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The First  $L^2$ -Betti Number of  
Classifying Spaces for Variations of  
Hodge Structures

by

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homogeneous complex manifolds arise naturally from the period matrices of general algebraic varieties. Those manifolds are also classifying spaces.

We know that  $D = G/V$ , where  $G$  is a semi-simple Lie group and  $V$  is a certain compact subgroup.  $G$  is called a group of Hodge type. As  $V$  typically is a proper subgroup of the maximal compact subgroup  $K$  of  $G$ ,  $D$  is usually not a symmetric space, but only a homogeneous one. The properties of the projection from  $D = G/V$  to  $G/K$  were studied by Griffiths and Schmid[7]. In fact, if one does not attach a marking to the underlying manifold  $M$ , that is, fix a homology basis, then instead of  $G/V$ , one rather needs to look at the quotient of that space by the action of a lattice, that is, a discrete subgroup  $\Gamma$  of  $G$  for which the quotient of  $G/V$  by  $\Gamma$  has finite volume. Usually, however, that quotient  $\Gamma \backslash G/V$  is not compact.

The Hodge decomposition may be interpreted as an action of the real algebraic group  $C^*$  on the cohomology group  $H^k(M, C)$  given by

$$z \circ \omega = z^p \bar{z}^q \omega$$

for  $z \in C^*$  and  $\omega \in H^{p,q}(M, C)$ .

Non-Abelian analogues of the Hodge structure are given by the action of  $C^*$  on moduli spaces of flat bundles. By an important result of Simpson[12], non-Abelian Hodge theory is also related to the classifying spaces for variations of Hodge structures.

For understanding variations of Hodge structures, it is then important to study the topology of the spaces  $G/V$  and  $\Gamma \backslash G/V$ . The present paper does so in the context of  $L^2$ -cohomology, by extending previous results for the case of  $\Gamma \backslash G/K$ ,  $K$  being a maximal compact subgroup of  $G$  as above.

$L^2$ -cohomology is the appropriate extension of Hodge theory for harmonic forms to the case of a noncompact (complete) manifold  $X$  inasmuch as here every  $L^2$ -cohomology class can be represented by an  $L^2$ -harmonic form, see [1]. Besides offering the possibility to extend Hodge theory to the noncompact case, it can be used to obtain topological information about compact quotients of  $X$  by the  $L^2$ - index theorem of Atiyah [1]. This is based on the fact that the operation of a discrete group  $\Gamma$  by isometries on  $X$  commutes with the Laplacian. Therefore, spaces of harmonic  $k$ -forms become  $\Gamma$ -modules, and by constructions from the theory of von-Neumann algebras, they can be assigned dimensions  $B_{\Gamma}^k(X)$ . While these dimensions need not be integers in general, the corresponding Euler characteristic is and coincides with the standard one of the quotient  $\Gamma \backslash X$  (assuming certain natural assumptions so that the latter is defined). This is the content of Atiyah's theorem. For a detailed discussion, we refer to the comprehensive volume on  $L^2$ -cohomology by W. Lück [11].

It turns out that  $L^2$ -cohomology is useful for studying a conjecture of Hopf. Let  $\bar{M}^{2m}$  be a compact manifold of dimension  $2m$  with negative sectional curvature. Hopf conjectured

$$(1.1) \quad (-1)^m \chi(\bar{M}^{2m}) > 0.$$

Dodziuk [4] and Singer [13] suggested to use  $L^2$ -cohomology to approach this problem as follows. Take the universal covering  $M \rightarrow \bar{M}^{2m}$  and show

$$(1.2) \quad \mathcal{H}^q(M) = \{0\} \quad \text{for } q \neq m$$

and

$$(1.3) \quad \mathcal{H}^m(M) \neq \{0\}.$$

In other words, in contrast to ordinary cohomology,  $L^2$ -cohomology should concentrate in the middle dimension (for spaces of negative curvature). So far, this approach has given partial answers to the Hopf conjecture, and it is fair to say that these represent the best attack on the problem to date. More precisely, a verification of the conjecture of Dodziuk and Singer has been possible in the following cases:

- \* Symmetric spaces of noncompact type and rank one by A. Borel [2].
- \* Kähler hyperbolic manifolds (including quotients of Hermitian symmetric spaces) by M. Gromov [8], extended to the Kähler non-elliptic case (that is, allowing also zero curvature (in which case of course only the first part of the Dodziuk-Singer conjecture can hold) by Jost-Zuo [10] and Cao-Xavier [3].
- \* For negatively pinched manifolds, Jost-Xin [9] improved the previous results of Donnelly-Xavier [5].

