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## Abstract

We derive general Novikov-Morse type inequalities in a Conley type framework for flows carrying cocycles, therefore generalizing our results in [FJ2] derived for integral cocycle. The condition of carrying a cocycle expresses the nontriviality of integrals of that cocycle on flow lines. Gradient-like flows are distinguished from general flows carrying a cocycle by boundedness conditions on these integrals.

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## 1. Introduction

In the 80's, S. P. Novikov [No1, No2] considered Morse type inequalities relating the zeros of a closed Morse 1-form  $\omega$  to the topology of the underlying space  $X$ . By the Poincaré lemma, a closed 1-form is locally exact. For instance, we can assume that  $\omega = df_p$  near a zero point  $p$  of  $\omega$ . So closed Morse 1-form means that each zero point  $p$  of  $\omega$  is a non-degenerate critical point of  $f_p$  and has a Morse index from  $f_p$ . Define  $S(\omega)$  to be the union of the sets  $S_i(\omega)$  of the zero points of  $\omega$  with index  $i$ . Define

$$c_i := \#S_i(\omega)$$

Then the Morse type inequalities are

$$c_i \geq b_i([\omega])$$
$$\sum_{j=0}^i (-1)^{i-j} c_j \geq \sum_{j=0}^i (-1)^{i-j} b_j([\omega]) \quad (1.1)$$

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for  $i = 0, 1, \dots, m$ . Here the Novikov numbers  $b_i([\omega])$  are determined only by  $X$  and the cohomology class  $[\omega]$ . These Morse type inequalities involving Novikov numbers are called Novikov inequalities.

Novikov inequalities were obtained by constructing the Novikov complex  $(C_*, \partial)$  with respect to the closed Morse 1-form  $\omega$ . If  $[\omega]$  is an integral cohomology class, then each  $C_i$  is a free  $\mathbb{Z}((t))$ -module generated by the set of all the zeros of  $\omega$  with index  $i$ . Here  $\mathbb{Z}((t)) := \mathbb{Z}[[t]][t^{-1}]$  is the Novikov ring. For the detailed construction of the boundary operators and some of their properties, see the discussion in [BF], [Pa], [Ra] and the references there.

M. Farber and A.A. Ranicki [FR] have used a noncommutative localization method to construct a universal complex for a rank 1 Morse closed 1-form. Subsequently, M. Farber [Fa2] applied the method to the general case, which can induce many kinds of Novikov complexes. In a series of papers [Fo1, Fo2, Fo3], R. Forman developed the discrete Morse theory and Novikov theory in a combinatorial category. Those Novikov-Morse inequalities are proved also by constructing certain combinatorial Morse(Novikov) complexes, e.g., using the Witten deformation technique. In order to use the techniques from the smooth category, he introduced combinatorial vector fields and differential forms.

Novikov theory for closed 1-forms is very important, since many functionals, such as the symplectic action functional, the Chern-Simons functional and many Hamiltonian systems from electromagnetism and fluid mechanics are multi-valued functionals. Though these are multi-valued functionals in infinite dimensional spaces, people hope that the Novikov theory in finite dimension can provide a model for the infinite dimensional case, and that it can provide a strong topological method to find the critical points of the multi-valued functionals.

Though Novikov theory for closed 1-forms has been studied for many years, the various proofs of Novikov inequalities were all based on constructing various (analytic or topological) Novikov complexes. Hence the closed 1-form needs some non-degeneracy assumptions. One could not handle the case if the closed 1-form  $\omega$  has general degenerate zero points (zero locus). On the other hand, the behavior of the dynamical systems generated by a closed 1-form has not been carefully studied. For instance, the previous research was limited to the smooth category. Though there should be an analogue in the continuous category, if so, then comes the natural question, what is the relation between this analogue and the well-known gradient-like dynamical systems?

In [FJ2], we introduced a concept of “flow carrying a cocycle  $\alpha$  ( $\alpha$ -flow and generalized  $\alpha$ -flow)”, where  $\alpha$  is a 1-dimensional cocycle in the bounded Alexander-Spanier cohomology (see i.e., [Sp]). This is the analogue of the flow generated by a closed 1-form. Actually, as shown in [FJ2], (generalized)  $\alpha$ -flows characterize most important dynamical systems, in particular the following:

- The gradient flow of a Morse function  $f$  is a  $df$ -flow.
- The gradient flow of a closed 1-form  $\omega$  is an  $\omega$ -flow and certain perturbation

flows are also  $\omega$ -flows.

- If an  $\alpha$ -flow has no fixed point, then this flow is a flow “carrying a cohomology class” as introduced in [Ch].
- An  $\alpha$ -Morse-Smale flow as defined in [FJ2] which is a generalization of the famous Morse-Smale flow; the generalization here essentially consists in allowing the existence of ”cycles”
- If  $\alpha$  is a coboundary, then this flow is a gradient-like flow. Conversely, if a flow is a gradient-like flow, then this flow carries a coboundary  $\delta g$  where  $g$  is a Lyapunov function.

A flow carrying a cocycle  $\alpha$  has a  $\pi$ -Morse decomposition (Theorem 3.4.4 in [FJ2]). If  $\alpha$  is an integral cocycle, we can obtain the Novikov-Morse type inequalities, Theorem 4.4.1 in [FJ2]. This theorem uses the Conley index to get the “local” topological information of the isolated invariant sets.

Those inequalities in [FJ2] are a generalization of the Novikov inequalities for an integral closed Morse 1-form. Starting from these inequalities, we can recover many Novikov type inequalities found before. For example, we can recover the Novikov inequalities for an integral closed 1-form  $\omega$  having Bott type nondegenerate zero sets, which was given in [Fa2]. In addition, our theory has some new features:

- Novikov inequalities now hold in the continuous category (i.e. for (generalized)  $\alpha$ -flows).
- New Novikov inequalities (Theorem 4.4.1 in [FJ2]) for a general closed 1-form without any non-degeneracy requirement.
- New Novikov inequalities for  $\alpha$ -Morse Smale flows.
- Vanishing of the Novikov numbers  $b_i([\alpha])$ , if the manifold allows the existence of a flow carrying a cohomology class  $\alpha$ .

However, [FJ2] is mainly focused on the explanation of the flow carrying a cocycle  $\alpha$ , and on the proof of the Novikov-Morse inequalities for this flow under the assumption that  $\alpha$  is an integral cocycle.

This paper is the second one. There are two aims. The first aim is to give a sufficient and necessary description of the relation between  $\alpha$ -flows and the gradient-like flows. This will be given in section 3. The second aim is to generalize the Novikov-Morse inequalities for integral cocycles  $\alpha$  to the analogous results for higher rank cocycles  $\alpha$ . This is discussed in sections 4,5. To do this, one should prove an analogue of Theorem 4.2.1 in [FJ2]. This is a nontrivial generalization, since the exit set of the lifting flow becomes more complicated when compared

with the integral cocycle case. One should handle carefully the deformation complex in the corners. As long as we have the analogous theorem, we can easily generalize those theorems in [FJ2] by following proofs in [FJ2].

In section 2, we recollect some basic definitions and examples of flows carrying a cocycle  $\alpha$ .

Recently, M. Farber [Fa3](2001) also proposed the concept: "Lyapunov one-form"  $\omega$  for a pair  $(\Phi, Y)$ , where  $\Phi$  is a continuous flow and  $Y$  is a closed invariant set in  $\Phi$ . Since a flow  $\Phi$  allowing the existence of a Lyapunov one-form  $\omega$  is actually a flow carrying a cocycle  $\omega$  introduced in [FJ2], our definition seems more general than his. In [Fa3, Fa4], Farber constructed the corresponding Ljusternik-Schnirelman theory. In [FKLZ], the authors discussed the existence of "Lyapunov one-forms".

The results presented in this paper were obtained in 2000 and circulated in preliminary preprint form in [FJ1].

## 2. Review of flows carrying a cocycle

In [FJ2], we have defined the dynamical systems, flows carrying a cocycle. In this section we will recollect some facts about such dynamical systems.

Let  $(X, d)$  be a compact metric space with metric  $d$ . Let  $v$  be a flow on  $X$ , whose fixed point set  $S_v$  has finitely many connected components.

In this paper, we always assume that  $\mathcal{A}$  is an isolated invariant set of  $v$  containing  $S_v$  and has finitely many connected components. Denote its connected components (including  $S_v$ ) by  $A_i, i = 1, 2, \dots, n$ .

Let  $A_i \in \mathcal{A}$  be a component with an isolating neighborhood  $U_i$ , then for any given closed neighborhood  $V_i$  of  $A_i$  in  $U_i$ , there is a flow neighborhood (see [FJ2] for definition)  $A_i(r) := A_i(V_i, r)$ . These are compact sets in  $X$ .

Consider the flow  $v$  restricted to the open space  $X / \cup_{i=1}^n A_i(r)$ . Then all the trajectories can be classified into three types:

- (1) trajectories with domain  $(a, b)$ ,  $-\infty < a < b < +\infty$ ;
- (2) trajectories with domain  $(a, +\infty)$  or  $(-\infty, b)$ ,  $-\infty < a, b < +\infty$ ;
- (3)  $(-\infty, +\infty)$ .

We denote the sets of three type trajectories by  $\Gamma_1^A(r), \Gamma_2^A(r)$  and  $\Gamma_3^A(r)$ , respectively. If  $A_i$  is a point  $p_i$  in  $X$  for  $i = 1, \dots, n$ , then we denote  $\Gamma_\gamma^A(r)$  by  $\Gamma_\gamma(r), \gamma = 1, 2, 3$ .

If  $A \subset X$  is an open set such that  $[\alpha]|_A = 0$ , then there is a continuous function  $\beta$  such that  $\alpha = \delta\beta$ . Define  $I_\alpha(x, y) := \beta(y) - \beta(x)$ , for  $x, y \in A$ . It is obvious that  $I_\alpha(x, y)$  depends only on  $[\alpha]$ .

**Definition 2.1** *The flow  $v$  defined on a compact metric space  $(X, d)$  is said to be a generalized  $\alpha$ -flow with respect to an isolated invariant set  $\mathcal{A} = \{A_n, \dots, A_1\}$ , if there exist a continuous 1-cocycle  $\alpha$ , a small  $r > 0$  and a  $T_0 > 0$  such that for some  $\rho > 0$  and  $0 \leq \lambda < 1$ , the following conditions are satisfied:*

- 1)  $[\alpha]|_{A_i} = 0$ ,  $\max_{(x,y) \in A_i(r) \times A_i(r)} |I_\alpha(x,y)| \leq \lambda\rho$ , for  $1 \leq i \leq n$ .  
2) for any trajectory  $\gamma \in \Gamma_1^A(r)$ ,

$$\int_\gamma \alpha \geq \rho$$

3) if  $\gamma(T_0)$  denotes any sub-trajectory of  $\gamma \in \Gamma_2^A(r) \cup \Gamma_3^A(r)$  with time interval  $T_0$ , then

$$\int_{\gamma(T_0)} \alpha \geq \rho$$

**Remark 2.1** In fact, we can take the constant  $\rho$  in condition (3) to be different from the one in conditions (1) and (2). This broader assumption will not change any proofs and conclusions in [FJ2]. Since  $\alpha$  is a continuous cocycle, the three conditions in Definition 2.1 also hold for  $r'$  sufficiently close to  $r$ .

**Definition 2.2** If the set  $\mathcal{A}$  in the definition of a generalized  $\alpha$ -flow  $v$  consists only of points, then  $v$  is called an  $\alpha$ -flow. If  $v$  is an  $\alpha$ -flow or generalized  $\alpha$ -flow, we simply call it a flow carrying a cocycle  $\alpha$ .

The definition of a continuous 1-cocycle in bounded Alexander-Spanier cohomology theory and its integration along a curve or a chain was already defined in [Ch, FJ2].

Some interesting and important (generalized)  $\alpha$ -flows are given below, more examples can be found in [FJ2].

**Example 2.1** In classical Novikov theory, the closed Morse 1-forms or the Bott-type closed 1-form were considered frequently. Now in the present example, we consider the flow  $V$  generated by a closed 1-form  $\omega$  without any non-degeneracy condition, even without a Bott-type condition. This case will also introduce the Novikov type inequalities to be presented in the last section.

Take the set  $\mathcal{A}$  to be the zero locus of  $\omega$  and to have connected components  $\{A_n, \dots, A_1\}$ .

Firstly we show that any  $A \in \mathcal{A}$  is an isolated invariant set. Choose an open covering  $\mathcal{U} = \{U_\alpha\}$  of  $A$  such that  $\omega = df_\alpha$  on  $U_\alpha$ . Let  $U \subset \bigcup U_\alpha$  be a tubular neighborhood of  $A$  and  $\pi : U \rightarrow A$  be the projection. We want to show  $I(U) = A$ . Let  $p \in I(U)$ , then for any  $t_0, t_1, t_0 < t_1$ ,

$$\begin{aligned} \int_{t_0}^{t_1} |\omega(p \cdot t)|^2 dt &= \int_{[p \cdot t_0, p \cdot t_1]} \omega = \int_{[p \cdot t_0, \pi(p \cdot t_0)]} \omega \\ &+ \int_{[\pi(p \cdot t_0), \pi(p \cdot t_1)]} \omega + \int_{[\pi(p \cdot t_1), p \cdot t_1]} \omega \\ &\leq C \end{aligned}$$

We use the fact that the integration of  $\omega$  is only dependent on the relative homotopy class of  $[p \cdot t_0, p \cdot t_1]$ .

The above inequality implies that there exist sequences  $t_{0k} \rightarrow -\infty$  and  $t_{1k} \rightarrow +\infty$  such that  $|\omega(p \cdot t_{0k})| \rightarrow 0$  and  $|\omega(p \cdot t_{1k})| \rightarrow 0$  as  $k \rightarrow \infty$ . Now the above formula forces  $|\omega(p \cdot t)| = 0, \forall t \in \mathbb{R}$ . This proves that  $p \in A$ .

