v_i(y) \quad \text{for} \quad l_i \leq |y| \leq L_i.$$

But $\tilde{v}_i(y) = v_i(y)$ for $|y| = l_i$. Hence (2.23) yields $u_i^*(y) = u_i^*(y^{\lambda_0})$ for $y \in \partial(\tilde{Z} \cup Z)$, which is reduced to the previous case. Therefore $y_0 \notin Z$. But $y_0 \in \tilde{Z}$ also leads to (2.23) by the strong maximum principle, which in turn yields a contradiction again. Hence the claim (2.18) is proved.

Once (2.18) is established, it is easy to see $\lambda_0 \geq \min(-\frac{1}{4}, \frac{a_1}{(n-2)a_0})$ follows from Lemma 2.1 by the standard argument of the method of moving planes. We omit the details here.

By the Hopf boundary lemma, we have

$$\frac{\partial}{\partial y_1} u_i^*(y) \geq 0 \quad \text{for} \quad y_1 \leq \min(-\frac{1}{4}, \frac{a_1}{(n-2)a_0}).$$

We want to prove $\frac{\partial}{\partial y_1} u_i^*(y) > 0$ for $y_1 \leq -\frac{1}{4}$. If not, then there exists $y_0 = (y_{0,1}, y_0')$ such that $y_{0,1} \leq -\frac{1}{4}$ and $\frac{\partial}{\partial y_1} u_i^*(y_0) = 0$. Then we do the Kelvin transformation u_i^{**} as,

$$u_i^{**}(y) = \left(\frac{r_0}{|y|}\right)^{n-2} u_i^* \left(\frac{r_0^2 y}{|y|^2} + y_0\right), \tag{2.24}$$

where $r_0 = \frac{1}{2}|y_0|$. Obviously, the singular set of u_i^{**} is in the half-space $\{y \mid y_1 > 0\}$. Then we can apply the method of moving planes to show

$$u_i^{**}(y^\lambda) < u_i^{**}(y) \quad \text{for } y \in \Sigma_\lambda \text{ and } \lambda < 0, \tag{2.25}$$

by Lemma 2.1 and by the fact $\frac{\partial u_i^*}{\partial y_1}(y_0) = 0$. The same argument as the proof of (2.18) yields that (2.25) holds for $\lambda = 0$ too. This implies

$$u_i^*(y^\lambda) < u_i^*(y) \quad \text{for } y \in \Sigma_\lambda \text{ and } \lambda = y_{0,1}.$$

But it yields a contradiction to $\frac{\partial}{\partial y_1} u_i^*(y_0) = 0$. Hence $\frac{\partial}{\partial y_1} u_i^*(y) > 0$ for $y_1 \leq -\frac{1}{4}$.

By the expression of (2.11), using (2.12) and (2.13), we then have

$$-\frac{\partial}{\partial y_1}\tilde{\tilde{v}}_i(y) \leq O(\delta^2)\tilde{\tilde{v}}_i(y) + O(\delta^2)\sum_{k=2}^n \left|\frac{\partial}{\partial y_k}\tilde{\tilde{v}}_i\right| \tag{2.26}$$

for $|y - \bar{y}_i| \leq \frac{1}{2}$. We can repeat the process by taking $e_\delta = (-\delta^2, 0, \ldots, 0)$. In this case, u_i^* has singularity near $(-\frac{1}{\delta^2}, 0, \ldots, 0)$. So, we can move the plane from the right-hand side and obtain the following inequality,

$$\frac{\partial}{\partial y_1}\tilde{\tilde{v}}_i(y) \leq O(\delta^2)\tilde{\tilde{v}}_i(y) + O(\delta^2)\sum_{k=2}^n \left|\frac{\partial}{\partial y_k}\tilde{\tilde{v}}_i\right| \tag{2.27}$$

for $|y - \bar{y}_i| \leq \frac{1}{2}$. Note that $v_i(\bar{y}_i) = \max_{|y|\leq 1} v_i(y) = 1$. Since u_i^* is increasing in y_1, we obtain

$$v_i(y) \leq 2 \quad \text{for} \quad |y - \bar{y}_i| \leq \frac{1}{2}. \tag{2.28}$$

Thus, together with (2.26) and (2.27), (2.28) yields

$$\left|\frac{\partial}{\partial y_1} v_i(y)\right| \leq O(\delta^2)\left(1 + \sum_{k=2}^n \frac{\partial}{\partial y_k} v_i(y)\right)$$

for $|y - \bar{y}_i| \leq \frac{1}{2}$. Therefore (2.9) is proved.

