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films

by

*Matthias Kurzke*

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# Boundary vortices in thin magnetic films

Matthias Kurzke\*

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## Abstract

We study the asymptotic behavior of a family of functionals describing the formation of topologically induced boundary vortices in thin magnetic films. We obtain convergence results for sequences of minimizers and some classes of stationary points, and relate the limiting behavior to a finite dimensional problem, the renormalized energy associated to the vortices.

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## 1 Introduction

In this article we analyze the behavior as  $\varepsilon \rightarrow 0$  of the functionals

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin^2(u - g) d\mathcal{H}^1, \quad (1.1)$$

where  $\Omega$  is a simply connected domain in  $\mathbb{R}^2$ , and  $g : \partial\Omega \rightarrow \mathbb{R}$  a function such that  $e^{ig} : \partial\Omega \rightarrow S^1$  is a map of degree  $D \neq 0$ . We show convergence results for sequences of minimizers and stationary points of not too high energy. The limit functions are harmonic functions with boundary singularities. In certain cases, in particular for minimizers we give an asymptotic expansion for the energy, showing that the singular part of the energy depends only on the number of such singularities, while their interaction is described by a renormalized energy occurring as the first nonsingular term in the expansion that only depends on the vortex positions, similar to results obtained by Bethuel-Brezis-Hélein for the Ginzburg-Landau energy.

The motivation to study the functionals (1.1) comes from micromagnetism. Kohn and Slastikov [3] were able to show that it arises as a thin-film limit of the micromagnetic energy functional given by

$$E(m) = w^2 \int_{\Omega_h} |\nabla m|^2 + \int_{\mathbb{R}^3} |\nabla u|^2, \quad (1.2)$$

where  $\Omega_h = \Omega \times (0, h)$  is a Lipschitz domain in  $\mathbb{R}^3$ ,  $m : \Omega_h \rightarrow S^2$ , and  $u$  is related to  $m$  via the static Maxwell equation  $\Delta u = \operatorname{div}(m\chi_{\Omega_h})$ . The number  $w$  is a material parameter, called the *exchange length*. We have neglected crystal anisotropy here, which amounts to considering so-called *soft* magnetic films, and have not included the interaction with an external magnetic field.

Depending on the relation between the length scales  $w$ ,  $h$  and  $\ell = \operatorname{diam} \Omega$  that we assume to be 1 by choice of units, many scaling limits of (1.2) can be considered. We concentrate here on thin films, i.e.  $h \rightarrow 0$ . One of the first results in this direction is due to Gioia and James [2], who studied the case where  $h \rightarrow 0$  while  $w$  stays constant. The resulting limiting theory predicts the limit magnetization to be constant.

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\*Max-Planck-Institute for Mathematics in the Sciences, Inselstraße 22–26, D-04103 Leipzig, Germany, mkurzke@mis.mpg.de

Kohn and Slustikov [3] studied the regimes  $\frac{w^2}{h|\log h|} \rightarrow \infty$  and  $\frac{w^2}{h|\log h|} \rightarrow \alpha \in (0, \infty)$  and could show  $\Gamma$ -convergence to limiting theories. In the first case, the limit energy is finite only on constant in-plane magnetizations  $m \equiv \bar{m} \in S^1$ , and given by

$$\frac{1}{2\pi} \int_{\partial\Omega} (\bar{m} \cdot \nu)^2 d\mathcal{H}^1,$$

where  $\nu$  denotes the outer normal to  $\partial\Omega$ . In the second case, the magnetization is still forced to be in-plane and unit length, but need not be constant. The energy is given by

$$\mathcal{E}^\alpha(m) = \alpha \int_{\Omega} |\nabla m|^2 + \frac{1}{2\pi} \int_{\partial\Omega} (m \cdot \nu)^2. \quad (1.3)$$

In the borderline case where  $\frac{w^2}{h}$  is constant, Moser [7, 8] was able to show a convergence result for minimizers of (1.2) and could show the formation of boundary vortices.

We investigate the behavior as  $\alpha \rightarrow 0$  (i.e.  $\frac{w^2}{h|\log h|} \rightarrow 0$ ) of  $\frac{1}{\alpha} \mathcal{E}^\alpha$  which can be seen as connecting the results of Kohn and Slustikov to that of Moser. The functionals (1.1) correspond to those of (1.3) after the substitutions  $m = e^{iu}$  and  $\nu = ie^{ig}$ .

Our main results are Theorem 4.2, where we prove subconvergence of minimizers and isolation of vortices, Theorem 5.4 where we obtain subconvergence for stationary points satisfying a natural logarithmic energy bound, and finally Theorem 7.8 where we give an asymptotic expansion of the energy along a converging sequence with isolated vortices, in particular for minimizers. The energy is given by a singular part depending only on the number of the vortices, and an  $O(1)$  part that depends on the position of the vortices, and can be calculated via the solution of a linear boundary value problem.

Our approach to convergence theorems for minimizers follows the ideas of Bethuel-Brezis-Hélein [1] and Struwe [9]. There are also similarities to the approach of Moser [7] who combined interior and boundary vortices, but without calculating a renormalized energy. A different view of (1.1) was pursued in [4], where the functional was reduced to a nonlocal one on the boundary, and a  $\Gamma$ -convergence theorem for the natural scaling was proved.

## 2 Conventions and basic results

We will use the expression “a sequence  $\varepsilon \rightarrow 0$ ” meaning any sequence  $\varepsilon_j \rightarrow 0$  that will then be regarded as fixed, and subsequences will be taken from this fixed sequence.