From the above conclusion, we can take a closed neighborhood  $V_i$  of  $A_i$  contained in an isolated neighborhood  $U_i$  s.t.  $d(\partial V_i, \partial U_i) > r_0, \min_{x \in \overline{U_i - V_i}} |\omega(x)| \geq \varepsilon_0$ . Now take the flow neighborhood  $A_i(r) := A_i(B_{2r}(A_i), r)$  for small  $r$ . Since any  $\gamma \in \Gamma_1^A(r)$  has to go across  $U_i - V_i$  for some  $i$ , we have

$$\begin{aligned} \int_{\gamma} \omega &= \int_a^b |\omega(\gamma(t))|^2 dt \\ &\geq \min_{1 \leq i \leq n} \left( \min_{x \in \overline{U_i - V_i}} |\omega(x)| \cdot d(\partial V_i, \partial U_i) \right) \geq r_0 \varepsilon_0 \end{aligned}$$

Now if we take  $r$  sufficiently small, then (1) and (2) holds.

To prove (3) in the definition of flows carrying a cocycle., we fix  $r$  and then choose  $T_0$  sufficiently large such that  $\forall \gamma \in \Gamma_2^A(r) \cup \Gamma_3^A(r)$ ,

$$\int_{\gamma(T_0)} \omega = \int |\omega(\gamma(t))|^2 dt \geq \frac{\min_{x \in X - B_r(A_i)}}{|\omega(x)|^2} T_0.$$

As a direct corollary, the flow generated by a closed 1-form with Bott-type non-degenerate zero sets is a generalized  $\omega$ -flow w.r.t. the zero locus of  $\omega$ .

**Example 2.2** A flow  $v$  on a manifold  $M$  is called a Morse-Smale flow (after [Sm]) if it satisfies:

(1) The chain recurrent set of  $v$  consists of a finite number of hyperbolic closed orbits and hyperbolic fixed points .

(2) The unstable manifold of any closed orbit or fixed point has transversal intersection with the stable manifold of any closed orbit or fixed point.

Smale [Sm] proved that such flows have “global” gradient-like structures and have a Morse decomposition which induces the Morse inequalities.

However, in some cases, although the nonwandering set of the flow contains only the hyperbolic periodic orbits and the hyperbolic fixed points, the flow is not a Morse-Smale flow because of the existence of “cycles” which consist of some orbits “connecting” different invariant sets and form a closed curve. We can give a definition of such flows when restricted to the category of flows carrying a cocycle.

**$\alpha$ -Morse-Smale flow** *Let  $v$  be a generalized  $\alpha$ -flow with respect to an isolated invariant set  $\mathcal{A} = \{A_n, \dots, A_1\}$ . If  $\mathcal{A}$  contains only the hyperbolic orbits or hyperbolic fixed points, then  $v$  is called an  $\alpha$ -Morse-Smale flow.*

Example 3.3.9 in [FJ2] provides concrete  $\alpha$ -Morse-Smale flows.



**Example 2.3 (Flows carrying a cohomology class)** We consider an extreme case namely that  $\mathcal{A}$  is empty in the definition of an  $\alpha$ -flow  $v$ . In this case, the set  $\Gamma_1(r) \cup \Gamma_2(r) = \emptyset$  and the only condition that makes  $v$  an  $\alpha$ -flow is that there exist constants  $\rho > 0$  and  $T_0 > 0$  such that for any trajectory  $\gamma(T_0)$  with time interval  $T_0$ , we have

$$\int_{\gamma(T_0)} \alpha \geq \rho$$

Now the  $\alpha$ -flow  $v$  becomes a so called “flow carrying a cohomology class” as introduced by R.C.Churchill [Ch]. The reason that the flow is called “carrying a cohomology class” is that the above condition is independent of the choice of the representative in the cohomology class  $[\alpha]$ . In fact, if  $\alpha_1 \in [\alpha]$  is another cocycle, then there exists a coboundary  $\delta\beta$ , such that

$$\alpha_1 - \alpha \simeq \delta\beta$$

and so

$$\begin{aligned} \int_{\gamma(kT_0)} \alpha_1 &= \int_{\gamma(kT_0)} \alpha + \beta(e(\gamma(kT_0))) - \beta(s(\gamma(kT_0))) \\ &\geq k\rho - 2M_\beta \end{aligned}$$

where  $s(\gamma)$  and  $e(\gamma)$  are the start point and the end point of the trajectory  $\gamma$ , and  $M_\beta$  is the bound of  $\beta$ . Hence if we choose  $k > \lceil \frac{2M_\beta + \rho}{\rho} \rceil + 1$ , then we have

$$\int_{\gamma(kT_0)} \alpha_1 \geq \rho.$$

The existence of a flow carrying a cohomology class in a manifold will induce the vanishing of the Novikov numbers. This result will be given in the last section.

### 3. $\alpha$ -flows and gradient-like flows

After defining the  $\alpha$ -flows, it is natural to find the relation between  $\alpha$ -flows and the well-known gradient-like flows. It seems that an  $\alpha$ -flow for  $\alpha$  being a non-trivial cocycle should be a nongradient-like flow. However the following simple example shows that this is not true.

**Example 3.1** Let  $S^1$  be the unit circle with the standard metric. Let  $\varphi_1$  and  $\varphi_2$  be two strictly monotonically increasing functions in the interval  $[0, \pi]$  satisfying the following conditions

$$\begin{aligned} \varphi_1(0) = 0, \varphi_1'(0) = 0; \varphi_1(\pi) = 3\pi, \varphi_1'(\pi) = 0 \\ \varphi_2(0) = 0, \varphi_2'(0) = 0; \varphi_2(\pi) = \pi, \varphi_2'(\pi) = 0. \end{aligned}$$

Define a smooth circle-valued function

$$f(x) = \begin{cases} e^{i\varphi_1(x)} & 0 \leq x \leq \pi \\ e^{i\varphi_2(2\pi-x)} & \pi \leq x \leq 2\pi. \end{cases}$$

Consequently,  $f(x)$  induces a cocycle  $\alpha(x, y) := J(f)(x, y) : S^1 \times S^1 \rightarrow \mathbb{R}$  such that if  $y$  lies in a small neighborhood of  $x$ , then

$$\int_{[x,y]} \alpha = \arg f(y) - \arg f(x).$$

Here the map  $J$  is defined in proposition 3.1.1 of [FJ2]. This cocycle is a nontrivial cocycle. Since if we let  $\gamma$  be the oriented curve starting from the point  $\theta = 0$  and then going around the circle in the anticlockwise direction, then for any integer  $l$ , we have

$$\int_{l\gamma} \alpha = 2\pi l.$$

Now we consider the gradient-like flow  $v$  on  $S^1$  that has two fixed points at  $\theta = 0, \pi$ , and flows from the point  $\theta = 0$  to the point  $\theta = \pi$ . Then it is easy to check that  $v$  carries the cocycle  $\alpha$  with parameter  $\lambda = 0, \rho = 1$  and  $\Gamma_2(r) \cup \Gamma_3(r) = \emptyset$  in the definition 2.1.

Thus, this example demonstrates that a gradient-like flow can carry a non-trivial cocycle.

**Remark 3.1** It is easy to see that for any  $x \in S^1$ , and  $t > 0$ ,  $\int_{[x,x-t]} \alpha > 0$ . Hence the pull-back form  $f^*(d\theta)$  is a Lyapunov 1-form for the pair  $(-v, \theta = 0, \pi)$ , which is defined in [Fa3, Fa4]. Though the smooth closed 1-form  $f^*(d\theta)$  is a nontrivial smooth closed 1-form,  $-v$  is a gradient-like flow.

We will give a series of theorems and examples to formulate the relation between  $\alpha$ -flows and gradient-like flows. The following theorem is already given in [FJ2].

**Theorem 3.1** *Let  $v$  be an  $\alpha$ -flow on the compact metric space  $(X, d)$ . If  $\alpha$  is a trivial cocycle, then  $v$  is a gradient-like flow. Conversely, if  $v$  is a gradient-like flow, then  $v$  is a  $\delta g$ -flow for some continuous function  $g$ .*

We will give necessary and sufficient conditions to distinguish  $\alpha$ -flows and gradient-like flows. Firstly we study the local structure of a gradient-like flow.

**Local structure of gradient-like flows** Let  $p$  be a fixed point of a gradient-like flow  $v$ . Then there exists an associated Lyapunov function  $g(x)$  on a small neighborhood  $\bar{B}_r(p)$ .

Let  $0 < s < \frac{r}{2}$  and let  $B_{\frac{r}{2}}(p)$  be a closed ball centered at  $p$  with radius  $\frac{r}{2}$ .

Define two sets on the sphere  $\partial B_r(p)$ ,

$$\begin{aligned} B_{r,s}^+(p) &= \{x \in \partial B_r(p) \mid [x, x \cdot t] \cap \partial B_s(p) \neq \emptyset \text{ for some } t > 0\} \\ B_{r,s}^-(p) &= \{x \in \partial B_r(p) \mid [x, x \cdot t] \cap \partial B_s(p) \neq \emptyset \text{ for some } t < 0\} \end{aligned}$$

For any point  $x \in B_{r,s}^+(p)$ , let  $t_x$  denotes the reach time of the trajectory  $[x, x \cdot t](t > 0)$  to the sphere  $\partial B_{\frac{r}{2}}(p)$ . We have the following lemma.

**Lemma 3.1** (1)  $B_{r,s}^\pm(p)$  are closed sets on  $\partial B_r(p)$ .  
(2)  $t_x$  is a lower semicontinuous function on  $B_{r,s}^+(p)$ .

Proof.  $B_{r,s}^\pm(p)$  are actually the boundaries of the flow neighborhoods  $A^\pm(B_r, s)$  defined in [FJ2]. Hence they are closed. For convenience, we give a proof here. To prove (1), we need only to prove that the set  $\partial B_r(p) - B_{r,s}^+(p)$  is open in  $\partial B_r(p)$  and will finally drop into another fixed point  $q$ . In terms of the choice of  $g(x)$ , we can choose  $a < b$  satisfying

$$B_r(p) \subset g^{-1}([b, +\infty]); B_r(q) \subset g^{-1}((-\infty, a]) \quad (3.1)$$

Since  $x \cdot t(t > 0)$  drops into  $q$ , there exists a  $T_x > 0$  such that  $x \cdot T_x$  is in the interior of  $B_r(q)$ . Therefore there is a closed ball  $D_\delta(x)$  such that for any  $y \in D_\delta(x) \cap \partial B_r(p)$ ,  $y \cdot T_x$  is in the interior of  $B_r(q)$ . We can choose  $\delta$  small enough such that the set  $D_\delta(x) \cdot [0, T_x] \cap \partial B_s(p) = \emptyset$ . Since  $y \cdot T_x \in g^{-1}((-\infty, a])$ ,  $y \cdot [T_x, +\infty) \in g^{-1}((-\infty, a])$ . Therefore  $D_\delta(x) \cdot [0, +\infty) \cap \partial B_s(p) = \emptyset$  in view of (3.1). This shows that  $\partial B_r(p) - B_{r,s}^+(p)$  is open in  $\partial B_r(p)$ .

To prove (2), let  $x \in B_{r,s}^+(p)$  and let  $t_x < +\infty$  be the arrival time. We need only to prove that for any small  $\epsilon > 0$ , there exists a neighborhood  $D_\delta(x)$  of  $x$  such that for any  $y \in D_\delta(x) \cap B_{r,s}^+(p)$ ,

$$t_y \geq t_x - \epsilon \quad (3.2)$$

Since  $t_x$  is the arrival time of the trajectory  $x \cdot t(t > 0)$  to  $\partial B_{\frac{r}{2}}(p)$ ,  $x \cdot [0, t_x - \epsilon]$  has a positive distance from  $\partial B_{\frac{r}{2}}(p)$ . By the continuity of the flow  $v$ , there is a neighborhood  $D_x$  such that  $\bar{D}_x \cdot [0, t_x - \epsilon]$  has a positive distance from  $\partial B_{\frac{r}{2}}(p)$  as well. Hence for  $y \in D_x \cap B_{r,s}^+(p)$ , (3.2) is true.  $\blacksquare$

Since the Lyapunov function  $g(x)$  is strictly decreased along any nonconstant trajectory, there exists a  $\delta_x > 0$  such that  $g(x) - g(x \cdot t_x) = 2\delta_x$ . By lemma 3.1, (2) and the property of  $g(x)$ , the function  $g(y) - g(y \cdot t_y)$  is lower semicontinuous with respect to  $y$  on  $B_{r,s}^+(p)$ . Hence for each  $x \in B_{r,s}^+(p)$ , there exists a neighborhood  $U_x$  in  $B_{r,s}^+(p)$  such that  $\forall y \in U_x$ ,  $g(y) - g(y \cdot t_y) > \delta_x$ . This implies that  $\delta_{r,s}^+(p) := \min_{x \in B_{r,s}^+(p)} \delta_x$  is positive and  $\forall x \in B_{r,s}^+(p)$ ,  $g(x) - g(x \cdot t_x) > \delta_{r,s}^+(p)$ . In the same way, we can obtain a  $\delta_{r,s}^-(p) > 0$  such that  $\forall x \in B_{r,s}^-(p)$ , there is

$$g(x \cdot (-t_x)) - g(x) > \delta_{r,s}^-(p).$$

Let  $0 < s_1 < s_2 < \frac{r}{2}$ , then we have the following conclusions:

- (1)  $B_{r,s_1}^+(p) \subset B_{r,s_2}^+(p); B_{r,s_1}^-(p) \subset B_{r,s_2}^-(p)$
- (2)  $\delta_{r,s_2}^\pm(p) \leq \delta_{r,s_1}^\pm(p)$
- (3) There exists  $s_0 > 0$ , such that for any  $0 < s < s_0$ ,  $B_{r,s}^+(p) \cap B_{r,s}^-(p) = \emptyset$

The first two conclusions are obvious. For (3), we can choose  $s_0 > 0$  satisfying  $\text{osc}_{x \in B_{s_0}(p)} g(x) < \min \delta_{r,s_0}^\pm(p)$ . If there is a  $0 < s < s_0$  such that  $B_{r,s}^+(p) \cap B_{r,s}^-(p) \neq \emptyset$ , then this means that there is a trajectory in  $\bar{B}_r(p)$  which has non empty intersection with  $\partial B_r(p)$  and  $\partial B_s(p)$ . But this is absurd, because  $g(x)$  is strictly decreasing along any trajectory. After travelling from  $\partial B_s(p)$  to  $\partial B_r(p)$ , then back to  $\partial B_s(p)$ , the value of  $g(x)$  will decrease at least  $\delta_{r,s}^+(p) + \delta_{r,s}^-(p)$  which contradicts the fact that  $\text{osc}_{x \in B_s(p)} g(x) \leq \text{osc}_{x \in B_{s_0}(p)} g(x) < \min \delta_{r,s}^\pm(p)$ . Hence for  $0 < s < s_0$ , (3) holds.