After (2.9) is established, we have $v_i(y)$ uniformly converges to the constant 1 in C^1 for $|y - \bar{y}_i| \leq \frac{1}{2}$. This gives $\sigma_1(S(v_i))$ convergent weakly to 0 in $|y - \bar{y}_i| \leq \frac{1}{2}$. On the other hand, by (3.11) in Lemma 3.1, $\sigma_1(S(v_i)) \geq C > 0$ in $|y - \bar{y}_i| \leq \frac{1}{2}$ as $f(S(v_i)) = 1$. This yields a contradiction. The proof of Theorem 1.2 is complete.

We note that we only used (3.11) in our proof, not the full concavity condition (1.7). Though (1.7) implies (3.11) by Lemma 3.1. □

Remark 2.1. *In* [19]*, Li-Li proved the Harnack type inequality in Theorem 1.2 under some further complicated conditions. The main reason for Li-Li to impose these conditions is that their proof of the inequality relies on local gradient estimates for equation (1.1). Such kind of local gradient estimates was established in* [11] *for $f(S_g) = \sigma_k(S_g)$ prior to* [19]*. The proof of the local gradient estimates for σ_k was deduced to* Claim (18) *in* [11]*. The implication of* Claim (18) *to local gradient estimates is simple, and that part of deduction can be easily adapted for general elliptic f. The main part of* [11] *is to prove* Claim (18) *for σ_k, which is quite delicate.* Claim (18) *in* [11]*, together with some other conditions, was renamed as H_α condition in* [19]*.*

Remark 2.2. *The local gradient estimates for quotient $\frac{\sigma_k}{\sigma_l}$ have also been established in* [13, 9] *recently. In the locally conformally flat case, we suspect that the concave assumption (1.7) and the conformal invariance of f should be sufficient to get the local gradient estimates. This is partially confirmed in the proof of Theorem 1.2, where a local gradient estimate for a sequence of solutions is obtained via the method of moving planes.*

3. Global a priori bounds and the existence

In this section, we will establish the global gradient estimate of $\log u$ via the method of moving planes. We first state the main estimates in the following proposition.

Proposition 3.1. *Let (M, g) be an n-dimensional smooth compact locally conformally flat manifold with g admissible. Suppose that f satisfies (1.6), (1.7) and (1.10), and (M, g) is not conformally equivalent to the standard n-sphere. Then there exists a positive constant $C > 0$, such that*

$$\max_M u \le C, \quad \|\nabla \log u\|_{L^\infty} + \|\nabla^2 \log u\|_{L^\infty} \le C. \tag{3.1}$$

Theorem 1.3 is a consequence of the proposition.

Proof of Theorem 1.3. First we prove the C^2 bound of the solutions. By Proposition 3.1 we only need o prove u has a positive lower bound. It is sufficient to prove $\max_M u$ has a positive lower bound. We now use an observation from Viaclovsky [25]. We would like to note that this is the only place where the admissible condition of S_g is used. At any maximum point x_0 of u, $u^{-\frac{4}{n-2}} S_{\hat g}(x_0) \ge u^{-\frac{4}{n-2}} S_g(x_0)$. Therefore,

$$1 = f(u^{-\frac{4}{n-2}}(x_0) g^{-1}(x_0) S_{\hat g}(x_0)) \ge f(u^{-\frac{4}{n-2}}(x_0) g^{-1}(x_0) S_g(x_0)).$$

Since $g^{-1} \cdot S_g(x_0)$ is admissible, and $K = \{g^{-1} \cdot S_g(x) | x \in M\}$ is compact, by (3.12), $u^{-\frac{4}{n-2}}(x_0) \le C_0$ for some constant C_0. Therefore, the C^0 and C^1 estimates are proved. By Lemma 3.2, we have C^2 estimates. Then it follows from the second condition in (1.10) that f is uniformly elliptic. The higher-derivatives follow from the Krylov-Evans Theorem and standard elliptic theory. So, the a priori estimate (1.11) is proved for the case when M is not comformally equivalent to $\mathbf{S}^n$.