We will use  $B_R^+(z_0)$  with  $z_0 = (x_0, y_0)$  to denote the half-ball  $\{z \in \mathbb{R}^2 : |z - z_0| < R, y > y_0\}$ , and abbreviate  $B_R^+ = B_R^+(0)$ . The symbol  $\Gamma_R$  will usually denote the flat part of  $\partial B_R^+$ .

We usually omit to explicitly mention the measure when writing integrals, unless there is possibility of confusion. Integrals over 2-dimensional sets like  $B_R^+$ ,  $\Omega$  etc. are thus implicitly meant to be w.r.t. 2-dimensional Lebesgue measure, while integrals over 1-dimensional sets such as  $\Gamma_R$ ,  $\partial\Omega$  or  $\partial B_R \cap \Omega$  are w.r.t. 1-dimensional Hausdorff measure  $\mathcal{H}^1$ .

For the convenience of the reader, we collect some results on existence and regularity results for minimizers and stationary point of (1.1) whose proofs are relatively straightforward.

**Proposition 2.1.** For all  $\varepsilon > 0$ , the functional  $E_\varepsilon$  attains its minimum.

**Proposition 2.2.** Stationary points of  $E_\varepsilon$  satisfy the equation

$$\int_{\Omega} \nabla u \cdot \nabla \varphi + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin(2(u - g)) \varphi = 0 \quad (2.1)$$

for all  $\varphi \in H^1(\Omega)$ . Any solution  $u$  of (2.1) is of class  $H^2(\Omega)$ , and is a strong solution of the equations

$$\Delta u = 0 \quad \text{in } \Omega \quad (2.2)$$

$$\frac{\partial u}{\partial \nu} = -\frac{1}{2\varepsilon} \sin 2(u - g) \quad \text{on } \partial\Omega. \quad (2.3)$$

If in addition  $\partial\Omega \in C^{k+1}$  and  $g \in C^k$  (i.e.  $e^{ig} \in C^k$ ), then  $u \in H^{k+1}(\Omega)$ . In particular, if  $\partial\Omega$  and  $g$  are  $C^\infty$ , then  $u \in C^\infty(\bar{\Omega})$ . If  $\partial\Omega$  and  $g$  are real analytic, then also  $u$  is real analytic up to the boundary.

PROOF. The  $H^k$  regularity can be proved by a difference quotient argument. The claim about the analyticity follows from [6].  $\square$

### 3 Localization of vortices

In this section we show that for sequences  $(u_\varepsilon)$  of stationary points of  $E_\varepsilon$  that satisfy an energy bound

$$E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}, \quad (3.1)$$

the *approximate vortex set*  $S_\varepsilon := \{z \in \partial\Omega : \sin^2(u(x) - g(x)) \geq \frac{1}{4}\}$  can be covered by a bounded number of  $\varepsilon$ -balls.

In order to see that the assumption (3.1) is reasonable, we show that it holds true for minimizers:

**Proposition 3.1.** There is a constant  $C_1 = C_1(\Omega, g)$  such that any sequence of minimizers  $(u_\varepsilon)$  of  $E_\varepsilon$  satisfies

$$E_\varepsilon(u_\varepsilon) \leq \pi D \log \frac{1}{\varepsilon} + C_1. \quad (3.2)$$

PROOF. It suffices to construct one sequence of functions  $(v_\varepsilon)$  satisfying this bound. To this end, choose  $2D$  distinct points  $a_1, \dots, a_{2D} \in \partial\Omega$  and let  $0 < R < \frac{1}{2} \min_{i \neq j} |a_i - a_j|$ . We construct the comparison function  $v_\varepsilon$  separately inside  $B_R(a_i) \cap \Omega$  and in the rest of the domain. Setting  $a_i = 0$  without loss of generality, we can assume  $R$  to be so small that  $\Omega \cap B_R = \{re^{i\vartheta} : \vartheta_1(r) < \vartheta < \vartheta_2(r), 0 < r < R\}$  with  $|\vartheta'_j| \leq c$  and so  $|\vartheta_2(r) - \vartheta_1(r) - \pi| \leq cr$ . With  $h_1(r) = g(e^{i\vartheta_1(r)}) + k\pi$  and  $h_2(r) = g(e^{i\vartheta_2(r)}) + (k-1)\pi$ ,  $k \in \mathbb{Z}$ , we define  $v_\varepsilon$  in  $\Omega \cap (B_R \setminus B_\varepsilon)$  as

$$v_\varepsilon(re^{i\vartheta}) = \frac{h_2(r) - h_1(r)}{\vartheta_2(r) - \vartheta_1(r)} (\vartheta - \vartheta_1(r)) + h_1(r).$$

Note that this function satisfies  $\sin^2(v_\varepsilon - g) = 0$  on  $B_R \cap \partial\Omega$ . Expressing the Dirichlet integral in polar coordinates, it is then easy to see that the part corresponding to the radial derivative is bounded independently of  $\varepsilon$ . The tangential derivative yields the term

$$\frac{1}{2} \int_\varepsilon^R \int_{\vartheta_1}^{\vartheta_2} \frac{1}{r^2} \left( \frac{h_2 - h_1}{\vartheta_2 - \vartheta_1} \right)^2 r dr d\vartheta = \frac{1}{2} \int_\varepsilon^R \frac{(h_2 - h_1)^2}{r(\vartheta_2 - \vartheta_1)} dr \leq \frac{1}{2} \int_\varepsilon^R \frac{(\pi + cr)^2}{\pi - cr} dr,$$

and this can be estimated by  $\frac{\pi}{2} \log \frac{R}{\varepsilon} + C$ . Inside  $B_\varepsilon \cap \Omega$ , we will have to violate the condition  $\sin^2(v_\varepsilon - g) = 0$  in order to obtain a function with bounded Dirichlet energy. By scaling, it is easy to see that a continuation of  $v_\varepsilon$  with uniformly bounded Dirichlet integral exists, and since  $\mathcal{H}^1(\partial\Omega \cap B_\varepsilon) \leq c\varepsilon$ , this shows  $E_\varepsilon(v_\varepsilon; B_R \cap \Omega) \leq \frac{\pi}{2} \log \frac{R}{\varepsilon} + c$ . Choosing the constants  $k = k_i$  near each  $a_i$  appropriately and using a harmonic continuation of  $v_\varepsilon|_{\partial B_R(a_i)}$  and  $g + k_i\pi$  to  $\Omega \setminus \bigcup B_R(a_i)$ , we finally can combine everything to a comparison function satisfying (3.2).  $\square$

