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**The Epstein-Glaser approach to pQFT:
Graphs and Hopf algebras**

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Abstract

The paper aims at investigating perturbative quantum field theory (pQFT) in the approach of Epstein and Glaser (EG) and, in particular, its formulation in the language of graphs and Hopf algebras (HAs). Various HAs are encountered, each one associated with a special combination of physical concepts such as normalization, localization, pseudo-unitarity, causal(ity and an associated) regularization, and renormalization. The algebraic structures, representing the perturbative expansion of the S -matrix, are imposed on the operator-valued distributions which are equipped with appropriate graph indices. Translation invariance ensures the algebras to be analytically well-defined and graded total symmetry allows to formulate bialgebras. The algebraic results are given embedded in the physical framework, which covers the two recent EG versions by Fredenhagen and Scharf that differ with respect to the concrete recursive implementation of causality. Besides, the ultraviolet divergences occurring in Feynman's representation are mathematically reasoned. As a final result, the change of the renormalization scheme in the EG framework is modeled via a HA which can be seen as the EG-analog of Kreimer's HA.

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1 Introduction

In perturbative Quantum Field Theory (pQFT) interaction of elementary particles is modeled to describe the outcomes of scattering experiments. In the approach of Epstein and Glaser (EG) the S (cattering)-matrix is investigated as a formal power series,

$$S(a\tau) = \sum_{n=0}^{\infty} \frac{a^n}{n!} T_n(\tau^{\otimes n}), \quad \begin{cases} a & \text{coupling (constant),} \\ \tau & \text{IR regularization (test function),} \end{cases} \quad (1)$$

representing an operator-valued distribution acting on the (Fock) space of free quantum fields, where in the adiabatic limit, i.e. $\tau \rightarrow 1$, the square of the corresponding scalar product, $|\langle \text{final} | S(a\tau) | \text{initial} \rangle|^2$, is supposed to yield the scattering probability. Based on the interaction term T_1 , which is assumed to be local, the higher order terms T_n , $n > 1$, of the perturbative expansion can be deduced by 'only' applying causality. In general the series does not converge (in any norm), but in a few cases it is at least Borel summable, cf. [43]. Despite this fact, conceptually, the theory is determined only up to the so-called renormalization freedom. However, in this approach, formulated in terms of distributions, renormalization in the usual sense, as subtraction of counterterms, does not have to

be performed. This actually reflects the original motivation of EG [15], namely, to give a proof of locality (i.e. renormalizability by local counterterms) just by proceeding the sketched way which can be seen as a parallel road to the well-known BPHZ approach [3, 20, 42] (although, referring to the abstract initial setting (1), coming from the same junction).

The specified title of this paper refers to terminology which so far has not properly been implemented into the considered approach. Graphs do not appear in the EG framework and its recent versions [38, 9], and Hopf algebras (HAs), from the perspective of physics, constitute a well-developed concept only in the subject of quantum groups, having decent applications in statistical physics (cf., e.g., Sects. 3.7, 7.6, 8.7 in [22]). Here, graphs will serve as indices for the n -point distributions T_n , and the inductive method the latter will be obtained with is modeled by various HAs.

In 1997 Kreimer [26] discovered that the counterterms in Zimmermann's forest formula [42], which underlies renormalization in the Feynman graph approach, can be interpreted as an antipode condition of a HA. By then one has obviously overcome the mathematical lack which had been associated with renormalization before. But the finding of the mathematical structure alone could not fully eliminate the mysticism involved — the occurrence of a combinatorial HA in this context requires further explanation. The method developed by EG would give a closer look at the physics involved. From a similar point of view (cf. [27], p. 24), in 2000 Kreimer [28], p. 9, formulated the research problem of investigating the EG approach under the aspect of HAs; the content of [18] was not considered as the solution.

A brief history of contributions to the topic of EG-relevant HAs in pQFT reads as follows. In 2000 Fauser [16] describes a HA that combines normal and time ordering. In the same year, based on the work of Fredenhagen and coworkers [10, 8, 12, 9], Pinter [34] discusses the question of uniqueness of EG's n -point distributions to establish a link to Kreimer's HA. In 2002 Brouder et al [6, 7] have taken up the previous two results again; they revive the work of Rota and Stein [35, 36] about combinatorial HAs and present a link to quantum groups. The most recent publication [5] along this line appeared at the end of 2003.

In the presented paper, which uses a very general formulation of pQFT, Kreimer's HA is reconstructed in the EG approach via a scheme of HAs associated with different physical concepts, such as normalization (referring to the normal product construction), localization, pseudo-unitarity, causal(ity with an appropriate) regularization, and eventually renormalization, cf. Fig. 1. The overall strategy has been to represent iterated solutions (e.g. of the perturbative expansion) in terms of antipode conditions. At the beginning of this project, which has been the topic of the author's thesis [31], it was not clear what the involved physical instances would be and neither whether one would be able to realize the aimed correspondence at all. Due to recent results, the starting point for the construction of the sketched HAs can be fixed by a bialgebra which most naturally models the quantum character of the considered physical objects. The algebra formalizes products of free quantum fields and the coalgebra — normal products of the latter. By turning the coalgebra into an algebra and adding a shuffling coalgebra which respects graded total symmetry (gtS) one can easily construct a HA which encodes normalization. Going over to a ring of scalars containing delta distributions, one can as well include the concept of localiza-

p Q F T	H o p f / b i - a l g e b r a		
physical concepts	product	coproduct	antipode
quantization	\circ	Δ^N	none
	T U R N		
normalization	N	Δ	$(-1)^{ \cdot \sim}$
localization	L		
	T W I S T		
pseudo-untarity	\circ	Δ	Σ
causal regularization	$\hat{\circ}^\exists$		$(-1)^{ \cdot \sim}$
	$\hat{\circ}^F$		
	$\hat{\circ}^T$		
renormalization	$\underline{\circ}^*$	Σ	

Figure 1: Hopf algebras modeling pQFT in the EG approach

tion. Extending this ring by propagators that satisfy the spectral condition and by translation invariant (tI) distributions, one obtains an algebra which formalizes the Wick expansion. The product is again the composition (i.e. the circle product, which defines a so-called twist w.r.t. to the localized normal product). With the shuffling coproduct and an antipode (given by the Schmitt formula) which implements pseudo-unitarity, the resulting (Laplace¹) HA models the perturbative expansion. This structure will have to be extended again to explicitly implement causality. In accordance with the well-known ultraviolet problems of pQFT, causality requires an adequate regularization. For the three considered approaches, i.e. the two recent EG versions of Fredenhagen (EGF) and Scharf (EGS) as well as the Feynman approach of Bogoliubov-Shirkov (BS), which vary with respect to the concrete product and antipode, this will be realized by localization on causally determined spacetime regions. In order to finally re-establish the concept of renormalization in the language of HAs one has to consider graded commutative versions of the causally regularizing HAs. Then one can, in analogy to Kreimer's achievements, via some kind of deformation, describe the change of the renormalization scheme in terms of HAs.

The content has been arranged according to items just given. This organizes the paper as follows. In Sect. 2 one introduces free quantum fields, incl. their algebraic structures with respect to normal products and localization. Including interaction, Sect. 3 is about the formal expansion of the S -matrix and associated algebras. In Sect. 4 causality and renormalization get explicitly implemented into pQFT, leading to various HAs. More detailed summaries are given at the beginning of each (sub)section.

Some technical remarks: Independent of its mathematical importance, only propositions which immediately refer to the table are called theorems. To keep

¹This is the terminology used by Rota and Stein [35, 36]. The sketched transfer (i.e. twist) of a HA into a Laplace HA is referred to as Cliffordization.

the number of pages in reasonable bounds, most of the proofs are only sketched (especially when they can be found in [31] or in some other reference). There is also an appendix with proofs and illustrations that might not really be necessary for understanding the text.

2 Local interaction between free quantized fields

The basic physical objects in pQFT are free quantized fields. They are introduced in Sect. 2.1, emphasizing physical but also mathematical aspects. For instance, the free fields can be shown to form a bialgebra where the normal product serves as a coproduct. Two HAs are obtained in Sect. 2.2, describing normalization and localization. The localization of free fields is realized via an appropriate ring of scalars. In Sect. 2.3 monomial interaction of free fields is modeled. This requires to apply the adiabatic limit and to implement the concept of classical background fields into the quantum formalism.

2.1 Free quantized fields

The two sections of this first subsection present, one after the other, a physically and a mathematically motivated approach to the subject. In Sect. 2.1.1 one introduces (derived anti) particles, (derived) quantum fields, as well as associated propagators, and one formulates the causal and the spectral conditions. In Sect. 2.1.2 one constructs a bialgebra modeling compositions of free quantized fields and normal (co)products of the them.

2.1.1 The physical set up

The inner product of the one particle Hilbert space \mathcal{H}_p ,

$$\frac{1}{i\hbar}(f, g)_p^+ \propto \Delta_p^+(f, g) := (\Delta_p^+ * g)(f^*), \quad f, g \in \mathcal{D}_p, \quad (2)$$

is induced by the fundamental solutions of a wave (i.e. relativistically covariant *translation invariant* linear hyperbolic) operator, where Δ_p^+ , given via Fourier transformation $(F\Delta_p^+)(k) = \theta(k^0)(F\Delta_p)(k)$, is the positive frequency part of the difference $\Delta_p = \Delta_p^R - \Delta_p^A$ between the retarded and advanced Green's functions (fulfilling $\text{supp } \Delta_p^{R|A}(\cdot - y) \subseteq y + \overline{V}_{+|-}$, $\forall y \in \mathbb{R}^{1+d}$, where $\overline{V}_{+|-}$ is the closed forward|backward lightcone, i.e. $\overline{V}_+ = \{x \in \mathbb{R}^{1+d} \mid \sqrt{x_1^2 + \dots + x_d^2} \leq x_0\} = -\overline{V}_-$). The appropriate test function space, of Schwartz class say, is denoted by \mathcal{D}_p . In addition to the adjoint f^* of a test function $f \in \mathcal{D}_p$, one will also have to apply the transposed of the previous, $\tilde{f} := f^{*t}$. Notice, Planck's constant \hbar has already been implemented and the velocity of light $c := 1$.

Following a notation similar² to [2], general indices, e.g. ν, μ , are supposed to determine derived (anti) particles, $p = \llbracket \nu \rrbracket$ (and $\tilde{p} = \llbracket \mu \rrbracket$, resp.), representing propagators including derivatives in terms of the Pauli Jordan function, i.e.

$$\Delta_p = (\Delta^{\nu\mu})_{\llbracket \nu \rrbracket=p, \llbracket \mu \rrbracket=\tilde{p}} \quad \text{where} \quad \Delta^{\nu\mu} \stackrel{\partial \dots}{\sim} D_{m_\nu} = D_{m_\nu}^+ + D_{m_\nu}^- \quad (3)$$

²e.g., covariance is not simulated here

and $D_{m_\nu}^\pm(x) = \pm \frac{i}{(2\pi)^{1+d}} \int d\vec{k} \frac{e^{\mp i k x}}{2\omega_\nu(\vec{k})}$ where $\omega_\nu(\vec{k}) = \sqrt{\vec{k}^2 + \frac{m_\nu^2}{\hbar^2}}$. Antiparticles $\tilde{p} = [\tilde{\nu}]$ are always assumed to exist, at least formally, defined by

$$\Delta_{\tilde{p}}^+(x) = -s_p \Delta_p^-(x)^t, \quad \text{where } s_p := (-1)^{2\text{spin}(p)} \quad (4)$$

and where Δ_p^- symbolizes the negative frequency part.

Quantum fields φ_p associated with (anti) particles p are defined as operators on (the dense subspace of) Fock space $\mathcal{D}_p^F = \bigoplus_{n \geq 0} S_n^{\sigma_p} \mathcal{D}_p^{\otimes n} \subset \mathcal{H}_p^F$, i.e. in terms of annihilation and creation operators,

$$\begin{pmatrix} \varphi_p(f) \\ \varphi_{\tilde{p}}(f) \end{pmatrix} = A_p^- \begin{pmatrix} a_p(f) \\ a_{\tilde{p}}(f) \end{pmatrix} + A_p^+ \begin{pmatrix} a_p^*(f) \\ a_{\tilde{p}}^*(f) \end{pmatrix}, \quad f \in \mathcal{D}_p, \quad (5)$$

where for the considered theories

$$A_p^- = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_p \end{pmatrix} \quad \text{and} \quad A_p^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (6)$$

so that by the choice of $\gamma_p = -1 \in \{\pm 1\}$ one can also model ghost fields. As usual, on $\mathcal{D}_p^F \ni |h\rangle = (h_0, h_1(\cdot), h_2(\cdot, \cdot), \dots)$,

$$(a_p(f)|h\rangle)_n = \sqrt{n+1} (f, h_{n+1}(\cdot, \dots))_p^+, \quad (7)$$

$$(a_p^*(f)|h\rangle)_n = \sqrt{n} S_n^{\sigma_p} (f \otimes h_{n-1}), \quad (8)$$

where (anti) symmetrization characterizes bosons (fermions, resp.),

$$S_n^{\sigma_p} h_n = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \sigma_p(\pi) h_n \circ \pi, \quad (9)$$

and $\sigma_p(\cdot)$ defines the *grading* w.r.t. the statistics σ_p ,

$$\sigma_p(\pi) = \begin{cases} 1 & \text{if } \sigma_p = \begin{cases} +1 \\ (-1, \text{ resp.}) \end{cases} \\ (\text{sgn } \pi) & \end{cases} \quad (10)$$

Derived fields $\varphi_{p'} = \partial^{\dots} \varphi_p$ are as well, in order to be treated as free, considered to be given by (derived anti) particles $p' \in A'(p)$,

$$\left(\bigcirc_{\mu \in I} \partial^\mu \right) \varphi^\nu(f) := \varphi_\nu \left((-1)^{|I|} \left(\bigcirc_{\mu \in I} \partial^\mu \right) f \right), \quad I \text{ depends on } (\nu, p'). \quad (11)$$

The graded commutators of free fields have been defined to fulfill the *causal property*, i.e. $\gamma_p = \frac{1}{s_p \sigma_p} \equiv s_p \sigma_p$, which imposes them,

$$[\varphi^\nu(f), \varphi^\mu(\tilde{g})]_{\sigma_p}^\circ := \varphi^\nu(f) \circ \varphi^\mu(\tilde{g}) - \sigma_p \varphi^\mu(\tilde{g}) \circ \varphi^\nu(f) \quad (12)$$

$$= i\hbar \gamma_p \Delta^{\nu\mu}(f, g) \text{id}_{\mathcal{F}} =: i\hbar \Delta_{\gamma}^{\nu\mu}(f, g), \quad (13)$$

to have a causal support, i.e.

$$[\varphi^\nu(f), \varphi^\mu(\tilde{g})]_{\sigma_p}^\circ = 0 \quad \text{if} \quad \text{supp } f \subseteq \mathbb{R}^{1+d} \setminus (\text{supp } g + (\overline{V}_+ \cup \overline{V}_-)), \quad (14)$$

where $\llbracket \nu \rrbracket = p$ and $\llbracket \mu \rrbracket = \tilde{p}'$. Notice, whenever spin and statistics do not coincide the particle is a ghost [25]. The annihilation and creation part commutators read (in density notation, with $x, y \in \mathbb{R}^{1+d}$)

$$[\varphi_{\mp}^{\nu}(x), \varphi_{\pm}^{\mu}(y)]_{\sigma_p}^{\circ} = i\hbar \Delta_{\pm\gamma}^{\nu\mu}(x-y), \quad (15)$$

where the upper case will be applied by Wick's theorem. The positive frequency part reflects the *spectral condition*, which ensures the propagators' multiplication (as distributions) to be well-defined. Namely, as $F\Delta_i^+ \in \mathcal{S}'(\overline{V}_+)$, i.e.

$$\text{supp } F\Delta_i^+ \subseteq \overline{V}_+, \quad \forall i \leq n, \quad (16)$$

the so-called Fourier product [33]

$$\prod_{i \leq n} \Delta_i^+(x) := F^{-1}\left(F(\Delta_1^+(x)) * \dots * F(\Delta_n^+(x))\right) \in \mathcal{S}'(\mathbb{R}^{1+d}). \quad (17)$$

Moreover, cf. [41], the (associative and commutative) convolution lies in $\mathcal{S}'(\overline{V}_+)$, i.e., the resulting product satisfies the spectral condition as well.

The relevant counter example is *Feynman's (time-ordered) propagator*

$$\Delta^{\text{F}} := \mathbb{T} \Delta^+. \quad (18)$$

Applying the causality condition, this leads to the well-known representation,

$$\Delta_p^{\text{F}\gamma}(x) = \mathbb{T} \Delta_p^+(x) := \theta(x^0) \Delta_p^{+\gamma}(x) + \sigma_p \theta(-x^0) \Delta_p^{+\gamma}(-x) \quad (19)$$

$$= \theta(x^0) \Delta_p^{+\gamma}(x) - \theta(-x^0) \Delta_p^{-\gamma}(x), \quad (20)$$

which shows that the spectral condition does not hold true.

2.1.2 The initial bialgebra

Starting with the algebraic considerations for an arbitrarily chosen pQFT, let P denote the finite set of particles and $A'(p)$ the class of derived (anti) particles associated with $p \in P$. Then the common Fock space is formed by the following symmetrized tensor product, $\mathcal{H}_P^{\text{F}} := (S_{|P|} \otimes_{p \in P}) (\otimes_{q \in A'(p)} \mathcal{H}_q^{\text{F}})$. Compatible with this definition, cf. (13), the composition of field operators $\mathcal{F}_P := \{\phi^i \equiv \varphi_{p_i}(x_i) \mid i \in \mathbb{N}\}$ that are fixed by a given surjective map $p : \mathbb{N} \ni i \mapsto p_i \in P$ which models the whole scattering scenario, is commutative only if the associated particles do not belong to the same class of derived (anti) particles, i.e., $[\varphi_{p_1}(x_1), \varphi_{p_2}(x_2)]^{\circ} = 0$ if $p_2 \notin A'(p_1)$.

The admissible compositions $\mathcal{F} = \mathbb{C}\langle \mathcal{F}_P, +, \circ \rangle / (\dots \circ \phi \circ \dots \circ \phi \circ \dots)$ of field operators \mathcal{F}_P , excluding multiple occurrences, generate an algebra $(\mathcal{F}, \mathbb{C}, +, \circ, \eta)$ with unity (map) $\eta := \cdot \text{id}_{\mathcal{H}_P^{\text{F}}}$. According to the determined brackets' resolution, i.e. ' $\circ_{i \leq n} \phi^i$ with brackets' = $\phi^1 \circ (\dots \circ (\phi^{n-1} \circ \phi^n) \dots)$, the composition is always assumed to be associative. In the same spirit, just as the field operators act (linearly) on their domain, which is a (dense) subspace of the Fock space \mathcal{H}_P^{F} , distributivity is implemented. However, repeating the statement from above, the composition algebra of free fields is not graded commutative.

In order to model local interaction the composition, as the following example shows, is not a good product. In the limit of coinciding spacetime points the vacuum expectation is not defined,

$$\langle 0 | \varphi^\nu(x) \circ \varphi^\mu(y) | 0 \rangle = \underbrace{\langle 0 | \varphi^\nu(x) \mathbb{N} \varphi^\mu(y) | 0 \rangle}_{=0, \forall x, y} + i\hbar \underbrace{\Delta_{+\gamma}^{\nu\mu}(x-y)}_{\rightarrow_{x \rightarrow y} (?)} \equiv 0 + (?). \quad (21)$$

The vanishing expectation, i.e. the first expression on the r.h.s., is induced by the so-called normal product,

$$\varphi^\nu(x) \mathbb{N} \varphi^\mu(y) = \varphi_+^\nu(x) \circ \varphi_-^\mu(y) + \sigma_p \varphi_+^\mu(y) \circ \varphi_-^\nu(x). \quad (22)$$

As already applied in (15, l.h.s.), the subscript sign $+|-$ denotes the projection $\text{pr}_\pm : \phi \mapsto \phi_\pm$ onto the creation|annihilation part, cf. (5). For more than two factors the normal product is defined by

$$\mathbb{N}\phi^{\otimes J} := \sum_{J_\pm \in \mathcal{P}_2^0(J)} \sigma_J(J_\pm) \circ \phi_+^{\otimes J_+} \circ \phi_-^{\otimes J_-}, \quad (23)$$

where σ_J denotes the grading sign formulated on tuples $\otimes J \equiv (J_1, \dots, J_n) \in \mathbb{N}_{\neq}^n$ of $n \in \mathbb{N}$ distinct numbers (i.e., $J_i \neq J_j$ if $i \neq j, \forall i, j \leq n$) that represent (indices of) particles p_{J_i} , where $J_\pm = (J_+, J_-) \in \mathcal{P}_2^{(0)}(J)$ denotes a 2-partition of J (resp., incl. empty parts), and where $\phi^{\otimes J} := \otimes_{j \in J} \phi^j$. The grading $\sigma : \mathcal{S}_n \rightarrow \mathbb{Z}_2, (J \mapsto K) = \pi \mapsto \sigma(\pi) = \sigma_J(K)$ is a (non-trivial) representation of the permutation group (\mathcal{S}_n, \circ) on (\mathbb{Z}_2, \cdot) . In accordance with the definition of the common Fock space \mathcal{H}_p^F it can be given in terms of the statistical coefficients σ_p , i.e.

$$\sigma(\pi) := \prod_{p \in P} \sigma_p(\pi|_p) = \prod_{\substack{p \in P \\ \text{sgn}(\pi|_p) = \sigma_p = -1}} \sigma_p(\pi|_p) \quad \text{and} \quad \sigma_p(\emptyset) := 1, \quad (24)$$

where $\pi|_p : J|_p \mapsto \pi(J|_p)$ denotes a subpermutation of π which exclusively applies to derived (anti) particles $A'(p)$ w.r.t. p , defined on the subtuple $J|_p := (J_i)_{(\exists i \leq n) p_{J_i} \in A'(p)}$.