The existence of solutions can be obtained by using the degree theory following the argument of Li-Li in [19]. We define a deformation

$$f_t(\lambda) = \begin{cases} f((1-t)\lambda + t\sigma_1(\lambda)e), & \text{for } t \in [0,1], \\ (2-t)f(\sigma_1(\lambda)e) + \frac{t-1}{nt_0}\sigma_1(\lambda), & \text{for } t \in [1,2] \end{cases}$$

with the corresponding cone

$$\Gamma_t = \begin{cases} \{\lambda \in \Gamma_1^+ \mid (1-t)\lambda + t\sigma_1(\lambda)e \in \Gamma\}, & \text{for } t \in [0,1], \\ \Gamma_1^+, & \text{for } t \in [1,2], \end{cases}$$

where $e = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ in Γ. Obviously, f_t in the deformation satisfies the assumptions of Theorem 1.3 and $f_t(t_0, \dots, t_0) = 1$, where t_0 as in (1.8). By a priori estimates (1.11), the C^3-norms of solutions are uniformly bounded. Therefore, the degree remains the same during the deformation. Since the degree for the Yamabe problem (i.e. for f_2) is -1 (see [22]), the degree for our equation is -1. The existence of solutions follows. □

Now we prove Theorem 1.1. The idea to prove Theorem 1.1 is to seek an admissible solution of the following fully nonlinear equation:

$$\sigma_k(g) = \text{constant}, \tag{3.2}$$

for g in the conformal class. In [2], Chang-Gursky-Yang proved that if Y_1 and Y_2 (note that Y_1 positive implies $\mathcal{C}_1 \neq \emptyset$, and in the case $n = 4$, $\int_M \sigma_2(g) = Y_2$ for all g in the conformal class) are positive, then equation (3.2) is solvable for $n = 4, k = 2$. This is an important result because the existence is obtained without the assumption on $\mathcal{C}_k \neq \emptyset$. Here we will deal with the case for higher dimension, but on the locally conformally flat manifolds. The key is to obtain some appropriate a priori estimates for (3.2) using Proposition 3.1 and a result in [13] concerning a sharp Sobolev type inequality. We list it in the following proposition, which is a special case of Theorem 1 in [13].

Proposition 3.2. *Let (M, g_0) be a locally conformally flat manifold with $g \in \mathcal{C}_l$ for some $l < n/2$. Then there is a constant $C_S(M) > 0$ such that for any metric $g \in \mathcal{C}_l$.*

$$\int_M \sigma_l(g) vol(g) \geq C_S(M) vol(g)^{\frac{n-2l}{n}}.$$

The equality holds if and only if $\sigma_l(g^{-1}S_g) = c$ for some positive constant c. Moreover,

$$C_S(M) \leq C_S(\mathbf{S}^n) = \binom{n}{l}^{\frac{1}{n-2l}} (\frac{\omega_n^2}{2^n})^{\frac{l}{n(n-2l)}},$$

where ω_n is the volume of the standard sphere $\mathbf{S}^n$.