As in the proofs for corresponding results in Ginzburg-Landau vortices [1], [9], a central point in obtaining estimates is a Rellich-Pohožaev identity. We state it in the following form:

**Lemma 3.2.** Assume that  $\Omega \subset \mathbb{R}^2$  is a Lipschitz domain,  $u \in H^2(\Omega)$  is harmonic, and  $w \in C^1(\overline{\Omega}, \mathbb{C})$  is holomorphic inside  $\Omega$ . Then

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} (w \cdot \nabla u) = \frac{1}{2} \int_{\partial\Omega} (w \cdot \nu) |\nabla u|^2, \quad (3.3)$$

where  $\nu$  denotes the outer normal to  $\partial\Omega$ .

PROOF. For any  $u \in H^2(\Omega)$ , it is easy to prove by direct calculation and using the Cauchy-Riemann equations for  $w$  that

$$\nabla u \cdot \nabla (w \cdot \nabla u) = \frac{1}{2} \operatorname{div}(w |\nabla u|^2).$$

Integrating by parts  $\int_\Omega \Delta u (w \cdot \nabla u) = 0$  and using the last identity, (3.3) now follows easily from the Gauß-Green theorem.  $\square$

We note the following important consequence of (3.3):

**Lemma 3.3.** Let  $\Omega$  be a *strongly star-shaped* Lipschitz domain, i.e. assume there exists a  $p \in \Omega$  and  $k > 0$  such that  $(z - p) \cdot \nu \geq k|z - p|$  for all  $z \in \partial\Omega$ . Assume  $u \in H^2(\Omega)$  is harmonic. Then there exist constants  $0 < c < C$  depending only on  $k$  such that

$$c \int_{\partial\Omega} \left| \frac{\partial u}{\partial \tau} \right|^2 \leq \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \leq C \int_{\partial\Omega} \left| \frac{\partial u}{\partial \tau} \right|^2, \quad (3.4)$$

where  $\frac{\partial u}{\partial \tau}$  denotes the tangential derivative.

PROOF. With  $p$  being a star point as above that we assume to be 0 without loss of generality, we use the Rellich-Pohožaev identity (3.3) with  $w(z) = z$ . This shows

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} z \cdot \nabla u = \frac{1}{2} \int_{\partial\Omega} (z \cdot \nu) |\nabla u|^2.$$

From the decomposition  $\nabla u = \frac{\partial u}{\partial \nu} \nu + \frac{\partial u}{\partial \tau} \tau$  we obtain

$$\int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 z \cdot \nu + \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial \tau} z \cdot \tau = \frac{1}{2} \int_{\partial\Omega} \left( \left| \frac{\partial u}{\partial \tau} \right|^2 + \left| \frac{\partial u}{\partial \nu} \right|^2 \right) z \cdot \nu$$

from which it follows that

$$\int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 z \cdot \nu = \int_{\partial\Omega} \left| \frac{\partial u}{\partial \tau} \right|^2 z \cdot \nu - 2 \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial \tau} \right| z \cdot \tau,$$

and now we can use the lower bound  $z \cdot \nu \geq k|z|$ ,  $|\tau| = 1$  and the inequality  $2AB \leq \alpha A^2 + \alpha^{-1} B^2$  to finish the proof.  $\square$

In the following we will derive estimates relating the penalty term and the following radial derivative of the energy:

**Definition 3.4.** For  $z_0 \in \partial\Omega$ ,  $\varepsilon > 0$  and  $u \in H^2(\Omega)$  define for all  $\rho > 0$

$$A(\rho) = A_{u, \varepsilon, z_0}(\rho) = \rho \int_{\partial B_\rho(z_0) \cap \Omega} |\nabla u|^2 d\mathcal{H}^1 + \frac{\rho}{\varepsilon} \int_{\partial B_\rho(z_0) \cap \partial\Omega} \sin^2(u - g) d\mathcal{H}^0. \quad (3.5)$$

**Proposition 3.5.** There exist  $\varepsilon_0 > 0$  and  $C_2 > 0$  depending only on  $\Omega$  and  $g$  such that for all  $\varepsilon < \varepsilon_0$ ,  $\rho < \varepsilon^{3/4}$ , any stationary point  $u$  of  $E_\varepsilon$ , and any  $z_0 \in \partial\Omega$ , the following inequality holds:

$$\frac{1}{2\varepsilon} \int_{\Gamma_\rho(z_0)} \sin^2(u - g) \leq A_{u, \varepsilon, z_0}(\rho) + C_2 \sqrt{\varepsilon}, \quad (3.6)$$

where  $\Gamma_\rho(z_0) = \partial\Omega \cap B_\rho(z_0)$ .