Important for the (Hopf/bi-) algebraic scenario going to be presented is the fact that the normal product can be denoted with the help of a coproduct,

$$\Delta^{\mathbb{N}} := (\text{pr}_+ \otimes \text{pr}_-) \circ \Delta, \quad (25)$$

which is based on (the gtS version, cf. right below, of) the shuffling coproduct,

$$\Delta(\phi^{\circ J}) = \sum_{J_\pm \in \mathcal{P}_2^0(J)} \sigma_J(J_\pm) \phi^{\circ J_+} \otimes \phi^{\circ J_-}. \quad (26)$$

(Notice, the abbreviation $\phi^{\square J} := \square_{j \in J} \phi^j$ is used for any kind of product ‘ \square ’.) Immediately one observes that

$$\mathbb{N} = \circ \circ \Delta^{\mathbb{N}} \circ \circ, \quad (27)$$

and Lemma A.2 in the Appendix verifies the implicit claim about $\Delta^{(\mathbb{N})}$ being (two) coproducts. Therefore, as one easily verifies, $(\mathcal{F}, \mathbb{C}, +, \Delta^{(\mathbb{N})})$ forms a (non)

cocommutative coalgebra where only Δ comes with a counit,

$$\varepsilon(\phi) = \begin{cases} c & \text{if } \phi = c1, \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

Moreover, $\Delta^{(\mathbf{N})}$ as well as ε , can be shown to define homomorphisms w.r.t. the composition product. This leads to the main result of this subsection.

Theorem 2.1 $(\mathcal{F}, \mathbb{C}, +, \circ, \eta, \Delta^{(\mathbf{N})}, \varepsilon)$ forms a bialgebra (without counit, resp.).

Proof. All what is left to show is an immediate consequence of Lemma A.3. \square

Remark 2.2 Because of (21), only the structure in parenthesis seems to be relevant for local pQFT.

Furthermore, one observes that the latterly introduced objects are equipped with the very quantum theoretic property of graded total symmetry (gtS), which models indistinguishable particles governed by a particular statistics.

Proposition 2.3 The normal product \mathbf{N} , as well as the associated (and, resp., the shuffling) coproduct $\Delta^{(\mathbf{N})}$, and the Feynman propagator $\Delta^{\mathbf{F}}$ are all gtS.

Proof. The gtS of $\Delta^{(\mathbf{N})}$, which holds true because of Lemma A.1, immediately implies gtS of \mathbf{N} . And, applying (19) one gets

$$\Delta_p^{\mathbf{F}\gamma}(x) = \sigma_p \Delta_{\bar{p}}^{\mathbf{F}\gamma}(-x) \quad (29)$$

which, due to the propagator's translation invariance, just expresses gtS. \square

2.2 Normalization and localization

In the two Sects., 2.2.1 and 2.2.2, HAs modeling the structure underlying the normal and local products are constructed. With respect to the initial bialgebra, the obtained HAs and the introduced ring of scalars, resp., mean a weakening and a specialization of the mathematical structure.

2.2.1 An appropriate weakening of structure

As already indicated by (21), the composition algebra of free fields does not provide the adequate algebraic structure for modeling local interaction. Only the normal product has been identified to be suitable so that, cf. (27), one would prefer to make the latter defining the physically relevant algebra in the considered context of local pQFT.

In fact, one can drop the 'co' and *turn* the previously constructed coalgebra into an algebra. As $\Delta^{\mathbf{N}}$ is coassociative the normal product \mathbf{N} is associative, and therefore, the normal \mathbb{C} -compositions $\mathcal{N} := \{\circ \circ \Delta^{\mathbf{N}}(\phi) \mid \phi \in \mathcal{F}\}$ constitute a graded commutative algebra with unity, i.e. $(\mathcal{N}, \mathbb{C}, +, \mathbf{N}, \eta)$. Notice, graded cocommutativity has switched to graded commutativity. Because of its shared gtS property, the right candidate for an additional coalgebra is the one with the shuffling coproduct, i.e. $(\mathcal{N}, \mathbb{C}, +, \Delta, \varepsilon)$. Together with the appropriate antipode, as the following result states, the structure of the normally ordered free field operators \mathcal{N} turns out to be a Hopf algebra (HA).

Theorem 2.4 $(\mathcal{N}, \mathbf{N}, \eta, \Delta, \varepsilon)$ forms a graded [co]commutative bialgebra which moreover, with the antipode $S := (-1)^{|\cdot|} \sim$, i.e.

$$S(\phi^{\mathbf{N}J}) = (-1)^n \phi^{\mathbf{N}\tilde{J}}, \quad \forall J \in \mathbb{N}_{\neq}^n, \forall n \in \mathbb{N} \quad (30)$$

(where $\tilde{J} := (J_n, \dots, J_1)$), defines a HA.

Proof. The bialgebra part follows from Lemma A.3. Notice that graded [co] commutativity is nothing else than gtS, thus realized for \mathbf{N} [and $\Delta^{\mathbf{N}}$]. It remains to prove the *antipode condition*, i.e. $\mathbf{N} \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \varepsilon = \mathbf{N} \circ (\text{id} \otimes S) \circ \Delta$. For the nontrivial case, i.e. $n > 0$, where $\eta \circ \varepsilon = 0$, one can write the r.h.s. (and similarly the l.h.s.) as a sum over vanishing pairs,

$$\begin{aligned} & \mathbf{N} \circ (\text{id} \otimes S) \circ \Delta(\phi^{\circ J}) \\ &= \sum_{\substack{J_{\pm} \in \mathcal{P}_2^0(J) \\ |J_{-}| \text{ odd}}} \left(\sigma_J(J_{\pm}) \phi^{\circ J_{+}} \mathbf{N} (-1) \phi^{\circ \tilde{J}_{-}} + \underbrace{\sigma_J(K_{\pm}) \phi^{\circ K_{+}} \mathbf{N} \phi^{\circ \tilde{K}_{-}}}_{\dots + \underbrace{\sigma_J(K_{\pm}) \sigma_{J_{\pm}}(K_{\pm})}_{=\sigma_J(J_{\pm})} \phi^{\circ J_{+}} \mathbf{N} \phi^{\circ \tilde{J}_{-}}} \right) \\ &= 0, \end{aligned}$$

where $K_{+} := J'_{+} \otimes (j) \otimes J''_{+}$ and $K_{-} := J'_{-} \otimes J''_{-}$ provided $J_{+} = J'_{+} \otimes J''_{+}$ and $J_{-} = J'_{-} \otimes (j) \otimes J''_{-}$. The first brace results from the gtS of \mathbf{N} and the second from the composition laws of the grading σ (as a group representation). \square

Going over from the bialgebra of free field operators to the bialgebra of normally ordered field operators, i.e. $\mathcal{F} \rightarrow \mathcal{N}$,

$$\phi^{\circ J} \mapsto \phi^{\mathbf{N}J}, \quad \circ \mapsto \mathbf{N} \quad \text{and} \quad \Delta^{\mathbf{N}} \mapsto \Delta \quad (31)$$

(as well as $\eta \mapsto \eta$), referred to as *normalization* here, this defines a functor that obviously weakens the algebraic structure, turning from a non-(graded)[co] commutative to a graded [co]commutative bialgebra.

2.2.2 Another ring of scalars

In order to model local interaction one has to describe different free particles localized at the same spacetime point. In the chosen mathematical context of distributions that can be realized by applying Dirac's delta distribution. At the current stage of developing the (Hopf) algebraic formalism the latter can be implemented by changing to another ring of scalars.

With tuples $J \in \mathbb{N}^n$ of $n \in \mathbb{N}$ numbers (representing particles) one associates the following products of delta distributions,

$$\delta^{\Pi J} \equiv \Pi \delta^{\otimes J} := \begin{cases} \prod_{i < n} \delta(x_{J_{i+1}} - x_{J_i}) & \text{if } J \in \mathbb{N}_{\neq}^n, \\ 0 & \text{otherwise.} \end{cases} \quad (32)$$

Generated by these distributions and its derivatives, one obtains an algebra of localizing of scalars, i.e.

$$\mathbb{L} := \mathbb{C} \oplus \mathbb{C} \langle \{ \partial^{\beta} \delta^K \mid \beta \in \mathbb{N}^2, K \in \mathbb{N}_{\neq}^2 \}, +, \cdot \rangle / (\delta^K \cdot \partial^{\beta} \delta^K, \delta^K - \delta^{\tilde{K}}), \quad (33)$$

having excluded multiple occurrences of $\delta^{\Pi J}$, even w.r.t. permutations of its indices, and added the unity. The localized quantum field operators $\phi^{\mathbb{L}J} := \delta^{\Pi J} \phi^{\mathbb{N}J}$ are introduced as a \mathbb{C} -submodule,

$$\mathcal{L} := \bigoplus_{J \in \mathbb{N}_{\neq}^n, n \geq 0} \mathbb{C}\{\phi^{\mathbb{L}J} \mid \phi^j \in \mathcal{F}_{\mathbb{P}}, \forall j \in J\}, \quad (34)$$

of $\mathbb{L} \otimes \mathcal{N} =: \mathcal{N}_{\mathbb{L}}$ equipped with the (associative and distributive) product $\mathbb{L} : \phi^{\mathbb{L}J} \otimes \phi^{\mathbb{L}K} \mapsto \phi^{\mathbb{L}(J \otimes K)}$ which varies from the one that is naturally given on $\mathcal{N}_{\mathbb{L}}$, i.e. $\Pi \otimes \mathbb{N} \circ (\text{id} \otimes \tau \otimes \text{id}) =: \mathbb{N}$ (where $\tau : J \otimes K \mapsto K \otimes J$ denotes the flip operation). The latter two spaces carry as well the over all announced structure.

Theorem 2.5 ($\mathcal{N}_{\mathbb{L}}, \mathbb{L}, +, \mathbb{N}, \eta, \Delta, \varepsilon, \mathbb{S}$) *and its substructure* $(\mathcal{L}, \mathbb{C}, +, \mathbb{L}, \eta, \Delta, \varepsilon, \mathbb{S})$ *form graded [co]commutative HAs over the ring \mathbb{L} and, resp., the field \mathbb{C} .*

Proof. Except of the modifications, this is again the content of Theorem 2.4. \square

2.3 Interaction

This last subsection contains necessary supplements. In Sect. 2.3.1, which covers the more mathematical ones, monomial couplings and the adiabatic limit are explained. Sect. 2.3.2 deals with the more physical supplements, introducing background fields and defining the interaction Hamiltonian.

2.3.1 Monomial couplings and the adiabatic limit

The Hamiltonian modeling local interaction will be formulated by the adiabatic limit of coupled localized field operators. In the considered physical approach the coupling Σ_c is given by a monomial in \mathcal{L} . Let $M \in \mathbb{N}_{\neq}^m$ symbolize a tuple of $m \in \mathbb{N}$ particles. Then

$$\Sigma_b(\phi^{\mathbb{L}M}) = \sum_{\bar{\mu} \in M} \sum_{|\alpha| \leq \omega_b} b^{\alpha, \bar{\mu}} (\partial^\alpha \delta^{\Pi \bar{\mu}}) \phi^{\mathbb{N} \bar{\mu}} \quad (35)$$

formalizes a coupling monomial, b labels the associated coupling coefficients $b^{\alpha, \bar{\mu}} \in \mathbb{C}$ where α serves as a suitable multi-index, and $\omega_b \geq 0$ denotes the order (of singularity). Spelled out explicitly, $\Sigma_b \phi^{\mathbb{L}M} \equiv \Sigma_b \varphi^{\mathbb{L}M}(x)$ formally depends on m spacetime variables $x \equiv (x_1, \dots, x_m) \in \mathbb{R}^{(1+d) \times m}$, even though only one of them is independent (because of the localization which uses delta distributions).

The adiabatic limit replaces those m variables by a single one, i.e.

$$A_{x_i} \Sigma_b \phi^{\mathbb{L}M} := \lim_{\tau \rightarrow 1} \langle \Sigma_b \varphi^{\mathbb{L}M}(x), \tau(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \rangle_{m-1} \quad (36)$$

$$= \langle \Sigma_b \varphi^{\mathbb{L}M}(\dots, x_i, \dots), 1(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \rangle_{m-1}, \quad (37)$$

where, as well as in general, $\langle t(x), \tau(x) \rangle_m := \int_{\mathbb{R}^{(1+d) \times m}} t(x) \tau(x) dx_1 \cdots dx_m$ denotes the distributions' dual bracket and $1(x)$ symbolizes an arbitrary test function in $\mathcal{D}_{\mathbb{P}M}^{\otimes} := \bigotimes_{i \leq m} \mathcal{D}_{\mathbb{P}M_i}$ with value '1' in a neighborhood of the origin $x = 0 \in \mathbb{R}^{(1+d) \times m}$. Of course, one may introduce another variable $x_{\bullet} \in \mathbb{R}^{1+d}$. Then

$$A_{x_{\bullet}} \Sigma_b \phi^{\mathbb{L}M} = \delta(x_{\bullet}) * A_{x_1} \Sigma_b \phi^{\mathbb{L}M} = \langle \delta(x_{\bullet} - x_1) \Sigma_b \varphi^{\mathbb{L}M}(x), 1^{\Pi M}(x) \rangle_m, \quad (38)$$

which yields the prototype of a so-called interaction Hamiltonian.

In the algebraic context of normalization and localization the composition of monomial couplings and adiabatic limits is quite straightforward. For any $i \leq n$, let $\overline{M}_i \in \mathbb{N}_{\neq}^{m_i}$ denote an $m_i \in \mathbb{N}$ tuple of particles and b_i the associated coupling with order $\overline{\omega}_{b_i}$. Furthermore, form tuples of those latter objects and let $y \in \mathbb{R}^{(1+d) \times n}$. Then

$$\prod_{i \leq n} A_{y_i \Sigma_{b_i}} \phi^{\overline{L}\overline{M}_i} = A_y^{\otimes \overline{M}} \Sigma_{\Pi b} \phi^{\text{NoL}\overline{M}}, \quad (39)$$

where $A_y^{\otimes \overline{M}} = A_{y_1}^{\overline{M}_1} \cdots A_{y_n}^{\overline{M}_n}$ specifies the application of the adiabatic limit (in the obvious way) and

$$\Sigma_{\Pi b} \phi^{\text{NoL}\overline{M}} = \sum_{\hat{\mu} \in \overline{M}} \sum_{|\overline{\alpha}| \leq \overline{\omega}_b} \prod_{i \leq n} b_i^{\overline{\alpha}_i, \hat{\mu}_i} (\partial^{\Pi \overline{\alpha}} \delta^{\Pi^2 \hat{\mu}}) \phi^{\text{No}^2 \hat{\mu}}. \quad (40)$$

2.3.2 Physical supplements

To make the coupling monomial a physical magnitude it might be supplemented by a further coupling constant, and to get a reasonable physical theory some of those prototype Hamiltonians with vanishing order of singularity might be additively combined.

Moreover, the concrete physical model, fixed by the interaction Hamiltonian, may contain classical background fields which are given by scalar (e.g. real or Grassmann) valued test functions $\tau_p \in \mathcal{D}_p$, supposed to smear the operator valued distributions. Those are non-quantized fields that, in order to be equipped with the required indices, are as well labeled by particles $p \in C \equiv P \setminus Q$, referred to as virtual ones. The particles considered so far are renamed $Q \subseteq P$. Notice that virtual particles $p \neq (\emptyset) \in C$, i.e. if not just representing a non-indexed real valued test function $\tau_{(\emptyset)}$, may also have to follow a non-trivial statistics.

Let the disjoint union $\mathcal{I} = \mathcal{C} \dot{\cup} \mathcal{Q} \subset \mathcal{P}(\mathbb{N})$ denote a system of subsets of basic indices, i.e. $\{c_j\} \in \mathcal{C}$ and $Q_j \in \mathcal{Q}$ where $j \leq m \in \mathbb{N}$, supposed specify virtual particles $p_j = p_{c_j} \in C$, that label the associated couplings a_{p_j} , b_{p_j} , as well as tuples of particles $Q_j = (p_i)_{i \in Q_j} \subset Q$ and their coordinates, x_{c_j} and $x_{Q_j} = (x_i)_{i \in Q_j}$. Then the smeared abstract interaction Hamiltonian reads

$$H_{\mathcal{I}}(\tau^{\oplus \mathcal{C}}) = \sum_{j \leq m} a_{p_j} \left\langle A_{x_{c_j}} \left(\Sigma_{b_{p_j}} \varphi^{\text{L}Q_j}(x_{Q_j}) \right), \tau_{c_j}(x_{c_j}) \right\rangle_1. \quad (41)$$

Example 2.6 Quantum electrodynamics (QED) in an external electromagnetic field is, using the conventional notions (e.g., $\overline{\psi}, \psi$ for the spinorial positron and electron, A^{cl}, A for the classical and quantized electromagnetic field, all equipped with localized spacetime indices, and colons for the normal product), described by the interaction Hamiltonian, $\frac{e}{i\hbar} : \overline{\psi}(x) \gamma^\mu \psi(x) : (A_\mu(x) + A_\mu^{\text{cl}}(x))$. Therefore, one has to identify $a = \frac{e}{i\hbar}$, $c = \gamma$, and $\varphi \in \{\overline{\psi}, \psi, A\}$. The two sorts of test functions are real and Grassmann valued, i.e. $\tau \in \{\tau_{(\emptyset)}, A^{\text{cl}}\}$.

3 Perturbative expansion

There is a HA underlying the expansion of the S -matrix. Its antipode condition represents the property of pseudo-unitarity. In order to achieve this result at the

end of Sect. 3.2, one has to construct algebras of operator-valued distributions that model the Wick expansion. The main physical ingredients that ensure a well-defined analytical behavior (relevant for the scalar parts of the n-point distributions) are translation invariance and the spectral condition. Rather unusual for the EG approach, in addition to vertex tuples, graphs are chosen as indices for n-point distributions. They are introduced in Sect. 3.1.

3.1 The concept of graphs

Graphs, vertices, and their constituents in the context of pQFT are defined in Sect. 3.1.1. Moreover, a correspondence between configuration space variables and vertex tuples of graphs is pointed out. Algebraic properties of graphs and their symmetric versions are introduced in Sect. 3.1.2.

3.1.1 Vertices and graphs

The elements of the index set \mathcal{I} in Sect. 2.3.2, and correspondingly the prototype interaction Hamiltonians, will be interpreted as vertices, cf. Fig. 2.

Starting from an initial set U of vertices, the perturbative expansion will lead to objects that can be interpreted as graphs $G_U = \mathcal{G}(U)$ formed by those vertices. Furthermore, renormalization will allow to reinterpret graphs as vertices, i.e. $\Omega : G_U \rightarrow W$, which produces new vertices in $W \setminus U$. With all these vertices one will as well form graphs in $G_W = \mathcal{G}(W)$ that can again, by forgetting about the inner graph structure, be shrunken to vertices via $\Omega : G_W \rightarrow W$. This rough description of the induction, cf. Fig. 3,

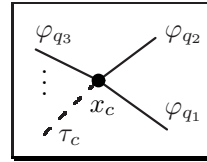


Figure 2: A vertex.

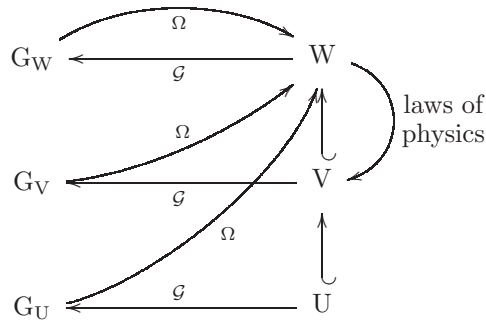


Figure 3: Generation of graphs and (of new) vertices.

has to be supplemented by the remark that physical requirements, e.g. symmetries, restrict the number of physically relevant vertices V .

Formally, the pairs $v \equiv (P, b) \in \mathbb{Q}^n \times \mathbb{C}^{|\mathcal{P}_n(\omega_b)| \times \Pi|P|}$ of particle tuples $P = P(v)$ and coupling coefficients $b = b(v)$, where $\mathcal{P}_n(\omega_b) = \{\alpha \in \mathbb{N}^n \mid \sum_{i \leq n} \alpha_i \equiv |\alpha| \leq \omega_b\}$ and $n \equiv |P|$, are supposed to define vertices $v \in \{U, V, W\}$. Graphs are defined as pairs $\Gamma \equiv (\bar{v}, l)$ of a vertex tuple $\bar{v} = V(\Gamma)$ and a $|\bar{v}| \times |\bar{v}|$

matrix $l = l(\Gamma)$ of (sets of internal) lines $l_{rs} \subseteq \{(i, j) \in \mathbb{N}_{|\bar{v}|} \times \mathbb{N}_{|\bar{v}|} \mid \gamma_{\bar{v}_i, \bar{v}_j} \neq 0, \bar{p} \equiv \bar{v}_r, \bar{q} \equiv \bar{v}_s\}$, forming a collection of index pairs that represent two matching particles, where so-called tadpoles are excluded, i.e. $l_{rr} = \emptyset$.