The present paper is devoted to prove

**Theorem 1.1.** *Let  $N$  be a classifying space for variations of Hodge structures,  $\Gamma$  a lattice on  $N$ . Then*

$$B_\Gamma^1(N) = 0$$

This result is conceptually different from the ones just quoted because in general  $G/V$  is not a space of non-positive sectional curvature since the compact group  $K$  and therefore also its (non-trivial) quotient  $K/V$  carry

some positive curvature. The proof will be accomplished by showing that any  $L^2$ -harmonic 1-form vanishes. An intermediate result will be that the squared norm of any such 1-form is horizontal for the natural Riemannian submersion  $G/V \rightarrow G/K$  and only depends on the symmetric space  $G/K$ . In that sense, the fiber  $K/V$  with its positive curvature disappears from the picture, and the situation is reduced to the one of non-positive curvature after all.

## 2 Preliminaries

Let  $G$  be a semi-simple Lie group all of whose simple factors are non-compact. Let  $\mathfrak{g}$  be its Lie algebra of all left invariant vector fields on  $G$  and  $K \subset G$  the Lie subgroup of  $G$  whose image in the adjoint group  $\text{ad } G$  is a maximal compact subgroup of  $\text{ad } G$ . Let  $\mathfrak{k}$  be the subalgebra of  $\mathfrak{g}$  corresponding to  $K$  and  $\mathfrak{m}$  the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$  with respect to the Killing form  $B(X, Y)$  of  $\mathfrak{g}$ . Then

$$(2.1) \quad \mathfrak{g} = \mathfrak{m} + \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}.$$

It is known that the restriction of  $B$  to  $\mathfrak{m}$  (resp.  $\mathfrak{k}$ ) defines a positive (resp. negative) definite bilinear form on  $\mathfrak{m}$  (resp.  $\mathfrak{k}$ ). Hence we can choose a base  $\{X, \dots, X_n\}$  of  $\mathfrak{m}$  and a base  $\{X_{n+1}, \dots, X_{n+r}\}$  of  $\mathfrak{k}$  with

$$(2.2) \quad \begin{aligned} B(X_i, X_j) &= \delta_{ij}, \\ B(X_\alpha, X_\beta) &= -\delta_{\alpha\beta}; \end{aligned}$$

here and in the sequel we employ the following range of indices

$$\begin{aligned} 1 \leq i, j, \dots &\leq n \\ n+1 \leq \alpha, \beta, \gamma, \dots &\leq n+r \\ 1 \leq a, b, c, \dots &\leq n+r. \end{aligned}$$

Let

$$[X_a, X_b] = c_{ab}^c X_c.$$

By (2.1), among the structure constants  $c_{ab}^c$ , only  $c_{\alpha\beta}^\gamma$ ,  $c_{ij}^\alpha$ ,  $c_{j\alpha}^i$ ,  $c_{\alpha j}^i$  can be  $\neq 0$ .

Let  $B(X, Y)$  be the Killing form of  $\mathfrak{g}$ . It is defined by

$$(2.3) \quad B_{ab} = B(X_a, X_b) = \text{trace}(\text{ad } X_a \text{ ad } X_b) = c_{ae}^f c_{bf}^e.$$

Multiplying the Jacobi identity

$$c_{ab}^e c_{ce}^f + c_{ca}^e c_{be}^f + c_{bc}^e c_{ae}^f = 0$$

by  $c_{df}^c$  and summing over the index  $f$ , we have

$$\begin{aligned} & c_{df}^c c_{ab}^e c_{ce}^f + c_{df}^c c_{ca}^e c_{be}^f + c_{df}^c c_{bc}^e c_{ae}^f \\ &= -c_{ab}^e B_{de} + c_{df}^c c_{ca}^e c_{be}^f + c_{df}^c c_{bc}^e c_{ae}^f = 0. \end{aligned}$$