PROOF. We choose  $\varepsilon_0$  so small that for all  $\rho < \varepsilon_0^{3/4}$  and all  $z_0 \in \partial\Omega$ ,  $\omega_\rho(z_0) = \Omega \cap B_\rho(z_0)$  is strongly star-shaped in the sense of Lemma 3.3 with respect to some  $p_\rho \in \omega_\rho(z_0)$ , with a  $k > 0$  that can be chosen uniformly in  $\rho$  and  $z_0$ . In addition, we assume by using  $\partial\Omega \in C^2$  and choosing  $\varepsilon_0$  sufficiently small that there exists a vector field  $Z \in C^1(\overline{\Omega}, \mathbb{R}^2)$  with the property that for  $|z - z_0| < \varepsilon_0^{3/4}$ , there hold  $Z \cdot \nu = 0$  on  $\partial\Omega$  and the inequalities  $|Z - z| \leq C|z - z_0|^2$  and  $|\nabla Z - \text{id}| \leq C|z - z_0|$ . Setting  $z_0 = 0$  for convenience, we multiply  $\Delta u = 0$  with  $z \cdot \nabla u$  and obtain by integration by parts over  $\omega_\rho$  the relation

$$\int_{\omega_\rho} \nabla u \cdot \nabla(z \cdot \nabla u) = \int_{\partial\omega_\rho} \frac{\partial u}{\partial \nu} z \cdot \nabla u.$$

We use (3.3) and split  $z = Z + (z - Z)$  on  $\Gamma_\rho$ . This yields

$$\frac{1}{2} \int_{\partial\omega_\rho} (z \cdot \nu) |\nabla u|^2 = \rho \int_{\partial B_\rho \cap \Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 + \int_{\Gamma_\rho} \frac{\partial u}{\partial \nu} Z \cdot \nabla u + \int_{\Gamma_\rho} \frac{\partial u}{\partial \nu} (z - Z) \cdot \nabla u.$$

Noting that  $Z \cdot \nabla u = (Z \cdot \tau) \frac{\partial u}{\partial \tau}$ , where  $\tau$  is a tangent field to  $\partial\Omega$ , we can integrate the term involving  $Z$  by parts and obtain using (2.3)

$$\begin{aligned} \int_{\Gamma_\rho} \frac{\partial u}{\partial \nu} Z \cdot \nabla u &= -\frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin 2(u-g) \frac{\partial u}{\partial \tau} (Z \cdot \tau) \\ &= -\frac{1}{2\varepsilon} \int_{\partial\Omega \cap \partial B_\rho} \sin^2(u-g) (Z \cdot \tau) d\mathcal{H}^0 \\ &\quad + \frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin^2(u-g) \frac{\partial}{\partial \tau} (Z \cdot \tau) - \frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin 2(u-g) \frac{\partial g}{\partial \tau} (Z \cdot \tau). \end{aligned}$$

Combining this with the results above shows

$$\begin{aligned} \frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin^2(u-g) \frac{\partial}{\partial \tau} (Z \cdot \tau) &= \frac{1}{2\varepsilon} \int_{\partial\Omega \cap \partial B_\rho} \sin^2(u-g) Z \cdot \tau d\mathcal{H}^0 \\ &\quad + \frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin 2(u-g) Z \cdot \tau g' + \frac{1}{2} \int_{\partial\omega_\rho} z \cdot \nu |\nabla u|^2 + \int_{\partial\omega_\rho} \frac{\partial u}{\partial \nu} (Z-z) \cdot \nabla u. \end{aligned}$$

Using the assumptions on  $Z$  and a  $C^1$  bound on  $g$ , this shows using also  $|z \cdot \nu| \leq C\rho^2$  on  $\Gamma_\rho$  that

$$\begin{aligned} (1-C\rho) \frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin^2(u-g) &\leq (1+C\rho) \frac{1}{2\varepsilon} \int_{\partial\Omega \cap \partial B_\rho} \sin^2(u-g) d\mathcal{H}^0 \\ &\quad + \frac{1}{\varepsilon} c(g)\rho^2 + \left(\rho^2 + \frac{\rho}{2}\right) \int_{\partial B_\rho \cap \Omega} |\nabla u|^2 + C\rho^2 \int_{\Gamma_\rho} |\nabla u|^2. \end{aligned}$$

By the star-shapedness of  $\omega_\rho$  and Lemma 3.3 we have the estimate

$$\int_{\Gamma_\rho} |\nabla u|^2 \leq C \left( \int_{\Gamma_\rho} \left| \frac{\partial u}{\partial \nu} \right|^2 + \int_{\partial B_\rho \cap \Omega} |\nabla u|^2 \right)$$

and by the differential equation (2.3), we can estimate

$$\int_{\Gamma_\rho} \left| \frac{\partial u}{\partial \nu} \right|^2 = \frac{1}{4\varepsilon^2} \int_{\Gamma_\rho} 4 \sin^2(u-g) \cos^2(u-g) \leq \frac{2}{\varepsilon} \left( \frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin^2(u-g) \right).$$

Combining terms, we obtain

$$\begin{aligned} \left(1 - C\rho - \frac{C\rho^2}{\varepsilon}\right) \frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin^2(u-g) \\ \leq (1+C\rho) \frac{1}{2\varepsilon} \int_{\partial\Omega \cap \partial B_\rho} \sin^2(u-g) d\mathcal{H}^0 + \left(C\rho^2 + \frac{\rho}{2}\right) \int_{\partial B_\rho \cap \Omega} |\nabla u|^2 + \frac{C\rho^2}{\varepsilon}, \end{aligned}$$

and from this we can deduce the claim for  $\varepsilon < \varepsilon_0$  sufficiently small and  $\varepsilon < \rho < \varepsilon^{3/4}$ .  $\square$