Proof of Theorem 1.1. Let $g = e^{-2v}g_0 \in \Gamma_{k-1}^+$. We modify the approaches in [15] and [19] to consider the following equation

$$f_t(v) = \sigma_k(tg^{-1}S_g + (1-t)\sigma_{k-1}^{1/(k-1)}(g^{-1}S_g)g) = 1. \tag{3.3}$$

Let

$$\Gamma_t = \{\Lambda \in \Gamma_{k-1}^+ \,|\, t\Lambda + (1-t)\sigma_{k-1}^{1/(k-1)}(\Lambda)I \in \Gamma_k^+\}.$$

It is clear that $\Gamma_0 = \Gamma^+_{k-1}$ and $\Gamma_1 = \Gamma^+_k$. And for any $t \in [0,1]$, f_t satisfies the conditions in Proposition 3.1 uniformly in t. From the proof of Theorem 1.3, we may take $g_0 \in \mathcal{C}_{k-1}$ with $\sigma_{k-1}(g_0) = 1$ and the degree of $\sigma_{k-1}(g) = 1$ is -1. From degree argument (e.g., see [19]), we only need to show a priori bound on solutions of equation (3.3) for all $0 \le t \le 1$.

For $g = e^{-2v} g_0$, and for any local orthonormal frame (with respect to g_0), we let S_{ij} be the Schouten tensor of g_0 and let $W_v = (v_{ij} + v_i v_j - \frac{|\nabla v|^2}{2}\delta_{ij} + S_{ij})$. Equation (3.3) then can be expressed as:

$$\sigma_k(tW_v + (1-t)\sigma_{k-1}^{1/(k-1)}(W_v)I) = e^{-2kv}. \tag{3.4}$$

By (3.1) in Proposition 3.1, there is C independent of t such that

$$\inf_M v \ge C, \quad \max_M |\nabla v| \le C, \quad \text{and} \quad \max_M |\nabla^2 v| \le C. \tag{3.5}$$

We now only need to obtain an upper bound of v. Set $\tilde v = v - \max_M v$. We have $W_{\tilde v} = W_v$. By (3.5), $\|\tilde v\|_{C^2(M)} \le \tilde C$ for some $\tilde C$ independent of t. $\tilde v$ satisfies equation

$$\sigma_k(tW_{\tilde v} + (1-t)\sigma_{k-1}^{1/(k-1)}(W_{\tilde v})I) = e^{-2k\max_M v} e^{-2k\tilde v}. \tag{3.6}$$

Expand

$$\sigma_k(tW_{\tilde v} + (1-t)\sigma_{k-1}^{1/(k-1)}(W_{\tilde v})I) = \sum_{i=0}^{k} \binom{n-i}{n-k} t^i (1-t)^{k-i} \sigma_i(W_{\tilde v}) \sigma_{k-1}^{\frac{k-i}{k-1}}(W_{\tilde v}). \tag{3.7}$$

Since $W_{\tilde v} \in \Gamma_{k-1}$, we have

$$e^{-2k\max_M v} e^{-2k\tilde v} = \sigma_k(tW_{\tilde v} + (1-t)\sigma_{k-1}^{1/(k-1)}(W_{\tilde v})I) \ge t^k \sigma_k(W_{\tilde v}) + (1-t)^k \sigma_{k-1}^{\frac{k}{k-1}}(W_{\tilde v}).$$

That is

$$e^{-2k\max_M v} \ge t^k \sigma_k(\tilde g^{-1} S_{\tilde g}) + (1-t)^k \sigma_{k-1}^{\frac{k}{k-1}}(\tilde g^{-1} S_{\tilde g}). \tag{3.8}$$

Since $\tilde v$ is bounded, integrating the above formula over M with respect to the metric $\tilde g = e^{-2\tilde v} g_0$, together with the Hölder inequality, yields

$$e^{-2k\max_M v} \ge c(t^k Y_k + (1-t)^k (\int_M \sigma_{k-1}(\tilde g^{-1} S_{\tilde g}) dvol(\tilde g))^{\frac{k}{k-1}}) \tag{3.9}$$

for a positive constant $c > 0$ independent of t. By Proposition 3.2, for $l < n/2$,

$$\inf_{g \in \mathcal{C}_l} (Vol(g))^{\frac{2l-n}{n}} \int_M \sigma_l(g^{-1} S_g) dvol(g) \ge C_S > 0.$$