Keeping the notations, the iterated tuple $I(\Gamma) \equiv (I_r)_{r \leq |\bar{v}|}$ of particles that support the internal lines of a graph Γ , referred to as representing the internal half-lines, can be reobtained by $I_r = (P_i(\bar{v}_r))_{i \in \bar{l}_r}$ where $\bar{l}_r = \{\text{pr}_1 \lambda \mid \lambda \in l_{rs}, s \leq |\bar{v}|\}$. Thus, the tuple $E(\Gamma) \equiv (E_r)_{r \leq |\bar{v}|}$ representing the external half-lines is given by $E_r = (P_i(\bar{v}_r))_{i \in \bar{l}_r}$ where $\bar{l}_r = \mathbb{N}_{|\mathbb{P}(\bar{v}_r)|} \setminus \bar{l}_r$.

By introducing graph related cardinalities Euler's formula is stated, i.e.

$$|\Gamma| = \#\Gamma - |l| + |\bar{v}|, \quad \begin{cases} |\Gamma| \\ \#\Gamma \\ |l| \\ |\bar{v}| \end{cases} \text{ denotes the } \begin{cases} \text{connected components,} \\ \text{independent loops,} \\ \text{(internal) lines,} \\ \text{vertices.} \end{cases} \quad (42)$$

where

The number of lines can be obtained by $|l| = \sum_{r < s \leq |\bar{v}|} |l_{rs}|$.

The map p which identifies particles p_i via basic indices $i \in I \subset \mathbb{N}$, and thus provides a correspondence with spacetime variables $x_i \in \mathbb{R}^{1+d}$, can consistently be defined to as well identify vertex tuples and associated classes of graphs. In this context one will have to consider a bigger system $\mathcal{J} := \mathcal{C} \dot{\cup} \mathcal{Q} \subseteq \mathcal{P}(\mathbb{N})$ of subsets of indices than \mathcal{I} in Sect. 2.3.2, i.e. $c_J := \dot{\bigcup}_{j \in J} \{c_j\}$ and $Q_J := \dot{\bigcup}_{j \in J} Q_j$ satisfying $c_J \cap c_K = \emptyset$ and $Q_J \cap Q_K = \emptyset$, for $J, K \in \mathcal{P}(\mathbb{N})$.

Lemma 3.1 *There are a system \mathcal{J} and a surjective map $p : \mathbb{N} \rightarrow P$,*

$$n \mapsto \begin{cases} p_n^C \in C \\ p_n^Q \in Q \end{cases} \quad \text{if} \quad n \in \begin{cases} \mathbb{N}_C, \\ \mathbb{N}_Q, \end{cases} \quad \text{where} \quad \mathbb{N}_C \cap \mathbb{N}_Q = \emptyset, \quad \text{such that}$$

$$(\forall \bar{v} \in \bigcup_{n \geq 0} W^n) (\exists J \in \mathbb{N}_{\neq}^n) (\forall j \in J) b(\bar{v}_j) = b_{p^C(c_j)} \wedge P(\bar{v}_j) = (p^Q(i))_{i \in Q_j}.$$

Proof. Decompose the operation of forming graphs, i.e. $\mathcal{G} = \bigcup_{n \geq 0} \mathcal{G}_n$, so that $\mathcal{G}_n : \mathcal{P}(W) \rightarrow \mathcal{P}(G_W)$, $W_k \mapsto \mathcal{G}_n(W_k)$ yields graphs Γ with $n = |\bar{V}(\Gamma)|$ vertices, $V(\Gamma) \in W_k^{\times n}$. Then the existence of \mathcal{J} and p is guaranteed by the inductive construction of $W = \bigcup_{k \geq 1} W_k$, and thus $G_W = \bigcup_{n \geq 1} \mathcal{G}_n(W_n)$, from finite sets $W_{k+1} = \{\Omega(\Gamma) \in W \mid \Gamma \in \mathcal{G}_k(W_k)\}$ starting with $W_1 = U$. \square

Notice that vertices and virtual particles immediately correspond via p^C . In Sect. 4 one will have to indicate spacetime variables x_i associated with vertices $\bar{v}_j = \Gamma_j \sim p_i^C$ of tuples \bar{v} and graphs $\Gamma \equiv (\bar{v}, l)$. This will be denoted by

$$x_\bullet : \bar{v}_j \mapsto x_{v_j} := x_i \quad \text{and} \quad \Gamma_j \mapsto x_{\Gamma_j} := x_i, \quad \text{resp.} \quad (43)$$

3.1.2 Algebras of graphs

For the algebraic considerations, graphs are always assumed to form a complex vector space which respects the \mathbb{N} -grading corresponding to the number of vertices, i.e. $(G_V, \mathbb{C}, +)$ where $G_V = \bigoplus_{n \geq 0} \mathbb{C} \mathcal{G}_n(V)$.

Let $\Gamma^i \equiv (\bar{v}^i, l^i)$, $i \leq 2$, denote two graphs. There are $|\bar{v}^1| \times |\bar{v}^2|$ matrices $l^{12} \in \mathbb{M}_{|\bar{v}^1| \times |\bar{v}^2|}(\mathcal{P}(\mathbb{N}^{\times 2}))$ that yield a new graph from the previous ones, i.e.

$$\Gamma^1 \sqcup_{l^{12}} \Gamma^2 := \left(\bar{v}^1 \otimes \bar{v}^2, \begin{pmatrix} l^1 & l^{12} \\ \tilde{l}^{12} & l^2 \end{pmatrix} \right), \quad (44)$$

so that the required symmetry, $l_{rs} = \tilde{l}_{sr} := \{(\text{pr}_2 \lambda, \text{pr}_1 \lambda) \mid \lambda \in l_{sr}\}$, is realized. The admissible ones, i.e. $l^{12} \in l(\Gamma^1, \Gamma^2)$, are called concatenation matrices. Cf. Example A.4. In contrast to the matrices of lines, $|l^{12}| = \sum_{r \leq |\bar{v}^1|, s \leq |\bar{v}^2|} |l_{rs}^{12}|$.

This composition leads to the Wick product of graphs which is defined by $\sqcup_\Sigma = \sum_{l^{12} \in l(\cdot, \cdot)} \circ \sqcup_{l^{12}}$. Moreover, by linear extension one [as well] considers gtS versions of graphs, $\sqcap \Gamma := \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \sigma(\pi) \pi(\Gamma) \in \bigoplus_{n \geq 0} \mathcal{S}_n^\sigma \mathbb{C} \mathcal{G}_n(V) =: \mathbb{G}_V^\sigma$, where the permutation $\pi(\Gamma) := (\pi(\bar{v}), \pi(l))$ acts on the vertex tuple and on the matrix of lines, i.e. $\pi(\bar{v}) := (\bar{v}_{\pi(r)})_{r \leq n}$ and $\pi(l) := (l_{\pi(r)\pi(s)})_{r, s \leq n}$, and the grading is reduced to the graph's vertex tuple, i.e. $\sigma_\Gamma(\pi(\Gamma)) := \sigma_{V(\Gamma)}(V \circ \pi(\Gamma))$, discussed right below in Sect. 3.2.1.

Lemma 3.2 $(\mathbb{G}_V^{[\sigma]}, \mathbb{C}, +, \sqcup_\Sigma, \emptyset)$ forms an algebra [in both cases] where the empty graph serves as unity.

Proof. Cf. Sect. 3.2.2 in [31], pp. 75ff. \square

Associativity can be illustrated introducing the inverse concept of concatenation, i.e. separation into subgraphs that, w.r.t. the reference graph, are completely determined by their vertices. Let $(\Gamma^1, \dots, \Gamma^k) \in \mathcal{P}_k^{(0)}(\Gamma)$ denote a k -partition (resp., with possibly empty subgraphs) of the graph $\Gamma \in \mathbb{G}_V^{[\sigma]}$ iff there are a permutation³ $\pi \in \mathcal{S}_k$ and appropriate concatenation matrices, l^1, \dots, l^{k-1} , such that $\pi(\Gamma) = \Gamma^1 \sqcup_{l^1} (\dots \sqcup_{l^{k-1}} \Gamma^k) \dots$ [appropriately extended].

Let $\Gamma \in \mathbb{G}_V^{[\sigma]}$. Then $\mathbb{G}_\Gamma^{[\sigma]} := \mathcal{P}(\Gamma) = \{\Gamma' \mid (\Gamma', \dots) \in \mathcal{P}_2^0(\Gamma)\} \subset \mathbb{G}_V^{[\sigma]}$ denotes the powerset which contains all subgraphs of Γ . Graphs in $\mathbb{G}_V^{[\sigma]}$ can be composed as Γ -restricted graphs, i.e. via $\sqcup_\Gamma = (\sigma_\Gamma^0 \circ \sqcup_\Sigma) \sqcup_\Sigma$, where

$$\sigma_\Gamma^0 : \mathbb{G}_V^{[\sigma]} \rightarrow \{-1, 0, 1\}, \quad \Gamma' \mapsto \begin{cases} \sigma_\Gamma(\Gamma') & \text{if } \Gamma' \in \mathbb{G}_\Gamma^{[\sigma]}, \\ 0 & \text{otherwise,} \end{cases} \quad (45)$$

and $\sigma_\Gamma(\Gamma') = \sigma_{\Gamma'}(\Gamma) \equiv \sigma(\pi)$, regarding $\pi : \Gamma' \mapsto \Gamma$ as a permutation in $\mathcal{S}_{|V(\Gamma')|}$.

Proposition 3.3 $(\mathbb{G}_V^{[\sigma]}, \mathbb{C}, +, \sqcup_\Gamma, \emptyset)$ forms an algebra, and $(\mathbb{G}_\Gamma^{[\sigma]}, \dots)$ identifies its largest zero divisor free subalgebra $(\mathbb{G}_{\Gamma'}^\sigma, \dots)$, $\Gamma' \in \mathbb{G}_V^{[\sigma]}$.

Proof. Associativity of \sqcup_Σ implies associativity of \sqcup_Γ . The other statements are obvious. \square

3.2 The expansion of the S -matrix

In the EG approach, the S -matrix is assumed to be expanded in terms of operator-valued gtS n -point distributions indexed by vertex tuples. This ansatz and its property of being an infrared regularization are content of Sect. 3.2.1. In contrast to the usual pQFT approach, EG have not mixed Wick and time ordering. Therefore, in Sect 3.2.2, one forms an algebra which models Wick ordering without an explicite implementation of causality. Translation invariance (tI) of the n -point distributions' scalar parts and the spectral condition for the propagators are the properties that ensure the algebraic constructions underlying the

³which depending on the context, e.g. when showing associativity of \sqcup_Σ , might be fixed by $\pi = \text{id}$

Wick expansion to be built on well-defined analytical grounds. In contrast to the EG approach, in Sect. 3.2.3, one introduces n -point distributions indexed by graphs. The usual ones, indexed by vertex tuples, are reproduced from the latter. Algebras of both these distributions, each one equipped with the shuffling coproduct and an antipode which represents the Schmitt formula, form HAs that represent the perturbative expansion and especially, as it is shown in Sect. 3.2.4, the property of pseudo-unitarity.

3.2.1 Infrared regularization, the ansatz, and gtS

In the considered approach the S -matrix is supposed to depend on coupling constants a_p and normalized test functions $\tau_p \in \mathcal{D}_p$ (i.e. $\tau_p(0) = 1$) that, representing external classical fields $p \in \mathcal{C}$, yield an infrared regularization. To obtain concrete physical quantities the adiabatic limit has to be taken, i.e.

$$S((a\tau)_C^\oplus) = \lim_{\epsilon \rightarrow 0} S((a\tau_\epsilon)_C^\oplus), \quad \forall \tau : \tau \circ \epsilon \equiv \tau_\epsilon \xrightarrow{L^\infty(K)}_{\epsilon \rightarrow 0} 1, \quad \forall K \in \mathbb{R}^{1+d}. \quad (46)$$

However, this task will not be regarded in the paper. Notice, as the vertices V correspond to the set \mathcal{C} in Sect. 2.3.2, which yields the virtual particles $C = \{p_i \mid i \in \mathcal{C}\}$, they will be used as indices as well.

The expansion of the S -matrix is given w.r.t. a renormalization scheme \mathcal{R} (only discussed in Sect. 4.2) and a (somehow related) set V of vertices. The ansatz, chosen in accordance with the original ones of Bogoliubov and Shirkov (BS) [4], EG, and Scharf [37], reflects the Dyson series (cf. Example A.5), i.e.

$$S_V^{\mathcal{R}}((a\tau)^{\oplus V}) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\bar{v} \in V^n} a^{\Pi \bar{v}} \left\langle T_{\bar{v}}^{\mathcal{R}}(y), \tau^{\Gamma \bar{v}}(y) \right\rangle_n, \quad (47)$$

where $T_{\bar{v}} \in C(S_{|\bar{v}|}^\sigma \mathcal{D}_{\bar{v}}^{\otimes}, \mathcal{H}_p^F)$ are operator-valued n -point distributions, i.e. strongly continuous mappings, indexed by vertices. The interaction Hamiltonian prototypes can be reidentified as $H_p = i\hbar a_p T_{(p)}$.

The vertex distributions are assumed to be gtS, i.e. $T_{\bar{v}} = \sigma(\pi) T_{\pi(\bar{v})} \circ \pi$, $\forall \pi \in S_n$. Here, σ symbolizes the grading on vertex tuples, given again by (24), but $\pi|_p : \bar{v}|_p \mapsto \pi(\bar{v}|_p)$ denotes a subpermutation of π defined on $\bar{v}|_p = (\bar{v}_k)_{k' \text{ odd}}$ where $k' = \sum_{q \in A'(p)} 1_{q=v_i \wedge v=\bar{v}_k}$ (cf. [31], p. 45). This is again a representation of S_n on \mathbb{Z}_2 , and one also writes $\sigma_U(V) := \sigma(U \mapsto V)$, for vertex tuples $U, V \in \mathbb{N}_{\neq}^n$.

Lemma 3.4 *The two definitions of σ are consistent, i.e. $\sigma_U(V) = \sigma_{[U]}([V])$ where, here, $[\cdot]$ symbolizes the formal transfer from a vertex tuple into a particle tuple by simply omitting brackets and couplings.*

Proof. Cf. Lemma 2.1.7 in [31], p. 45. □

Of course, the test functions associated with vertex tuples are gtS as well, $\tau^{\Gamma \bar{v}} \equiv \tau^{\Gamma \bar{c}} := S_n^\sigma \tau^{\otimes \bar{c}} \in S_n^\sigma \mathcal{D}_{\bar{c}}^{\otimes} \equiv S_n^\sigma \mathcal{D}_{\bar{v}}^{\otimes}$, where $\bar{c} \in \mathbb{C}_{\neq}^n$ denotes the corresponding tuple of virtual particles. Again, the definitions are consistent.

Lemma 3.5 $\sigma_{\bar{v}}(\pi(\bar{v})) = \sigma_{\bar{c}}(\pi(\bar{c}))$, $\forall \pi \in S_n$.

Proof. Applying the permutation π to both sides of the dual bracket in (47), this yields $\sigma_{\bar{v}}(\pi(\bar{v})) \sigma_{\bar{c}}(\pi(\bar{c})) = 1$. Hence, the statement is due to the fact that $\sigma(\pi) \in \{\pm 1\}$. □

3.2.2 Algebras of n-point distributions

Founded on Wick's theorem, referring to the expansion rule of normal products w.r.t. the composition product (which can also be considered as twist or so-called Cliffordization), one is going to construct an algebra of graph distributions over a carefully chosen ring of translation invariant (tI) scalar distributions. The tI distributions $t^J \in \mathcal{S}'(\mathbb{R}^{(1+d) \times n})$, $J \in \mathbb{N}_{\neq}^n$, i.e.

$$t^J \equiv t_J(\tau) = t_J(\tau(\cdot - z^{\times n})), \quad \forall \tau \in \mathcal{S}(\mathbb{R}^{(1+d) \times n}), \quad \forall z \in \mathbb{R}^{1+d}, \quad (48)$$

are supposed to form an \mathbb{N} -graded algebra w.r.t. the direct product, i.e.

$$\mathbb{T} := \mathbb{C} \oplus \mathbb{C} \langle \{t^K \mid K \in \mathbb{N}_{\neq}^n, n \in \mathbb{N}\}, +, \cdot \rangle / (t^{(\dots k, \dots)} \cdot t^{(\dots k, \dots)}), \quad (49)$$

covering the (algebra of) localizing scalars \mathbb{L} and the positive frequency part propagators $\Delta_+^J \equiv \Delta_{p_{J_1}}^{+\gamma}(x_{J_1} - x_{J_2})$ with derivatives, i.e.

$$\mathbb{P} := \mathbb{C} \oplus \mathbb{C} \langle \{\partial^\beta \Delta_+^K \mid \beta \in \mathbb{N}^2, K \in \mathbb{N}_{\neq}^2, \gamma_K \neq 0\}, +, \cdot \rangle / (\Delta_+^{\tilde{K}} \cdot \partial^\beta \Delta_+^K), \quad (50)$$

that, w.r.t. the Fourier product and in accordance with the spectral condition (cf. end of Sect. 2.1.1), form an algebra as well. Precisely, \mathbb{L} and \mathbb{P} are ideals in \mathbb{T} . Combining the two previously defined algebras, one gets another commutative algebra,

$$\mathbb{T}_{\mathbb{P}} := \mathbb{T} \otimes \mathbb{P} / (t \otimes p - p \otimes t, \dots, \underbrace{t^{(\dots, K_{1|2}, \dots, K_{2|1}, \dots)}}_{\neq t^{(\dots, K_{1|2}, \dots)} t^{(\dots, K_{2|1}, \dots)}} \otimes \dots \Delta_+^K) \quad (51)$$

which, on the level of distributions, encodes the Fourier and the direct product, and additionally the well-defined product of distributions with continuous functions. Including localization, the related commutative algebra $(\mathbb{T}_{\mathbb{P}\mathbb{L}}, \mathbb{C}, +, \cdot, 1)$ is given by

$$\mathbb{T}_{\mathbb{P}\mathbb{L}} := \mathbb{T}_{\mathbb{P}} \otimes \mathbb{L} / (t \otimes l - l \otimes t, \dots, \underbrace{t^{(\dots, K_{1|2}, \dots, K_{2|1}, \dots)}}_{\text{irreducible in } \mathbb{T}_{\mathbb{P}}} \otimes \dots \delta^K - \dots \otimes \dots \delta^K). \quad (52)$$

Again, the substructures are ideals, i.e. $\mathbb{T}_{\mathbb{P}\mathbb{L}} \subset \mathbb{T}_{\mathbb{P}} \subset \mathbb{T}$. Similar to Sect. 2.2.2, those algebras will mainly be applied as rings.

Turning to the operator side, let $\mathcal{T}_{\mathbb{V}} := \bigoplus_{n \geq 0} \bigcup \{\mathcal{T}_{\Gamma} \mid \Gamma \in \mathbb{G}_{\mathbb{V}}, |\mathbb{V}(\Gamma)| = n\}$ be the \mathbb{N} -graded space of graph-indexed n -point distributions

$$\mathcal{T}_{\Gamma} = \left\{ \sum_{\mathbf{b}(\Omega\Gamma)} t_{\Gamma}(\bullet) \mathbf{A}_{\bullet}^{\otimes \mathbb{V}(\Gamma)} \phi^{\mathbb{N} \circ \text{LE}(\Gamma)} \in C(S_n^{\sigma} \mathcal{D}_{\mathbb{V}(\Gamma)}^{\otimes}, \mathcal{H}_{\mathbb{P}}^{\mathbb{F}}) \mid t_{\Gamma} \in \mathbb{T} \right\}, \quad (53)$$

where, in terms of the couplings at the vertices of $\mathbb{V}(\Gamma)$, the coupling at the vertex $\Omega\Gamma$ reads

$$\mathbf{b}(\Omega\Gamma) = \prod_{r < s \leq n} \prod_{(i,j) \in l_{rs}(\Gamma)} \sum_{\llbracket \mu \rrbracket \in P_{ri}} \sum_{\llbracket \nu \rrbracket \in P_{sj}} \prod_{k \leq n} b_{\mathbb{V}_k(\Gamma)}^{\alpha_k, P_k}. \quad (54)$$

Then $\mathcal{T}_{\mathbb{V}}$ is obviously a submodule of $\mathbb{T}_{\mathbb{P}\mathbb{L}} \otimes \mathcal{N}$, and it is algebraically closed w.r.t. the composition product.

Proposition 3.6 $(\mathcal{T}_V, \mathbb{T}_{\text{PL}}, +, \circ, \eta)$ forms an algebra.