This leads to the following criterion for vortex-free parts of the boundary:

**Proposition 3.6.** There are constants  $\gamma > 0$  and  $C_3 > 0$  depending on  $\Omega$  and  $g$  such that for every  $z_0 \in \partial\Omega$ ,  $\varepsilon < \varepsilon_0$  (with  $\varepsilon_0$  from Proposition 3.5),  $\rho < \varepsilon^{3/4}$ , and every stationary point  $u$  of  $E_\varepsilon$  satisfying  $A_{u,\varepsilon,z_0}(\rho) < \gamma$ , there holds

$$\sup_{\Gamma_{\rho/2}(z_0)} \sin^2(u-g) < \frac{1}{4} \quad (3.7)$$

and

$$\frac{1}{2\varepsilon} \int_{\Gamma_\rho(z_0)} \sin^2(u-g) \leq C_3. \quad (3.8)$$

PROOF. By Lemma 3.3, we can estimate

$$\int_{\Gamma_\rho} \left| \frac{\partial u}{\partial \tau} \right|^2 \leq C \int_{\partial \omega_\rho} \left| \frac{\partial u}{\partial \nu} \right|^2 \leq C \int_{\partial B_\rho \cap \Omega} |\nabla u|^2 + C \int_{\Gamma_\rho} \left| \frac{\partial u}{\partial \nu} \right|^2. \quad (3.9)$$

From (3.6), the definition of  $A$  in (3.5) and  $A(\rho) < \gamma$ , we thus can estimate, using (2.3) as in the last proof and Sobolev embedding in one dimension

$$[u]_{C^{0,1/2}(\Gamma_\rho)}^2 \leq C \int_{\Gamma_\rho} \left| \frac{\partial u}{\partial \tau} \right|^2 \leq C \left( \frac{1}{\rho} A(\rho) + \frac{1}{\varepsilon^2} \int_{\Gamma_\rho} \sin^2(u - g) \right) \leq \frac{C}{\varepsilon} (2\gamma + C_2 \sqrt{\varepsilon_0}).$$

Assume now that  $\sin^2(u(z) - g(z)) \geq \frac{1}{4}$  for some  $z \in \Gamma_{\rho/2}$ . Then by the last equation and the differentiability of  $g$ , there holds  $\sin^2(u(z') - g(z')) \geq \frac{1}{8}$  at least for  $|z - z'| \leq \frac{\varepsilon}{C(\gamma + \sqrt{\varepsilon_0})}$ , where the latter term is  $\geq \frac{\varepsilon}{2}$  if we choose  $\varepsilon_0$  and  $\gamma$  sufficiently small. We estimate  $\int_{\Gamma_\rho} \sin^2(u - g)$  from below:

$$\frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin^2(u - g) \geq \frac{1}{2\varepsilon} \frac{1}{8} \frac{\varepsilon}{2} \geq \frac{1}{32}.$$

On the other hand, we have by Proposition 3.5 the upper bound  $\gamma + C_2 \sqrt{\varepsilon_0}$ , and now choosing  $\gamma$  and  $\varepsilon_0$  sufficiently small leads to a contradiction.  $\square$

**Lemma 3.7.** Let  $(u_\varepsilon)$  be a sequence of stationary points of  $E_\varepsilon$  satisfying the logarithmic energy bound  $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ . Then for any  $z_0 \in \partial\Omega$ , the function  $A(\rho) = A_{u_\varepsilon, \varepsilon, z_0}(\rho)$  defined as in (3.5) satisfies

$$\inf_{\varepsilon^{6/7} \leq \rho \leq \varepsilon^{5/6}} A(\rho) \leq \frac{84}{\log \frac{1}{\varepsilon}} E_\varepsilon(u_\varepsilon; \Omega \cap B_{\varepsilon^{5/6}}(z_0)) \leq 84M \quad (3.10)$$

and

$$\inf_{5\varepsilon^{5/6} \leq \rho \leq 5\varepsilon^{4/5}} A(\rho) \leq 60M. \quad (3.11)$$

PROOF. The first claim follows from the calculation

$$M \log \frac{1}{\varepsilon} \geq E_\varepsilon(u_\varepsilon; \Omega \cap B_{\varepsilon^{5/6}}) \geq \frac{1}{2} \int_{\varepsilon^{6/7}}^{\varepsilon^{5/6}} \frac{A(\rho)}{\rho} d\rho \geq \frac{1}{2} (\inf A) \log \frac{\varepsilon^{5/6}}{\varepsilon^{6/7}} = \frac{\inf A}{84} \log \frac{1}{\varepsilon}.$$

The inequality (3.11) follows in a similar manner.  $\square$

**Lemma 3.8 ( $\eta$ -compactness).** There exist constants  $\eta_0, \varepsilon_0 > 0$  such that for any  $\varepsilon < \varepsilon_0$  and  $\rho < \varepsilon^{3/4}$ , and every stationary point  $u$  of  $E_\varepsilon$  satisfying for some  $z_0 \in \partial\Omega$  the inequality

$$E_\varepsilon(u; B_\rho(z_0) \cap \Omega) \leq \eta \log \frac{\rho}{\varepsilon},$$

there holds  $\sin^2(u - g) < \frac{1}{4}$  on  $B_{\rho/2}(z_0) \cap \partial\Omega$ .