**Integration of cocycles in a gradient-like flow** In this part we assume that  $v$  is a gradient-like flow with finitely many fixed points  $\{p_j; j = 1, 2, \dots, n\}$ . Let  $\alpha$  be a cocycle on  $X$ . Then there is an associated  $r$ -covering  $\mathcal{U}(r)$  (covering consisting of radius  $r$ -balls) such that when restricted to the closure of each open ball in  $\mathcal{U}(r)$ ,  $\alpha$  is a coboundary  $\delta\beta$ , where  $\beta$  is a bounded function. Let  $r$  be small enough such that the Lyapunov function  $g(x)$  corresponding to the flow  $v$  has mutually non-intersecting images  $g(B_r(p_j))$  for  $j = 1, 2, \dots, n$ .

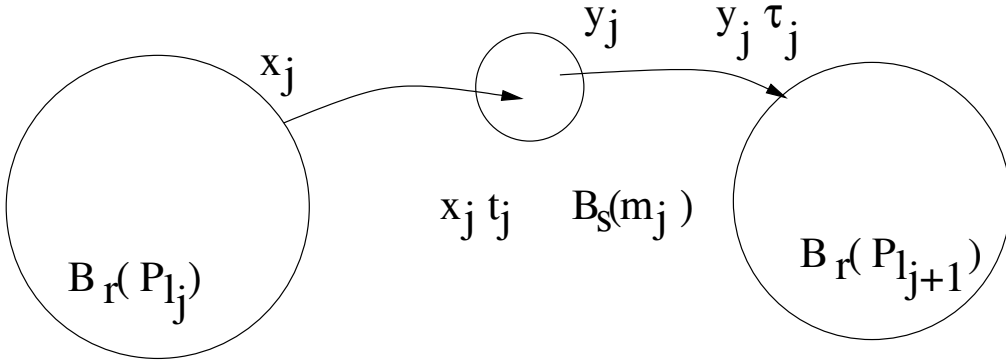
Define  $\delta_0(r) = \min_{1 \leq j \leq n} \delta_{r, \frac{r}{4}}^\pm(p_j)$ . Since  $g(x)$  is uniformly continuous, there exists  $s_0 \leq \frac{r}{4}$  satisfying  $\text{osc}_{B_{s_0}} g(x) < \frac{1}{3} \delta_0(r)$ . By our choice and in view of the analysis of the local structure of gradient-like flows, it is easy to see that the following inequality holds:

$$3 \max_x \text{osc}_{y \in B_{s_0}(x)} g(y) < \delta_0(r) < \delta_{r, \frac{r}{4}}^\pm(p_j) \leq \delta_{r, s_0}^\pm(p_j) \quad (3.3)$$

for  $j = 1, 2, \dots, n$ .

**Theorem 3.2** *Let  $v$  be a gradient-like flow with finitely many fixed points.  $\alpha$  is a cocycle with an associated  $r$ -covering. Let  $\mathcal{U}(s)$  be an  $s$ -covering with  $s < s_0 < \frac{r}{4}$  for some  $s_0$  satisfying (3.3). Then there exist  $M > 0$  and  $T(s_0) > 0$  depending on  $r, n, \alpha$  but not on  $s$  such that for any  $(\mathcal{U}(s), T(s_0))$ -chain  $\tilde{\gamma}$ , we have*

$$\left| \int_{\tilde{\gamma}} \alpha \right| \leq M \quad (3.4)$$



$(r, s_0)$ -chain

Recall that a  $(\mathcal{U}(s), T)$ -chain from  $x$  to  $x'$  is a sequence  $\{x = x_1, \dots, x_{n+1} = x' | t_1, \dots, t_n\}$  such that  $t_i \geq T$  and each pair  $(x_i \cdot t_i, x_{i+1}) (i = 1, \dots, n)$  belongs to a ball with radius  $s$  in the covering  $\mathcal{U}$ .

**Proof** Firstly we will prove that given any  $s \leq r$ , for any trajectory  $\gamma \in \Gamma_1(s)$  with domain  $[a, b]$ , there exists a constant  $T(s)$  satisfying  $b - a \leq T(s)$ .

Let  $x \in \partial B_s(p_j)$ , then the trajectory  $x \cdot t$  or  $x \cdot (-t)$  for  $t \geq 0$  will flow into some different fixed point of  $v$ . (There does not exist any trajectory joining one point to itself.) Without loss of generality, we assume that  $(x \cdot t) (t \geq 0)$  flows into the point  $p_k$  for  $k \neq j$ . (It is possible that the trajectory may pass through some ball  $B_s(p_l)$ , but this does not influence the result below.) Hence there exists a  $T_x > 0$  such that  $x \cdot T_x \cap B_{\frac{s}{2}}(p_k) \neq \emptyset$ . By the continuity of the flow, there exists a small closed ball  $D_{s_x}(x)$  which satisfies that for any point  $y \in D_{s_x}(x) \cap \partial B_s(p_j)$ , the trajectory  $[y, y \cdot T_x] \cap \partial B_s(p_k) \neq \emptyset$ . This shows that for any trajectories starting from  $D_{s_x}(x) \cap \partial B_s(p_j)$  their time intervals are not greater than  $T_x$ , and covering the sphere  $\partial B_s(p_j)$  by finitely such small closed balls, then it is easy to see that any trajectory in  $\Gamma_1(s)$  starting from  $\partial B_s(p_j)$  has time interval not greater than a constant  $T_j(s)$ . Let  $T(s) = \max_{1 \leq j \leq n} T_j(s)$ , then all the trajectories in  $\Gamma_1(s)$  have time interval not greater than  $T(s)$ .

Let  $\gamma = \{x_0, y_0, x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_k | t_0, \tau_0, \dots, t_{k-1}, \tau_{k-1}\}$  be a chain. It is called an  $(r, s_0)$ -chain if it satisfies the following conditions:

- (1)  $x_j \in \partial B_r(p_{l_j})$  for  $j = 0, 1, \dots, k - 1$  and  $x_k \in B_r(p_{l_k})$ ;
- (2)  $y_j \cdot \tau_j \in \partial B_r(p_{l_{j+1}})$  for  $j = 0, 1, \dots, k - 1$ ;
- (3) the point pair  $(x_j \cdot t_j, y_j) \in B_s(m_j)$ , for  $j = 0, 1, \dots, k - 1$ , where those  $B_s(m_j)$ 's are elements in the  $s$ -covering  $\mathcal{U}(s)$  of  $X$ .
- (4)  $B_s(p_{l_j}), j = 0, 1, \dots, k$  are  $k$  different balls, hence  $k \leq n - 1$ .

(5)  $t_j, \tau_j \leq T(s_0)$ , where  $T(s_0)$  is an upper bound for the time interval of all the trajectories in  $\Gamma_1(s_0)$ .

Now compute the integral of  $\alpha$  along the  $(r, s_0)$ -chain  $\gamma$ .

$$\begin{aligned} \left| \int_{\tilde{\gamma}} \alpha \right| &\leq \sum_{j=0}^{k-1} \left| \int_{[x_j, x_j \cdot t_j]} \alpha \right| + \sum_{j=0}^{k-1} \left| \int_{[y_j, y_j \cdot \tau_j]} \alpha \right| \\ &\quad + \sum_{j=0}^{k-1} \left| \int_{[x_j \cdot t_j, y_j]} \alpha \right| + \sum_{j=0}^{k-1} \text{osc}_{x \in B_r(p_{l_{j+1}})} \beta_{p_{l_{j+1}}} \\ &= I + II + III + M \end{aligned} \tag{3.5}$$

Since  $0 < t_j, \tau_j \leq T(s_0)$ , applying corollary 3.2.3 in [FJ2],

$$I + II \leq 2kC_1$$

where  $C_1$  is a constant depending only on  $\max |\dot{v}(t)|, \alpha$  and  $r$ .

Connect the point pair  $(x_j \cdot t_j, y_j)$  with a line segment, then its length is at most  $2s$ . Using proposition 3.2.2 in [FJ2], we have

$$III \leq kC_2$$

Here  $C_2$  depends only on  $s_0, \alpha$  and  $r$ .

For  $M$ ,

$$M \leq 2kM_\alpha(r)$$

Combining the above estimates, we have

$$\left| \int_{\tilde{\gamma}} \alpha \right| \leq kC \tag{3.6}$$

where  $C$  depends on  $r, s_0, \alpha$  and  $\max |\dot{v}(t)|$ .

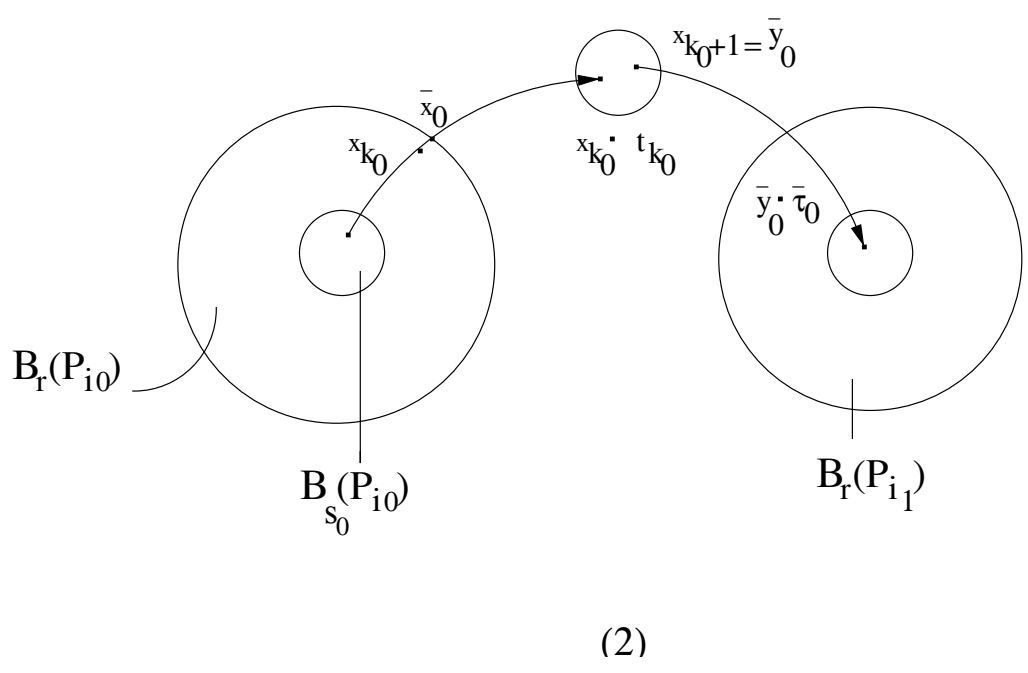
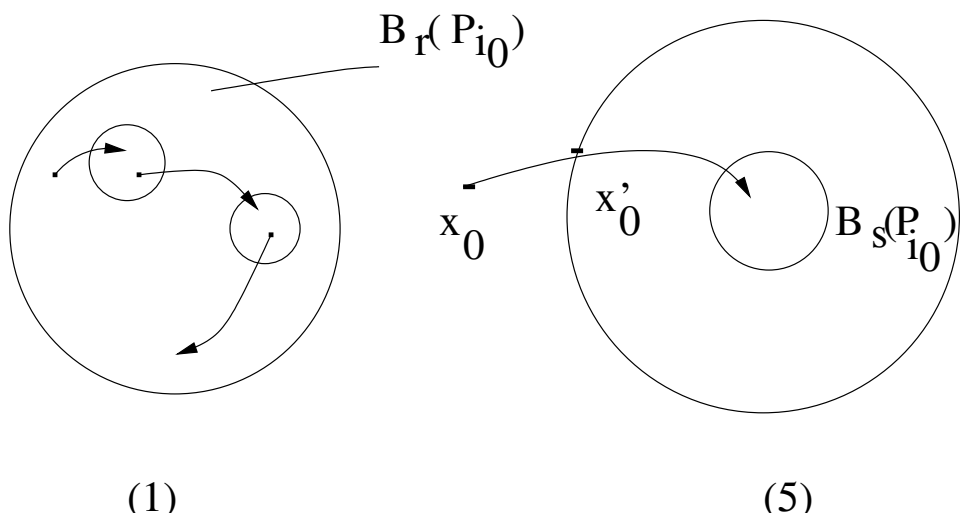
Now the proof of the proposition is changed to the problem to reduce each  $(\mathcal{U}(s), T(s_0))$ -chain to an  $(r, s_0)$ -chain, while keeping the integral of  $\alpha$  invariant.

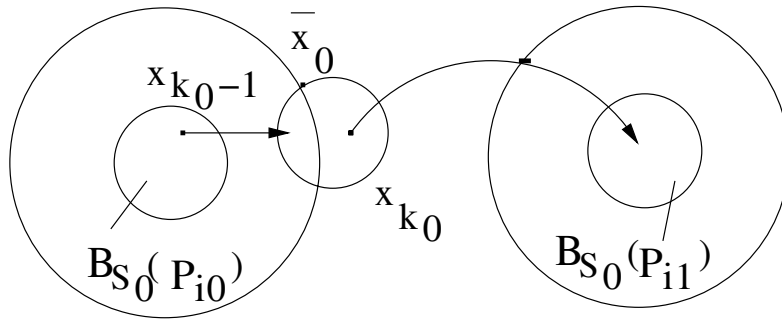
Let  $\tilde{\gamma} = \{x_0, x_1, \dots, x_k | t_0, \dots, t_{k-1}\}$  be a  $(\mathcal{U}(s), T(s_0))$ -chain. Consider the following cases.