Proof. This is the content of Prop. 3.3.2 in [31], p. 85, which in the presented approach serves as the alternative to EG's Theorem 0 in [15], p. 229. In order to illustrate the crucial points, let $T^i \in \mathcal{T}_{\Gamma^i}$ with scalars $t^i \equiv t_{\Gamma^i} \in \mathbb{T}$, for $\Gamma^i \in \mathbf{G}_V$ and $i \leq 2$, and use the abbreviation $N(\bullet) := \mathbf{A}_{x_V(\bullet)}^{\otimes V(\bullet)} \phi^{\mathbf{N} \circ \text{LE}(\bullet)}$. Then

$$T^1 \circ T^2 = \sum_{l \in l(\Gamma^1, \Gamma^2)} (i\hbar)^{|l|} \sum_{\mathbf{b}(\Omega(\Gamma^1 \sqcup_l \Gamma^2))} t^l N(\Gamma^1 \sqcup_l \Gamma^2) \in \mathcal{T}_V, \quad (55)$$

having an well-defined associated scalar distribution which is again tI, i.e.

$$t^l(y^1 \otimes y^2) = t^1(y^1) t^2(y^2) \Pi_{(\Gamma^1, \Gamma^2)}^l(y^1, y^2) \in \mathbb{T}_{\mathbb{P}}. \quad (56)$$

The most r.h.s. expression abbreviates a certain product of positive frequency propagators (that, cf. below, are also referred to as Wick contractions⁴),

$$\Pi_{(\Gamma^1, \Gamma^2)}^l(y^1, y^2) := \prod_{r \leq |P^1|, s \leq |P^2|} \prod_{(i,j) \in l_{rs}} \Delta_{+\gamma}^{P_{ri}^1 P_{sj}^2}(y_r^1 - y_s^2) \in \mathbb{P}, \quad (57)$$

being well defined due to the spectral condition (16). Besides tI — for the analytical part, the proof of the algebraic part rests upon \square

Lemma 3.7 (Wick's theorem). *Keeping the notations, then*

$$N(\Gamma^1) \circ N(\Gamma^2) = \sum_{l \in l(\Gamma^1, \Gamma^2)} (i\hbar)^{|l|} \Pi_{(\Gamma^1, \Gamma^2)}^l N(\Gamma^1 \sqcup_l \Gamma^2). \quad (58)$$

Proof. On the level of free field operators the expansion reads

$$\phi^{\mathbf{N}I} \otimes \phi^{\mathbf{N}J} = \sum_{\Lambda \subseteq \mathbb{N}_{|I|} \times \mathbb{N}_{|J|}} \left(\prod_{\lambda \in \Lambda} [\phi_{-}^{I_{\lambda_1}}, \phi_{+}^{J_{\lambda_2}}]_{\sigma}^{\otimes} \right) \phi^{\mathbf{N}(I \otimes_{\Lambda} J)}, \quad (59)$$

provided $[\cdot, \cdot]_{\sigma}^{\otimes} \propto \text{id}_{\mathcal{H}_{\mathbb{P}}}$, where $I \otimes_{\Lambda} J = (I_i)_{i \in \mathbb{N}_{|I|} \setminus \{\lambda_1 | \lambda \in \Lambda\}} \otimes (J_i)_{i \in \mathbb{N}_{|J|} \setminus \{\lambda_2 | \lambda \in \Lambda\}}$, for tuples I, J . This is proven by successive insertion of Wick contractions,

$$\phi_{-}^i \otimes \phi_{+}^j = [\phi_{-}^i, \phi_{+}^j]_{\sigma}^{\otimes} + \sigma_{(i,j)}((j, i)) \phi_{+}^j \otimes \phi_{-}^i,$$

moving the ϕ_{+}^j 's to the left and the ϕ_{-}^i 's to the right. Due to (15), the bracket in (59) can be replaced by propagators. In accordance with the definition of \mathbb{T}_{PL} , the localizing delta distribution's are supposed to dominate (i.e. erase) the propagators, which reduces (58) to (59). \square

Remark 3.8 The associativity of the algebra, i.e. $N(\Gamma^1) \circ (N(\Gamma^2) \circ N(\Gamma^3)) = (N(\Gamma^1) \circ N(\Gamma^2)) \circ N(\Gamma^3)$, is equivalent to the 2-cocycle condition for Π , i.e. $\Pi_{(\Gamma^1, \Gamma^2 \sqcup_{\Gamma} \Gamma^3)} \Pi_{(\Gamma^2, \Gamma^3)} = \Pi_{(\Gamma^1, \Gamma^2)} \Pi_{(\Gamma^1 \sqcup_{\Gamma} \Gamma^2, \Gamma^3)}$, $\forall \Gamma^i \in \mathbf{G}_{\Gamma}$, $i \leq 3$, which is obviously fulfilled. In the context of [35, 36] and [5], $\Pi_{(\cdot, \cdot)}$ is called a Laplace pair. There, the relation (58) is regarded as a so-called Cliffordization or twist, resp.

⁴The symbol 'II' is chosen in accordance with BS' notation of a Wick contraction,

$$\mathbf{N}(\dots \otimes \underbrace{\phi^i \otimes \dots \otimes \phi^j}_{\text{Wick contraction}} \otimes \dots) = [\phi_{-}^i, \phi_{+}^j]_{\sigma}^{\otimes} \mathbf{N}(\dots \otimes \dots \otimes \dots).$$

3.2.3 Graph and vertex distributions

In this subsection the physically relevant solution of the perturbation expansion is assumed to be given — formally, without having introduced the essential concepts yet. This will only be done in Sect. 4.

Graph distributions $\mathcal{R}_{G_V}^T := \bigoplus_{n \geq 0} \mathbb{C}\{T_\Gamma^R \mid \Gamma \in G_V, |\mathbb{V}(\Gamma)| = n\}$, supposed to solve the perturbative expansion by respecting causality, are introduced via a vector space homomorphism,

$$T_\bullet^R : G_V \rightarrow \mathcal{T}_\Gamma, \quad \Gamma \mapsto T_\Gamma^R := (i\hbar)^{\#\Gamma - |\Gamma|} \sum_{b(\Omega\Gamma)} t_{I(\Gamma)}^R N(\Gamma), \quad (60)$$

fixing the associated scalar valued n-point distributions $t_{I(\Gamma)}^R \in S_n^\sigma \mathcal{D}_{\mathbb{V}(\Gamma)}^\otimes \subset \mathbb{T}$ which are supposed to be obtained w.r.t. a renormalization scheme \mathcal{R} , and via a gtS requirement (which can only be motivated in Sect. 4, cf. Remark 4.17),

$$T_{\pi(\Gamma)}^R = \sigma(\pi) T_\Gamma^R, \quad \forall \pi \in \mathcal{S}_{|\mathbb{V}(\Gamma)|}. \quad (61)$$

Remark 3.9 Due to gtS, one may alternatively have introduced T_\bullet^R as a vector space isomorphism, $G_V^\sigma \rightarrow \mathcal{T}_\Gamma$.

Lemma 3.10 *Let Γ^1, Γ^2 be independent, i.e. $\nexists c \in \mathbb{C}$ s.t. $\sqcap \Gamma^1 = c \sqcap \Gamma^2$. Then $c_1 T_{\Gamma^1} + c_2 T_{\Gamma^2} = 0$, $\forall c_1, c_2 \in \mathbb{C}$, implies that $T_{\Gamma^1}^R = T_{\Gamma^2}^R = 0$.*

Proof. This is an immediate consequence of the previous remark. \square

Defining the homomorphism T_\bullet^R only for graphs $G_\Gamma \subset G_V$, this leads to the space $\mathcal{R}_{G_\Gamma}^T$ of Γ -restricted graph distributions with a finite grading. But, not only on this vector space, the product is given by

$$\circ_\Gamma^T : T_{\Gamma^1}^R \otimes T_{\Gamma^2}^R \mapsto T_{\Gamma^1 \sqcup_\Gamma \Gamma^2}^R. \quad (62)$$

Lemma 3.11 *Let $\Gamma, \Gamma' \in G_V$ and $*$ $\in \{\mathbb{V}, \Gamma'\}$. Then $(\mathcal{R}_{G_*}^T, \mathbb{C}, +, \circ_\Gamma^T, \eta)$ forms an algebra (which is free of zero divisors only if $* \subseteq \Gamma$).*

Proof. Obviously, $T_\bullet^R : G_* \rightarrow \mathcal{R}_{G_*}^T$ defines an algebra homomorphism. (Cf. (45), if $\Gamma^1 \not\subseteq \Gamma$ or $\Gamma^2 \not\subseteq \Gamma$ then $T_{\Gamma^1} \circ_\Gamma^T T_{\Gamma^2} \equiv T_{\Gamma^1 \sqcup_\Gamma \Gamma^2} = T_0 \equiv 0$.) \square

Consider the following restrictions of the composition product, obtained by redefining the r.h.s. of (55),

$$T^1 \circ T^2 =: \sum_{l \in l(\Gamma^1, \Gamma^2)} T^1 \circ_l T^2 =: \sum_{\Upsilon \in G_V, (\Gamma^1, \Gamma^2) \in \mathcal{P}_2^0(\Upsilon)} T^1 \circ_\Upsilon T^2, \quad (63)$$

where $T^1 \circ_l T^2 = T^1 \circ_\Upsilon T^2$ iff $\exists \pi \in \mathcal{S}_{|\Upsilon|}$ s.t. $l = (\mathbb{I} \ \emptyset) \cdot \mathbb{I}(\pi \circ \Upsilon) \cdot (\emptyset \ \mathbb{I})^t$. For any fixed graph $\Gamma \in G_V$, this induces a Γ -restricted product on the tI distributions \mathcal{T}_V , i.e. $\circ_\Gamma : \mathcal{T}_{\Gamma^1} \otimes \mathcal{T}_{\Gamma^2} \rightarrow \mathcal{T}_{\Gamma^1 \sqcup_\Gamma \Gamma^2}$,

$$T^1 \circ_\Gamma T^2 := \begin{cases} \sigma_\Upsilon(\Gamma) T^1 \circ_\Upsilon T^2 & \text{if } \exists! \Upsilon \in \mathcal{P}(\Gamma) \text{ s.t. } (\Gamma^1, \Gamma^2) \in \mathcal{P}_2^0(\Upsilon), \\ 0 & \text{otherwise.} \end{cases} \quad (64)$$

$\mathcal{R}_{G_\Gamma} := \bigoplus_{n \geq 0} \mathbb{C}\{ \bigcirc_\Gamma \otimes_{i \leq r} T_{\Gamma^i}^R \mid \Gamma^i \in G_\Gamma, \sum_{i \leq r} |\mathbb{V}(\Gamma^i)| = n \}$ denotes the corresponding space of distributions defining a \mathbb{C} -submodule of \mathcal{T}_V and hence a \mathbb{C} -algebra, i.e. $(\mathcal{R}_{G_\Gamma}, \mathbb{C}, +, \circ_\Gamma, \eta)$.

Remark 3.12 The consistency of the choice of the scalar $(i\hbar)^{\#\Gamma-|\Gamma|}$ in (60) can easily be checked with the help of Euler's formula (42), cf. Corollary 3.3.8 in [31], p. 90.

In this context the vertex distributions used in the ansatz, cf. (47), reappear as a sum over graph distributions, i.e.

$$T_{\bar{v}}^{\mathcal{R}} = \sum_{\Gamma \in \mathcal{G}_V, V(\Gamma) = \bar{v}} T_{\Gamma}^{\mathcal{R}}, \quad \forall \bar{v} \in V, \quad (65)$$

so that the space $\mathcal{R}_V^{\mathcal{I}} := \bigoplus_{n \geq 0} \mathbb{C}\{T_{\bar{v}}^{\mathcal{R}} \mid \bar{v} \in V, |\bar{v}| = n\}$ of vertex distributions can be introduced restricting the homomorphism $T_{\bullet}^{\mathcal{R}}$ to be defined on the vector space of vertex tuples, i.e. $\bigoplus_{n \geq 0} \mathbb{C}\{\bar{v} \mid |\bar{v}| = n\} \equiv V \rightarrow \mathcal{T}_V, \bar{v} \mapsto T_{\bar{v}}^{\mathcal{R}}$.

Lemma 3.13 *Let $\circ^{\mathcal{I}} : T_{\bar{v}^1}^{\mathcal{R}} \otimes T_{\bar{v}^2}^{\mathcal{R}} \mapsto T_{\bar{v}^1 \otimes \bar{v}^2}^{\mathcal{R}}$. Then $(\mathcal{R}_V^{\mathcal{I}}, \mathbb{C}, +, \circ^{\mathcal{I}}, \eta)$ forms an algebra.*

Proof. Obviously, $T_{\bullet}^{\mathcal{R}} : V \rightarrow \mathcal{T}_V$ defines an algebra homomorphism. \square

Furthermore, $\mathcal{R}_V := \bigoplus_{n \geq 0} \mathbb{C}\{\bigcirc_{i \leq r} T_{\bar{v}^i}^{\mathcal{R}} \mid \bar{v}^i \in V, \sum_{i \leq r} |\bar{v}^i| = n\}$ denotes the space of vertex distributions which defines a \mathbb{C} -submodule of \mathcal{T}_V and hence a \mathbb{C} -algebra, i.e. $(\mathcal{R}_V, \mathbb{C}, +, \circ, \eta)$.

3.2.4 HAs underlying pseudo-unitarity

The perturbative expansion is now going to be modeled by a HA. In accordance with the power series ansatz (47), let

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{\bar{v} \in V^n} a^{\#\bar{v}} \langle \tilde{T}_{\bar{v}}^{\mathcal{R}}(y), \tau^{\#\bar{v}}(y) \rangle_n := S_V^{\mathcal{R}}((a\tau)^{\oplus V})^{-1}, \quad (66)$$

where the inverse n-point distributions $\tilde{T}_{\bar{v}} \in C(S_{|\bar{v}|}^{\sigma} \mathcal{D}_{\bar{v}}^{\otimes}, \mathcal{H}_{\bar{v}}^{\mathbb{F}})$, supposed to satisfy gtS as well, are denoted by a tilde. Formally, the inverse of the S -matrix is characterized by relation of pseudo-unitarity, i.e.

$$S_V^{\mathcal{R}}(\xi)^{-1} \circ S_V^{\mathcal{R}}(\xi) = \text{id}_{\mathcal{H}_{\mathbb{F}}^{\oplus}} = S_V^{\mathcal{R}}(\xi) \circ S_V^{\mathcal{R}}(\xi)^{-1}, \quad \forall \xi \in \mathcal{D}_V^{\oplus}(\mathbb{R}^{1+d}). \quad (67)$$

Remark 3.14 The probabilistic interpretation of quantum theory is ensured by requiring unitarity. But, for gauge theories with their unphysical states, unitarity can only be pseudo (i.e. true on a subspace of of the Fock space $\mathcal{H}_{\mathbb{F}}^{\oplus}$), i.e. $S(\xi)^{-1} = S(\xi)^K$, where K is a pseudo-adjoint (i.e. adjoint for a subdomain). That has been investigated extensively by Scharf and collaborators in the nineties, cf. [13, 24, 19].

In terms of n-point distributions the relation of pseudo-unitarity reads as follows.

Lemma 3.15 *Let $T_Y, T_Z \in \mathcal{R}_V$, where $Y, Z \in V$, and let $\emptyset \neq X \in V$. Then*

$$\sum_{(Y,Z) \in \mathcal{P}_2^{\circ}(X)} \sigma_X(Y \otimes Z) \tilde{T}_Y \circ T_Z = 0 = \sum_{(Y,Z) \in \mathcal{P}_2^{\circ}(X)} \sigma_X(Y \otimes Z) T_Y \circ \tilde{T}_Z. \quad (68)$$

Furthermore, $\tilde{T}_Y, \tilde{T}_Z \in \mathcal{R}_V$ and therefore the inverse distributions are gtS.

Proof. To obtain (68), perform the power expansion (cf. Lemma 3.3.15 in [31], p. 94). The other claim is a side result. \square

This statement can be rewritten assuming the tilde to be linear, i.e. assigning the tilde to graph distributions in the following straightforward way,

$$\sum_{\Gamma \in G_V, V(\Gamma) = \bar{v}} \tilde{T}_\Gamma^{\mathcal{R}} := \tilde{T}_{\bar{v}}^{\mathcal{R}}. \quad (69)$$

Corollary 3.16 *Let $\emptyset \neq \Gamma' \in G_\Gamma$ and $\Gamma \in G_V$. Then*

$$\sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2^0(\Upsilon)} \sigma_\Gamma(\Gamma^1 \sqcup_\Gamma \Gamma^2) \tilde{T}_{\Gamma^1}^{\mathcal{R}} \circ_\Gamma T_{\Gamma^2}^{\mathcal{R}} = 0 = \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2^0(\Upsilon)} \sigma_\Gamma(\Gamma^1 \sqcup_\Gamma \Gamma^2) T_{\Gamma^1}^{\mathcal{R}} \circ_\Gamma \tilde{T}_{\Gamma^2}^{\mathcal{R}}, \quad (70)$$

Proof. Insert the expansions (65) and (69) into (68) and apply Lemma 3.10 (cf. Corollary 3.3.17 in [31], p. 97). \square

The previous two equations, if interpreted as antipode conditions, motivate to consider \mathcal{R}_V and \mathcal{R}_{G_Γ} , resp., as HAs.

Theorem 3.17 *($\mathcal{R}_*, \mathbb{C}, +, \circ_{(\Gamma)}, \eta, \Delta, \varepsilon, S_{(\circ_\Gamma)}$), where $*$ = V (resp., = G_Γ , for any $\Gamma \in G_V$), forms a HA. The tilde serves as antipode, i.e. $S_{(\circ_\Gamma)} T := \tilde{T}$, which, e.g. in \mathcal{R}_{G_Γ} , is explicitly given by*

$$S_{\circ_\Gamma} T_{\Gamma'}^{\mathcal{R}} = \sum_{r=1}^{|\mathcal{V}(\Gamma')|} (-1)^r \sum_{(\Gamma^1, \dots, \Gamma^r) \in \mathcal{P}_r(\Gamma')} \sigma_\Gamma(\Gamma^1 \sqcup_\Gamma \dots \sqcup_\Gamma \Gamma^r) T_{\Gamma^1}^{\mathcal{R}} \circ_\Gamma \dots \circ_\Gamma T_{\Gamma^r}^{\mathcal{R}}. \quad (71)$$

Proof. It only remains to show the representation of the antipode. This will be done by induction over the number of vertices. Assume (71) is fulfilled for $|\mathcal{V}(\Gamma')| = n$, which is true for $n = 1$. Then, starting with the antipode condition, one reproduces (71),

$$\begin{aligned} S_{\circ_\Gamma} T_\Upsilon^{\mathcal{R}} &= -T_\Upsilon^{\mathcal{R}} - \sum_{(\Gamma', \Gamma'') \in \mathcal{P}_2(\Upsilon)} \sigma_\Gamma(\Gamma' \sqcup_\Gamma \Gamma'') S_{\circ_\Gamma} T_{\Gamma'}^{\mathcal{R}} \circ_\Gamma T_{\Gamma''}^{\mathcal{R}} \\ &= -T_\Upsilon^{\mathcal{R}} - \sum_{r=1}^n (-1)^r \sum_{(\Gamma', \Gamma'') \in \mathcal{P}_2(\Upsilon)} \sum_{(\Gamma^1, \dots, \Gamma^r) \in \mathcal{P}_r(\Gamma')} \\ &\quad \sigma_\Gamma(\Gamma' \sqcup_\Gamma \Gamma'') \sigma_\Gamma(\Gamma^1 \sqcup_\Gamma \dots \sqcup_\Gamma \Gamma^r) T_{\Gamma^1}^{\mathcal{R}} \circ_\Gamma \dots \circ_\Gamma T_{\Gamma^r}^{\mathcal{R}} \circ_\Gamma T_{\Gamma'}^{\mathcal{R}} \circ_\Gamma T_{\Gamma''}^{\mathcal{R}} \\ &= - \sum_{r=0}^n (-1)^r \sum_{(\Gamma^1, \dots, \Gamma^r, \Gamma'') \in \mathcal{P}_{r+1}(\Upsilon)} \sigma_\Gamma(\Gamma^1 \sqcup_\Gamma \dots \sqcup_\Gamma \Gamma^r \sqcup_\Gamma \Gamma'') \dots \\ &= \sum_{r=1}^{n+1} (-1)^r \sum_{(\Gamma^1, \dots, \Gamma^r, \Gamma'') \in \mathcal{P}_r(\Upsilon)} \dots, \end{aligned}$$

for the graph Υ with $|\mathcal{V}(\Upsilon)| = n + 1$ and hence for any such graph. \square

Remark 3.18 Rota and Stein [35, 36] have obtained this kind of antipode for their *plethystic HAs* and refer to the general version of (71) as *Schmitt formula*.

4 Renormalized causal pQFT

The main concepts that determine the S -matrix of a considered pQFT, causality and renormalization, will enter the (algebraic) formalism in this section. The presentation is given in terms of graph distributions, in contrast to vertex distributions used in the original papers. In Sect. 4.1, which is devoted to causality, the required regularization of distributions is always performed as a localization on an appropriate causally determined spacetime region in configuration space, excluding so-called diagonals. In Sect. 4.2 renormalization is treated. As the EG approach is a recursive method, it is free of subdivergences. A change of the renormalization scheme, however, requires all the structure that is contained in Kreimer's HA of renormalization.

4.1 Causal regularization

The basic notions of (non) (weak) causally determined spacetime regions (resp.) are introduced in Sect. 4.1.1. Furthermore, causality of the S -matrix is characterized in terms of n -point distributions. In Sect. 4.1.2, the two recent EG versions by Fredenhagen und Scharf (which vary with respect to the concrete recursive implementation of causality) are presented in the framework of causal regularization, and in Sect. 4.1.3, they are reformulated in terms of HAs. Sect. 4.1.4 deals with the conventional, the Feynman approach to pQFT which, because of mixing Wick with time ordering, yields an explicit representation of graph distributions. However, the required coarse localization motivates the occurrence of subdivergences.