Denoting

$$(2.4) \quad c_{ab}^e B_{de} = c_{dab},$$

we have

$$\begin{aligned} c_{dab} &= c_{df}^c c_{ca}^e c_{be}^f + c_{df}^c c_{bc}^e c_{ae}^f \\ &= c_{de}^f c_{fa}^c c_{bc}^e + c_{df}^c c_{bc}^e c_{ae}^f \\ &= c_{bc}^e (c_{de}^f c_{fa}^c + c_{df}^c c_{ae}^f), \end{aligned}$$

which is anti-symmetric in  $a, d$ . Hence  $c_{abc}$  is anti-symmetric in all indices.

(2.2), (2.3) and (2.4) give

$$(2.5) \quad \sum_{\alpha, k} c_{\alpha ik} c_{\alpha jk} = \frac{1}{2} \delta_{ij}$$

and

$$(2.6) \quad \sum_{i, j} c_{i\alpha j} c_{i\beta j} + \sum_{\gamma, \delta} c_{\gamma\alpha\delta} c_{\gamma\beta\delta} = \delta_{\alpha\beta}.$$

Now let  $\{\omega^a\}$  be invariant forms dual to  $\{X_a\}$ . We have the Maurer-Cartan equations

$$(2.7) \quad d\omega^a = -\frac{1}{2} c_{be}^a \omega^b \wedge \omega^e.$$

Let us consider the case when the Cartan subgroup  $H \subset G$  is compact. Let  $S^1 \subset H$ . Then the centralizer of  $S^1$  in  $G$  is denoted by  $V$  and  $H \subset V \subset K$ . We then have a complex homogeneous space  $G/V$ . Denote  $\dim H = r_1 + 1$ ,  $\dim V = r_1 + r_2 + 1$  and assume that  $r_1 + r_2 + 1 < r$ . We also agree on the range of indices

$$\begin{aligned} n_1 &= n + (r - r_1 - r_2 - 1), \\ s, t &= n + 1, \dots, n_1, \\ i_1 &= 1, \dots, n_1, \\ \alpha_1 &= n_1, \dots, n + r. \end{aligned}$$

We define the Riemannian metric  $ds^2$  on  $G$  by

$$ds^2 = \sum (\omega^a)^2.$$

This induces the canonical metric on  $G/V$  for which

$$\Pi : G \rightarrow G/V$$

is a Riemannian submersion. We see that  $\{X_{i_1}\} = \{X_i, X_s\}$  are horizontal vector fields and  $\{X_{\alpha_1}\}$  are vertical vector fields on  $G$ .  $\{\Pi_*X_i, \Pi_*X_s\} \in TG/V$  are orthonormal vector fields, and the dual coframe is  $\{\underline{\omega}^i, \underline{\omega}^s\}$ . The Riemannian metric on  $G/V$  is

$$ds^2 = \sum (\underline{\omega}^i)^2 + \sum (\underline{\omega}^s)^s.$$

Let  $\omega$  be a  $L^2$ -harmonic 1-form on  $G/V$

$$\omega = u_i \underline{\omega}^i + u_s \underline{\omega}^s.$$

By definition

$$\begin{aligned} d\omega(\Pi_*X_{i_1}, \Pi_*X_{i_2}) &= (\nabla_{\pi_*X_{i_1}} \omega) \Pi_*X_{i_2} - (\nabla_{\pi_*X_{i_2}} \omega) \Pi_*X_{i_1} \\ &= \Pi_*X_{i_1}(u_{i_2}) - \Pi_*X_{i_2}(u_{i_1}) + \omega([\Pi_*X_{i_2}, \Pi_*X_{i_1}]) \\ &= \Pi_*X_{i_1}(u_{i_2}) - \Pi_*X_{i_2}(u_{i_1}) + C_{i_2 i_1}^a \omega(\Pi_*X_a) \end{aligned}$$

and

$$\begin{aligned} \delta\omega &= -(\nabla_{\pi_*X_{i_1}} \omega) \Pi_*X_{i_1} \\ &= -\Pi_*X_{i_1}(u_{i_1}) + \omega(\nabla_{\pi_*X_{i_1}} \Pi_*X_{i_1}) \\ &= -\Pi_*X_{i_1}(u_{i_1}) + \omega(C_{i_1 i_1}^{j_1} \Pi_*X_{j_1}) \\ &= -\Pi_*X_{i_1}(u_{i_1}). \end{aligned}$$