(1) If the chain  $\tilde{\gamma}$  is contained in a ball  $\bar{B}_r(p_{i_0})$  for  $0 \leq i_0 \leq n$ , then

$$\begin{aligned} \left| \int_{\tilde{\gamma}} \alpha \right| &= |\beta_{p_{i_0}}(x_k) - \beta_{p_{i_0}}(x_0)| \\ &\leq \text{osc}_{x \in B_r(p_{i_0})} \beta_{p_{i_0}}(x) \leq 2M_\alpha(r) \end{aligned}$$

(2) There is  $1 \leq k_0 \leq k$  such that the subchain  $\tilde{\gamma}_{0k_0} = \{x_0, x_1, \dots, x_{k_0} | t_0, \dots, t_{k_0-1}\}$  is contained in  $B_r(p_{i_0})$  but the point  $x_{k_0} \cdot t_{k_0}$  is not in  $\cup_{j=0}^n \bar{B}_r(p_j)$ . Hence there is a first intersection point  $\bar{x}_0$  of  $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$  and  $\partial B_r(p_{i_0})$ . Let  $\bar{t}_0$  satisfy





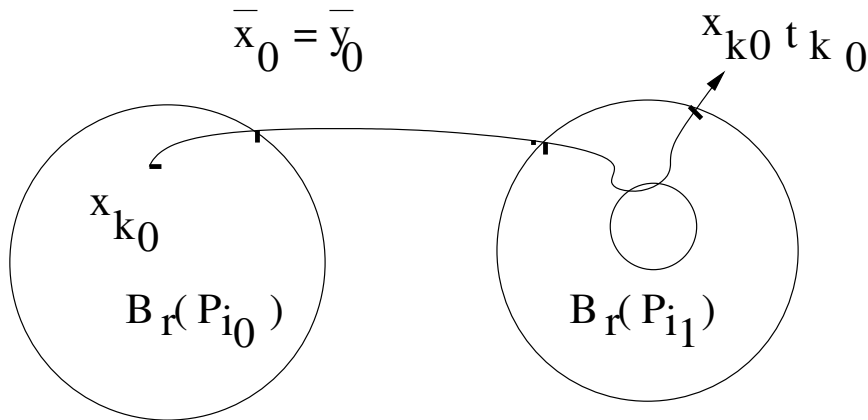
(3)

$\bar{x}_0 \cdot \bar{t}_0 = x_{k_0} \cdot t_{k_0}$  and let  $\bar{y}_0 = x_{k_0+1}$ . Denote the ball containing the point pair  $(\bar{x}_0 \cdot \bar{t}_0, \bar{y}_0)$  by  $B_s(m_0)$ . Since  $t_{k_0} \geq T(s_0)$  and  $\tilde{\gamma}_{0k_0} \subset B_r(p_{i_0})$ , this implies that  $x_{k_0} \in B_{s_0}(p_{i_0})$ , hence the trajectory  $[\bar{x}_0, \bar{x}_0 \cdot \bar{t}_0]$  does not intersect with  $B_{s_0}(p_{i_0})$ . Otherwise,  $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$  starts from  $B_{s_0}(p_{i_0})$ , intersects  $\partial B_r(p_{i_0})$ , then goes back to  $B_{s_0}(p_{i_0})$ . This conclusion contradicts (3.3) when we check the change of the Lyapounov function  $g(x)$  along  $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$ . Therefore  $[\bar{x}_0, \bar{x}_0 \cdot \bar{t}_0]$  is part of a trajectory in  $\Gamma_1(s_0)$ , and we have  $\bar{t}_0 \leq T(s_0)$ . Since  $\bar{x}_0 \cdot \bar{t}_0 = x_{k_0} \cdot t_{k_0}$  is not in  $\bar{B}_r(p_{i_0})$ , by the choice of  $s$ , we know that  $\bar{y}_0 := x_{k_0+1} \notin B_{s_0}(p_{i_0})$ . Hence  $\bar{y}_0 \cdot t_{k_0+1}$  meets firstly  $B_{s_0}(p_{i_1})$  for some fixed point  $p_{i_1}$ . Let  $\bar{y}_0 \cdot \bar{\tau}_0$  be the last intersection point of  $[\bar{y}_0, \bar{y}_0 \cdot t_{k_0+1}]$  with  $\partial B_r(p_{i_1})$ . In the same way, we can prove that  $\bar{\tau}_0 \leq T(s_0)$ . Joining  $x_0$  and  $\bar{x}_0$  by a curve  $l_{x_0\bar{x}_0}$ , then the subchain  $\tilde{\gamma}_{0k_0}$  is reduced to a curve  $l_{x_0\bar{x}_0}$  combined with a chain  $\{\bar{x}_0, \bar{y}_0 | \bar{t}_0, \bar{\tau}_0\}$  and it is clear that the reduction keeps the integral of  $\alpha$  invariant.

(3) There is a  $2 \leq k_0 \leq k$  such that the subchain  $\tilde{\gamma}_{0,k_0-1} = \{x_0, x_1, \dots, x_{k_0-1} | t_0, t_1, \dots, t_{k_0-2}\}$  and the trajectory  $[x_{k_0-1}, x_{k_0-1} \cdot t_{k_0-1}]$  is contained in  $B_r(p_{i_0})$  but  $x_{k_0} \notin B_r(p_{i_0})$ . Then we take a point  $\bar{x}_0 \in \partial B_r(p_{i_0}) \cap B_s(m_0)$ , where  $B_s(m_0)$  is the ball containing  $(x_{k_0-1} \cdot t_{k_0-1}, x_{k_0})$ . Since  $t_{k_0} \geq T(s_0)$ , the trajectory  $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$  will meet  $B_{s_0}(p_{i_1})$ . We denote  $x_{k_0}$  by  $\bar{y}_0$  and represent the last intersection point of  $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$  with  $\partial B_r(p_{i_1})$  by  $\bar{y}_0 \bar{\tau}_0$  for some  $\bar{\tau}_0 > 0$ . Due to the same argument as in (2), we know that  $\bar{\tau}_0 \leq T(s_0)$ . So  $\tilde{\gamma}_{0k_0}$  is reduced to the combination of a curve  $l_{x_0\bar{x}_0}$  and a chain  $\{\bar{x}_0, \bar{y}_0 | 0, \bar{\tau}_0\}$ . It is clear that such inductions also keep the integral of  $\alpha$  invariant. Note that  $x_{k_0-1}$  must be in  $B_{s_0}(p_{i_0})$ .

(4) There is a subchain  $\tilde{\gamma}_{0k_0} = \{x_0, x_1, \dots, x_{k_0} | t_0, \dots, t_{k_0-1}\}$  contained in  $B_r(p_{i_0})$  and the trajectory  $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$  goes through  $B_{s_0}(p_{i_1})$ . Let  $\bar{x}_0 = \bar{y}_0$  be the first intersection point of  $[x_{k_0}, x_{k_0} \cdot t_{k_0}]$  with  $\partial B_r(p_{i_1})$ . Also we can prove that  $\bar{\tau}_0 \leq T(s_0)$  as in (2) and (3). Then the chain  $\tilde{\gamma}_{0k_0}$  followed by  $[x_{k_0}, \bar{y}_0 \cdot \bar{\tau}_0]$  is reduced to the combination of  $l_{x_0\bar{x}_0}$  with the chain  $\{\bar{x}_0, \bar{y}_0 | 0, \bar{\tau}_0\}$ , while the integral of  $\alpha$  is invariant under change.





(4)

(5) If  $x_0 \notin \cup_{j=1}^n B_r(p_j)$ , then  $x_0 \cdot t_0$  will firstly meet some  $B_s(p_{i_0})$ . Let  $x'_0 = x_0 \cdot t'_0$  be the last intersection point of  $[x_0, x_0 \cdot t_0]$  with  $\partial B_r(p_{i_0})$ . In the same way, we can prove  $t'_0 \leq T(s_0)$ .

Now using the above steps (2)-(5) repeatedly, any  $(\mathcal{U}(s), T(s_0))$ -chain  $\tilde{\gamma} = \{x_0, x_1, \dots, x_k | t_0, \dots, t_{k-1}\}$  can be reduced to the combination of a curve  $l_{x_0 \bar{x}_0}$  (or  $[x_0, x_0 \cdot t'_0]$  in case (5)) with an  $(r, s_0)$ -chain  $\bar{\gamma} = \{\bar{x}_0, \bar{y}_0, \dots, \bar{x}_l | \bar{t}_0, \bar{\tau}_0, \dots, \bar{\tau}_{l-1}\}$  if we can prove that (4) in the definition of a  $(r, s_0)$ -chain holds.

If (4) is not true, then there exists an  $(r, s_0)$ -cycle  $\bar{\gamma}_c = \{\bar{x}_{i_0}, \bar{y}_{i_0}, \dots, \bar{y}_{i_l}, \bar{x}_{i_l} | \bar{t}_{i_0}, \bar{\tau}_{i_0}, \dots, \bar{\tau}_{i_l}\}$  where the point pair  $(\bar{y}_{i_l} \cdot \bar{\tau}_{i_l}, \bar{x}_{i_0}) \in \partial B_r(p_{i_0})$ . Now extend the trajectory  $\bar{y}_{i_j} \cdot t (t \geq 0)$  for  $j = 0, \dots, l$ . Since  $\bar{\tau}_c$  is obtained by reducing the  $(\mathcal{U}(s), T(s_0))$ -chain, hence  $\bar{y}_{i_j} \cdot t$  will go through  $B_{s_0}(p_{i_{j+1}})$ . We let  $\bar{y}_{i_j} \cdot \tau_{i_j} (\tau_{i_j} > \bar{\tau}_{i_j})$  be a point in  $B_{s_0}(p_{i_{j+1}})$ . Take the inverse process corresponding to (2)-(4), then there is a trajectory starting from some point  $x_{i_{j-1}} \in B_{s_0}(p_{i_{j-1}})$  to some point  $x_{i_{j-1}} \cdot t_{i_{j-1}}$  in the  $s$ -ball containing  $\bar{x}_{i_{j-1}} \cdot \bar{t}_{i_{j-1}}$  and  $\bar{y}_{i_j}$ . Therefore from the cycle  $\bar{\gamma}_c$  we get a  $(\mathcal{U}(s), T(s_0))$ -chain  $\tilde{\gamma}_c = \{x_{i_0}, \bar{y}_{i_0}, \dots, \bar{y}_{i_l}, x_{i_0} | t_{i_0}, \tau_{i_0}, \dots, t_{i_l}, \tau_{i_l}\}$ .

Consider the change of the function  $g(x)$  along  $\tilde{\gamma}_c$ . On one hand, since the two ends of  $\tilde{\gamma}_c$  are  $x_{i_0}$  and  $\bar{y}_{i_l} \cdot \tau_{i_l}$ , by (3.3)

$$\left| \int_{\tilde{\gamma}_c} \delta g \right| \leq \text{osc}_{x \in B_{s_0}(p_{i_0})} g(x) < \frac{1}{3} \delta_0(r)$$

On the other hand, each  $\bar{y}_{i_j} \cdot t (0 < t \leq \tau_{i_j}) (j = 0, \dots, l)$  goes across  $B_r(p_{i_j}) \setminus B_{\frac{r}{4}}(p_{i_j})$ , we have

$$\int_{\tilde{\gamma}_c} \delta g \geq (l+1) \delta_{r, \frac{r}{4}}^+ - 2(l+1) \max_x \text{osc}_{y \in B_s(x)} g(y) > \delta_0(r) - \frac{2}{3} \delta_0(r) = \frac{1}{3} \delta_0(r)$$

This is absurd. The contradiction shows that the chain  $\bar{\gamma}$  we get from a  $(\mathcal{U}(s), T(s_0))$ -chain  $\tilde{\gamma}$  is indeed an  $(r, s_0)$ -chain. Let  $\bar{\gamma} = \{\bar{x}_0, \bar{y}_0, \dots, \bar{x}_l | \bar{t}_0, \bar{\tau}_0, \dots, \bar{\tau}_{l-1}\}$ . Since

$v$  is gradient-like, the index  $l$  is less than  $n$ . Therefore applying the estimate (3.6) and the fact that the reduction from  $\tilde{\gamma}$  to  $\bar{\gamma}$  keeps the integral of  $\alpha$ , we have

$$\left| \int_{\tilde{\gamma}} \alpha \right| = \left| \int_{\bar{\gamma}} \alpha \right| \leq nC.$$

where  $C$  depends on  $r, s_0, \alpha$  and  $\max |\dot{v}(t)|$ . Theorem 3.2 now is proved.  $\blacksquare$

**Theorem 3.3** *Let  $v$  be an  $\alpha$ -flow on the compact metric space  $(X, d)$ . Let  $\{s_i\} \rightarrow 0$  as  $i \rightarrow \infty$ . If  $\alpha$  is a nontrivial cocycle and for any  $M > 0$  and  $T > 0$ , there is a  $(\mathcal{U}(s_i), T)$ -chain  $\tilde{\gamma}_i$  for each  $s_i$  such that*

$$\left| \int_{\tilde{\gamma}_i} \alpha \right| \geq M, \quad (3.7)$$

*then  $v$  is not a gradient-like flow. Furthermore, if it is known that for any  $M > 0$ , there is a trajectory  $\tilde{\gamma}$  satisfying (3.7) or there is an oriented cycle  $\tilde{\gamma}_0$  consisting of some orbits joining fixed points where the direction is determined by the forward direction of the flow  $v$  such that*

$$\left| \int_{\tilde{\gamma}} \alpha \right| \geq 1, \quad (3.8)$$

*then  $v$  is not a gradient-like flow.*

*Proof.* The first conclusion is a direct corollary of Theorem 3.2 and the third one is obvious. We only consider the second case. It is easy to see that if the time intervals of the trajectories have an upper bound, then for any cocycles on  $X$  the absolute value of the integral,  $|\int_{\tilde{\gamma}} \alpha|$  has a uniform bound. Therefore the trajectory  $\tilde{\gamma}$  with respect to the arbitrary large  $M$  has arbitrary large time interval and it is the chain needed for the hypothesis in the first conclusion.  $\blacksquare$

**Theorem 3.4** *Let  $v$  be an  $\alpha$ -flow with a nontrivial cocycle  $\alpha$  on a compact metric space  $(X, d)$ . If  $v$  is not a gradient-like flow, then for any small sequence  $\{s_i\}$  which tends to zero as  $i \rightarrow \infty$ , and for any  $M > 0, T > 0$  there is a  $(\mathcal{U}(s_i), T)$ -chain  $\tilde{\gamma}_i$  for each  $s_i$  such that*

$$\left| \int_{\tilde{\gamma}_i} \alpha \right| \geq M \quad (3.9)$$

*Proof* If  $v$  is not a gradient-like flow, then there is a non fixed point  $x_0$  in the chain recurrent set of  $v$ . Therefore for any  $s$  and  $T > 0$  there is a  $(\mathcal{U}(s), T)$ -chain  $\tilde{\gamma}_i = \{x_0, \dots, x_k = x_0 | t_0, \dots, t_{k-1}\}$ .