PROOF. By virtually the same argument as above, we obtain around any  $z \in B_{\rho/2}(z_0) \cap \partial\Omega$  that

$$\eta \log \frac{\rho}{\varepsilon} \geq \frac{1}{2} \int_{\varepsilon/2}^{\varepsilon^{3/4}/2} \frac{A(r)}{r} dr \geq \frac{1}{8} (\inf A) \log \frac{1}{\varepsilon},$$

hence

$$\inf_{\varepsilon/2 < \sigma < \varepsilon^{3/4}/2} A(\sigma) \leq 8 \frac{\log \frac{\rho}{\varepsilon}}{\log \frac{1}{\varepsilon}} < 8\eta_0.$$

We can now choose  $\eta_0$  sufficiently small so that Proposition 3.6 implies the claim.  $\square$

**Proposition 3.9.** There is a constant  $N = N(g, \Omega, M)$  such that for any sequence of stationary points  $u_\varepsilon$  satisfying the energy bound  $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ , the approximate vortex set  $S_\varepsilon$  can be covered by at most  $N$  balls of radius  $\varepsilon$ , such that the  $\varepsilon/5$  balls around the same centers are disjoint.



PROOF. For  $z \in S_\varepsilon$ , we choose by virtue of Proposition 3.6 and Lemma 3.7 a radius  $\rho \in [\varepsilon^{6/7}, \varepsilon^{5/6}]$  such that

$$\frac{84}{\log \frac{1}{\varepsilon}} E_\varepsilon(u_\varepsilon; \Omega \cap B_{\varepsilon^{4/5}}(z)) \geq A_{u_\varepsilon, \varepsilon, z}(\rho) \geq \gamma. \quad (3.12)$$

Choose by Vitali's  $5r$  covering lemma  $z_j = z_j^\varepsilon \in S_\varepsilon$ ,  $j \in J_\varepsilon$ , such that  $S_\varepsilon \subset \bigcup_{j \in J_\varepsilon} B_{5\varepsilon^{4/5}}(z_j)$ , and such that the  $B_{\varepsilon^{4/5}}(z_j)$  are disjoint. Then (3.12) shows that

$$|J_\varepsilon| \leq \frac{84M}{\gamma}. \quad (3.13)$$

We now choose radii  $\rho_j \in [5\varepsilon^{5/6}, 5\varepsilon^{4/5}]$  such that  $A_{u_\varepsilon, \varepsilon, z_j}(\rho_j) \leq 60M$ . Using Proposition 3.6 we obtain

$$\frac{1}{2\varepsilon} \int_{\partial\Omega \cap B_{\rho_j}(z_j)} \sin^2(u_\varepsilon - g) \leq C,$$

and now by the same argument as in the proof of Proposition 3.6, we see

$$[u_\varepsilon]_{C^{0,1/2}(\partial\Omega \cap B_{\rho_j}(z_j))} \leq \frac{C}{\sqrt{\varepsilon}},$$

and this again implies

$$\frac{1}{2\varepsilon} \int_{\partial\Omega \cap B_{\varepsilon/5}(z_j)} \sin^2(u_\varepsilon - g) \geq c > 0. \quad (3.14)$$

Using once more the  $5r$  lemma, we can choose  $z_k = z_k^\varepsilon$ ,  $k \in K_\varepsilon$  such that  $B_{\varepsilon/5}(z_k)$  are disjoint and  $B_\varepsilon(z_k)$  cover  $S_\varepsilon$ . By (3.13) and (3.14) we now have

$$c|K_\varepsilon| \leq \sum_{k \in K_\varepsilon} \frac{1}{\varepsilon} \int_{\partial\Omega \cap B_{\varepsilon/5}(z_j)} \sin^2(u_\varepsilon - g) \leq \sum_{j \in J_\varepsilon} \frac{1}{\varepsilon} \int_{\partial\Omega \cap B_{\rho_j}(z_j)} \leq \frac{84CM}{\gamma},$$

which implies the claim.  $\square$

For comparison arguments we shall need the following lower bound for the energy on half-annuli:

**Proposition 3.10.** Let  $0 < \rho < R \leq R_0$ ,  $R_0$  sufficiently small,  $z_0 \in \partial\Omega$ , w.l.o.g.  $z_0 = 0$ . Assume  $D_{R,\rho} = (B_R \setminus \overline{B_\rho}) \cap \Omega = \{re^{i\vartheta} : \vartheta_1(r) < \vartheta < \vartheta_2(r), \rho < r < R\}$  with  $|\vartheta_2(r) - \vartheta_1(r) - \pi| \leq Cr$ . Assume also that for  $j = 1, 2$  there holds  $(u - g)(re^{i\vartheta_j(r)}) \in (k_j\pi - \delta, k_j\pi + \delta)$  for some  $k_j \in \mathbb{Z}$  and  $\delta \in (0, \frac{\pi}{2})$ . Then there is a constant depending on  $R_0$  (which in turn depends on  $\Omega$ ) and  $g$  so that for any such function  $u$ , its energy is bounded below as

$$E_\varepsilon(u; D_{R,\rho}) \geq \frac{\pi}{2}(k_2 - k_1)^2 \log \frac{R}{\rho} - C(k_2 - k_1)^2(R + \frac{\varepsilon}{\rho}). \quad (3.15)$$

PROOF. We will use the abbreviations  $u_j(r) = u(re^{i\vartheta_j(r)})$  (and similar for  $g$ ) for the functions on the two boundary components. We also assume w.l.o.g.  $k_1 = k$  and  $k_2 = 0$ . Using polar coordinates, disregarding the radial derivative and by Hölder's inequality, we calculate

$$\begin{aligned} \int_{D_{R,\rho}} |\nabla u|^2 &\geq \int_\rho^R \frac{1}{r} \int_{\vartheta_1}^{\vartheta_2} \left| \frac{\partial u}{\partial \vartheta} \right|^2 d\vartheta dr \\ &\geq \int_\rho^R \frac{1}{\vartheta_2 - \vartheta_1} \left( \int_{\vartheta_1}^{\vartheta_2} \left| \frac{\partial u}{\partial \vartheta} \right| \right)^2 \geq \int_\rho^R \frac{(u_1 - u_2)^2}{r(\pi + cr)} dr. \end{aligned}$$