We then have

$$(2.8) \quad \Pi_*X_{i_1}(u_{i_2}) - \Pi_*X_{i_2}(u_{i_1}) = C_{i_1 i_2}^{j_1} u_{j_1}$$

and

$$(2.9) \quad \Pi_*X_{i_1}(u_{i_1}) = 0.$$



### 3 Proof of the main theorem

Let  $\mathfrak{L}_{\Pi_* X_s}$  be the Lie derivative with respect to the vector field  $\Pi_* X_s$ ,  $g$  the metric tensor on  $G/V$ . Then

$$\begin{aligned}
(\mathfrak{L}_{\Pi_* X_s} g)(\Pi_* X_{i_1}, \Pi_* X_{j_1}) &= \Pi_* X_s(\delta_{i_1 j_1}) \\
&- g(\mathfrak{L}_{\Pi_* X_s} \Pi_* X_{i_1}, \Pi_* X_{j_1}) - g(\Pi_* X_{i_1}, \mathfrak{L}_{\Pi_* X_s} \Pi_* X_{j_1}) \\
&= -g([\Pi_* X_s, \Pi_* X_{i_1}], \Pi_* X_{j_1}) - g(\Pi_* X_{i_1}, [\Pi_* X_s, \Pi_* X_{j_1}]) \\
&= -g(\Pi_* C_{s i_1}^{k_1} X_{k_1}, \Pi_* X_{j_1}) - g(\Pi_* X_{i_1}, \Pi_* C_{s j_1}^{k_1} X_{k_1}) \\
&= -C_{s i_1}^{j_1} + C_{s j_1}^{i_1}
\end{aligned}$$

which is zero. This means that  $\mathfrak{L}_{\Pi_* X_s} g = 0$  and  $\Pi_* X_s$  is a Killing vector field on  $G/V$ . Noting that  $\omega$  is harmonic and by a result in [16]

$$\mathfrak{L}_{\Pi_* X_s} \omega = 0.$$

By using the formula

$$\mathfrak{L}_{\Pi_* X_s} \omega = (d \circ i_{\Pi_* X_s} + i_{\Pi_* X_s} \circ d)\omega,$$

we have

$$d u_s = 0,$$

and  $u_s$  is constant.  $\omega$  is an  $L^2$ -harmonic 1-form and

$$|\omega|^2 = \sum_i u_i^2 + \sum_s u_s^2,$$

is integrable. It turns out that each constant  $u_s$  must be zero. Furthermore,

$$\begin{aligned}
\Pi_* X_s \sum_i u_i^2 &= 2u_i \Pi_* X_s u_i \\
&= 2u_i (\Pi_* X_i u_s + c_{s i}^j u_j) \\
&= 2c_{s i}^j u_i u_j = 2c_{j s i} u_i u_j = 0,
\end{aligned}$$

noting that the  $c_{abc}$  are anti-symmetric in all indices.

We see that  $\Pi_1 : G/V \rightarrow G/K$  is also a Riemannian submersion whose fiber is a compact submanifold  $K/V$  in  $G/V$ . Notice that  $K/V$  is a totally geodesic submanifold in  $G/V$ .

In summary, we have shown that the squared norm of any  $L^2$ -harmonic 1-form  $\omega$  is horizontal and depends only on the symmetric space  $G/K$ .

Let  $\omega$  be a  $L^2$ -harmonic 1-form in  $N$ . We have the Riemannian submersion

$$\Pi_1 : N \rightarrow M$$

with totally geodesic fibers, where  $M$  is the corresponding symmetric space. Furthermore,  $|\omega|^2$  is horizontal. By  $L^2$ -Hodge theory it is only necessary to prove that  $\omega$  vanishes.