4.1.1 Causality

On sets of spacetime points, two versions of causality will be introduced. Let $X, Y \subseteq \mathbb{R}^{1+d}$. The relation of causal ordering \lesssim is given by

$$X \cap (Y + \overline{V}_+) = \emptyset \Leftrightarrow X \lesssim Y \Leftrightarrow Y \gtrsim X, \quad (72)$$

and the relation of weak causal ordering \lesssim^\exists (i.e., $\supseteq \lesssim$) is given by

$$\exists(x, y) \in X \times Y \text{ s.t. } \{x\} \lesssim \{y\} \Leftrightarrow X \lesssim^\exists Y \Leftrightarrow Y \gtrsim^\exists X. \quad (73)$$

These two relations are not reflexive, i.e. $\exists X \not\lesssim^\exists X$. Therefore, coinciding spacetime points cannot be compared with respect to causality. This fact, however, makes renormalization look acceptable from a physics point of view.

Accordingly, one distinguishes two sorts of configuration space regions. For the one sort, the causal cones, the spacetime points are causally related, whereas for the other, the diagonals, which somehow model coinciding spacetime points, they are not. Let $P \in \mathcal{P}_r(\Gamma)$, where $\Gamma \in G_V$ and $r \leq n \equiv |V(\Gamma)|$. Then

$$\text{cone}_P^{(\exists)} := \{y \in \mathbb{R}^{(1+d) \times n} \mid \{y_{P_1}^\otimes\} \gtrsim^{(\exists)} \dots \gtrsim^{(\exists)} \{y_{P_r}^\otimes\}\} \quad (74)$$

define (two) tI open cones on the configuration space $\mathbb{R}^{(1+d) \times n} := \times_{i \leq n} \mathbb{R}^{1+d}$. Furthermore, the sets of r coinciding points,

$$\text{diag}_P := \{y \in \mathbb{R}^{(1+d) \times n} \mid y_i = y_j, \forall i, j \in P_s, \forall s \leq r\}, \quad (75)$$

are called the (sub)diagonals w.r.t. P (if $P \neq (\Gamma)$, resp.). Subtracting

$$\text{diag}_r^n := \bigcup_{s < r} \bigcup_{P \in \mathcal{P}_s(I)} \text{diag}_P \quad (76)$$

from the configuration space, one obtains the causally determined region which contains at least r non-coinciding spacetime points, i.e.

$$\mathbb{R}_{r \neq}^{(1+d) \times n} := \mathbb{R}^{(1+d) \times n} \setminus \text{diag}_r^n. \quad (77)$$

One uses the abbreviations, $\mathbb{R}_x^{(1+d) \times n} := \mathbb{R}_{2 \neq}^{(1+d) \times n}$ and $\mathbb{R}_{\neq}^{(1+d) \times n} := \mathbb{R}_{n \neq}^{(1+d) \times n}$.

Lemma 4.1 *Let $\Gamma \in G_V$ and $r \leq n \equiv |V(\Gamma)|$. Then*

$$\bigcup_{P \in \mathcal{P}_r(\Gamma)} \text{cone}_P = \mathbb{R}_{r \neq}^{(1+d) \times n}. \quad (78)$$

Let $\Gamma, \Gamma_1, \Gamma'' \in G_V$ where $(\Gamma_1, \Gamma'') \in \mathcal{P}_2(\Gamma)$ and $|V(\Gamma_1)| = 1$. Then

$$\text{cone}_{(\Gamma_1, \Gamma'')}^{\exists} \cup \text{cone}_{(\Gamma'', \Gamma_1)}^{\exists} = \mathbb{R}_x^{(1+d) \times n} \quad (79)$$

where the two cones are represented by

$$\text{cone}_{\left\{ \begin{array}{l} (\Gamma_1, \Gamma'') \\ (\Gamma'', \Gamma_1) \end{array} \right\}}^{\exists} = \bigcup_{(\Gamma_Q, \Gamma_P) \in \mathcal{P}_2(\Gamma), \Gamma_1 \subseteq \left\{ \begin{array}{l} \Gamma_Q \\ \Gamma_P \end{array} \right\}} \text{cone}_{(\Gamma_Q, \Gamma_P)}^{\exists}. \quad (80)$$

Proof. These relations follow immediately from the definitions above. \square

In the EG approach, the causality assumption of the S -matrix is crucial. Causality holds true, i.e. the S -matrix is causal, iff $(\forall \xi^i \in \mathcal{D}_V^{\oplus}(\mathbb{R}^{1+d}), i \leq 2)$

$$\text{supp } \xi^1 \gtrsim \text{supp } \xi^2 \Rightarrow S_V^{\mathcal{R}}(\xi^1 + \xi^2) = S_V^{\mathcal{R}}(\xi^1) \circ S_V^{\mathcal{R}}(\xi^2). \quad (81)$$

There is an equivalent definition of causality in terms of graph distributions.

Proposition 4.2 *The S -matrix is causal iff $\text{supp } \xi^1 \gtrsim \text{supp } \xi^2$ implies*

$$T_{\Gamma'}^{\mathcal{R}}(\xi^1 \sqcap \xi^2) = \sigma_{\Gamma'}(\Gamma^1 \sqcup_{\Gamma} \Gamma^2) T_{\Gamma^1}^{\mathcal{R}}(\xi^1) \circ_{\Gamma} T_{\Gamma^2}^{\mathcal{R}}(\xi^2), \quad (82)$$

$\forall (\Gamma^1, \Gamma^2) \in \mathcal{P}_2(\Gamma'), \forall \Gamma' \subseteq \Gamma \in G_V$, and $\forall \xi^i \in \mathcal{D}_{V(\Gamma^i)}^{\otimes}, i \leq 2$.

Proof. This is due to power expansion (cf. Lemma 4.1.4 in [31], p. 103). \square

From now on, one assumes that the S -matrix is causal.

4.1.2 Regularization of two EG versions

EG's method can be understood as an induction over subgraphs where the implementation of causality leads to localized n -point distributions (which are gtS and have tI scalar parts) of the form $T_{\Gamma'}^{\mathcal{R}} := T_{\Gamma'}^{\mathcal{R}}|_{\mathbb{R}_{r \neq}^{(1+d) \times n}} \in \mathcal{R}_{G_{\Gamma}}^{\mathcal{I}}(\mathbb{R}_{r \neq}^{(1+d) \times n})$, $r \leq n$. For this subsection, which introduces the EG versions by Fredenhagen und Scharf, let $\Gamma \in G_V$ and $\Gamma' \in \mathcal{P}(\Gamma)$ where $n \equiv |V(\Gamma')|$.

EG à la Fredenhagen (EGF) For each graph Γ and each $r \leq n$ one chooses a partition of unity $(\chi_P)_{P \in \mathcal{P}_r(\Gamma)}$ of class C^∞ subordinate to the finite open covering $(\text{cone}_P)_{P \in \mathcal{P}_r(\Gamma)}$ of the metric space $\mathbb{R}_{r \neq}^{(1+d) \times n}$, i.e.

$$\sum_{P \in \mathcal{P}_r(\Gamma)} \chi_P(x) = 1, \quad \forall x \in \mathbb{R}_{r \neq}^{(1+d) \times n}, \quad (83)$$

$$\text{where } 0 \leq \chi_P \in C^\infty(\mathbb{R}_{r \neq}^{(1+d) \times n}) \quad (84)$$

$$\text{and } \text{supp } \chi_P \subset \text{cone}_P. \quad (85)$$

It is well-known that partitions of this kind exist (e.g., cf. [11], p. 214).

Proposition 4.3 *The graph distributions are (uniquely) determined by*

$$T_{\Gamma'}^\chi := \sum_{(\Gamma^1, \dots, \Gamma^r) \in \mathcal{P}_r(\Gamma')} \chi_{(\Gamma^1, \dots, \Gamma^r)} \sigma_\Gamma(\Gamma^1 \sqcup_\Gamma \dots \sqcup_\Gamma \Gamma^r) T_{\Gamma^1}^{\mathcal{R}} \circ_\Gamma \dots \circ_\Gamma T_{\Gamma^r}^{\mathcal{R}} \quad (86)$$

(independently of χ), i.e. $T_{\Gamma'}^{r \neq} = T_{\Gamma'}^\chi|_{\mathbb{R}_{r \neq}^{(1+d) \times n}} \in \mathcal{R}_{\Gamma'}^{\mathcal{I}}(\mathbb{R}_{r \neq}^{(1+d) \times n})$.

Proof. By applying (78) one can straightforwardly generalize [9], where only the case of $r = 2$ is considered (cf. Prop. 4.2.6 in [31], p. 110). \square

EG à la Scharf (EGS) The genuine approach is based on two expressions,

$$\text{the } \begin{cases} \text{retarded} \\ \text{advanced} \end{cases} \text{ distribution } \begin{cases} R_{(\Gamma_1, \Gamma'')} := T_{\Gamma'} + R'_{(\Gamma_1, \Gamma'')}, \\ A_{(\Gamma_1, \Gamma'')} := T_{\Gamma'} + A'_{(\Gamma_1, \Gamma'')}, \end{cases} \quad (87)$$

where, applying Scharf's notation,

$$R'_{(\Gamma_1, \Gamma'')} := \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2(\Gamma'), \Gamma_1 \subseteq \Gamma^1} \sigma_\Gamma(\Gamma^1 \sqcup \Gamma^2) T_{\Gamma^1}^{\mathcal{R}} \circ_\Gamma \tilde{T}_{\Gamma^2}^{\mathcal{R}}, \quad (88)$$

$$A'_{(\Gamma_1, \Gamma'')} := \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2(\Gamma'), \Gamma_1 \subseteq \Gamma^2} \sigma_\Gamma(\Gamma^1 \sqcup \Gamma^2) \tilde{T}_{\Gamma^1}^{\mathcal{R}} \circ_\Gamma T_{\Gamma^2}^{\mathcal{R}}. \quad (89)$$

These expressions vanish when restricted to appropriate cones.

Lemma 4.4 *Let $\Gamma, \Gamma_1, \Gamma'' \in \mathbf{G}_V$ where $(\Gamma_1, \Gamma'') \in \mathcal{P}_2(\Gamma)$ and $|\mathbf{V}(\Gamma_1)| = 1$. Then*

$$A_{(\Gamma_1, \Gamma'')}|_{\text{cone}_{(\Gamma'', \Gamma_1)}^\exists} = 0 = R_{(\Gamma_1, \Gamma'')}|_{\text{cone}_{(\Gamma_1, \Gamma'')}^\exists}. \quad (90)$$

Proof. One will have to apply the property (80) of the cones of weak causality, cf. Lemma 4.2.12 in [31], pp. 113 f. \square

Again, but formally varying from the named approach, one considers a partition of unity. For each triple of graphs $\Gamma, \Gamma_1, \Gamma'' \in \mathbf{G}_V$ where $(\Gamma_1, \Gamma'') \in \mathcal{P}_2(\Gamma)$ and $|\mathbf{V}(\Gamma_1)| = 1$ one chooses a partition of unity $(\chi_s^\exists)_{s \leq 2}$ of class C^∞ subordinate to the finite open covering $\text{cone}^\exists := (\text{cone}_{(\Gamma_1, \Gamma'')}^\exists, \text{cone}_{(\Gamma'', \Gamma_1)}^\exists)$ of the metric space $\mathbb{R}_x^{(1+d) \times n}$, i.e.

$$\chi_1^\exists(x) + \chi_2^\exists(x) = 1, \quad \forall x \in \mathbb{R}_x^{(1+d) \times n}, \quad (91)$$

$$\text{where } 0 \leq \chi_s^\exists \in C^\infty(\mathbb{R}_x^{(1+d) \times n}) \quad (92)$$

$$\text{and } \text{supp } \chi_s^\exists \subset \text{cone}_s^\exists, \quad s \leq 2. \quad (93)$$

Example 4.5 One may use the partitions of unity χ introduced for the EGF approach to define such a $\chi^{(\exists)}$, i.e. $\chi_s^{(\exists)} := \sum_{P \in \mathcal{P}_2(\Gamma'), \Gamma_1 \subseteq P_s} \chi_P$, $s \leq 2$.

Proposition 4.6 *The graph distributions are (uniquely) determined by*

$$T_{\Gamma'}^{(\exists)} := -\chi_1^{(\exists)} R'_{(\Gamma_1, \Gamma'')} - \chi_2^{(\exists)} A'_{(\Gamma_1, \Gamma'')} \quad (94)$$

(independently of $\chi^{(\exists)}$), i.e. $T_{\Gamma'}^{\times} = T_{\Gamma'}^{(\exists)}|_{\mathbb{R}_{\times}^{(1+d) \times n}} \in \mathcal{R}_{G_{\Gamma}}^{\Gamma}(\mathbb{R}_{\times}^{(1+d) \times n})$.

Proof. This is a consequence of (80) and Lemma 4.4 (cf. Prop. 4.2.15, p. 116). \square

Remark 4.7 In this EG version the n-point distribution is actually obtained from a so-called splitting of the causal distribution $D := R' - A' = R - A$. Here, the mapping $D \mapsto \chi_2^{(\exists)} D$ defines the regularized splitting so that

$$T_{\Gamma'}^{\times} = -\chi_2^{(\exists)} D_{(\Gamma_1, \Gamma'')}|_{\mathbb{R}_{\times}^{(1+d) \times n}} - R'_{(\Gamma_1, \Gamma'')}|_{\mathbb{R}_{\times}^{(1+d) \times n}}. \quad (95)$$

4.1.3 Regularizing HAs

As the causal regularization is understood to be a localization of distributions to subregions of non-coinciding spacetime points of configuration spaces, it can be modeled algebraically via A^0 -modules $\overline{A} = A^0 \overline{\mathbb{1}} \oplus \bigoplus_{n \geq 0} \overline{A}^n$ that are extensions of \mathbb{N} -graded (configuration space depending) modules $A = \bigoplus_{n \geq 0} A^n$, where $\overline{A}^n = \bigoplus_{r \leq n} A^n(\mathbb{R}_{r \neq}^{(1+d) \times n})$ and $\overline{\mathbb{1}}$ denotes the formally defined unity. Moreover, the concrete structures will be algebras. The instances for the latter modules (resp., algebras) are the (algebra of) n-point distributions $\mathcal{R}_{G_{\Gamma}}$ as well as their tensor product structure, $C^{\infty} \otimes \mathcal{R}_{G_{\Gamma}}$, with C^{∞} functions (multiplied pointwise).

The concrete regularization for the EGF (resp., EGS) version is given as a subalgebra of $\overline{C^{\infty} \otimes \mathcal{R}_{G_{\Gamma}}}$ which is isomorphic to $\overline{\mathcal{R}_{G_{\Gamma}}}$ as an algebra, i.e. $(\widehat{\mathcal{R}}_{G_{\Gamma}}^{(\exists)}, \mathbb{C}, +, \widehat{\circ}_{\Gamma}^{(\exists)}, \overline{\mathbb{1}})$, where

$$\overline{\mathcal{R}}_{G_{\Gamma}} \stackrel{\iota}{\cong} \widehat{\mathcal{R}}_{G_{\Gamma}}^{(\exists)} \xrightarrow{\vartheta^{(\exists)}} \overline{C^{\infty} \otimes \mathcal{R}_{G_{\Gamma}}} \quad (96)$$

and, for at least $r \in \mathbb{N}$ non-coinciding spacetime points, the embedding algebra homomorphism

$$\vartheta^{(\exists)} : \widehat{\circ}_{\Gamma}^{(\exists)} \iota \left(T_P^{\otimes} |_{\mathbb{R}_{r \neq}^{(1+d) \times n}} \right) =: \widehat{\circ}_{\Gamma}^{(\exists)} \widehat{T}_P^{\otimes} \mapsto \vartheta_P^{(\exists)} \otimes \circ_{\Gamma} T_P^{\otimes} |_{\mathbb{R}_{r \neq}^{(1+d) \times n}} \quad (97)$$

is realized via functions $\vartheta_P^{(\exists)} \in C^{\infty}(\mathbb{R}_{r \neq}^{(1+d) \times n})$ indexed with r -partitions $P \in \mathcal{P}_r^0(\Gamma')$ of graphs $\Gamma' \in G_{\Gamma}$ where $r \leq n \equiv |V(\Gamma')|$. Those causally ordering functions (coF) will be used to model the considered regularizing algebra. One of their general properties is due to the defining isomorphism, i.e. $\vartheta_{(\Gamma')}^{(\exists)} = 1$, $\forall \Gamma' \in G_{\Gamma}$. The co-structure is introduced by the gtS shuffling coproduct where, e.g., $\Delta(\overline{\mathbb{1}}) = \overline{\mathbb{1}} \otimes \overline{\mathbb{1}}$ and $\varepsilon(\overline{\mathbb{1}}) = 1$. Moreover, the (two) algebras above define bialgebras, i.e. $(\widehat{\mathcal{R}}_{G_{\Gamma}}^{(\exists)}, \mathbb{C}, +, \widehat{\circ}_{\Gamma}^{(\exists)}, \eta, \Delta, \varepsilon)$.

Notice, one will always use the following abbreviation, $T_{\Gamma'} = \widehat{T}_{(\Gamma')} \in \widehat{\mathcal{R}}_{G_{\Gamma}}^{(\exists)}$.

Modeling EGF The coFs are given in terms of the partition of unity, i.e.

$$\vartheta_P := \frac{1}{2}\chi_{\check{P}}, \quad \forall P \in \mathcal{P}_r^0(\Gamma'), \forall \Gamma' \in \mathbf{G}_\Gamma, \forall r \leq |\mathbf{V}(\Gamma')|, \quad (98)$$

where $\check{P} \in \mathcal{P}_s(\Gamma')$, for $s \leq r$, denotes the tuple P reduced by empty graph entries. This motivates a further general property of ϑ , i.e.

$$\vartheta_P = \vartheta_Q \quad \text{if } \check{P} = \check{Q} \text{ and } |P| = |Q|. \quad (99)$$

Theorem 4.8 $(\widehat{\mathcal{R}}_{\mathbf{G}_\Gamma}, \mathbb{C}, +, \widehat{\circ}_\Gamma, \eta, \Delta, \varepsilon, \mathbf{S}_{\widehat{\circ}_\Gamma})$ forms a HA where

$$\mathbf{S}_{\widehat{\circ}_\Gamma} T_{\Gamma'} = \begin{cases} T_\emptyset & \text{if } \Gamma' = \emptyset, \\ -T_{\Gamma'} & \text{otherwise.} \end{cases} \quad (100)$$

Proof. It remains to check the antipode condition. This can be done by rewriting EGF's induction (Prop. 4.3), i.e. $0 = \frac{1}{2}T_{\Gamma'}^\times - T_{\Gamma'}^{\chi/2}$. Let $L := \mathbb{R}_\times^{(1+d) \times n}$, then

$$\begin{aligned} 0 &= \left(1 - \frac{1}{2}\right)\chi_{(\Gamma')} T_{\Gamma'} \Big|_L - \frac{1}{2} \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2(\Gamma')} \chi_{(\Gamma^1, \Gamma^2)} \sigma_{\Gamma'}(\Gamma^1 \sqcup_\Gamma \Gamma^2) T_{\Gamma^1} \circ_\Gamma T_{\Gamma^2} \Big|_L \\ &= \vartheta_{(\Gamma', \emptyset)} \otimes T_{\Gamma'} \Big|_L - \vartheta_{(\emptyset, \Gamma')} \otimes T_{\Gamma'} \Big|_L - \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2(\Gamma')} \vartheta_{(\Gamma^1, \Gamma^2)} \otimes \sigma_{\Gamma'}(\dots) \dots \Big|_L \\ &= T_{\Gamma'} \widehat{\circ}_\Gamma T_\emptyset + T_\emptyset \widehat{\circ}_\Gamma (-T_{\Gamma'}) + \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2(\Gamma')} \sigma_{\Gamma'}(\Gamma^1 \sqcup_\Gamma \Gamma^2) T_{\Gamma^1} \widehat{\circ}_\Gamma (-T_{\Gamma^2}) \\ &= \widehat{\circ}_\Gamma \circ (\text{id} \otimes \mathbf{S}_{\widehat{\circ}_\Gamma}) \circ \Delta T_{\Gamma'}. \end{aligned}$$

The other side of the antipode condition follows similarly. \square

Modeling EGS There are two extra properties of ϑ^\exists , i.e.

$$\vartheta_P^\exists = \begin{cases} \vartheta_{(\dots \sqcup_\Gamma P_{i-1}, P_i, P_{i+1} \sqcup_\Gamma \dots)}^\exists & \text{if } \exists \Gamma_1 \subseteq P_i \text{ s.t. } y_1 = x_{\Gamma_1}, \\ \vartheta_{\sqcup_\Gamma P}^\exists & \text{otherwise,} \end{cases} \quad (101)$$

and

$$\vartheta_P^\exists = \begin{cases} \vartheta_{(\Gamma_1, \Gamma'')}^\exists & \text{if } \exists \Gamma_1 \subseteq \begin{cases} P_1 \\ P_r \end{cases} \text{ s.t. } y_1 = x_{\Gamma_1}, \\ \vartheta_{(\Gamma'', \Gamma_1)}^\exists & \end{cases} \quad (102)$$

so that the coFs can again be given in terms of the partition of unity, i.e.

$$\vartheta_P^\exists := \begin{cases} \chi_s^\exists & \text{if } \exists \Gamma_1 \subseteq P_s \text{ s.t. } y_1 = x_{\Gamma_1} \text{ and } P \in \mathcal{P}_2(\Gamma'), \\ 0 & \text{otherwise,} \end{cases}$$

$\forall P \in \mathcal{P}_r^0(\Gamma'), \Gamma' \in \mathbf{G}_\Gamma, r \leq |\mathbf{V}(\Gamma')|$.