We rewrite  $u_1 - u_2 = k\pi - (u_1 - g_1 - k\pi) - (u_2 - g_2) - (g_1 - g_2)$ . Using the lower bound  $\sin^2(t - k_i\pi) \geq \sigma t^2$  valid for  $|t| < \delta$  with some  $\sigma = \sigma(\delta)$ , we can thus estimate

$$\begin{aligned} E_\varepsilon(u; D_{R,\rho}) &\geq \frac{1}{2} \int_\rho^R \frac{1}{r(\pi + cr)} (k\pi - (u_1 - g_1 - k\pi) - (u_2 - g_2) - (g_1 - g_2))^2 \\ &\quad + \frac{\sigma}{\varepsilon} ((u_1 - g_1 - k\pi)^2 + (u_2 - g_2)^2) dr. \end{aligned}$$

On the last term, we use the inequality  $a^2 + b^2 \geq \frac{1}{2}(a+b)^2$  with  $a = u_1 - g_1 - k\pi$  and  $b = u_2 - g_2$ . Then we use the inequality  $\alpha(A-B)^2 + \beta B^2 \geq \frac{1}{\alpha+\beta}A^2$  that can be obtained by optimizing over  $B$  on  $A = (k\pi - (g_1 - g_2))$  and  $B = (u_1 - g_1 - k\pi) + (u_2 - g_2)$ . This yields using also a  $C^1$  bound on  $g$

$$E_\varepsilon(u; D_{R,\rho}) \geq \frac{1}{2} \int_\rho^R \frac{(k\pi - cr)^2}{r(\pi + cr) + \frac{4\varepsilon}{\sigma}} dr.$$

After subtraction of  $\frac{k^2\pi}{2r}$ , the integral of the difference can then be estimated by

$$-C|k|(R - \rho) - Ck^2\varepsilon \left( \frac{1}{\rho} - \frac{1}{R} \right)$$

which implies the claim.  $\square$

## 4 Convergence results by comparison arguments

In this section, we assume  $u_\varepsilon$  to be stationary points of  $E_\varepsilon$  satisfying an upper bound

$$E_\varepsilon(u_\varepsilon) \leq \pi D \log \frac{1}{\varepsilon} + C_0 \quad (4.1)$$

for some constant  $C_0$ , where  $D$  is the degree of  $e^{ig}$ . This bound holds true for minimizers by Proposition 3.1. We will use the following notation. By Proposition 3.9, there exist  $a_j^\varepsilon \in \partial\Omega$ ,  $1 \leq j \leq N_\varepsilon \leq N$  such that the approximate vortex set  $S_\varepsilon$  satisfies  $S_\varepsilon \subset \bigcup_{1 \leq j \leq N_\varepsilon} B_\varepsilon(a_j^\varepsilon)$ . Passing to a subsequence of  $\varepsilon \rightarrow 0$ , we can assume that  $N_\varepsilon = N_0$  is constant and  $a_j^\varepsilon \rightarrow a_j^0$  as  $\varepsilon \rightarrow 0$ . Note that the  $a_j^0$  need not be distinct. We define for  $0 < \sigma < \frac{1}{2} \min_{a_j^0 \neq a_{j'}^0} \text{dist}(a_j^0, a_{j'}^0)$  the sets  $\Omega_\sigma^\varepsilon = \Omega \setminus \bigcup_j B_\sigma(a_j^\varepsilon)$  and  $\Omega_\sigma^0 = \Omega \setminus \bigcup_j B_\sigma(a_j^0)$ . With this setup (and this subsequence) we have the following bounds:

**Proposition 4.1.** There is a constant  $C = C(g, \Omega, C_0)$  such that  $E_\varepsilon(u_\varepsilon; \Omega_\sigma^\varepsilon) \leq \pi D \log \frac{1}{\sigma} + C$ .

**PROOF.** We follow closely the proof of Proposition 3.3 in [9]. We write  $x_j$  for  $a_j^\varepsilon$  and set  $\mathcal{N} = \{1, \dots, N_0\}$ . Let  $\mathcal{R}_\varepsilon^\sigma$  denote the set of radii in  $[\varepsilon, \sigma]$  such that  $\partial B_R(x_j) \cap B_\varepsilon(x_\ell) = \emptyset$  for  $j \neq \ell$  and such that there exists for  $R \in \mathcal{R}_\varepsilon^\sigma$  a  $\mathcal{N}_R \subset \mathcal{N}$  with the properties that  $(B_R(x_j))_{j \in \mathcal{N}_R}$  is disjoint,  $\mathcal{N}_R \subset \mathcal{N}_{R'}$  for  $R' \leq R$  and  $\bigcup_{j \in \mathcal{N}} B_\varepsilon(x_j) \subset \bigcup_{j \in \mathcal{N}_R} B_R(x_j)$ . It is possible to show that  $\mathcal{R}_\varepsilon^\sigma = \bigcup_{m=1}^M [\alpha_m, \beta_m]$ , where for  $R = \alpha_m$ , there exists  $\ell \notin \mathcal{N}_R$  with  $\overline{B_\varepsilon(x_\ell)} \setminus \bigcup_{j \in \mathcal{N}_R} B_R(x_j) \neq \emptyset$ , and for  $R = \beta_m$ , there exist  $j \neq \ell \in \mathcal{N}_R$  with  $\partial B_R(x_j) \cap \overline{B_R(x_\ell)} \neq \emptyset$ . Then  $\mathcal{N}_R = \mathcal{N}^m$  is constant for  $R \in [\alpha_m, \beta_m]$  and  $\mathcal{N}^{m+1} \subsetneq \mathcal{N}^m$  so  $M \leq N_0$ . In addition, there exists a constant  $K = K(N_0)$  such that  $\alpha_1 \leq K\varepsilon$ ,  $\beta_M \geq \frac{\sigma}{K}$  and  $\alpha_{m+1} \leq K\beta_m$ , since never more than  $N$  balls can touch.