Take any unit vector field  $n$  in  $M$  and any function  $b$  in  $M$ . We have the horizontal lift  $\bar{X}$  of  $bn$ .  $\bar{X}$  is the normal vector field of the fiber submanifold whose length is constant along the fibers. Choose an orthonormal frame field  $\{e_i\}$  in  $M$ , and call its horizontal lift  $\{\bar{e}_i\}$ .  $\{\bar{e}_s\}$  is an orthonormal frame on the fiber. Thus,  $\{\bar{e}_i, \bar{e}_s\}$  is an orthonormal frame field on  $N$ . Therefore  $\langle \nabla_{\bar{e}_s} \bar{X}, \bar{e}_s \rangle$  is a multiple of the mean curvature with respect to the normal direction  $n$ . It is zero since the fibers are totally geodesic.  $\operatorname{div} \bar{X}$  can be computed in the base manifold  $M$ . We also have

$$\langle \omega \odot \omega, \nabla \bar{X} \rangle = u_i u_j \langle \nabla_{e_i} X, e_j \rangle.$$

Hence

$$\langle S_\omega, \nabla X \rangle = \frac{1}{2} |\omega|^2 \operatorname{div} X - \langle \omega \odot \omega, \nabla X \rangle$$

can be computed in  $M$ , provided  $X$  is of the above type, where  $S_\omega$  is the stress-energy tensor of  $\omega$ .

Choose  $D = B_R(x_0)$ , a geodesic ball in  $M$  of radius  $R$  with center in  $x_0 \in M$ . Its boundary is a geodesic sphere  $S_R(x_0)$  in  $M$ . Let  $\bar{D} = \Pi_1^{-1}(D) \subset N$ .  $\partial \bar{D}$  is compact since the fiber is compact. Let  $X = r \frac{\partial}{\partial r}$ , which is a smooth vector field in  $M$ . Let  $\bar{X}$  be the horizontal lift of  $X$ . Since the fiber submanifold is orthogonal to the horizontal vector field,  $\bar{X}$  is also a normal vector field on  $\partial \bar{D}$ . Its length is equal to  $r$ . Thus, for any  $L^2$ -harmonic 1-form  $\omega$  in  $N$ , we have

$$\begin{aligned} (3.1) \quad & \int_{\partial \bar{D}} \frac{1}{2} |\omega|^2 \langle \bar{X}, n \rangle * 1 - \int_{\partial \bar{D}} \langle i_{\bar{X}} \omega, i_n \omega \rangle * 1 \\ & = \int_{\partial \bar{D}} \frac{1}{2} R |\omega|^2 * 1 - \int_{\partial \bar{D}} R \langle i_{\frac{\partial}{\partial r}} \omega, i_{\frac{\partial}{\partial r}} \omega \rangle \leq \frac{1}{2} R \int_{\partial \bar{D}} |\omega|^2 * 1. \end{aligned}$$

On the other hand,

$$\begin{aligned}\nabla_{\frac{\partial}{\partial r}} X &= \frac{\partial}{\partial r}, \quad \nabla_{e_{s'}} X = r \operatorname{Hess}(r)(e_{s'}, e_{t'}) e_{t'}, \\ \operatorname{div} X &= 1 + r \operatorname{Hess}(r)(e_{s'}, e_{s'}),\end{aligned}$$

where  $\{e_i\} = \{e_{s'}, \frac{\partial}{\partial r}\}$  ( $s', t' = 1, \dots, n-1$ ) is an orthonormal frame field in  $D$ . Therefore

$$\langle \omega \odot \omega, \nabla X \rangle = \left| i_{\frac{\partial}{\partial r}} \omega \right|^2 + \langle i_{e_{s'}} \omega, i_{e_{t'}} \omega \rangle r \operatorname{Hess}(r)(e_{s'}, e_{t'}),$$