Theorem 4.9 $(\widehat{\mathcal{R}}_{\mathbf{G}_\Gamma}^\exists, \mathbb{C}, +, \widehat{\circ}_\Gamma^\exists, \eta, \Delta, \varepsilon, \mathbf{S}_{\widehat{\circ}_\Gamma^\exists})$ forms a HA where

$$\mathbf{S}_{\widehat{\circ}_\Gamma^\exists} T_{\Gamma'} = \begin{cases} 0 & \text{if } \exists \Gamma_1 \subseteq \Gamma' \text{ s.t. } x_1 = x_{\Gamma_1}, \\ \mathbf{S}_\Gamma T_{\Gamma'} & \text{otherwise.} \end{cases} \quad (103)$$

Proof. Again, it remains to check the antipode condition. By rewriting the R -part of EGS' induction (cf. Lemma 4.4), i.e. $0 = \chi_1^{\exists} R_{(\Gamma^1, \Gamma'')} \Big|_L = \chi_1^{\exists} T_{\Gamma'}^{\exists} + \chi_1^{\exists} R'_{(\Gamma^1, \Gamma'')} \Big|_L$, where $L := \mathbb{R}_x^{(1+d) \times n}$, one obtains

$$\begin{aligned}
0 &= \chi_1^{\exists} T_{\Gamma'} \Big|_L + \chi_1^{\exists} \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2(\Gamma'), \Gamma_1 \subseteq \Gamma^1} \sigma_{\Gamma}(\Gamma^1 \sqcup_{\Gamma} \Gamma^2) T_{\Gamma^1} \circ_{\Gamma} S_{\circ_{\Gamma}} T_{\Gamma^2} \Big|_L \\
&= \vartheta_{(\Gamma', \emptyset)}^{\exists} \otimes T_{\Gamma'} \Big|_L + \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2(\Gamma'), \Gamma_1 \subseteq \Gamma^1} \vartheta_{(\Gamma^1, \Gamma^2)}^{\exists} \otimes \sigma_{\Gamma'}(\Gamma^1 \sqcup_{\Gamma} \Gamma^2) T_{\Gamma^1} \circ_{\Gamma} S_{\Gamma} T_{\Gamma^2} \Big|_L \\
&= \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2^{\circ}(\Gamma'), \Gamma_1 \subseteq \Gamma^1} \vartheta_{(\Gamma^1, \Gamma^2)}^{\exists} \otimes \sigma_{\Gamma'}(\Gamma^1 \sqcup_{\Gamma} \Gamma^2) T_{\Gamma^1} \circ_{\Gamma} S_{\Gamma} T_{\Gamma^2} \Big|_L \\
&= \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2^{\circ}(\Gamma')} \sigma_{\Gamma'}(\Gamma^1 \sqcup_{\Gamma} \Gamma^2) T_{\Gamma^1} \widehat{\circ}_{\Gamma}^{\exists} S_{\widehat{\circ}_{\Gamma}^{\exists}} T_{\Gamma^2} \\
&\equiv \widehat{\circ}_{\Gamma}^{\exists} \circ (\text{id} \otimes S_{\widehat{\circ}_{\Gamma}^{\exists}}) \circ \Delta T_{\Gamma'}.
\end{aligned}$$

Similarly, the other side of the antipode condition follows from the A -part. \square

4.1.4 BS' Feynman approach

In the conventional approach to pQFT Feynman propagators are used. This leads to an explicite representation of graph distributions in momentum space which, without (iterated) renormalization, suffers from so-called ultraviolet (sub) divergences. The distributions, when going over to configuration space in this subsection, are shown to be determined only on $L := \mathbb{R}_{\neq}^{(1+d) \times n}$. The aspect of causality has already been investigated by BS [4], but only the work of BPHZ [3, 20, 42] could verify the correctness of the common method of curing (sub)divergences via renormalization (over so-called forests of subgraphs, nowadays encoded by the antipode condition of Kreimer's HA [26], resp.).

Lemma 4.10 *Let $\Gamma' = \Gamma^1 \sqcup_l \Gamma^2$ be a graph and $P' \equiv P^1 \otimes P^2 = (P \circ V_i(\Gamma'))_{i \leq n}$, $n \equiv |V(\Gamma')|$, the associated tuples of particles. Localized on L , time-ordered (cf. (135)) Wick contractions can be represented by Feynman contractions, i.e.⁵*

$$(\Gamma \Pi)_{(\Gamma^1, \Gamma^2)}^l \Big|_L = \Pi_{(\Gamma^1, \Gamma^2)}^l \Big|_L \quad \text{where} \quad \Pi_{(\Gamma^1, \Gamma^2)}^l := \prod_{\substack{(i,j) \in l_{rs} \\ r \leq |P^1|, s \leq |P^2|}} \Delta_{F\gamma}^{P_{ri}^1 P_{sj}^2}. \quad (104)$$

Proof. As chronological cycles $C \subset M$ can be excluded by localization on L

⁵The symbol ' Π ' is chosen in accordance with BS' notation of a Feynman contraction,

$$\mathbf{N}(\dots \otimes \overbrace{\phi^i \otimes \dots \otimes \phi^j} \otimes \dots) = \mathbf{T}[\phi_-^i, \phi_+^j]_{\sigma}^{\otimes} \mathbf{N}(\dots \otimes \dots \otimes \dots).$$

(cf. third line), one confirms

$$\begin{aligned}
\Pi_{(\Gamma^1, \Gamma^2)}^l|_L &= \prod_{\substack{(i,j) \in l_{rs} \\ r \leq |P^1|, s \leq |P^2|}} \left(\theta_{(r,s)} \Delta_{+\gamma}^{P_{ri}^1 P_{sj}^2} + \sigma \theta_{(s,r)} \Delta_{+\gamma}^{P_{sj}^2 P_{ri}^1} \right) \Big|_L \\
&= \sum_{\substack{M \subseteq M^{12} \cup M^{21} \\ (u,v) \in M \Leftrightarrow (v,u) \notin M}} \prod_{\substack{(i,j) \in \bar{l}_{uv} \\ (u,v) \in M}} \theta_{(u,v)} \sigma_{\mathbb{I}(\Gamma')}(P'_u, P'_v) \Delta_{+\gamma}^{P'_{ui} P'_{vj}} \Big|_L \\
&= \sum_{\substack{M \subseteq M^{12} \cup M^{21} \\ \exists \bar{C} \subset M \text{ cycle}}} \prod_{(u,v) \in M} \theta_{(u,v)} \sigma_{\mathbb{I}(\Gamma')}(P'_u \sqcup_{\emptyset} P'_v) \Pi_{(\Gamma'_u, \Gamma'_v)}^{\bar{l}_{uv}} \Big|_L \\
&= \sum_{\pi \in \mathcal{S}_n} \theta_{\pi(\Gamma')} \sigma_{\mathbb{I}(\Gamma^1 \sqcup_i \Gamma^2)}(\pi(\Gamma')) \prod_{i < j \leq n} \Pi_{(\pi \circ \Gamma'_i, \pi \circ \Gamma'_j)}^{\pi \circ \bar{l}_{ij}} \Big|_L \\
&= (\mathbb{T} \Pi)_{(\Gamma^1, \Gamma^2)}^l|_L,
\end{aligned}$$

where the (omitted) variables in $\Delta_{+\gamma}^{**}(y_{**})$ are related as follows,

$$y_{uv} := y'_u - y'_v = \begin{cases} y_r - y_s =: y_{rs} \\ y_s - y_r =: y_{sr} \end{cases} \quad \text{if } (u, v) = \begin{cases} (r, s + |\nu^1|), \\ (s + |\nu^1|, r). \end{cases}$$

The sets $\begin{cases} M^{12} \\ M^{21} \end{cases}$ collect the corresponding pairs $(u, v) \in \mathbb{N}_n \times \mathbb{N}_n$ of indices, and the bar over the concatenation matrix $l \in \mathbb{I}(\Gamma^1, \Gamma^2)$ denotes its conversion into a matrix of lines, $\bar{l} := \begin{pmatrix} \emptyset & l \\ \bar{l} & \emptyset \end{pmatrix}$. \square

Remark 4.11 Restricted to $x_{uv} \in \mathbb{R}_{\neq}^{1 \times n} \times \mathbb{R}^{d \times n} \subset L$, the products of step functions vanish whenever their variables form a chronological cycle C , i.e.,

$$\prod_{(u,v) \in C} \theta_{(u,v)}(x_{uv}) \equiv \prod_{(u,v) \in C} \theta(x_{uv}^0) = 0 \quad \text{if } \sum_{(u,v) \in C} x_{uv}^0 = 0.$$

The localization on L is sufficient, namely $\text{supp } \Delta_{+\gamma}^{**} \subseteq \bar{V}_+ \cup \bar{V}_- =: \bar{V}$, which yields $\text{supp } \theta_{(*,*)} \Delta_{+\gamma}^{**} \subseteq L \cap \bar{V} \subset \mathbb{R}_{\neq}^{1 \times n} \times \mathbb{R}^{d \times n}$.

Without localization, a simple consideration about the role of cycles, which correspond to so-called loops in Feynman diagrams, can motivate the phenomenon of renormalization freedom.

Example 4.12 Let (r, s, t) be a cycle, i.e. $\theta(x_{rs}^0) \theta(x_{st}^0) \theta(x_{tr}^0)|_{\mathbb{R}_{\neq}^{1 \times 3} \times \mathbb{R}^{d \times 3}} = 0$, and apply tI. Then, on the subspace generated by $x_r = x_t$,

$$\theta(x^0) \theta(-x^0)|_{(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^d} = 0 \quad \text{where } x := x_{rs} = -x_{st}.$$

The restriction cannot be dropped, as one realizes by applying the l.h.s. to certain distributions t . E.g., let $t = \delta$ be the Dirac distribution, and let the

product be defined via a regularization of θ , e.g. $\theta_n(x^0) := \frac{1}{2}(1 + \tanh(nx^0))$. Then, induced by this particular regularization,

$$\theta(x^0) \theta(-x^0) t(x) := \lim_{n \rightarrow \infty} \theta_n(x^0) \theta_n(-x^0) t(x) = \frac{1}{4} \delta(x^0),$$

which obviously spoils associativity (i.e. the algebraic structure),

$$\frac{1}{4} \delta(x) = \theta(x^0) \theta(-x^0) \delta(x) \neq (\theta(x^0) \theta(-x^0)) \delta(x) = 0 \delta(x).$$

This is consistent with the no-go theorem of Schwartz [39].

Proposition 4.13 *Localized on L , graph distributions can be represented by time-ordered products $T_{\Gamma'}^{\neq} = T_{\Gamma^1}^{\mathcal{R}} \circ_{\Gamma}^{\top} \cdots \circ_{\Gamma}^{\top} T_{\Gamma^r}^{\mathcal{R}}|_L \in \mathcal{R}_{G_{\Gamma}}^{\top}(L)$, where*

$$T_{\Gamma'}^{\neq} = (i\hbar)^{\#\Gamma' - |\Gamma'|} \sum_{b(\Omega_{\Gamma'})} t_{I(\Gamma^1)}^{\mathcal{R}} t_{I(\Gamma^2)}^{\mathcal{R}} \Pi_{(\Gamma^1, \Gamma^2)}^l N(\Gamma')|_L, \quad (105)$$

for $r := 2$ and $\Gamma' \equiv \Gamma^1 \sqcup_l \Gamma^2$.

Proof. The scalar part rests upon Lemma 4.10 and the structural part upon \square

Lemma 4.14 (Wick's theorem of time-ordering). *Keeping the notations,*

$$N(\Gamma^1) \circ_{\Gamma}^{\top} N(\Gamma^2)|_L = \sum_{\substack{l \in \{1, \Gamma^1, \Gamma^2\} \\ \Gamma^1 \sqcup_l \Gamma^2 = \Gamma^1 \sqcup_{\Gamma} \Gamma^2}} (i\hbar)^{|l|} \Pi_{(\Gamma^1, \Gamma^2)}^l N(\Gamma^1 \sqcup_l \Gamma^2)|_L. \quad (106)$$

Proof. This follows from (58) and (104) (cf. Lemma 4.2.25 in [31], p. 122). \square

Remark 4.15 Associativity of \circ_{Γ}^{\top} corresponds with $\Pi_{(\cdot, \cdot)}$ being a 2-cocycle. In addition to Π , also Π can be considered as a Laplace pair.

Explicitely, the graph distributions (60) can be characterized by

Proposition 4.16 (Feynman diagrams). *Localized on L , graph distributions are (uniquely) determined by their scalar parts,*

$$t_{I(\Gamma')}^{\mathcal{R}}|_L = \Pi_{\Gamma'}^{\Gamma}|_L \quad \text{where} \quad \Pi_{\Gamma'}^{\Gamma} = \prod_{r < s \leq n} \prod_{\Gamma'_r \sqcup_{\Gamma} \Gamma'_s \in \mathcal{P}(\Gamma')} \Pi_{(\Gamma'_r, \Gamma'_s)}^{l(\Gamma')_{rs}} \quad (107)$$

(independent of the renormalization scheme \mathcal{R}).

Proof. This is a consequence of Prop. 4.13. The calculation of (107) is similar to the one in the proof of Lemma 4.10 (cf. (4.139) in [31], p. 127). \square

Remark 4.17 Immediately, (29), (104), and (107) imply gtS of $t_{I(\Gamma')}|_L$.

For completion, one will as well present an algebraic formulation of the BS regularization in the sense of Sect. 4.1.3. The the corresponding coFs read

$$\vartheta^{\Gamma} : \widehat{\circ}_{\Gamma}^{\top} \widehat{T}_P^{\otimes} \mapsto \vartheta_P^{\Gamma} \otimes \circ_{\Gamma}^{\top} T_P^{\otimes}|_L \quad \text{where} \quad \vartheta_P^{\Gamma} := 1|_L, \quad (108)$$

$\forall P \in \mathcal{P}_r^0(\Gamma'), \forall \Gamma' \in G_{\Gamma}, \forall r \leq |V(\Gamma')|$.

Theorem 4.18 $(\widehat{\mathcal{R}}_{\Gamma}^{\top}, \mathbb{C}, +, \widehat{\circ}_{\Gamma}^{\top}, \eta, \Delta, \varepsilon, S_{\widehat{\circ}_{\Gamma}^{\top}})$ forms a HA where

$$S_{\widehat{\circ}_{\Gamma}^{\top}} T_{\Gamma'} = (-1)^{|\mathbb{V}(\Gamma')|} T_{\overline{\Gamma'}}. \quad (109)$$

Proof. It remains to check the antipode condition. Due to the claimed form of the antipode, the calculation is analogous to the one that proves Theorem 2.4. Formally, one would only have to substitute $\mathbb{N} := \mathbb{T}$ and $\phi^{\circ J} := T_{\Gamma'}$. \square

4.2 Renormalization

Using EGF's method [9] of extending localized distributions to the (smallest) diagonal, as explained in Sect. 4.2.1, the concept of renormalization can be implemented for both EG versions. Sect. 4.2.2 briefly reminds of the physical meaning of renormalization. Finally, by weakening the structure of causally regularizing HAs one is able to construct EG counterparts of Kreimer's HA in Sect. 4.2.3, modeling the change of the renormalization scheme. For the whole subsection, let $*$ $\in \{\mathbb{F}, \exists, \mathbb{T}\}$ symbolize the reference to one of the three causal regularizations where, to include EGF, $\widehat{\mathcal{R}}_{\Gamma}^{\mathbb{F}} := \widehat{\mathcal{R}}_{\Gamma}$ and $\widehat{\circ}_{\Gamma}^{\mathbb{F}} := \widehat{\circ}_{\Gamma}$.

4.2.1 Implementation via an extension of distributions

Renormalization can be understood as a method to (re)obtain elements of $\widehat{\mathcal{R}}_{\Gamma}^*$ which are not determined by causality. Here it is introduced, without reference to the concrete regularization, via (two) projective maps,⁶

$$\widehat{\text{ren}} \equiv \widehat{\circ} \circ \text{ren} : \widehat{\mathcal{R}}_{\Gamma}^{*n}(\mathbb{R}_{r \neq}^{(1+d) \times n}) \rightarrow \mathcal{R}_{\Gamma}^{*n} \hookrightarrow \widehat{\mathcal{R}}_{\Gamma}^{*n}. \quad (110)$$

In particular, when performing the (causally regularizing) induction step, this map (as indicated below) implements the renormalization scheme \mathcal{R} depending on the underlying graph Γ' ,

$$\text{ren}_{\Gamma'}^{\mathcal{R}} : T_{\Gamma'}^{r \neq} \mapsto T_{\Gamma'}^{\mathcal{R}}, \quad \forall r \leq n. \quad (111)$$

For both EG versions, renormalization is performed as an extension of (the) graph distributions (in (86) and (94), where $r := 2$) to the diagonal diag_2^n , determined by the scalar parts of the operator-valued distributions, i.e.

$$T_{\Gamma'}^{\mathcal{R}} := \text{ren}_{\Gamma'}^{\mathcal{R}} T_{\Gamma'}^{\times} \equiv (i\hbar)^{\#\Gamma' - |\Gamma'|} \sum_{b(\Omega_{\Gamma'})} \text{ext}_{\Gamma'}^{\varrho, k}(t_{I(\Gamma)}^{\times}) N(\Gamma'). \quad (112)$$

Reflecting the involved renormalization freedom (i.e. ambiguity), the extension map $\text{ext}_{\Gamma'}^{\varrho, k} : \mathcal{D}'_{I(\Gamma')}^{\otimes}(\mathbb{R}_{\times}^{(1+d) \times n}) \rightarrow \mathcal{D}'_{I(\Gamma')}^{\otimes}$ is indexed by an auxiliary function $\varrho \in \mathcal{S}(\mathbb{R}^{(1+d) \times n})$ and a set (abbr. by k) of complex constants $k_{\Gamma'}^{\alpha, I(\Gamma')}$ where $|\alpha| \leq \omega_{\Gamma'}$. Depending on the associated graph Γ' , the instances for ϱ and k can up to a few restrictions be chosen arbitrarily. The concrete choice of $(\varrho_{\Gamma'}, k_{\Gamma'})$ fixes a particular renormalization scheme, i.e. $\mathcal{R} \simeq \{(\varrho_{\Gamma'}, k_{\Gamma'}) \mid \Gamma' \text{ divergent}\}$. According to the assumed property of tI, the relevant dimension reduces to $\text{dim} = (1+d)(n-1)$, and hence, the diagonal $\text{diag}_2^n \subset \mathbb{R}^{(1+d) \times n}$ goes over into the origin $\{0\} \subset \mathbb{R}^{\text{dim}}$. For the BS approach, renormalization can only be given

⁶i.e., $\widehat{\text{ren}} \Big|_{\widehat{\mathcal{R}}_{\Gamma}^{*n}(\mathbb{R}^{(1+d) \times n})} = \text{id}$

formally, i.e. $T_{\Gamma'}^{\mathcal{R}} := \text{ren}^{\mathcal{R}}(T_{\Gamma_1}^{\mathcal{R}} \hat{\circ}_{\Gamma}^{\top} T_{\Gamma_2}^{\mathcal{R}}) \equiv T_{\Gamma_1}^{\mathcal{R}} \circ_{\Gamma}^{\top} T_{\Gamma_2}^{\mathcal{R}}$, for $\Gamma' \equiv \Gamma^1 \sqcup_{\Gamma} \Gamma^2$. This is because of its coarse regularization. For a concrete realization one will have to apply EG.

In the EG framework renormalization is as well based on powercounting. This can be realized, as in the EGF version, via the so-called scaling degree,

$$\text{sd}(t) := \inf \left\{ \rho \mid \lim_{\lambda \rightarrow 0} \lambda^{\rho - \dim} t(\tau(\lambda^{-1} \cdot)) = 0, \forall \tau \in \mathcal{D}(\mathbb{R}^{\dim}) \right\}, \quad (113)$$

which counts the inverse power of distributions $t \in \mathcal{D}'(\mathbb{R}^{\dim})$ at the origin of \mathbb{R}^{\dim} , i.e., $t(x) = O(|x|^{\text{sd}})$ as $|x| \rightarrow 0$. Then the extension of distributions is characterized as follows.