This gives

$$(\int_M \sigma_{k-1}(\tilde g^{-1} S_{\tilde g}) dvol(\tilde g))^{\frac{k}{k-1}} \ge (C_S)^{\frac{k}{k-1}} (Vol(\tilde g))^{\frac{k(n-2l)}{k-1}} \ge C > 0,$$

since $\tilde v$ is bounded. It follows that v has an upper bound independent of t. Hence $\|v\|_{C^2(M)}$ is bounded independent of t. By the Krylov-Evans theorem and standard elliptic theory, $\|v\|_{C^m(M)}$ is bounded for any m. The Theorem is proved for the case $Y_k > 0$.

If $Y_k = 0$, By (3.5), v is bounded from below, and the first and second derivatives of v are bounded independent of t. By (3.9), for any $t < 1$, v is bounded from above (depending on t). If $\sup v \to \infty$ for some sequence $t_j \to 1$, from (3.6) we obtain a $C^{1,1}$ solution $g \in \bar{\Gamma}_k^+$ with $\sigma_k(g) = 0$. If for some sequence $t_j \to 1$, $\sup v$ stay bounded, we obtain a solution $g \in \Gamma_k^+$ with $\sigma_k(g) = 1$. These two cases can not be happen at the same time by Lemma 2 in [13]. □

Proof of Corollary 1.1. Note that $\int_M \sigma_m(g)dvol(g)$ is a topological invariant. We have $Y_m > 0$. By Theorem 1.1, $\mathcal{C}_m \neq \emptyset$. Hence (M, g) is conformally equivalent to $\mathbf{S}^{2m}$, by Theorem 1 in [10]. □

Finally, we prove Proposition 3.1. It is well-known that once gradient estimates are available, C^2 estimates of $\log u$ will follow easily. And higher-order derivatives follow readily the Krylov-Evans theory. Though in [11] we dealt only with $f = \sigma_k^{1/k}$ and estimated $\Delta u + |\nabla u|^2$, but the argument works for general concave elliptic operator f and second derivative bounds can be obtained easily just by considering $T^2 u + |Tu|^2$ for any unit vector field T. This was noted in [19]. For the completeness, we will include the local C^2 estimates here following the same lines of proof in [11].

We list some properties stemmed from concavity assumption on f.

Lemma 3.1. *Suppose that f satisfies (1.7) and (1.8). Set $F^{ij}(U) = \frac{\partial f(U)}{\partial U_{ij}}$ for $U = (U_{ij}) \in \Gamma$.*

1. *Let t_0 be the number in (1.8), then for all $U \in \Gamma$ with $f(U) \leq 1$,*

$$\sum_{i,j} F^{ij}(U)U_{ij} \leq t_0 \sum_i F^{ii}(U). \tag{3.10}$$

2. *Suppose further that f satisfies (1.6), then there is $C > 0$ such that $\forall U \in \Gamma$ with $f(U) \geq 1$, the following is true:*

$$\sigma_1(U) \geq C f(U). \tag{3.11}$$

3. *If in addition, f satisfies condition (1.10), then $\sum_{i,j} F^{ij}(U)U_{ij} \geq 0$ for all $U \in \Gamma$. And for any compact set K in Γ, there is a $t_K > 0$, such that*

$$f(t\gamma) > 1, \quad \textit{for all} \quad \gamma \in K, t \geq t_K. \tag{3.12}$$

Moreover there is $\delta > 0$ such that for all $U \in \Gamma$ with $f(U) \leq 1$, the following is true

$$\delta \leq \delta + \sum_{i,j} F^{ij}(U)U_{ij} \leq 2t_0 \sum_i F^{ii}(U). \tag{3.13}$$

Proof. Let I be the identity matrix. By the concavity of f,

$$f(tI) \leq f(U) + \sum_{i,j} F^{ij}(U)(t\delta_{ij} - U_{ij}). \tag{3.14}$$

By (1.8), $f(t_0 I) \geq 1$. Since $f(U) \leq 1$, (3.10) follows from (3.14).