Now we consider two possibilities:

(1). If there exists  $x_i$  such that  $x_i \cdot t \notin \cup_{i=1}^n p_i(\frac{r}{4})$ , where  $p_i(r) := A_i(V_i, r)$  is the flow neighborhood of  $p_i$ .

There are two cases. The first case is that after meeting finite flow neighborhoods  $p_i(r)$ ,  $x_i \cdot t$  becomes a flow line in  $\Gamma_2 \cup \Gamma_3$ . Then

$$\int_{[x_i, x_i \cdot t]} \alpha \geq [\frac{t - \tau}{T_0}] \rho - C,$$

where  $\tau, C > 0$  are constants. Hence we can take the trajectory  $[x_i, x_i \cdot t]$  as the required  $(\mathcal{U}(s_i), T)$ -chain for  $s_i = 0$  and  $T$  arbitrary large. The second case is that  $x_i \cdot t$  can meet infinitely many flow neighborhoods  $\{p_{i_0}(r), \dots, p_{i_j}(r), \dots\}$ , then

$$\int_{[x_i, x_i \cdot t]} \alpha \geq \sum_{k=0}^{l(t)-1} \int_{[x_i, x_i \cdot t]_k} \alpha - \sum_{k=1}^{l(t)-1} \text{osc}_{x \in p_{i_k}(r)} \beta_{p_{i_k}}(x),$$

where  $l(t)$  is an integer representing the number of flow neighborhoods that  $[x_i, x_i \cdot t]$  has met and  $[x_i, x_i \cdot t]_k \in \Gamma_1$  is a trajectory of  $[x_i, x_i \cdot t]$  between  $p_{i_k}(r)$  and  $p_{i_{k+1}}(r)$ . We have

$$\int_{[x_i, x_i \cdot t]} \alpha \geq l(t)(1 - \lambda)\rho.$$

If  $t$  is large enough, this integral is larger than any given number. Hence we can take  $[x_i, x_i \cdot t]$  as the required chain.

(2). For any  $i$ ,  $x_i \cdot t \in \cup_{i=1}^n B_{\frac{r}{4}}(p_i)$  for some  $t$ .

Firstly there exists a  $T(\frac{r}{4})$  such that for any trajectory  $\gamma \in \Gamma_1(\frac{r}{4})$  with time interval  $[a, b]$ , where  $T(\frac{r}{4}) > b - a$  (one can use the same argument in the proof of Theorem 3.2). Consider the homeomorphism  $\varphi_{T(\frac{r}{4})} : X \rightarrow X$  which maps  $x$  to  $x \cdot T(\frac{r}{4})$ . Let  $B_s$  be any balls in  $\mathcal{U}(s)$ , then by compactness the upper bound of the diameter of  $\varphi_t(B_s), \forall t \in [0, T(\frac{r}{4})]$ , will be less than a uniform constant  $C_s < \frac{r}{4}$  if  $s$  is small enough.

Since  $x_0 \cdot t_0$  will meet the first flow neighborhood  $p_{i_0}(\frac{r}{4})$ , we can replace  $x_0$  by  $x_0 \cdot t_0$ , since  $x_0 \cdot t_0$  is also a point in the chain recurrent set of the flow  $v$ . So we assume that  $x_0 \in p_{i_0}(\frac{r}{4})$ . Now we want to modify the  $(\mathcal{U}(s), T)$ -chain (assume  $T > T(\frac{r}{4})$ )  $\gamma_i = \{x_0, \dots, x_k = x_0|t_0, \dots, t_{k-1}\}$  to another  $(\mathcal{U}(r), T(\frac{r}{4}))$ -chain  $\gamma' = \{x'_0, \dots, x'_k = x'_0|t'_0, \dots, t'_{k-1}\}$  satisfying that the pair  $(x'_j \cdot t'_j, x'_{j+1}) \in p_{i_{j+1}}(r)$  for  $j = 0, 1, \dots, k-1$ . If  $B_s(m_1)$  is contained in some  $p_{i_1}(r)$ , then we are done. Assume that  $B_s(m_1)$  is not contained in any  $p_i(r)$ . By the choice of  $T(\frac{r}{4})$ , the flow line  $[x_1, x_1 \cdot T(\frac{r}{4})]$  will intersect some flow neighborhood  $p_{i_1}(\frac{r}{4})$  at some point  $x'_1 = x_1 \cdot \tau_1 \in B_{\frac{r}{4}}(p_{i_1})$ . If  $s$  is small enough, then  $(x_0 \cdot (t_0 + \tau_1) \in B_{\frac{r}{2}}(p_{i_1}))$ , hence in  $p_{i_1}(r)$ . Let  $x'_0 = x_0, x'_1 = x_1 \cdot \tau_1, t'_0 = t_0 + \tau_1$  and  $t'_1 = t_1 - \tau_1$ . Hence we get the modified chain  $\{x'_0, x'_1, x_2, \dots, x_k|t'_0, t'_1, t_2, \dots\}$ . Now  $x'_1 \in p_{i_1}(\frac{r}{4})$ , we can continue

this operation until we get the modified chain  $\gamma'$ . If  $s$  is small, it is easy to see

$$\int_{\gamma} \alpha = \int_{\gamma'} \alpha \geq \sum_{j=0}^{k-1} \left( \int_{[x'_j, x'_j, t'_j]} -\text{osc}_{x \in p_{i_j}(r)} \beta_{p_{i_j}}(x) \right) \geq k(1 - \lambda)\rho > 0$$

So if  $n$  is large enough, the integral of  $\alpha$  along  $n\gamma$  will be larger than any given number. So (3.9) holds.  $\blacksquare$

In view of Theorem 3.2-3.4, the following corollary is obvious.

**Corollary 3.5** *Let  $v$  be an  $\alpha$ -flow on the compact metric space  $(X, d)$ . If for any chain  $\tilde{\gamma}$ ,*

$$\left| \int_{\tilde{\gamma}} \alpha \right| \leq M$$

*for some  $M > 0$ , then  $v$  is a gradient-like flow.*

**Example 3.2** Let  $v$  be a flow on  $S^1$  which has three fixed points at  $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ , and the forward direction of the flow is the anticlockwise direction. It is easy to see that  $v$  carries a nontrivial cocycle and is a non gradient-like flow.

**Example 3.3** Let  $v$  be the flow generated by a closed Morse 1-form  $\omega$  on a manifold  $X$ . Assume  $[\omega]$  is a nontrivial cohomology class, then there are some cases:

- (1)  $\omega$  has no zero point. So any trajectory is the one we want to find in Theorem 3.4, hence  $v$  is a non gradient-like flow.
- (2)  $\omega$  has only one zero point  $p$ . Then either  $\Gamma_1(r) \neq \emptyset$  or  $\Gamma_2(r) \cup \Gamma_3(r) \neq \emptyset$ . In the first case, there is a homoclinic orbit. In the second case, we can prove by the same argument as in Example 2.1 that for  $\gamma \in \Gamma_2(r) \cup \Gamma_3(r)$ , the integral  $\int_{\gamma(T)} \omega$  will be larger than any given number. Thus by Theorem 3.4,  $v$  is a non gradient-like flow.
- (3)  $\omega$  has several zero points with the same index. If  $\dim X = 1$ , and the orbits between them forms a heteroclinic chain, then  $v$  is non gradient-like. Otherwise, it is a gradient-like flow as shown in Example 3.1. If  $\dim X \geq 2$ , then there are also some cases: (i) if there exists a homoclinic orbit; (ii) if there exists a heteroclinic cycle; (iii) if  $\Gamma_2(r) \cup \Gamma_3(r) \neq \emptyset$ . All these three cases imply  $v$  is a non gradient-like flow.
- (4)  $\dim X \geq 2$ , and there are at least two zero points with different indices. Assume  $p, q$  are the two among them. Let  $X_p^u$  be the unstable manifold at  $p$

and  $X_p^s$  be the stable manifold at  $p$ . Denote  $p^+ = \dim X_p^u$  and  $p^- = \dim X_p^s$ . We have the relation

$$p^+ + p^- = \dim X, \quad q^+ + q^- = \dim X.$$

Then either  $p^+ + q^- > \dim X$  or  $p^- + q^+ > \dim X$ . In either cases, there exists trajectories  $\gamma_i$  connecting the two zero points such that  $\int_{\gamma_i} \omega \rightarrow \infty$  as  $i \rightarrow \infty$ . Hence  $v$  is a non gradient-like flow.

So in summary, if  $\dim X \geq 2$ , the flow  $v$  generated by a nontrivial closed Morse 1-form  $\omega$  is a non gradient-like flow.

#### 4. An algebraic theorem relative to $\pi$ -Morse decomposition

In this section, we let  $X$  be an  $m$ -dimensional compact polyhedron with metric. Let  $v$  be a  $\alpha$ -flow on  $X$ , where  $\alpha$  is a nontrivial cocycle with higher rank.

By Theorem 3.4.4 in [FJ2], the flow  $v$  has a  $\pi$ -Morse decomposition, where  $\pi$  is the deck transformation group on the covering space  $\tilde{X}$  determined by the cocycle  $\alpha$ . Assume that  $\pi$  is spanned by  $G_+ := \{l_1, \dots, l_s\}$ , where  $s$  is the rank of  $\alpha$ . By the proof of Theorem 3.4.3 in [FJ2], we can get  $s$  pair cross sections  $N_i^\pm$  that satisfy the relations:

$$l_i \cdot N_i^- = N_i^+, \quad i = 1, \dots, s.$$

The union  $\cup_{i=1}^s (N_i^+ \cup N_i^-)$  forms the boundary of a fundamental domain  $X_0$ . Since  $X$  is a compact polyhedron, we can take an  $m - 1$ -dimensional polyhedron approximating the cross sections  $N_i^-$  for  $i = 1, \dots, s$  such that the intersections  $N_\alpha^- = N_{i_1}^- \cap N_{i_2}^- \cap \dots \cap N_{i_r}^-$ ,  $\alpha = \{i_1, \dots, i_r\}$  are polyhedra with codimension  $r$ . So without loss of generality, we can assume the cross sections  $N_i^\pm$  and their intersections are polyhedra. For convenience, we denote  $N_i^- = N_i$ .

**Cell decomposition of the fundamental domain** We can retract  $N_i^+$  a small distance  $\epsilon$  into the inner part of the fundamental domain to get a homeomorphic codimension 1 polyhedron (see Figure 1. for  $s = 2$  case). We denote the obtained polyhedron as  $R_i$ . Let  $i_{j,+} : R_j \rightarrow X_0$ ,  $j = 1, \dots, s$  be the inclusion and  $i_{j,-} : R_j \rightarrow N_j$  be the homeomorphism.

Let  $I = (0, 1)$  be the 1-dimensional cell. Firstly we delete  $R_1 \times \epsilon I^1$  and get a polyhedron  $N_{0,1}$ . Denote  $X_{0,1} = N_{0,1}/N_1^+$ . We call  $G_1 = (X_{0,1}, N_{0,1}, R_1, i_{1,\pm})$  to be the first gluing data. Assume after  $r$ -th excision, one has the gluing data  $G_r = (X_{0,r}, N_{0,r}, R_1, \dots, R_r, i_{1,\pm}, \dots, i_{r,\pm})$ . Now deleting  $R_{r+1} \times \epsilon I^{r+1}$  from  $N_{0,r}$ , we get the polyhedron  $N_{0,r+1}$ . Define  $X_{0,r+1} = N_{0,r+1} - \cup_{i=1}^{r+1} N_i^+$ . Hence after  $r + 1$ -th excision, one has the gluing data  $G_{r+1}$ . Continue this process until we get the  $s$ -th gluing data  $G_s = (X_{0,s}, N_{0,s}, R_1, \dots, R_s, i_{1,\pm}, \dots, i_{s,\pm})$ .  $G_i$  is called

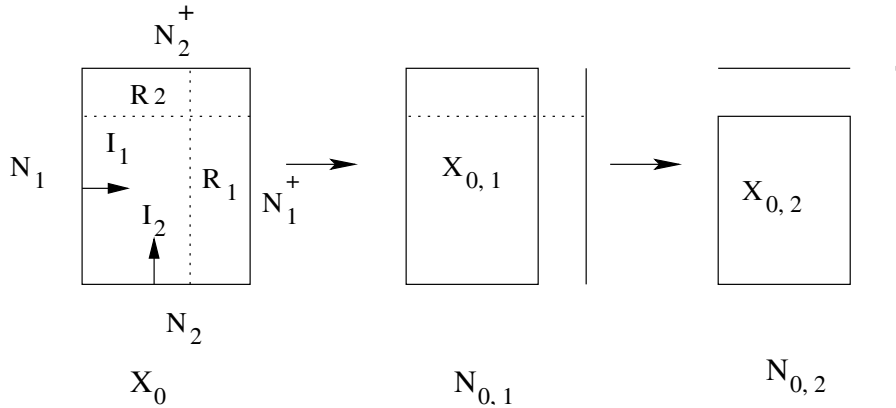


Figure 1: Cell decomposition

the  $i$ -th gluing data because starting from these data, one can reconstruct and give a compatible cell decomposition to the fundamental domain  $X_0$  through  $i$  times inverse attaching operations.

Now we want to construct a compatible regular cell decomposition of  $X_0$ . Firstly we chose an arbitrary cell decomposition of  $\cup_{i=1}^s (R_i^\circ \cap X_{0,s})$ , where  $R_i^\circ$  is the interior part of  $R_i$ . Then the homeomorphisms  $i_{j,-}$ ,  $j = 1, \dots, s$  provide a cell decomposition of the other half part of the boundary of  $X_{0,s}$ . In the interior of  $X_{0,s}$ , we choose an arbitrary compatible regular cell decomposition. Then we get a cell decomposition of  $X_{0,s}$ . Now the action of the group provides a cell decomposition of  $N_{0,s}$ . At this moment, we can use our gluing data to add the cylindrical cells to  $N_{0,s}$ . We add firstly the cell  $e \times \epsilon I^s$  for  $e \in R_s \cap N_{0,s}$  to  $N_{0,s}$  by the inclusion map  $i_{s,+}$  and by identifying the end  $e \times \epsilon\{1\}$  to  $l_s \cdot i_{s,-}(e)$ . So we obtain a cell decomposition of  $N_{0,s-1}$  with the gluing data  $G_{s-1}$ . Continuing this process, we can obtain a compatible regular cell decomposition (i.e., each closed sets appeared here are all cell subcomplexes) of  $X_0$ .