and hence

$$\begin{aligned}\langle S_\omega, \nabla X \rangle &= \frac{1}{2} |\omega|^2 (1 + r \operatorname{Hess}(r)(e_{s'}, e_{s'})) - \left| i_{\frac{\partial}{\partial r}} \omega \right|^2 \\ &\quad - \langle i_{e_{s'}} \omega, i_{e_{t'}} \omega \rangle r \operatorname{Hess}(r)(e_{s'}, e_{t'}) \\ &= \left( \frac{1}{2} \sum_{s'} r \operatorname{Hess}(r)(e_{s'}, e_{s'}) - \frac{1}{2} \right) \left| i_{\frac{\partial}{\partial r}} \omega \right|^2 \\ &\quad + \sum_{s'} \left( \frac{1}{2} + \frac{1}{2} \sum_{t'} r \operatorname{Hess}(r)(e_{t'}, e_{t'}) \right) \langle i_{e_{s'}} \omega, i_{e_{s'}} \omega \rangle \\ &\quad - \sum_{s', t'} r \operatorname{Hess}(r)(e_{s'}, e_{t'}) \langle i_{e_{s'}} \omega, i_{e_{t'}} \omega \rangle.\end{aligned}$$

Choose a local orthonormal frame field  $\{e_{s'}\}$  near  $x$  in  $S_r(x_0)$ , such that  $\operatorname{Hess}(r)$  is diagonalized at  $x$ . By parallel translating along the radial geodesics from  $x_0$  we have a local orthonormal frame field in  $M$ . We have at  $x$

$$(3.2) \quad \begin{aligned}\langle S_\omega, \nabla X \rangle &= \left( \frac{1}{2} \sum_{s'} r \operatorname{Hess}(r)(e_{s'}, e_{s'}) - \frac{1}{2} \right) \left| i_{\frac{\partial}{\partial r}} \omega \right|^2 \\ &\quad + \sum_{s'} \left( \frac{1}{2} + \frac{1}{2} \sum_{t'} r \operatorname{Hess}(r)(e_{t'}, e_{t'}) - r \operatorname{Hess}(r)(e_{s'}, e_{s'}) \right) \langle i_{e_{s'}} \omega, i_{e_{s'}} \omega \rangle.\end{aligned}$$

First of all, by the Hessian comparison theorem

$$(3.3) \quad \frac{1}{2} \sum_s r \operatorname{Hess}(r)(e_s, e_s) - \frac{1}{2} \geq \frac{n-2}{2} > 0.$$

To estimate the coefficients of the second term of (3.2) let

$$A_s = \sum_{t'} \operatorname{Hess}(r)(e_{t'}, e_{t'}) - 2 \operatorname{Hess}(r)(e_{s'}, e_{s'}).$$

Since

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial r}} \text{Hess}(r)(e_{s'}, e_{s'}) &= \langle \nabla_{\frac{\partial}{\partial r}} \nabla_{e_{s'}} \frac{\partial}{\partial r}, e_{s'} \rangle \\
&= -\langle R(\frac{\partial}{\partial r}, e_{s'}) \frac{\partial}{\partial r}, e_{s'} \rangle + \langle \nabla_{[\frac{\partial}{\partial r}, e_{s'}]} \frac{\partial}{\partial r}, e_{s'} \rangle \\
&= -\langle R(\frac{\partial}{\partial r}, e_{s'}) \frac{\partial}{\partial r}, e_{s'} \rangle - \langle \nabla_{e_{s'}} \frac{\partial}{\partial r}, \nabla_{e_{s'}} \frac{\partial}{\partial r} \rangle,
\end{aligned}$$

we have

$$\frac{d}{dr}(\Delta r) = -\text{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) - |\text{Hess}(r)|^2,$$

Moreover, if the sectional curvature of  $M$  satisfies  $-a^2 \leq K \leq 0$  and its Ricci curvature  $\text{Ric} \leq -b^2$ , then

$$\begin{aligned}
\frac{dA_s(r)}{dr} &= \left(-\text{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) - |\text{Hess}(r)|^2\right) \\
&\quad + 2\langle R(\frac{\partial}{\partial r}, e_{s'}) \frac{\partial}{\partial r}, e_{s'} \rangle + 2\langle \nabla_{e_{s'}} \frac{\partial}{\partial r}, \nabla_{e_{s'}} \frac{\partial}{\partial r} \rangle \\
&\geq b^2 - 2a^2 + 2\langle \nabla_{e_s} \frac{\partial}{\partial r}, \nabla_{e_s} \frac{\partial}{\partial r} \rangle - |\text{Hess}(r)|^2 \\
&= b^2 - 2a^2 + (\text{Hess}(r)(e_{s'}, e_{s'}))^2 \\
&\quad - \sum_{u', v' \neq s'} \text{Hess}(r)(e_{u'}, e_{v'}) \text{Hess}(r)(e_{u'}, e_{v'}).
\end{aligned}$$