To prove (3.11), we note $\sigma_1(U)$ is invariant under symmetrization (i.e., symmetrization of eigenvalues of U), while $f(U)$ is non-decreasing under symmetrization by the concavity of f. So we only need to check that if $f(t,\cdots,t)\ge 1$, then $\sigma_1(t,\cdots,t)\ge Cf(t,\cdots,t)$. By (1.6), $f(t,\cdots,t)\ge 1$ implies $t\ge t_0$. From the concavity of f,

$$f(t,\cdots,t)\le f(t_0I)+(t-t_0)\sum_i f_{\lambda_i}(t_0,\cdots,t_0)\le A\sigma_1(t,\cdots,t),$$

if we pick $A\ge \frac{f(t_0I)}{\sigma_1(t_0,\cdots,t_0)}+\sum_i f_{\lambda_i}(t_0,\cdots,t_0)$.

We note that by concavity assumption on f and the first condition in (1.10), for any $\gamma\in\Gamma$, $f(t\gamma)$ is an increasing function for $t>0$. This implies $\sum_{i,j}F^{ij}(U)U_{ij}\ge 0$. By the monotonicity of $f(t\gamma)$ and the first condition in (1.10), for any $\gamma\in\Gamma$, there is $t_\gamma<\infty$ such that $f(t\gamma)>1$ for all $t\ge t_\gamma$. Then (3.12) follows from the continuity of f and compactness of K in Γ.

By the first condition in (1.10) again, there exists $\delta>0$ such that $f(2t_0I)\ge 1+\delta$ (this also follows from the monotonicity condition (1.6)). Since $f(U)\le 1$, (3.13) follows from (3.14). □

Set $v=\frac{-2}{n-2}\log u$, then v satisfies equation

$$f(e^{2v}(\nabla^2 v+dv\otimes dv-\frac{|\nabla v|^2}{2}g+S_g))=1. \tag{3.15}$$

Lemma 3.2. *Suppose that f satisfies conditions (1.6), (1.7), and (1.8), and suppose that $v\in C^4$ is an admissible solution of (3.15) in B_r. Then, there exists a constant $c>0$ depending only on r, $\|g\|_{C^4(B_r)}$ and $\|\nabla v\|_{L^\infty(B_r)}$, such that*

$$|\nabla^2 v|(x)<c, \qquad \text{for } x\in B_{r/2}. \tag{3.16}$$

Proof. Choose r' small such that there is a local orthonormal frame in each geodesic ball $B_{r'}(x)$ for all $x\in B_{\frac{2r}{3}}$. We only need to verify (3.16) for such $B_{r'}(x)$, which we will still denote B_r. We may also assume $r=1$. Let ρ be a smooth nonnegative cut-off function in B_1, $\rho=1$ in $B_{\frac12}$ and $\rho=0$ in $B_1\setminus B_{\frac23}$. We only need to get an upper bound for $\rho(T^2v+|Tv|^2)$ for any unit vector field T. Since $\bar B_{\frac23}\times \mathbf{S}^{n-1}$ is compact, we may assume that the maximum is attained at some point $y_0\in B_{\frac23}$ and $T=e_1$ for some orthonormal frame $\{e_1,\cdots,e_n\}$ in B_1. Set $G=\rho(v_{11}+|v_1|^2)$. So y_0 is a local maximum point of G. By the C^1 bound assumption, we may assume $v_{11}\ge 1+|v_1|^2$ and $v_{11}(y_0)>\frac{1}{4n}|v_{ij}(y_0)|,\forall i,j$. Now at y_0, we have

$$0=\frac{\rho_j}{\rho}G+\rho(v_{11j}+2v_1v_{1j}), \qquad (\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G+\rho(v_{11ij}+2v_{1i}v_{1j}+2v_1v_{1ij}))\le 0. \tag{3.17}$$