Since  $N_{0,s}$  is a cell complex, we can choose arbitrary orientation of a cell  $e \in C_q(N_{0,s})$ . Hence fixing the incidence coefficients appearing in the boundary operator of the cell chain complex. For the orientation of the cylindrical cells  $e \times I^{i_1} \times \dots \times I^{i_k}$ , we choose the natural orientation, i.e., the order  $i_1 < i_2 < \dots < i_k$  defines the positive orientation. The orientation of the product cell  $e \times I^{i_1} \times \dots \times I^{i_k}$  is uniquely determined by the orientation of the factor cells. So the incidence coefficients of the boundary operator of  $C_*(X_0)$  are uniquely determined.

**Deformation of cell chain complexes** Let  $\alpha \in C_q(X_0)$ , then there is a unique decomposition

$$\alpha = (-1)^{q-1} (i_{1,+})_*(\alpha_1) \times \epsilon I^1 + \alpha'_1.$$

Define  $\beta_1(\alpha) = \alpha_1$ ,  $\beta'_1(\alpha) = \alpha'_1$  and  $\theta_1(\alpha_1) = (-1)^{q-1} (i_{1,+})_*(\alpha_1) \times \epsilon I^1$ . We have

the relation

$$(\theta_1, 1) \begin{pmatrix} \beta_1 \\ \beta'_1 \end{pmatrix} = Id.$$

Similarly, for  $\alpha \in C_q(N_{0,k-1})$  we have the decomposition

$$\alpha = (-1)^{q-1}(i_{k,+})_*(\alpha_k) \times \epsilon I^k + \alpha'_k.$$

Let  $\beta_k(\alpha) := \alpha_k \in C_{q-1}(R_k - \cup_{i=1}^{k-1} R_i \times I^i)$ ,  $\beta'_k(\alpha) := \alpha'_k \in C_q(N_{0,k})$  and  $\theta_k(\alpha_k) := (-1)^{q-1}(i_{k,+})_*(\alpha_k) \times \epsilon I^k$ . We have the equalities

$$(\theta_k, 1) \begin{pmatrix} \beta_k \\ \beta'_k \end{pmatrix} = Id, i = 2, \dots, s \quad (4.1)$$

**Lemma 4.1**  $\theta_k, \beta_k, \beta'_k$  are cellular chain maps and satisfy the following equalities:

- (1)  $d_{N_{0,k-1}} \circ \theta_k + \theta_k \circ d_{R_k} = l_k \cdot (i_{k,-})_* - (i_{k,+})_*$
- (2)  $\beta_k \circ d_{N_{0,k-1}} + d_{R_k} \circ \beta_k = 0$
- (3)  $\beta'_k \circ d_{N_{0,k-1}} - d_{N_{0,k}} \circ \beta'_k = (l_k \cdot (i_{k,-})_* - (i_{k,+})_*) \circ \beta_k$

The above relations can also be written in matrix form:

$$(4) \quad d_{N_{0,k-1}} \circ (\theta_k, 1) = (\theta_k, 1) \begin{pmatrix} -d_{R_k} & 0 \\ f_k & d_{N_{0,k}} \end{pmatrix}$$

$$(5) \quad \begin{pmatrix} \beta_k \\ \beta'_k \end{pmatrix} \circ d_{N_{0,k-1}} = \begin{pmatrix} -d_{R_k} & 0 \\ f_k & d_{N_{0,k}} \end{pmatrix} \begin{pmatrix} \beta_k \\ \beta'_k \end{pmatrix}$$

Here  $f_k := l_k \cdot (i_{k,-})_* - (i_{k,+})_*$ .

**Proof**

(1) Let  $\alpha \in C_{q-1}(R_k - \cup_{i=1}^{k-1} R_i \times I^i)$ , then

$$\begin{aligned} d_{N_{0,k-1}} \circ \theta_k(\alpha) &= d_{N_{0,k-1}}((-1)^{q-1}(i_{k,+})_*(\alpha) \times \epsilon I^k) \\ &= (-1)^{q-1}(i_{k,+})_*(d_{R_k} \alpha) \times \epsilon I^k + (l_k \cdot (i_{k,-})_* - (i_{k,+})_*)(\alpha) \\ &= -\theta_k \circ d_{R_k} \alpha + (l_k \cdot (i_{k,-})_* - (i_{k,+})_*)(\alpha) \end{aligned}$$

(2) Assume  $\alpha \in C_{q-1}(N_{0,k-1})$  has the form

$$\alpha = (-)^{q-2}(i_{k,+})_*(e_1) \times \epsilon I^k + e_2,$$

where  $e_1 \in C_{q-2}(R_k - \cup_{i=1}^{k-1} R_i \times \epsilon I^i)$  and  $e_2 \in C_{q-1}(N_{0,k})$ . Then

$$d_{N_{0,k-1}} \alpha = (-)^{q-2}(i_{k,+})_*(d_{R_k} e_1) \times \epsilon I^k + (l_k \cdot (i_{k,-})_* - (i_{k,+})_*)(e_1) + d_{N_{0,k}} e_2,$$

hence

$$\beta_k \circ d_{N_{0,k-1}} \alpha = -d_{R_k} \circ \beta_k(\alpha),$$

and

$$\beta'_k \circ d_{N_{0,k-1}} \alpha = (l_k \cdot (i_{k,-})_* - (i_{k,+})_*) \circ \beta_k(\alpha) + d_{N_{0,k}} \circ \beta'_k(\alpha)$$

■

**Lemma 4.2** *Let  $f_k := l_k \cdot (i_{k,-})_* - (i_{k,+})_*$ ,  $k = 1, \dots, s$  be the chain map from  $C_{*-1}(R_k - \cup_{i=1}^{k-1} R_i \times I^i)$  to  $C_{*-1}(N_{0,k})$ . Then  $(\theta_k, 1)$  is a chain isomorphism from the cone  $\text{con}(f_k) := (C_{*-1}(R_k - \cup_{i=1}^{k-1} R_i \times I^i) \oplus C_{*-1}(N_{0,k}), d_c^k)$  to  $(C_*(N_{0,k-1}), d_{N_{0,k-1}})$ , where the differential of the algebraic cone is*

$$d_c^k := \begin{pmatrix} -d_{R_k} & 0 \\ f_k & d_{N_{0,k}} \end{pmatrix}$$

Proof. By equalities (4) and (5) of Lemma 4.1,  $(\theta_k, 1)$  and  $\begin{pmatrix} \beta_k \\ \beta'_k \end{pmatrix}$  are chain maps with respect to corresponding cellular chain complexes. Furthermore, we have

$$(\theta_k, 1) \begin{pmatrix} \beta_k \\ \beta'_k \end{pmatrix} = Id$$

and

$$\begin{pmatrix} \beta_k \\ \beta'_k \end{pmatrix} (\theta_k, 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore  $(\theta_k, 1)$  is a chain isomorphism. ■

**Lemma 4.3** *Let*

$$D_q^k = C_{q-1}(R_1) \oplus C_{q-1}(R_2 - (i_{1,+})(R_1) \times \epsilon I^1) \oplus \dots \oplus C_{q-1}(R_k - \cup_{j=1}^{k-1} (i_{j,+})(R_j) \times \epsilon I^j) \otimes C_q(N_{0,k})$$

$$d_{c,k} = \begin{pmatrix} -d_{R_1} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ f_{k-1,1}^k & \cdots & -d_{R_{k-1}} & 0 & 0 \\ f_{k,1}^k & \cdots & f_{k,k-1}^k & -d_{R_k} & 0 \\ f_{k,1}^{k'} & \cdots & f_{k,k-1}^{k'} & f_{k,k}^{k'} & d_{N_{0,k}} \end{pmatrix} \quad (4.2)$$



Here

$$f_{m,n}^k = \beta_m \circ \beta'_{m-1} \cdots \beta'_{n+1} \circ f_n,$$

for  $1 \leq n < m \leq k$ ;

$$f_{k,n}^{k'} = \beta'_k \circ \beta'_{k-1} \cdots \beta'_{n+1} \circ f_n,$$

for  $1 \leq n \leq k-1$ ; and

$$f_{k,k}^{k'} = f_k.$$

Then  $(D_*^k, d_{c,k})$  is a chain complex.

Proof. We prove  $d_{c,k} \circ d_{c,k} = 0$  by using induction for  $k$ . If  $k = 1$ , then  $D_*^1 = C_{*-1}(R_1) \oplus C_*(N_{0,1})$  and

$$d_{c,1} = \begin{pmatrix} -d_{R_1} & 0 \\ f_1 & d_{N_{0,1}} \end{pmatrix}.$$

It is obviously a chain complex. Assume that  $(D_*^k, d_{c,k})$  is a chain complex, we want to show that  $(D_*^{k+1}, d_{c,k+1})$  is a chain complex. Notice that for  $1 \leq n < m \leq k$ , we have

$$f_{m,n}^{k+1} = f_{m,n}^k.$$

Therefore  $d_{c,k}$  and  $d_{c,k+1}$  can be expressed by block matrices:

$$d_{c,k} = \begin{pmatrix} \tilde{d}_k & 0 \\ \tilde{f}_k & d_{N_{0,k}} \end{pmatrix}$$

and

$$d_{c,k+1} = \begin{pmatrix} \tilde{d}_k & 0 & 0 \\ \beta_{k+1} \cdot \tilde{f}_k & -d_{R_{k+1}} & 0 \\ \beta'_{k+1} \cdot \tilde{f}_k & f_{k+1} & d_{N_{0,k+1}} \end{pmatrix}$$

According to the assumption, we have the relations:

$$\tilde{d}_k \circ \tilde{d}_k = 0,$$

and

$$\tilde{f}_k \tilde{d}_k + d_{N_{0,k}} \tilde{f}_k = 0.$$

Let  $\beta_{k+1}$  and  $\beta'_{k+1}$  act on the above equality and using the formulas (2) and (3) in Lemma 4.1, we obtain

$$\beta_{k+1} \cdot \tilde{f}_k \cdot \tilde{d}_k - d_{R_{k+1}} \cdot \beta_{k+1} \cdot \tilde{f}_k = 0,$$

and

$$\beta'_{k+1} \cdot \tilde{f}_k \cdot \tilde{d}_k + d_{N_{0,k+1}} \cdot \beta'_{k+1} \cdot \tilde{f}_k + f_{k+1} \cdot \beta_{k+1} \tilde{f}_k = 0.$$

combining the above two relations, the fact that  $\tilde{d}_k \circ \tilde{d}_k = 0$  and Lemma 4.2, we can deduce  $d_{c,k+1} \cdot d_{c,k+1} = 0$ .  $\blacksquare$

**Lemma 4.4**  $\forall k = 1, \dots, s$ , the cellular chain complex  $(D_*^k, d_{c,k})$  is chain isomorphic to  $(C_*(X_0), d_{X_0})$ .

Proof. Firstly we will show the following map

$$\begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \theta_{k+1} & 1 \end{pmatrix}_{(k+1) \times (k+2)} : C_{q-1}(R_1) \oplus C_{q-1}(R_2 - (i_{1,+})(R_1) \times \epsilon I^1) \oplus \cdots$$

$$\oplus C_{q-1}(R_{k+1} - \cup_{j=1}^k (i_{j,+}) \times \epsilon I^j) \otimes C_q(N_{0,k+1}) \rightarrow C_{q-1}(R_1) \oplus C_{q-1}(R_2 - (i_{1,+})(R_1) \times \epsilon I^1) \oplus \cdots \oplus C_{q-1}(R_k - \cup_{j=1}^{k-1} (i_{j,+}) \times \epsilon I^j) \otimes C_q(N_{0,k})$$

is a chain map, i.e., it should satisfy the commutation relation:

$$\begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \theta_{k+1} & 1 \end{pmatrix} \circ d_{c,k+1} = d_{c,k} \circ \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \theta_{k+1} & 1 \end{pmatrix}.$$

This is equivalent to the following equality:

$$\begin{pmatrix} \tilde{d}_k & 0 & 0 \\ \theta_{k+1} \beta_{k+1} \tilde{f}_k + \beta'_{k+1} \cdot \tilde{f}_k & -\theta_{k+1} d_{R_{k+1}} + f_{k+1} & d_{N_{0,k+1}} \end{pmatrix} = \begin{pmatrix} \tilde{d}_k & 0 & 0 \\ \tilde{f}_k & d_{N_{0,k}} \theta_{k+1} & d_{N_{0,k}} \end{pmatrix},$$

where  $\tilde{f}_k, \tilde{d}_k$  are defined in Lemma 4.3. Now this is true by Lemma 4.1 and the fact  $d_{N_{0,k}}|_{N_{0,k+1}} = d_{N_{0,k+1}}$ . Thus we proved

$$\begin{pmatrix} Id & 0 & 0 \\ 0 & \theta_{k+1} & 1 \end{pmatrix}$$

is a chain map. It is also an isomorphism, since it has an inverse chain map:

$$\begin{pmatrix} Id & 0 \\ 0 & \beta_{k+1} \\ 0 & \beta'_{k+1} \end{pmatrix}.$$

Therefore we proved  $(D_*^{k+1}, d_{c,k+1})$  is chain isomorphic to  $(D_*^k, d_{c,k})$  for any  $k = 1, \dots, s$ . Note the  $k = 1$  case was already proved in Proposition 4.1.1 in [FJ2]. ■

Define  $\pi_+$  as the monoid constructed by  $(G_+, e)$  with the group action from  $\pi$ . Let  $\rho_e : \pi_1(X) \rightarrow \pi$  be the extension of  $\pi$  by the normal group  $\pi_1(\bar{X})$ . Since  $\pi_+$

is a monoid in  $\pi$ , the set  $\rho_e^{-1}(\pi_+)$  is also a monoid of  $\pi_1(X)$ . We denote  $\rho_e^{-1}(\pi_+)$  by  $\pi_1(X)_+$ .  $\mathbb{Z}\pi_1(X)_+$  is a subring of  $\mathbb{Z}\pi_1(X)$ .