For the classifying spaces  $N$ , the base manifold  $M$  of the Riemannian submersion  $\Pi_1 : N \rightarrow M$  is  $SO(p, 2q)/SO(p) \times SO(2q)$  ( $q \geq 2$ ) or  $Sp(m+n)/Sp(m) \times Sp(n)$ . We know that [14]

In any case we have

$$\frac{dA_s(r)}{dr} \geq (\text{Hess}(r)(e_{s'}, e_{s'}))^2 - \sum_{u', v' \neq s'} \text{Hess}(r)(e_{u'}, e_{v'}) \text{Hess}(r)(e_{u'}, e_{v'}).$$

Noting that the sectional curvature of  $M$  is nonpositive and each  $\text{Hess}(r)(e_t, e_t) \geq \frac{1}{r} > 0$ ,

$$\begin{aligned}
(3.4) \quad \frac{dA_s(r)}{dr} &\geq (\text{Hess}(r)(e_s, e_s))^2 - \left(\sum_{t \neq s} \text{Hess}(r)(e_t, e_t)\right)^2 \\
&= \left(\text{Hess}(r)(e_s, e_s) - \sum_{t \neq s} \text{Hess}(r)(e_t, e_t)\right) \Delta r \\
&= -A_s(r) \Delta r.
\end{aligned}$$

Table 3.1:

Type	Sec. Curvature	Ricci Curvature
$SO(p, 2q)/SO(p) \times SO(2q), p > 1$	$-2 \leq K \leq 0$	$-(p + 2q - 2)$
$SO(p, 2q)/SO(p) \times SO(2q), p = 1$	$K = -1$	$-(p + 2q - 2)$
$Sp(m, n)/Sp(m) \times Sp(n)$ $m, n > 1$	$-4 \leq K \leq 0$	$-4(n + m + 1)$
$Sp(m, n)/Sp(m) \times Sp(n)$ $m = n = 1$	$K = -4$	$-4(n + m + 1)$
$Sp(m, n)/Sp(m) \times Sp(n),$ $\min(m, n) = 1$ and $m \neq n$	$-4 \leq K \leq -1$	$-4(n + m + 1)$

Since  $A_s(0) > 0$  because the dimension is at least 4, we may deduce from (3.4) that  $A_s(r) > 0$  for all  $r > 0$ . Altogether, we conclude that

$$\langle S_\omega, \nabla X \rangle \geq \text{const. } |\omega|^2$$

for a positive constant. If  $|\omega| \neq 0$  there exists  $R_0 > 0$  such that when  $R \geq R_0$

$$(3.5) \quad \int_{\bar{D}} \langle S_\omega, \nabla \bar{X} \rangle * 1 \geq C > 0.$$

We have the basic inequality for the stress energy tensor [15]

$$(3.6) \quad \int_{\partial \bar{D}} \frac{1}{2} |\omega|^2 \langle \bar{X}, n \rangle * 1 = \int_{\bar{D}} \langle S_\omega, \nabla \bar{X} \rangle * 1 + \int_{\partial \bar{D}} \langle i_{\bar{X}} \omega, i_n \omega \rangle * 1.$$

By (3.1), (3.5) and (3.6) we obtain

$$\int_{\partial \bar{D}} |\omega|^2 * 1 \geq \frac{2C}{R}$$

and

$$\int_N |\omega|^2 * 1 \geq \int_{R_0}^{\infty} dR \int_{\partial \bar{D}} |\omega|^2 * 1 = \infty$$

which contradicts the  $L^2$ -assumption on  $\omega$ .

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