For any fixed local orthonormal frame, we may view S_g and $S_{\hat g}$ as matrices. We denote S_{ij} and U_{ij} the entries of $g^{-1}S_g$ and $\hat g^{-1}S_{\hat g}$ respectively. By the ellipticity assumption on f, (F^{ij}) is positive definite at $U=\hat g^{-1}S_{\hat g}$. Since y_0 is a maximum point of G,

$$0\ge \sum_{i,j\ge 1}F^{ij}\{\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G+\rho(v_{ij11}+2v_{1i}v_{1j}+2v_1v_{ij1})\}-CG\sum_i F^{ii}, \tag{3.18}$$

where the last term comes from the commutators related to the curvature tensor of g and its derivatives. From the construction of ρ, $|\nabla\rho(x)| \le C\rho^{\frac{1}{2}}(x)$ for all $x \in B_1$. We have

$$\sum_{i,j\ge 1} F^{ij}\frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}G \ge -C\sum_{i\ge 1} F^{ii}\frac{1}{\rho}G.$$

By (3.17), (3.10), (3.18) and the concavity of f,

(3.19)

$$\begin{aligned}
0 &\ge \rho\sum_{i,j\ge 1} F^{ij}(v_{ji11} + 2v_{1i}v_{1j} + 2v_1v_{1ij}) - C\frac{G}{\rho}\sum_{i\ge 1} F^{ii} \\
&= \rho\sum_{i,j\ge 1} F^{ij}\{(e^{-2v}U_{ij} - v_iv_j + \frac{1}{2}|\nabla v|^2\delta_{ij} - S_{ij})_{11} \\
&\quad +2\rho v_{1i}v_{1j} + 2\rho v_1(e^{-2v}U_{ij} - v_iv_j + \frac{1}{2}|\nabla v|^2\delta_{ij} - S_{ij})_1\} - C\frac{G}{\rho}\sum_{i\ge 1} F^{ii} \\
&= \rho\sum_{i,j\ge 1} F^{ij}\{e^{-2v}(U_{ij})_{11} - 2v_1e^{-2v}(U_{ij})_1 + (\frac{1}{2}|\nabla v|^2\delta_{ij} - S_{ij})_{11} - 2v_{11}e^{-2v}U_{ij} \\
&\quad -v_iv_{j11} - v_jv_{i11} + 2v_1(-v_iv_j + \frac{1}{2}|\nabla v|^2\delta_{ij} - S_{ij})_1\} - C\frac{G}{\rho}\sum_{i\ge 1} F^{ii} \\
&\ge \rho e^{-2v}(f_{11} - 2v_1f_1) + \sum_{i\ge 1} F^{ii}[\rho v_{11}^2 - C(\frac{1}{\rho} + \frac{|\nabla\rho|}{\rho})G] - Ct_0\rho v_{11}\sum_i F^{ii},
\end{aligned}$$

where t_0 is the number in (1.8).

As f is a constant, $f_1 = f_{11} = 0$. By assumption $v_{11} \ge \frac{1}{2\rho}G$ at y_0. It follows from (3.19) that at y_0, $G \le C$. □

Proof of Proposition 3.1. The second derivative estimates have been proved in Lemma 3.2 assuming the gradient boundedness. We only need to get an upper bound of u and a gradient estimate for $\log u$. We should first use the theory of Schoen-Yau in [23] to set up the situation where the method of moving planes can work. Let $(\tilde{M}, \tilde{g})$ be the universal cover of M with $\tau : \tilde{M} \to M$ be a covering and $\tilde{g} = \tau^*(g)$ is the pull-back metric of g. By applying the theory of Schoen-Yau on locally conformally flat manifold, there exists a developing map $\Phi : (\tilde{M}, \tilde{g}) \to (\mathbf{S}^n, \sigma)$ where σ is the standard metric on $\mathbf{S}^n$. The map Φ is conformal and one to one. Let

$$\Omega = \Phi(\tilde{M}). \tag{3.20}$$

Then Ω is an open set of $\mathbf{S}^n$. In our case, the scalar curvature of g is positive. Then Schoen-Yau's Theorem tells us that the Hausdorff-dimension of $\partial\Omega$ is at most $\frac{n-2}{2}$.