Tensor  $(D_*^k, d_{c,k})$  with the ring  $\mathbb{Z}\pi_1(X)_+$ , then we have the  $\mathbb{Z}\pi_1(X)_+$ -module chain complex  $(\mathbb{Z}\pi_1(X)_+ \otimes D_*^k, I \otimes d_{c,k})$ .

On the other hand, we can lift the related quantities (subcomplexes, or maps) to the universal covering space  $\tilde{X}$  such that they become  $\pi_1(X)_+$ -equivariant. For instance, we can lift the  $m-1$ -dimensional polyhedra  $\cup_{g \in \pi_+} g \cdot R_i$  to  $\tilde{X}$  such that the obtained polyhedra  $\tilde{R}_i$  is  $\pi_1(X)_+$ -equivariant. If  $F$  is a map  $(\beta_k, \beta'_k, f_k, \text{etc.})$ , then the action of the lifted map  $\tilde{F}$  is defined as:  $\tilde{F}(g \cdot e) = g \cdot \tilde{F}(e)$ . Thus we can get a  $\mathbb{Z}\pi_1(X)_+$ -module cellular chain complexes  $(\tilde{D}_*^k, \tilde{d}_{c,k})$ , where

$$\tilde{D}_q^k := C_{q-1}(\tilde{R}_1) \oplus C_{q-1}(\tilde{R}_2 - (\tilde{i}_{1,+})(\tilde{R}_1) \times \epsilon I^1) \oplus \cdots \oplus C_{q-1}(\tilde{R}_k - \cup_{j=1}^{k-1} (\tilde{i}_{j,+}) \times \epsilon I^j) \oplus C_q(\tilde{N}_{0,k})$$

and  $\tilde{d}_{c,k}$  is defined by the same form of the matrix (4.2), and the difference is that the entries of (4.2) are replaced by their lifting homomorphisms.

It is easy to see that the two  $\mathbb{Z}\pi_1(X)_+$ -module chain complexes  $(\mathbb{Z}\pi_1(X)_+ \otimes D_*^k, I \otimes d_{c,k})$  and  $(\tilde{D}_*^k, \tilde{d}_{c,k})$  are chain equivalent.

Let  $\tilde{X}_+ = \cup_{g \in \pi_1(X)_+} g \cdot \tilde{X}_0$ . By Lemma 4.4, it is obvious that the following holds.

**Proposition 4.5** *The two  $\mathbb{Z}\pi_1(X)_+$ -module chain complexes  $(C_*(\tilde{X}_+), d_{\tilde{X}_+})$  and  $(\tilde{D}_*^s, \tilde{d}_{c,k})$  are chain equivalent.*

**Monodromy representations and Novikov numbers** In this part, we will generalize the crucial theorem, Theorem 4.2.1 in [FJ2] for an integral cocycle  $\alpha$  to a higher rank cocycle  $\alpha$ . To get such a theorem, one needs also to study the monodromy representation and the evaluation representation as in [FJ2].

Since  $\pi$  is a commutative group with rank  $s$ , we obtain a group homomorphism

$$\rho_{\pi_1} : \pi_1(X) \xrightarrow{\rho_e} \pi \xrightarrow{\rho_A} \mathbb{Z}^s.$$

Here  $\rho_A(l_1^{a_1} \cdots l_s^{a_s}) := (a_1, \cdots, a_s)$ . Thus we have a ring homomorphism

$$\rho_{\pi_1} : \mathbb{Z}[\pi_1(X)] \longrightarrow \mathbb{Z}[\mathbb{Z}^s].$$

This ring homomorphism can induce another ring homomorphism

$$\rho_q : \mathbb{Z}[\pi_1(X)] \rightarrow Q_s = \mathbb{Z}[t_i, t_i^{-1}; i = 1, 2, \cdots, s]$$

defined as, for  $g = \sum z_j (l_1^{a_{j1}} \cdot l_s^{a_{js}}) \in \mathbb{Z}[\pi]$ ,

$$\rho_q(g) = \sum z_j t_1^{a_{j1}} \cdots t_s^{a_{js}}.$$

In fact,  $\rho_q$  is fully determined by the group  $\pi$  and its representation. Restricting  $\rho_q$  to the subring  $\mathbb{Z}[\pi_1(X)_+]$ , we can get a ring homomorphism

$$\rho_p = \rho_q|_{\mathbb{Z}[\pi_1(X)_+]} : \mathbb{Z}[\pi_1(X)_+] \longrightarrow P_s = \mathbb{Z}[t_1, \dots, t_s]$$

Let  $\tilde{E}$  be a local system of free abelian groups on the compact polyhedron  $X$ , then  $\tilde{E}$  is determined by its monodromy representation  $\rho_{\tilde{E}}$ :

$$\rho_{\tilde{E}} : \pi_1(X, x_0) \longrightarrow \text{Aut}(\tilde{E}_0) = GL(k, \mathbb{Z})$$

where  $\tilde{E}_0$  is the fibre of the free abelian group at  $x_0$  and  $k = \text{rank}(\tilde{E}_0)$ . Let  $E = \tilde{E} \otimes \mathbb{C}$ , then  $E$  is a complex flat vector bundle with the holonomy  $\rho_E$

$$\rho_E : \pi_1(X, x_0) \longrightarrow GL(k; \mathbb{Z}) \otimes \mathbb{C}$$

Now the tensor product of the representations  $\rho_q \otimes \rho_{\tilde{E}}$  gives a representation of a  $\mathbb{Z}[\pi_1(X)]$ -ring to the linear space  $(Q_s)^k$ , where  $Q_s$  is the polynomial space with  $s$  variables over  $\mathbb{Z}$ .

Since  $\rho_{\tilde{E}}$  is an anti-homomorphism, i.e.,  $\forall g, g' \in \mathbb{Z}[\pi_1(X)], \rho_E(g \cdot g') = \rho_E(g') \cdot \rho_E(g)$ , hence  $\rho_P \otimes \rho_{\tilde{E}}$  gives a right  $\mathbb{Z}[\pi_1(X)]_+$ -module structure on  $P_s^k$ . With the  $P_s$ -module structure of itself,  $P_s^k$  becomes a  $(P_s, \mathbb{Z}[\pi_1(X)_+])$ -bimodule.

Define  $D_* = P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$ , then  $D_*$  is a  $P_s$ -module chain complex.

**Evaluation representations** Take any complex  $s$ -vector  $a = (a_1, \dots, a_s) \in \mathbb{C}^s$ . The complex number field  $\mathbb{C}$  can be given a  $P_s$ -module structure, whose module structure is provided by the action: for a polynomial  $P(t_1, \dots, t_s)$ ,  $P(t_1, \dots, t_s) \cdot x = P(a_1, \dots, a_s) \cdot x = P(a) \cdot x$  for  $x \in \mathbb{C}$ . We denote the  $P_s$ -module of  $\mathbb{C}$  evaluated at  $t = a$  by  $\mathbb{C}_a$ . Similarly for any  $a \in (\mathbb{C}^*)^s$ ,  $\mathbb{C}$  can be viewed as a  $Q_s$ -module. If  $p$  is a prime number, then the field  $\mathbb{Z}_p$  also has a  $P_s$ -module structure which is given by the evaluation at  $t = 0$ . We consider the complexes  $\mathbb{C}_a \otimes_{P_s} D_*$  and  $\mathbb{Z}_p \otimes_{P_s} D_*$ . The following theorem is the generalization of Theorem 4.2.1 in [FJ2] for higher rank  $\alpha$ .

**Theorem 4.6** *Assume  $\text{rank}\pi = s \geq 1$ . Let  $D_* = P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$  be defined by the above argument. We have*

- (1) *For any nonzero complex vector  $a \in (\mathbb{C}^*)^s$ , the homology  $H_*(\mathbb{C}_a \otimes_{P_s} D_*)$  is isomorphic to  $H_*(X, a^\pi \otimes E)$ , which is viewed as the homology of the presheaf  $a^\pi \otimes E$  on  $X$ .*
- (2) *Let  $p$  be a prime number and let  $\mathbb{Z}_p$  have the  $P_s$ -module structure which is provided by the evaluation at  $t = (t_1, \dots, t_s) = 0$ . Then the homology*

$H_*(\mathbb{Z}_p \otimes_{P_s} D_*)$  is isomorphic to  $H_*(X_{0,s}, \cup_{j=1}^s (i_{j,+})(R_j); \mathbb{Z}_p \otimes P_\pi^* \tilde{E})$ , where  $\tilde{E}$  is a local system on  $X$  and  $P_\pi^* \tilde{E}$  is the pull-back local system on  $X_{0,s}$  by the projection  $P_\pi : \tilde{X} \rightarrow X$ .

(3)  $H_*(C_0 \otimes_{P_s} D_*)$  is isomorphic to  $H_*(X_{0,s}, \cup_{j=1}^s (i_{j,+})(R_j); P_\pi^* E)$ .

Proof. Since the proof of (3) is the same as that of (2), we only give the proofs of (1) and (2).

(1). Since the  $\mathbb{Z}[\pi_1(X)_+]$ -basis of  $C_*(\tilde{X}_+)$  is finite, all the complexes related to  $C_*(\tilde{X}_+)$  are finitely generated, and hence all the homology groups are finitely generated.

For  $a \in (C^*)^s$ , we have the isomorphism

$$\begin{aligned} \mathbb{C}_a \otimes_{P_s} D^* &\cong \mathbb{C}_a \otimes_{P_s} ((P_s)^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)) \\ &\cong (\mathbb{C}_a \otimes_{Q(s)} (Q_s)^k) \otimes_{\mathbb{Z}[\pi_1(X)]} (\mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)) \\ &\cong (\mathbb{C}_a \otimes_{\mathbb{Z}} \mathbb{Z}^k) \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X}) \\ &\cong \mathbb{C}^k \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X}). \end{aligned}$$

Here the representation of  $\mathbb{Z}[\pi_1(X)]$  is given by

$$g \longrightarrow \rho_{P(a)}(g) \otimes \rho_E(g).$$

Hence the homology of  $\mathbb{C}^k \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X})$  is the same as the homology of the presheaf  $a^\pi \otimes E$  on  $X$  that corresponds to the flat vector bundle produced by the above holonomy representation. (1) is proved.

(2)

$$\mathbb{Z}_p \otimes_{P_s} D^* \cong \mathbb{Z}_p \otimes_{P_s} (P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)) \cong \mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+).$$

Here the representation of  $\mathbb{Z}[\pi_1(X)_+]$  on  $\mathbb{Z}_p^k$  is

$$g \longrightarrow \rho_{P(0)}(g) \otimes \rho_E(g). \quad (4.3)$$

Since  $\rho_{P(0)}(g) = \rho_{P(0)} \cdot \rho_A \rho_e(g)$ , except in the case that  $g \in \mathbb{Z}[\pi_1(X)_+]$  satisfies  $\rho_e(g) = 1$ , the evaluation representation will make the final representation vanish. Hence (4.3) becomes

$$\begin{aligned} g &\longrightarrow \rho_E(g), \text{ if } \rho_e(g) = 1 \\ g &\longrightarrow 0, \text{ if } \rho_e(g) \neq 1. \end{aligned}$$

By Proposition 4.5, in order to prove (2), we need to prove that  $\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} \tilde{D}_*^s$  is equivalent to  $(\mathbb{Z}_p \otimes P_\pi^* \tilde{E}) \otimes C_*(X_{0,s}, \cup_{j=1}^s (i_{j,+})(R_j))$ . Now we have

$$\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} \tilde{D}_q^s = (\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_{q-1}(\tilde{R}_1)) \oplus \cdots \oplus (\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} \oplus C_q(\tilde{N}_{0,s})).$$

If we take  $\tilde{E} \otimes C_*(R_t - \cup_{j=1}^{t-1} (i_{j,+})(R_j) \times \epsilon I^j, N^+)$  as the complexes with twisted coefficients  $\tilde{E}$ , then  $\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{R}_t - \cup_{j=1}^{t-1} (i_{j,+})(\tilde{R}_j) \times \epsilon I^j) \cong \mathbb{Z}_p \otimes (\tilde{E} \otimes C_*(R_t - \cup_{j=1}^{t-1} (i_{j,+})(R_j) \times \epsilon I^j, N^+))$  (Notice that the evaluation map at  $t = 0$  makes the cell lying in  $N^+$  vanishing). Similarly  $\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{N}_{0,t}) \cong \mathbb{Z}_p \otimes (P_\pi^* \tilde{E} \otimes C_*(X_{0,t}))$ . Therefore

$$\mathbb{Z}_p^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} \tilde{D}_*^s \cong (\mathbb{Z}_p \otimes (\tilde{E} \otimes C_{*-1}(R_1, N^+))) \oplus \cdots \oplus (\mathbb{Z}_p \otimes (P_\pi^* \tilde{E} \otimes C_*(X_{0,s})))$$

Denote this complex by  $D_*^{s,0}$ . Its differential becomes

$$d_c^{s,0} = \begin{pmatrix} -d_{R_1} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ i_{s-1,1}^s & \cdots & -d_{R_{s-1}} & 0 & 0 \\ i_{s,1}^s & \cdots & i_{s,s-1}^s & -d_{R_s} & 0 \\ i_{s,1}^{s'} & \cdots & i_{s,s-1}^{s'} & i_{s,s}^{s'} & d_{N_{0,s}} \end{pmatrix} \quad (4.4)$$

Here

$$i_{m,n}^s = -\beta_m \circ \beta'_{m-1} \cdots \beta'_{n+1} \circ i_{n,+},$$

for  $1 \leq n < m \leq s$ ,

$$i_{s,n}^{s'} = -\beta'_s \circ \beta'_{s-1} \cdots \beta'_{n+1} \circ i_{n,+},$$

for  $1 \leq n \leq s-1$ , and

$$i_{s,s}^{s'} = -i_{s,+}.$$

Now for convenience, we assume the coefficient ring is the integer ring.