Proposition 4.19 *Let $t^{\times} \in \mathcal{S}(\mathbb{R}^{\dim} \setminus \{0\})$ with scaling degree $\rho := \text{sd}(t^{\times}) < \infty$. Then there exist extensions $t^{\varrho, k} \equiv \text{ext}^{\varrho, k}(t^{\times}) \in \mathcal{S}(\mathbb{R}^{\dim})$ with the same finite scaling degree $\text{sd}(t^{\varrho, k}) = \rho$, i.e. $t^{\varrho, k}(\tau^{\times}) = t^{\times}(\tau^{\times})$, $\forall \tau^{\times} \in \mathcal{S}(\mathbb{R}^{\dim} \setminus \{0\})$. Two cases have to be distinguished. If the distribution t^{\times} is not singular, i.e. if its singular order ω is negative, then the extended distribution $t^{\varrho, k}$ is uniquely determined. Otherwise, if the singular order $\omega := \rho - \dim \geq 0$, uniqueness is violated. Then the extension depends, firstly, on an arbitrarily chosen tempered function $\varrho \in \mathcal{S}(\mathbb{R}^{\dim})$ which satisfies $\varrho(0) = 1$ and $\partial^{\alpha} \varrho(0) = 0$, and secondly, on an arbitrary set of constants $k^{\alpha} \in \mathbb{C}$ which defines the extended distribution on a set of testfunctions, $x \mapsto x^{\alpha} \varrho(x)$, i.e. $t^{\varrho, k}(\cdot^{\alpha} \varrho(\cdot)) = k^{\alpha}$, $\forall \alpha$, $|\alpha| \leq \omega$.*

Proof. Cf. Theorems 5.2 and 5.3 in [9], pp. 23ff. \square

Remark 4.20 More precisely, the extended distribution reads

$$t^{\varrho, k}(\tau) = t^{\times}(W_{\omega, \varrho} \tau) + \sum_{|\alpha| \leq \omega} \frac{k^{\alpha}}{\alpha!} \partial^{\alpha} \tau(0), \quad (114)$$

so that in the *second case*, where an associated graph would be called divergent, the following projection cannot be the identity, $W_{\omega, \varrho} : \mathcal{S}(\mathbb{R}^{\dim}) \rightarrow \mathcal{S}_{\omega}(\mathbb{R}^{\dim})$, $\tau \mapsto W_{\omega, \varrho} \tau$, where $(W_{\omega, \varrho} \tau)(x) = \tau(x) - \sum_{|\alpha| \leq \omega} \frac{\partial^{\alpha} \tau(0)}{\alpha!} x^{\alpha} \varrho(x)$ and where $\mathcal{S}_{\omega} = \{\xi \in \mathcal{S} \mid \partial^{\alpha} \xi(0) = 0, |\alpha| \leq \omega\}$ denotes test function space appropriate for the unique extension t in the *first case*, i.e. $t(\tau_{\omega}^{\times}) = t^{\times}(\tau_{\omega}^{\times})$, $\forall \tau_{\omega}^{\times} \in \mathcal{S}_{\omega}(\mathbb{R}^{\dim} \setminus \{0\})$.

As the singular order is supposed to be preserved by the extension, (114) and the following lemma allow to describe the difference of the scalar distributions $t_{\text{I}(\Gamma)}^{\mathcal{R}_i} := t_{\text{I}(\Gamma)}^{\varrho, k_i}$ w.r.t. renormalization schemes \mathcal{R}_i , $i \leq 2$, by a polynomial in derivatives of the delta distribution, i.e.

$$t_{\text{I}(\Gamma)}^{\mathcal{R}_2} - t_{\text{I}(\Gamma)}^{\mathcal{R}_1} = \sum_{|\alpha| \leq \omega_{\Gamma}} k_{\Gamma}^{\alpha, \text{I}(\Gamma)} \partial^{\alpha} \delta^{\text{IV}(\Gamma)}, \quad (115)$$

with coefficients $k^{\alpha} = k_2^{\alpha} - k_1^{\alpha}$ formed by the difference of the renormalization constants in \mathcal{R}_2 and \mathcal{R}_1 , provided that ϱ has been fixed.

Lemma 4.21 *Let $t \in \mathcal{S}'(\mathbb{R}^{\dim})$ and $\text{supp } t = \{0\}$. Then $\exists! \{k^{\alpha} \in \mathbb{C} \mid |\alpha| \leq \omega \equiv \lfloor \text{sd}(t) \rfloor\}$ s.t. $t(x) = \sum_{|\alpha| \leq \omega} k^{\alpha} \prod_{i \leq \dim} \partial^{\alpha_i} \delta(x_i)$.*

Proof. Cf. a textbook on distributions, e.g. §8.4 in [40]. \square

4.2.2 On the physical meaning

This subsection will explain the diagram in Fig. 3, especially the ‘laws by physics’ arrow, in more detail. It is divided in two paragraphs, devoted to the classification of renormalizable theories in pQFT and to the question of the renormalization ambiguity.

Classification of theories By applying the scaling degree’s homomorphic properties which, based on the scaling degree $\text{sd}(\Delta_{p'}^{+\gamma}) = 2|p'| + \text{sd}(\Delta_p^{+\gamma}) =: 2\rho_{p'}$ of the propagator $\Delta_{p'}^{+\gamma} \equiv (-1)^{|p'|} \partial_{p'} \partial_{\bar{p}'} \Delta_p^{+\gamma}$, lead to

$$\rho_v = \sum_{i \leq |P(v)|} \rho_{P_i(v)} \quad \text{and} \quad \rho_\Gamma = \sum_{k \leq |I(\Gamma)|} \rho_{I_k(\Gamma)}, \quad (116)$$

one determines the singular order $\omega_\Gamma = \rho_\Gamma - \dim$ of the graph Γ , i.e.

$$\omega_\Gamma = \sum_{k \leq |V(\Gamma)|} \omega_{V_k(\Gamma)} - \omega_{\Omega(\Gamma)} \quad \text{where} \quad \omega_v := \rho_v - (1 + d). \quad (117)$$

Powercounting therefore reproduces the well-known classes, called

$$\begin{cases} \text{non-renormalizable} \\ \text{renormalizable} \\ \text{super-renormalizable} \end{cases} \quad \text{if} \quad \begin{cases} (\exists v \in \mathbf{U}) \omega_v > 0, \\ (\forall v \in \mathbf{U}) \omega_v \leq 0, (\exists v \in \mathbf{U}) \omega_v = 0, \\ (\forall v \in \mathbf{U}) \omega_v < 0. \end{cases} \quad (118)$$

or as usual, in numbers of divergent graphs \mathbf{G}_{div} and vertices \mathbf{V}_{div} ,

$$\text{if} \quad \begin{cases} |\mathbf{V}_{\text{div}}| \\ |\mathbf{G}_{\text{div}}| \end{cases} \begin{cases} = \infty, \\ < \infty, \\ < \infty, \end{cases} \quad \text{where} \quad \begin{cases} \mathbf{V}_{\text{div}} = \{v \in \mathbf{W} \mid (\exists \Gamma \in \mathbf{G}_{\text{div}}) v = \Omega\Gamma\}, \\ \mathbf{G}_{\text{div}} = \{\Gamma \in \mathbf{G}_{\mathbf{W}} \mid \omega_\Gamma \geq 0\}. \end{cases} \quad (119)$$

This will be illustrated for a (toy model) pQFT with $|\mathbf{U}| = 1$ initial vertex which couples m scalar fields ϕ .

Example 4.22 Let $H_{\mathcal{I}} \propto : \phi^m :$. Since $\Delta_\phi^+(x) = O(|x|^{-2})$ as $x \rightarrow 0$, $\rho_\phi = 1$ and $\rho_{\phi^m} = m$. Then, for a graph Γ consisting (exclusively) of ϕ^m -vertices,

$$\omega_\Gamma = |V(\Gamma)|(m - (1 + d)) - |E(\Gamma)| + (1 + d). \quad (120)$$

Depending on the considered spacetime dimension $1+d$, i.e.

$$\begin{cases} < m, \\ = m, \\ > m, \end{cases} \quad \text{so that} \quad \begin{cases} \mathbf{V}_{\text{div}} = \mathbf{W}, \\ \mathbf{V}_{\text{div}} = \{v \in \mathbf{W} \mid |P(v)| \leq m\}, \\ \mathbf{G}_{\text{div}} \subseteq \{\Gamma \in \mathbf{G}_{\mathbf{W}} \mid |V(\Gamma)| \leq \frac{1+d}{1+d-m}\}, \end{cases} \quad (121)$$

one reproduces the above classes of renormalizable theories.

Renormalization freedom Besides translation invariance, which has been implemented explicitly, k_Γ will have to satisfy other symmetries. For instance, gtS is ensured by the condition $k_\Gamma^{\alpha, \mathbf{I}(\Gamma)} = \sigma_\Gamma(\mathbf{I}(\pi \circ \Gamma)) k_{\pi(\Gamma)}^{\pi(\alpha), \mathbf{I}(\pi \circ \Gamma)}$, $\forall \alpha, |\alpha| \leq \omega_\Gamma$. The requirements for Lorentz invariance have been investigated in [1]. Moreover, the physicists' strategy of selecting new vertices produced by renormalization is to implement further (physical) symmetries, e.g. C,P,T invariance, pseudo-unitarity, and gauge invariance; Scharf [38] calls this *the principle of perturbative gauge invariance*.

Example 4.23 (Scalar QED) Let $H_{\mathcal{I}} = e : \phi^* (\partial^\mu \phi) A_\mu : - e : (\partial^\mu \phi^*) \phi A_\mu :$. Then there are two initial vertices,

$$u_1 = \boxed{\begin{array}{c} \text{---} A_\mu \text{---} \\ | \\ \phi^* \text{---} \text{---} \partial^\mu \phi \end{array}}, \quad u_2 = \boxed{\begin{array}{c} \text{---} A_\mu \text{---} \\ | \\ \partial^\mu \phi^* \text{---} \text{---} \phi \end{array}}.$$

Consider $d = 3$. Then $\omega_\Gamma = -\omega_{\Omega(\Gamma)} \equiv 1 + d - \rho_{\Omega(\Gamma)}$, i.e. a graph Γ is divergent only if $E(\Gamma) \leq 4$ and $\partial\phi \notin E(\Gamma)$. Therefore, three candidates $V \setminus U \subseteq V_{\text{div}}$ of new physical vertices will be generated by renormalization,

$$v_1 = \boxed{\begin{array}{c} \phi^* \text{---} \text{---} \phi \\ \diagdown \quad \diagup \\ \phi^* \text{---} \text{---} \phi \end{array}}, \quad v_2 = \boxed{\begin{array}{c} \text{---} A_\mu \text{---} \text{---} \text{---} A_\mu \\ \diagdown \quad \diagup \\ \phi^* \text{---} \text{---} \phi \end{array}}, \quad w_1 = \boxed{\begin{array}{c} \text{---} A_\mu \text{---} \text{---} \text{---} A_\mu \\ \diagdown \quad \diagup \\ \text{---} A_\mu \text{---} \text{---} \text{---} A_\mu \end{array}}.$$

However, only v_1 and v_2 are relevant (cf. [37], pp. 335ff.). In accordance with pseudo-unitarity, $T_{(v_1)}(x) =: \phi^{*2}(x) \phi^2(x) :$ where $a_{v_1} = \lambda \in i\mathbb{R}$, and in accordance with gauge and C-invariance, $T_{(v_2)}(x) = \frac{1}{i\hbar} g^{\mu\nu} : \phi^*(x) \phi(x) A_\mu(x) A_\nu(x) :$ where $a_{v_2} = \frac{e^2}{i\hbar}$. In the conventional approach, one would then rewrite the interaction Hamiltonian by summing over V , i.e. $H_{\mathcal{I}} = i\hbar \sum_{v \in V} a_v T_v$.

4.2.3 EG counterparts of Kreimer's HA

The goal is to model the change of the renormalization scheme via a HA similar to Kreimer's [26]. Formally, this is in contrast with the literature [18, 34] where it has been claimed having extracted Kreimer's HA of renormalization in the EG approach (which should be impossible). Similar to Kreimer's approach in [28], one will somehow deform graded commutative versions of (the three) causally regularizing HAs in order to construct the aimed HA. However, the resulting HAs are still coassociative but not quasi as Kreimer's.

The graded commutative versions of the causally regularizing HAs are given as quotient algebras, $\underline{\mathcal{R}}_{G_\Gamma}^* = \widehat{\mathcal{R}}_{G_\Gamma}^* / \widehat{\mathcal{O}}_\Gamma^*[\cdot, \cdot]_\sigma^\otimes$, divided by the ideal which is generated by the graded commutator, i.e. $(\underline{\mathcal{R}}_{G_\Gamma}^*, \mathbb{C}, +, \underline{\mathcal{O}}_\Gamma^*, \eta, \Delta, \varepsilon, S_{\mathcal{O}_\Gamma^*})$ where the product reads

$$\underline{T}_{\Gamma^1} \underline{\mathcal{O}}_\Gamma^* \underline{T}_{\Gamma^2} = \frac{1}{2} [\widehat{T}_{\Gamma^1}, \widehat{T}_{\Gamma^2}]_{-\sigma}^{\widehat{\mathcal{O}}_\Gamma^*}, \quad \forall T_{\Gamma^1}, T_{\Gamma^2} \in \mathcal{R}_{G_\Gamma}^\Gamma. \quad (122)$$

Again, one will abbreviate the new representatives, $T_{\Gamma'} = \underline{T}_{(\Gamma')} \in \underline{\mathcal{R}}_{G_\Gamma}^*$. Notice, due to graded commutativity, $\underline{\mathcal{R}}_{G_\Gamma}^\top = \widehat{\mathcal{R}}_{G_\Gamma}^\top$ and $\underline{\circ}_\Gamma^\top = \widehat{\circ}_\Gamma^\top$.

As another prerequisite one introduces the algebra of tuples of strongly decreasing graphs, i.e. $(\mathcal{G}_\Gamma^\supset, \mathbb{C}, +, \cdot, \emptyset)$, which is defined as the quotient of the tensor algebra of graphs divided by some ideal of non-decreasing graphs,

$$\mathcal{G}_\Gamma^\supset = \bigoplus_{n \geq 0} G_\Gamma^{\otimes n} / ((\dots, \Gamma^2) \otimes (\Gamma^1, \dots), \sqcap \Gamma^2 \not\subseteq \sqcap \Gamma^1; \quad (123)$$

$$(\dots, \Gamma^2, \Gamma^1, \dots) - (\dots, \Gamma^1, \dots), \sqcap \Gamma^2 = \sqcap \Gamma^1). \quad (124)$$

The EG counterpart of Kreimer's HA, providing the transition $(\mathcal{R} \rightarrow \mathcal{R}') =: \mathcal{K}$ of renormalization schemes, will be a HA $\underline{\mathcal{K}}_{G_\Gamma}^*$ that defines an extension of $\underline{\mathcal{R}}_{G_\Gamma}^*$. This HA can be constructed in complete analogy to the latter, starting with the counterpart of $\mathcal{R}_{G_\Gamma}^\top$, i.e. $\mathcal{K}_{G_\Gamma}^\top := \bigoplus_{n \geq 0} \mathbb{C}\{k_{\overline{\Gamma}'} T_{\Gamma'}^\mathcal{R} \mid \overline{\Gamma}' \in \mathcal{G}_\Gamma^\supset, \overline{\Gamma}'|_{\overline{\Gamma}'} = \Gamma', |\mathbb{V}(\Gamma')| = n\}$ where, analogous to (60),

$$k_{\overline{\Gamma}'} T_{\Gamma'}^\mathcal{R} := (i\hbar)^{\#\Gamma' - |\Gamma'|} \sum_{b(\Omega\Gamma')} k_{\overline{\Gamma}'} t_{\Gamma'}^\mathcal{R} N(\Gamma') \quad (125)$$

which, chosen in accordance with (115), is determined by its scalar part,

$$\mathbb{T} \ni k_{\overline{\Gamma}'} t_{\Gamma'}^\mathcal{R} := \begin{cases} t_{\Gamma'}^\mathcal{R} & \text{if } \overline{\Gamma}' = \emptyset, \\ \sum_{|\alpha| \leq \omega_{\Gamma'}} k_{\overline{\Gamma}'}^{\alpha, \mathbb{I}(\Gamma')} \partial^\alpha \delta^{\mathbb{IV}(\Gamma')} & \text{otherwise.} \end{cases} \quad (126)$$

The komplex constants $k_{\overline{\Gamma}'}^{\alpha, \mathbb{I}(\Gamma')}$ with multi-index α and particle indices $\mathbb{I}(\Gamma')$, depending on the product of the graph Γ' with an appropriate tuple of decreasing graphs $\overline{\Gamma}'$, fix the transition \mathcal{K} . As an empty sum is zero, $k_{\overline{\Gamma}'} T_{\Gamma'}^\mathcal{R}$ vanishes for non-divergent graphs Γ' , i.e., if $\omega_{\Gamma'} < 0$.

The additional structure in \mathcal{K}_{G_Γ} (i.e. the HA analogous to \mathcal{R}_{G_Γ}) can be introduced as an action of an algebra, $k_\bullet : \mathcal{G}_\Gamma^\supset \rightarrow \text{End}(\mathcal{K}_{G_\Gamma})$ ($k_{\overline{\Gamma}'} T_{\Gamma'}^\mathcal{R} = 0$ if $\sqcap \overline{\Gamma}'|_{\overline{\Gamma}'} \not\subseteq \sqcap \Gamma'$), i.e., $k_{\overline{\Gamma}^2} \circ k_{\overline{\Gamma}^1} = k_{\overline{\Gamma}^1, \overline{\Gamma}^2}$, $\forall \overline{\Gamma}^1, \overline{\Gamma}^2 \in \mathcal{G}_\Gamma^\supset$, so that, $\forall \overline{\Gamma}' \in \mathcal{G}_\Gamma^\supset$, $k_{\overline{\Gamma}'}$ defines a HA homomorphism, i.e., $k_{\overline{\Gamma}'} \circ \circ_\Gamma = \circ_\Gamma \circ (k_{\overline{\Gamma}'} \otimes k_{\overline{\Gamma}'})$, $\Delta \circ k_{\overline{\Gamma}'} = (k_{\overline{\Gamma}'} \otimes k_{\overline{\Gamma}'}) \circ \Delta$, and $S_{\circ_\Gamma} \circ k_{\overline{\Gamma}'} = k_{\overline{\Gamma}'} \circ S_{\circ_\Gamma}$. Along this line, starting with Sect. 3.2.3, one repeats all the algebraic constructions for \mathcal{K} . As far as renormalization is involved, the scheme is still \mathcal{R} . One only has to require that transition and renormalization exchange, i.e. $k_{\overline{\Gamma}'} \circ \text{ren}_{\Gamma'}^\mathcal{R} := \text{ren}_{\Gamma'}^\mathcal{R} \circ k_{\overline{\Gamma}'}$, whenever $\overline{\Gamma}'|_{\overline{\Gamma}'} \supset \sqcap \Gamma'$.

Theorem 4.24 ($\underline{\mathcal{K}}_{G_\Gamma}^*, \mathbb{C}, +, \underline{\circ}_\Gamma^*, \eta, \Delta, \varepsilon, S_{\circ_\Gamma^*}$) forms a HA which extends $\underline{\mathcal{R}}_{G_\Gamma}^*$, and $k_\bullet : \mathcal{G}_\Gamma^\supset \rightarrow \text{End}(\mathcal{K}_{G_\Gamma}^*)$ defines an action of an algebra where, $\forall \overline{\Gamma}' \in \mathcal{G}_\Gamma^\supset$, $k_{\overline{\Gamma}'}$ is a HA homomorphism. More precisely, $(\forall \Gamma' \in \underline{\mathcal{K}}_{G_\Gamma}^*) (\exists! T \in \underline{\mathcal{R}}_{G_\Gamma}^*) (\exists! \overline{\Gamma}' \in \mathcal{G}_\Gamma^\supset)$ s.t. $k_{\overline{\Gamma}'} T = \Gamma'$, i.e. $k_\bullet : \mathcal{G}_\Gamma^\supset \rightarrow \text{Hom}(\underline{\mathcal{R}}_{G_\Gamma}^*, \underline{\mathcal{K}}_{G_\Gamma}^*)$ where $T' \in \underline{\mathcal{R}}_{G_\Gamma}^* \subset \underline{\mathcal{K}}_{G_\Gamma}^*$ requires $T = T' \equiv k_\emptyset T$.

Proof. This is the result of the constructions of the previous subsections, but in the context of \mathcal{K} instead of \mathcal{R} . \square

The theorem gets meaning when claiming the concrete embedding of the new scheme \mathcal{R}' , i.e. $\underline{\mathcal{R}}_{G_\Gamma}^* \stackrel{\iota}{\simeq} \underline{\mathcal{R}}_{G_\Gamma}^* \subset \underline{\mathcal{K}}_{G_\Gamma}^*$,

$$\iota : T_{\Gamma'}^\mathcal{R} \mapsto T_{\Gamma'}^{\mathcal{R}'} := T_{\Gamma'}^\mathcal{R} + S_{\underline{\circ}_\Gamma^*}^{k,1} T_{\Gamma'}^\mathcal{R}, \quad \forall \Gamma' \in G_\Gamma, \quad (127)$$

where $S_{\underline{\sigma}_\Gamma}^k T_{\Gamma'}^{\mathcal{R}} \equiv k_\bullet \circ S_{\underline{\sigma}_\Gamma} T_{\Gamma'}^{\mathcal{R}} := S_{\underline{\sigma}_\Gamma} (k_{(\Gamma')} T_{\Gamma'}^{\mathcal{R}})$, $n \equiv |\mathbb{V}(\Gamma')|$, and $S_{\underline{\sigma}_\Gamma}^{k,1} T_{\Gamma'}^{\mathcal{R}} = S_{\underline{\sigma}_\Gamma}^k T_{\Gamma'}^{\mathcal{R}}$, except for $n = 1$, where $S_{\underline{\sigma}_\Gamma}^{k,1} T_{\Gamma'}^{\mathcal{R}} = 0$. The correctness of this setting rests upon the requirement that $\text{supp}(T_{\Gamma'}^{\mathcal{R}'} - T_{\Gamma'}^{\mathcal{R}}) \subseteq \text{diag}_n^n$, which is guaranteed by the following fact.

Lemma 4.25 $\text{supp}(S_{\underline{\sigma}_\Gamma}^{k,1} T_{\Gamma'}^{\mathcal{R}}) \subseteq \text{diag}_n^n$.