If $\Omega = \mathbf{S}^n$, then M has an unique conformal structure, and solution always exists, which can be derived from the solutions on $\mathbf{S}^n$. Hence we consider $\partial\Omega$ is not empty. Now fix a point $p \in M$ and choose $\tilde{p} = \Phi \circ \tau^{-1}(p)$ such that $\text{dist}(\tilde{p}, \partial\Omega) \ge \delta_0 > 0$. By composing a conformal transformation on $\mathbf{S}^n$ and identifying $\mathbf{R}^n = \mathbf{S}^n \backslash \{\text{North pole}\}$ through the stereographic projection, we may assume $\tilde{p} = (-1, 0, \ldots, 0)$ and $\partial\Omega \subset \{x \mid |x| \geq \frac{1}{\delta}\}$ for some $\delta > 0$. For the simplicity, we assume $\infty \notin \partial\Omega$. We still denote the conformal map: $(\tilde{M}, \tilde{g}) \to (\mathbf{R}^n, |dx|^2)$ by Φ. Set $v(x)$ to be the conformal factor:

$$\Phi^*(|dx|^2) = v(\Phi^{-1}(x))^{\frac{4}{n-2}} \tilde{g}.$$

Then $\tilde{u}(x) = v(\Phi^{-1}(x)) u(\tau\Phi^{-1}(x))$ for $x \in \Omega$ is a solution of

$$\begin{cases} f(\lambda(S(\tilde{u}))(x)) = 1 \quad \text{and} \quad \lambda(S(\tilde{u}))(x) \in \Gamma \quad \text{for} \quad x \in \Omega, \\ \lim\limits_{x \to \partial\Omega} \tilde{u}(x) = +\infty. \end{cases} \tag{3.21}$$

Note that the boundary condition of (3.21) follows from [23], because M is compact. By composition with a rotation, we may assume

$$\begin{aligned} &\frac{\partial \tilde{u}}{\partial x_i}(-1, 0, \ldots, 0) = 0 \quad \text{if} \quad i \neq 1 \\ &\frac{\partial \tilde{u}}{\partial x_1}(-1, 0, \ldots, 0) > 0 \end{aligned} \tag{3.22}$$

Let u^* be the Kelvin transformation with respect to the unit ball, that is,

$$u^*(y) = |y|^{2-n} \tilde{u}\left(\frac{y}{|y|^2}\right).$$

Then $u^*(y)$ satisfies equation (3.21) in Ω^*, where Ω^* is the image of Ω under the inversion $y \to \frac{y}{|y|^2}$, and $\partial\Omega^* \subset B(0, \delta)$. Since $\infty \notin \partial\Omega$, $u^*(x)$ is C^2 at the origin and $\lim_{x \to \partial\Omega^*} u^*(x) = +\infty$. Because $u^*(x)$ has a harmonic expansion at ∞, by Lemma 2.1, we can start the method of moving planes. Since the argument is essentially the same as in section 2, we won't repeat it here. Hence, we may conclude that $u^*(y)$ is increasing in y_1 as long as $y_1 \leq -\frac{1}{2}$. Thus,

$$\frac{\partial u^*}{\partial y_1}(-1, 0, 0, \ldots, 0) > 0,$$

which, together with (3.22), implies

$$|\bigtriangledown \tilde{u}(-1, 0, \ldots, 0)| = \frac{\partial \tilde{u}}{\partial y_1}(-1, 0, \ldots, 0) < (n-2)\tilde{u}(-1, 0, \ldots, 0). \tag{3.23}$$

By noting $\tilde{u}(x) = v(x) u(\tau \circ \Phi^{-1}(x))$, we then obtain

$$|\bigtriangledown \log u(p)| \leq c \quad \text{for} \quad p \in M. \tag{3.24}$$

Clearly, the gradient estimate (3.24) yields

$$\frac{\max_M u}{\min_M u} \leq C. \tag{3.25}$$

Together with Theorem 1.2, we get

$$\max_M u \leq C.$$

□

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