Define maps

$$\begin{aligned} I_1 &:= (i_{2,1}^s, \cdots, i_{s,1}^s, i_{s,1}^{s'}) : C_*(R_1, N^+) \rightarrow \\ D_*^{s,1} &:= C_{*-1}(R_2 - (i_{1,+})(R_1) \times \epsilon I^1, N^+) \oplus \cdots \oplus C_*(X_{0,s}), \end{aligned}$$

and

$$P_1 := D_*^{s,1} \rightarrow \bar{D}_*^{s,1} := C_{*-1}(R_2 - (i_{1,+})(R_1) \times I^1, R_1 \cup N^+) \oplus \cdots \oplus C_*(X_{0,s}, R_1)$$

to be the quotient map. Let  $d_c^{s,1}$  be the matrix obtained from (4.4) by deleting the first row and the first column. We have  $d_c^{s,1} \circ d_c^{s,1} = 0$ . Hence  $(D_*^{s,1}, d_c^{s,1})$  is a chain complex. It induces the quotient chain complex  $(\bar{D}_*^{s,1}, \bar{d}_c^{s,1})$ . The following is a short exact sequence.

$$0 \rightarrow C_*(R_1) \xrightarrow{I_1} D_*^{s,1} \xrightarrow{P_1} \bar{D}_*^{s,1} \rightarrow 0 \quad (4.5)$$

To see this, we have to show the following three points:

- (i)  $P_1 \circ I_1 = 0$
- (ii)  $\ker P_1 \subset \text{im} I_1$
- (iii)  $I_1, P_1$  are chain maps.

(i) is obvious. Assume  $P_1(\alpha_2, \dots, \alpha_s, \alpha'_s) = 0$ , i.e.,  $\forall j = 2, \dots, s$ ,

$$\alpha_j \in R_1 \cap R_j - \cup_{t=1}^{j-1} R_t \times \epsilon I^t - N^+$$

and

$$\alpha'_s \in X_{0,s} - \cup_{j=2}^s R_j.$$

Let

$$\alpha = -\left(\sum_{j=2}^s \theta_j(\alpha_j) + \alpha'_s\right).$$

Then we have  $I_1(\alpha) = (\alpha_2, \dots, \alpha_s, \alpha'_s)$ . Hence (ii) is proved. To prove (iii), one need only to check the commutative diagram:

$$\begin{array}{ccc} C_q(R_1) & \xrightarrow{I_1} & D_q^{s,1} \\ \downarrow d_{R_1} & & \downarrow d_c^{s,1} \\ C_{q-1}(R_1) & \xrightarrow{I_1} & D_{q-1}^{s,1} \end{array}$$

i.e., check the equality:  $d_c^{s,1} \cdot I_1 = I_1 \cdot d_{R_1}$ . This holds in view of the relation  $d^{s,0} \cdot d^{s,0} = 0$ . Hence (iii) is proved.

From the short exact sequence (4.5), we obtain the long exact sequence:

$$\begin{aligned} \dots \rightarrow H_q(R_1) &\xrightarrow{I_1} H_q(D_*^{s,1}) \rightarrow H_q(\bar{D}_*^{s,1}) \rightarrow \\ &\rightarrow H_{q-1}(R_1) \xrightarrow{I_1} H_{q-1}(D_*^{s,1}) \rightarrow \dots \end{aligned} \quad (4.6)$$

Similarly, from the short exact sequence

$$0 \rightarrow C_*(D_*^{s,1}) \rightarrow C_*(D_*^{s,0}) \rightarrow (C_{*-1}(R_1), -d_{R_1}) \rightarrow 0$$

we get the long exact sequence

$$\dots \rightarrow H_q(R_1) \xrightarrow{I_1} H_q(D_*^{s,1}) \rightarrow H_q(D_*^{s,0}) \rightarrow H_{q-1}(R_1) \xrightarrow{I_1} H_{q-1}(D_*^{s,1}) \rightarrow \dots$$

The two long exact sequences yield the following two short exact sequences

$$\begin{aligned} 0 \rightarrow \text{Coker}(I_1 : H_q(R_1) \rightarrow H_q(D_*^{s,1})) &\rightarrow H_q(\bar{D}_*^{s,1}) \rightarrow \\ &\rightarrow \text{Ker}(I_1 : H_{q-1}(R_1) \rightarrow H_{q-1}(D_*^{s,1})) \rightarrow 0 \end{aligned}$$

and

$$0 \rightarrow \text{Coker}(I_1 : H_q(R_1) \rightarrow H_q(D_*^{s,1})) \rightarrow H_q(D_*^{s,0}) \rightarrow \\ \rightarrow \text{Ker}(I_1 : H_{q-1}(R_1) \rightarrow H_{q-1}(D_*^{s,1})) \rightarrow 0$$

Using the Five-Lemma, we get for any  $q \geq 0$

$$H_q(\bar{D}_*^{s,1}) \cong H_q(D_*^{s,0}).$$

Continuing our operation, finally we get

$$H_q(D_*^{s,0}) \cong H_q(\bar{D}_*^{s,s}) = H_q(X_{0,s}, \cup_{j=1}^s R_j).$$

Hence we finally proved the second conclusion of this theorem. ■

*Novikov numbers* In Theorem 4.4, we have considered the complex  $D_* = P_s^k \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$ . In this part, we always let the vector bundle  $E$  that appears in Theorem 4.6 be a trivial line bundle. Then the complex  $D_*$  there has the form  $D_* = P_s \otimes_{\mathbb{Z}[\pi_1(X)_+]} C_*(\tilde{X}_+)$ . Since the representation of  $\mathbb{Z}[\pi_1(X)_+]$  in  $P_s$  is completely determined by the cohomology class  $[\alpha]$ , we denote the homology group  $H_*(\mathbb{C}_a \otimes_{P_s} D_*)$  as  $H_*(X, a^\alpha)$ , or in other words, view the homology group  $H_*(\mathbb{C}_a \otimes_{P_s} D_*)$  as the homology group of the presheaf  $a^\alpha$  on  $X$  which is given by the monodromy representation  $\rho_P : \mathbb{Z}[\pi_1(X)_+] \rightarrow P_s = \mathbb{Z}[t_1, \dots, t_s]$ .

The following is essentially given in [No3].

**Proposition 4.7** *Let  $X$  be a compact polyhedron. Define a function for fixed  $i$  to be*

$$a \in (\mathbb{C}^*)^s \longrightarrow \dim_{\mathbb{C}} H_i(X, a^\alpha),$$

*then it has the following properties:*

(1) *It is generically constant, more precisely, except on a proper algebraic subvariety  $\mathcal{L}$  in  $(\mathbb{C}^*)^s$ , the dimension  $\dim_{\mathbb{C}} H_i(X, a^\alpha)$  is constant and this constant is just the Novikov number  $b_i([\alpha])$  we defined above.*

(2) *For any point  $\hat{a} \in \mathcal{L}$ ,*

$$\dim_{\mathbb{C}} H_i(X, \hat{a}^\alpha) > b_i([\alpha])$$

**Definition**  $b_i([\alpha]) := \text{rank}(H_i(D_*))$  for  $i = 0, 1, \dots, m$  are called the Novikov numbers.

**Remark** If  $s = 1$  and  $X$  is a closed manifold, then it was proved in [Fa5] that the Novikov number we defined here is the same as the Novikov number which is defined as the dimension of the homology group of the Novikov complex.



## 5. Novikov-Morse type inequalities for higher rank $\alpha$

Theorem 4.6 is crucial in the proof of Novikov-Morse type inequalities. As long as we obtain theorem 4.6 for flows carrying a higher rank cocycle  $\alpha$ , we can follow the same line as in the proof in [FJ2] to get the expected analogous inequalities. We omit the similar proof as in section 4.4 and 4.5 of [FJ2], and only list the results.

*Ideals* Let  $P_s = \mathbb{Z}[t_1, \dots, t_s]$  be the polynomial ring with  $s$  variables over the integers  $\mathbb{Z}$ . Let  $\langle t_1, \dots, t_s \rangle$  be the ideal generated by  $t_1, \dots, t_s$ . Define  $I = \{t \in \mathbb{C}^s; 1 + \langle t_1, \dots, t_s \rangle = 0\}$ . Let  $a \in (\mathbb{C}^*)^s$  be not in  $I$ . Define  $I_a$  to be the prime ideal in the polynomial ring  $P_s$  consisting of the polynomials vanishing at  $a$ . By the choice of  $a$ , the free terms of all the polynomials  $f(t) \in I_a$  are divisible by some prime number  $p$ . Therefore we obtain

$$I_a \subset I_p = \langle p \rangle + \langle t_1, \dots, t_s \rangle$$

**Theorem 5.1** *Let  $X$  be a compact polyhedron with a metric  $d$ . Let  $\alpha$  be an continuous cocycle and  $v$  be a generalized  $\alpha$ -flow w.r.t. an isolated invariant set  $\mathcal{A} = \{A_1, \dots, A_n\}$ . Let  $\tilde{E}$  be a local system of free abelian groups and let  $E = \mathbb{C} \otimes \tilde{E}$ . If  $a \in \mathbb{C}^*$  and  $a \notin I = 1 + \langle t_1, \dots, t_s \rangle \subset P_s$ , then there is a prime number  $p$  relative to  $a$  such that*

$$\begin{aligned} \sum_{A_i \in \mathcal{A}} p(h(A_i); t, \mathbb{Z}_p \otimes J_{A_i}^* E) &= p(X; t, a^\alpha \otimes E) \\ &+ (1+t)Q_1(I_a, I_p; t) + (1+t)Q_2(I_p; t) \end{aligned} \quad (5.1)$$

where  $J_{A_i}$  is the inclusion map from the isolating neighborhood of an isolated invariant set  $A_i$  to  $X$ , and  $Q_1(I_a, I_p; t)$  and  $Q_2(I_p; t)$  are all polynomials with nonnegative integer coefficients.

**Corollary 5.2 (Euler-Poincaré formula)** *Under the hypothesis of Theorem 5.1,*

$$\sum_{A \in \mathcal{A}} p(h(A); -1, \mathbb{Z}_p \otimes J_A^* E) = p(X, -1, a^\alpha \otimes E) \quad (5.2)$$

*In particular, if  $E$  is a trivial line bundle, then*

$$\sum_{A \in \mathcal{A}} p(h(A); -1, \mathbb{Z}_p) = \chi(X) \quad (5.3)$$

*for any prime number  $p$ . Here  $\chi(X)$  is the Euler characteristic number of the compact polyhedron.*

As explained in [FJ2], if the cocycle  $\alpha$  is a trivial cocycle, then the above Novikov-Morse type inequalities will induce the Conley-Morse type inequalities given in [CZ]. If  $\omega$  is a closed 1-form with higher rank, then Theorem 5.1 generalizes the rank one case which is proved in [FJ2]. We have

**Theorem 5.3** *Let  $\omega$  be a closed 1-form with critical set  $\mathcal{A}$  consisting of finitely many connected components. Let  $\tilde{E}$  be a local system of free abelian groups and  $E = \mathbb{C} \otimes \tilde{E}$ . Assume that  $a \in \mathbb{C}^*$  and  $a \notin I = 1 + \langle t_1, \dots, t_s \rangle \subset P_s$ . Then there is a prime  $p$  such that*

$$\begin{aligned} & \sum_{A \in \mathcal{A}} p(h(A); t, \mathbb{Z}_p \otimes J_A^* E) \\ &= p(X, t; a^\omega \otimes E) + (1+t)Q(E, t) \end{aligned}$$

where  $Q(E, t)$  is a polynomial with nonnegative coefficients.

If the closed 1-form  $\omega$  in Theorem 5.3 has only non-degenerate isolated zero points or Bott type non-degenerate zero locus, then the above theorem reduces to the theorems proven by M. Farber [Fa1, Fa2] (The reader can also see [FJ2] for a complete description of those cases).

**Corollary 5.4 (Classical Novikov inequality)** *Let  $X$  be an oriented closed smooth manifold and  $\omega$  a Morse closed 1-form. Then the numbers  $c_j(\omega)$  of zeros of  $\omega$  having index  $j$  satisfy*

$$\begin{aligned} c_j(\omega) &\geq b_j([\omega]) \\ \sum_{i=0}^j (-1)^i c_{j-i}(\omega) &\geq \sum_{i=0}^j (-1)^i b_{j-i}([\omega]) \end{aligned} \tag{5.4}$$

for  $j = 0, 1, \dots, m$ .

**Theorem 5.5** *Let  $v$  be an  $\alpha$ -Morse-Smale flow w.r.t.  $\mathcal{A} = \{A_n, \dots, A_1\}$ . Let  $c_j$  be the number of hyperbolic fixed points with index  $j$ ,  $a_j$  be the number of the hyperbolic periodic orbits with index  $j$ , and  $\mu_j = c_j + a_j + a_{j+1}$ . Then*

$$\begin{aligned} \mu_j &\geq b_j([\alpha]) \\ \sum_{i=0}^j (-1)^{j-i} \mu_i &\geq \sum_{i=0}^j (-1)^{j-i} b_j([\alpha]) \end{aligned} \tag{5.5}$$

for  $j = 0, 1, \dots, m$ .

**Theorem 5.6 (Vanishing theorem)** *Let  $X$  be a compact polyhedron with a metric  $d$ . If there exists a flow carrying a cohomology class  $[\alpha]$  on  $X$ , then*

$$b_i([\alpha]) = 0, \quad \forall i = 0, 1, \dots, m.$$

**Example 5.1** Let  $f : X \rightarrow X$  be a homeomorphism from the compact polyhedron  $X$  to itself, then we can define the mapping torus of  $f$ :

$$T_f := \{(x, t) | x \in X, t \in [0, 1]\} / (x, 0) \sim (f(x), 1).$$

Then the natural projection  $\pi : T_f \rightarrow S^1$  provides a nontrivial cocycle  $\alpha = J(\pi) : X \times X \rightarrow \mathbb{R}$  (where  $J$  is defined in proposition 3.1.1 of [FJ2]). Define the flow carrying the cohomology class  $\alpha$  as: for  $0 \leq s < 1$ , and  $\forall t \in \mathbb{R}$ ,

$$[x, s] \cdot t := [f^{\lfloor s+t \rfloor}(x), (s+t)],$$

where  $\lfloor \cdot \rfloor$  is the function taking the integer part and  $(\cdot)$  is the function taking the decimal part. According to Theorem 5.6, we have

$$b_i([\alpha]) = 0, \quad \forall i.$$

## References

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