Proof. This can be shown by expansion of the antipode, i.e.

$$S_{\underline{\sigma}_\Gamma}^k T_{\Gamma'}^{\mathcal{R}} = \text{ren}_{(\Gamma')}^{\mathcal{R}} \circ k_{(\Gamma')} \left(-T_{\Gamma'}^{\mathcal{R}} - \sum_{(\Gamma^1, \Gamma^2) \in \mathcal{P}_2(\Gamma')} \sigma_\Gamma(\Gamma^1 \sqcup_\Gamma \Gamma^2) S_{\underline{\sigma}_\Gamma}^k T_{\Gamma^1}^{\mathcal{R}} \circ_\Gamma^* T_{\Gamma^2}^{\mathcal{R}} \right), \quad (128)$$

and applying an adequate induction argument. E.g., if $K^i \in \underline{\mathcal{K}}_{\mathbb{G}_\Gamma}^*$ and $\text{supp} K^i \subseteq \text{diag}_{n_i}^{n_i}$, $i \leq 2$, where $n = n_1 + n_2$, then $\text{supp}(K^1 \circ_\Gamma^* K^2) \subseteq \text{diag}_n^n$. \square

For the case, $* = \mathbb{T}$, one can state the transition between the schemes explicitly.

Example 4.26 Let $P \in \mathcal{P}_r(\Gamma')$, $r \leq n$. Then R is called a P -tree iff $R \in \mathcal{G}_\Gamma^{\triangleright \times r}$, $(\forall j < r) (R_{ij})_{i \leq j} \in \mathcal{P}_j(\Gamma')$ and $\exists! i_j \in \{i_{j-1}, i_{j-1} + 1\}$ s.t. $(R_{i_j j+1}, R_{i_j+1 j+1}) \in \mathcal{P}_2(R_{i_j j})$ where $i_1 = 1$, and $(R_{i_r})_{i \leq r} = P$. Applying this notation, one obtains

$$T_{\Gamma'}^{\mathcal{R}'} = T_{\Gamma'}^{\mathcal{R}} + \sum_{r < n} \sum_{P \in \mathcal{P}_r(\Gamma')} \sigma_{\Gamma'}(P) \sum_{R \text{ } P\text{-tree}} \bigcirc_\Gamma^{\mathbb{T}} k_{R_i} T_{P_i}^{\mathcal{R}}. \quad (129)$$

For the (most) trivial transition (in this context), i.e. $k_{\Gamma'} := k_{(\Gamma)}$, $\forall \Gamma' \in \mathcal{G}_\Gamma^{\triangleright}$, this reduces to

$$T_{\Gamma'}^{\mathcal{R}'} = T_{\Gamma'}^{\mathcal{R}} + \sum_{r < n} \sum_{P \in \mathcal{P}_r(\Gamma')} \sigma_{\Gamma'}(P) \bigcirc_\Gamma^{\mathbb{T}} k_{P_i} T_{P_i}^{\mathcal{R}}, \quad (130)$$

corresponding to Pinter's result, cf. Eq. (14) in [34], p. 6.

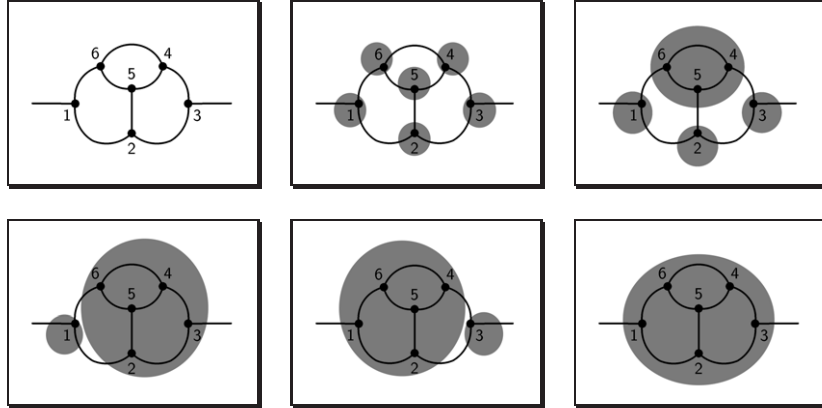


Figure 4: A sample graph and partitions P of divergent subgraphs P_i .

Starting the comparison with the conventional approach, the following two examples will illustrate how overlapping divergences are treated in the EG framework.

Example 4.27 Consider $H_{\mathcal{I}} \propto \phi^3$ for $d = 2$ and, especially, the graph in Fig. 4. Then the singular order of the divergent subgraphs $\Upsilon = P_i$ is given by $\omega_{\Upsilon} = 3 - |\mathbf{E}(\Upsilon)|$, i.e.

$$\omega_{\Upsilon} = \begin{cases} 1 & \text{if } \mathbf{V}(\Upsilon) \in \{(4, 5, 6), (1, \dots, 6)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice, the two subgraphs with loops, i.e. $((1, 2, 5, 6), \dots)$ and $((2, 3, 4, 5), \dots)$, which because of their 4 (> 3) external lines are not divergent, do not contribute to the so-called forest formula (129).

Here, loops obviously do not play the distinguished role they play in the conventional approach (cf. Example 4.12). That holds true for tree graphs as well.

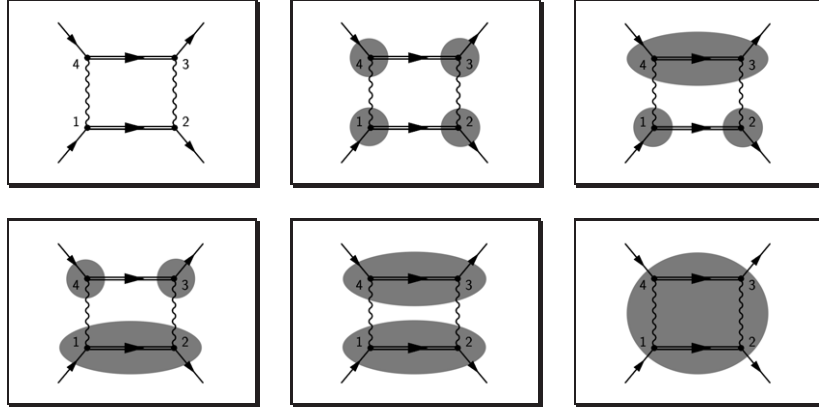


Figure 5: A sample graph and partitions P of divergent subgraphs P_i .

Example 4.28 Consider scalar QED for $d = 3$ and, especially, the graph in Fig. 5. Then the singular order of the divergent subgraphs $\Upsilon = P_i$ is given by $\omega_{\Upsilon} = 4 - |\mathbf{E}(\Upsilon)| = 0$. Notice, also tree graphs, i.e. $((1, 2), \dots)$ and $((3, 4), \dots)$, do represent divergent subgraphs.

Even if the structure of graphs presented here varies from Kreimer's (which is only pre-Lie), one can identify the following connection.

Remark 4.29 Let R be Kreimer's map which is supposed to leave the short distance singularities unaltered, e.g., if $\text{reg}(\varepsilon) = \sum_{n=-p}^{\infty} c_n \varepsilon^n$, $p > 0$, denotes a regularization for $\varepsilon \rightarrow 0$ then $R : \text{reg}(\varepsilon) \mapsto \text{reg}(\varepsilon) - \sum_{n \geq 0} c'_n \varepsilon^n$. Then one observes the following correspondence between the presented and Kreimer's HA,

$$k_{\bullet} \leftrightarrow R \quad (\text{i.e. } S_{\Sigma_r^k}^* \leftrightarrow S_R) \quad \text{and} \quad \text{ren}^{\mathcal{R}} \leftrightarrow \text{id} - R, \quad (131)$$

reflecting the algebraic structure and, resp., the function of renormalization.

In order to ensure that the deformed antipode S_R defines a homomorphism (which, for $S_{\Sigma_r^*}$, is fulfilled for free) the map R (and hence, $\text{id} - R$) has to satisfy a multiplicativity constraint, the so-called Rota-Baxter relation (cf. [14]),

$$p(XY) + p(X)p(Y) = p(Xp(Y)) + p(p(X)Y). \quad (132)$$

Moreover, one can motivate the quasiness of Kreimer's deformed HA.

Remark 4.30 In the spirit of Kreimer's presentation, e.g. in [28], one may interpret the functor $\text{Hom}(\underline{\mathcal{R}}_{\text{Gr}}^*, \underline{\mathcal{K}}_{\text{Gr}}^*)$ as a deformation of HAs induced by deforming

$$\begin{cases} \text{either the antipode, } S_{\circ_r^*} \mapsto S_{\circ_r^*}^k \equiv k_{\bullet} \circ S_{\circ_r^*}, \\ \text{or the coproduct, } \Delta \mapsto \Delta^k \equiv (k_{\bullet} \otimes k_{\bullet}) \circ \Delta, \end{cases} \quad (133)$$

while keeping the coproduct/antipode, resp., fixed. Now it is obvious why the deformed HA in Kreimer's original approach is only quasi; as one easily checks, $(\text{id} \otimes \Delta^k) \circ \Delta^k T_{\Gamma'}^{\mathcal{R}} = (\Delta^k \circ \text{id}) \circ \Delta^k T_{\Gamma'}^{\mathcal{R}}$ is true only if the transition k_{\bullet} is trivial.

A more detailed presentation of the content of this subsection will be given elsewhere [32].

5 Conclusions and outlook

About ten years ago HAs have entered the framework of pQFT. At that time Wick and Feynman contractions (i.e. Π and $\bar{\Pi}$) have been identified, by Rota and Stein [35, 36], as examples for Laplace pairs in the context of combinatorial HAs. Since then, the formal perturbative expansion could have been considered as given by a Laplace HA. However, only recently this has been done, cf. [16, 5]. The knowledge about it did not get that popular among physicists as Kreimer's Hopf algebraic description of renormalization [26] in the late nineties.

In this paper, both these HA approaches to pQFT (i.e. the forth/seventh and the eighth entry in the table of Fig. 1) get represented in the EG framework. The causally regularizing HAs (entries 5–7) can therefore be regarded as a bridge between them. Furthermore, the HAs that model normalization and localization of free fields (entries 2–3) appear as the structure which Cliffordization twists into the Laplace HA representing the perturbative expansion.

Supplementing the presented scenario of HAs, which apparently underlies the EG approach to pQFT, some topics for further research in the field turn up immediately (enumerated from the physical to the mathematical one):

- In the spirit of Kreimer's more recent work [29, 30], one might also think about implementing Scharf's principle [38] of perturbative gauge invariance into the HA context.
- Laplace pairs can be interpreted as objects in the theory of quantum groups (e.g., [6, 7, 5]), namely as so-called R-forms (cf. Chap. VIII in [21]). Investigations in this direction still seem to be fruitful.
- Following an idea of Kontsevich [23], one might try to understand renormalization as a dual procedure to the compactification of the configuration space [17].

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A Appendix

A.1 Calculations

Lemma A.1 $\Delta^{(\mathbb{N})}$, i.e. Δ and $\Delta^{\mathbb{N}}$, are *gtS*.

Proof. The two claims can be proven at once, where the possible empty alternative is symbolized by brackets. Let $X \subseteq \mathbb{P}^n$, $n \in \mathbb{N}$, and let $\pi \in S_n$ be a permutation on X . Then

$$\begin{aligned} \Delta^{(\mathbb{N})}(\varphi^{\circ\pi(X)}) &= \sum_{P_{\pm} \in \mathcal{P}_2^0(\pi(X))} \sigma_{\pi(X)}(P_{\pm}) \cdots \\ &= \sum_{P_{\pm} \in \mathcal{P}_2^0(X)} \sigma_{\pi(X)}(X) \sigma_X(P_{\pm}) \cdots = \sigma(\pi) \Delta^{(\mathbb{N})}(\varphi^{\circ X}), \end{aligned}$$

where the following identity for sets of partitions, i.e. $\mathcal{P}_2^0 = \mathcal{P}_2^0 \circ \pi$, $\forall \pi \in S_n$, and following the group representation rule for the grading sign, $\sigma_Y(Z) = \sigma_Y(X) \sigma_X(Z)$, $\forall X, Y, Z \in \mathbb{P}^n$, have been applied. \square

Lemma A.2 Δ and $\Delta^{\mathbb{N}}$ are *coassociative* where, for the first coproduct, ε also satisfies the *counit property*.

Proof. The three calculations below verify the claims. Let $X \in \mathbb{P}^n$, $n \in \mathbb{N}$. Firstly,

$$\begin{aligned} (\text{id} \otimes \Delta) \circ \Delta(\varphi^{\circ X}) &= \sum_{P \in \mathcal{P}_2^0(X)} \sigma_X(P) \varphi^{\circ P_1} \otimes \Delta(\varphi^{\circ P_2}) \\ &= \sum_{P \in \mathcal{P}_2^0(X)} \sigma_X(P) \varphi^{\circ P_1} \otimes \sum_{Q \in \mathcal{P}_2^0(P_2)} \sigma_{P_2}(Q) \varphi^{\circ Q_1} \otimes \varphi^{\circ Q_2} \\ &= \sum_{(P_1, Q_1, Q_2) \in \mathcal{P}_3^0(X)} \sigma_X(P_1, Q_1, Q_2) \varphi^{\circ P_1} \otimes \varphi^{\circ Q_1} \otimes \varphi^{\circ Q_2} \\ &\quad \vdots \\ &= (\Delta \otimes \text{id}) \circ \Delta(\varphi^{\circ X}), \end{aligned}$$

secondly,

$$\begin{aligned} (\text{id} \otimes \varepsilon) \circ \Delta(\varphi^{\circ X}) &= \sum_{P \in \mathcal{P}_2^0(X)} \sigma_X(P) \varphi^{\circ P_1} \otimes \varepsilon(\varphi^{\circ P_2}) \\ &= \sigma_X(X \otimes \emptyset) \varphi^{\circ X} \otimes 1 = \varphi^{\circ X} \\ &\quad \vdots \\ &= (\varepsilon \otimes \text{id}) \circ \Delta(\varphi^{\circ X}), \end{aligned}$$

which have been rather standard, and thirdly,

$$\begin{aligned}
& (\text{id} \otimes \Delta^{\mathbb{N}}) \circ \Delta^{\mathbb{N}}(\varphi^{\circ X}) \\
&= \sum_{P \in \mathcal{P}_2^0(X)} \sigma_X(P) \text{pr}_+ \circ \varphi^{\circ P_1} \otimes \text{pr}_- \circ \Delta^{\mathbb{N}}(\varphi^{\circ P_2}) \\
&= \sum_{P \in \mathcal{P}_2^0(X)} \sigma_X(P) \varphi_+^{\circ P_1} \otimes \sum_{Q \in \mathcal{P}_2^0(P_2)} \sigma_{P_2}(Q) \text{pr}_- \circ \varphi_+^{\circ Q_1} \otimes \text{pr}_- \circ \varphi_-^{\circ Q_2} \\
&= \sum_{P \in \mathcal{P}_2^0(X)} \sigma_X(P) \varphi_+^{\circ P_1} \otimes \sigma_{P_2}(\emptyset \otimes P_2) 1 \otimes \varphi_-^{\circ P_2} \\
&= \sum_{P \in \mathcal{P}_2^0(X)} \sigma_X(P) \varphi_+^{\circ P_1} \otimes 1 \otimes \varphi_-^{\circ P_2} \\
&\vdots \\
&= (\Delta^{\mathbb{N}} \otimes \text{id}) \circ \Delta^{\mathbb{N}}(\varphi^{\circ X}),
\end{aligned}$$

where the projection properties, $\text{pr}_{\pm}^2 = \text{pr}_{\pm}$ and $\text{pr}_{\mp} \circ \text{pr}_{\pm} = 0$, have been applied. \square

Lemma A.3 $\Delta^{(\mathbb{N})}$ and ε define homomorphisms w.r.t. the products $\square \in \{\circ, \mathbb{N}\}$.

Proof. Let $X^1, X^2 \subset \mathbb{P}^n$, $n \in \mathbb{N}$. Then

$$\varepsilon(\varphi^{\square X^1} \square \varphi^{\square X^2}) \equiv \varepsilon(\varphi^{\square X^1}) \varepsilon(\varphi^{\square X^2}) = \begin{cases} 1 & \text{if } X^1 = X^2 = \emptyset, \\ 0 & \text{otherwise,} \end{cases} \quad (134)$$

and

$$\begin{aligned}
\Delta^{(\mathbb{N})}(\varphi^{\square X^1} \square \varphi^{\square X^2}) &= \sum_{P \in \mathcal{P}_2^0(X^1 \otimes X^2)} \sigma_{X^1 \otimes X^2}(P) \varphi_{(+)}^{\square P_1} \otimes \varphi_{(-)}^{\square P_2} \\
&= \sum_{\substack{P \in \mathcal{P}_2^0(X^1 \otimes X^2) \\ P_i = P_i^1 \otimes P_i^2, i \leq 2 \\ P_i^j \in \mathcal{P}_2^0(X^j), j \leq 2}} \sigma_{X^1}(P^1) \sigma_{X^2}(P^2) \underbrace{\varphi_{(+)}^{\square P_1^1} \circ \varphi_{(+)}^{\square P_1^2} \otimes \varphi_{(-)}^{\square P_2^1} \circ \varphi_{(-)}^{\square P_2^2}}_{= (\varphi_{(+)}^{\square P_1^1} \otimes \varphi_{(-)}^{\square P_2^1}) \square (\varphi_{(+)}^{\square P_1^2} \otimes \varphi_{(-)}^{\square P_2^2})} \\
&= \Delta^{(\mathbb{N})}(\varphi^{\square X^1}) \square \Delta^{(\mathbb{N})}(\varphi^{\square X^2}),
\end{aligned}$$

where the composition laws for the grading sign (based on the group representation) have been applied. \square

A.2 Illustrations

Example A.4 Illustrated for two graphs of QED, cf. Example 2.6, e.g. for

$$\Gamma^1 \equiv (X^1, \Lambda^1) = \boxed{\begin{array}{c} \text{---} \nearrow \\ X_2^1 \\ \text{---} \searrow \\ \text{---} \downarrow \\ X_1^1 \\ \text{---} \nearrow \\ \text{---} \searrow \end{array}}, \quad \Gamma^2 \equiv (X^2, \Lambda^2) = \boxed{\begin{array}{c} X_3^2 \\ \text{---} \nearrow \\ \text{---} \searrow \\ X_1^2 \\ \text{---} \nearrow \\ \text{---} \searrow \end{array}}$$

with vertices $(\bar{\psi}, \psi, A)$, except of $X_2^2 = (\bar{\psi}, \psi, A^{\text{cl}})$, and matrices of lines

$$\Lambda^1 = \begin{pmatrix} \emptyset & \{(1, 2)\} \\ \{(2, 1)\} & \emptyset \end{pmatrix}, \quad \Lambda^2 = \begin{pmatrix} \emptyset & \{(2, 1)\} & \{(3, 3)\} \\ \{(1, 2)\} & \emptyset & \{(2, 1)\} \\ \{(3, 3)\} & \{(1, 2)\} & \emptyset \end{pmatrix},$$

the concatenation with the matrix

$$\Lambda^{12} = \begin{pmatrix} \{(2, 1)\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{(1, 2)\} \end{pmatrix},$$

results in the graph

$$\Gamma^1 \sqcup_{\Lambda^{12}} \Gamma^2 \equiv \left(X, \begin{pmatrix} \Lambda^1 & \Lambda^{12} \\ \tilde{\Lambda}^{12} & \Lambda^2 \end{pmatrix} \right) = \boxed{\begin{array}{c} \text{Diagram with vertices } X_1, X_2, X_3, X_4, X_5 \text{ and directed edges.} \end{array}}.$$

Example A.5 Consider the following quantum model, where the time evolution is determined by the Schrödinger equation, $i\hbar \partial_t \varphi(t, \vec{x}) = H(t)\varphi(t, \vec{x})$, and a interaction Hamiltonian, $H(t) = \int_{\mathbb{R}^d} dx H(t, \vec{x})$. The S -matrix $S = \lim_{t_{\pm} \rightarrow \pm\infty} U(t_+, t_-)$ is the limit of an unitary operator U which describes the time evolution of an incoming free field. By repeatedly applying the integrated Schrödinger equation, $\varphi(t, \vec{x}) = \varphi(s, \vec{x}) - \frac{i}{\hbar} \int_s^t dt_1 H(t_1)\varphi(t_1, \vec{x})$, one obtains the so-called *Dyson series*

$$U(t, s) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{\hbar^n} \int_s^t dt_1 \dots \int_s^{t_{n-1}} dt_n H(t_1) \dots H(t_n).$$

By inserting the time-ordering operation, i.e.

$$\mathsf{T}(h(t_1, \dots, t_n)) = \sum_{\pi \in \mathcal{S}_n} \theta_{(t_{\pi_1}, \dots, t_{\pi_n})} \sigma(\pi) h(t_{\pi_1}, \dots, t_{\pi_n}), \quad (135)$$

where

$$\theta_{(t_1, \dots, t_n)} := \theta(t_1 - t_2) \dots \theta(t_{n-1} - t_n) = \begin{cases} 1 & \text{if } t_1 > \dots > t_n, \\ 0 & \text{otherwise} \end{cases} \quad (136)$$

denotes a product of step functions, the limit can be performed, i.e.

$$S = 1 + \sum_{n=1}^{\infty} \frac{1}{n!(i\hbar)^n} \int_{\mathbb{R}^{1+d}} dt_1 dx_1 \dots \int_{\mathbb{R}^{1+d}} dt_n dx_n \mathsf{T}(H(t_1, \vec{x}_1) \dots H(t_n, \vec{x}_n)).$$

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