On the half-annuli  $D_{\beta_m, \alpha_m}(x_j)$  for  $j \in \mathcal{N}^m$ , we apply Proposition 3.10 with a jump height  $\varkappa_{m,j}$  that satisfies  $\sum_m \sum_{j \in \mathcal{N}^m} \varkappa_{m,j}^2 \geq \left| \sum_m \sum_{j \in \mathcal{N}^m} \varkappa_{m,j} \right| = 2D$ . This leads to the estimate

$$\begin{aligned} E_\varepsilon(u_\varepsilon; \Omega_\sigma^\varepsilon) &\leq E_\varepsilon(u; \Omega) - \sum_{m=1}^M \sum_{j \in \mathcal{N}^m} E_\varepsilon(u; D_{\beta_m, \alpha_m}(x_j)) \\ &\leq \pi D \log \frac{1}{\varepsilon} + C_0 - \sum_m \sum_j \frac{\pi}{2} \varkappa_{m,j}^2 \left( \log \frac{\beta_m}{\alpha_m} - C \right) \\ &\leq \pi D \log \frac{1}{\varepsilon} + C - \pi D \sum_m (\log \beta_m - \log \alpha_m) \\ &\leq \pi D \log \frac{1}{\sigma} + C. \end{aligned}$$

$\square$

**Theorem 4.2.** *Let  $(u_\varepsilon)$  be a sequence of critical points satisfying the energy bound  $E_\varepsilon(u_\varepsilon) \leq \pi D \log \frac{1}{\varepsilon} + C_0$ . Then there is a subsequence and  $N = 2D$  points  $a_1, \dots, a_N \in \partial\Omega$  such that*

$$\int_{\Omega'} |\nabla u_\varepsilon|^2 \leq M(\Omega') < \infty \quad (4.2)$$

for all open  $\Omega'$  with  $\overline{\Omega'} \subset \overline{\Omega} \setminus \{a_1, \dots, a_N\}$ . Additionally, there hold the bounds

$$\int_{\Omega} |\nabla u_\varepsilon|^p \leq C(p) \quad (4.3)$$

uniformly in  $\varepsilon$  for all  $1 \leq p < 2$ . In particular, after adding a suitable  $z_\varepsilon \in 2\pi\mathbb{Z}$ , a subsequence of  $(u_\varepsilon)$  converges weakly in  $H_{\text{loc}}^1$  and  $W^{1,p}$ ,  $p < 2$ , to a harmonic function  $u_*$ . The limit has the properties that  $(u_* - g)$  is piecewise constant on  $\partial\Omega \setminus \{a_1, \dots, a_N\}$ , with values in  $\pi\mathbb{Z}$ , and jumps by  $-\pi$  at the points  $a_j$ .

PROOF. We use the setup described at the beginning of this section. In particular, we use the points  $a_j^0$  as defined there. Note that for  $\varepsilon < \varepsilon_0(\sigma)$ , there holds  $\Omega_\sigma^0 \subset \Omega_{\sigma/2}^\varepsilon$  and so by Proposition 4.1,

$$\int_{\Omega_\sigma^0} |\nabla u_\varepsilon|^2 \leq 2E_\varepsilon(u_\varepsilon; \Omega_{\sigma/2}^\varepsilon) \leq 2\pi D \log \frac{2}{\sigma} + C, \quad (4.4)$$

which proves (4.2). To obtain the  $L^p$  bounds (4.3), fix a  $\sigma > 0$  and  $1 \leq p < 2$ . Then by Hölder's inequality and Proposition 4.1

$$\begin{aligned} \int_{\Omega} |\nabla u_\varepsilon|^p &\leq \int_{\Omega_\sigma^\varepsilon} |\nabla u_\varepsilon|^p + \sum_{\ell=1}^{\infty} \int_{\Omega_{2^{-\ell}\sigma}^\varepsilon \setminus \Omega_{2^{-\ell+1}\sigma}^\varepsilon} |\nabla u_\varepsilon|^p \\ &\leq C + \sum_{\ell=1}^{\infty} |\Omega_{2^{-\ell}\sigma}^\varepsilon \setminus \Omega_{2^{-\ell+1}\sigma}^\varepsilon|^{1-p/2} \left( \int_{\Omega_{2^{-\ell}\sigma}^\varepsilon} |\nabla u_\varepsilon|^2 \right)^{p/2} \\ &\leq C + c \sum_{\ell=1}^{\infty} 2^{-(1-p/2)\ell} \left( 2\pi D \log \frac{1}{2^\ell \sigma} + C \right)^{p/2} \\ &\leq C, \end{aligned}$$

since the sum converges by the root test. From this  $L^p$  gradient bound, we obtain the weak compactness up to translation by Poincaré's inequality. The weak limit  $u_*$  is harmonic since  $\int_{\Omega} \nabla u_* \cdot \nabla \varphi = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla u_\varepsilon \cdot \nabla \varphi = 0$  for all  $\varphi \in C_c^\infty(\Omega)$ . That the boundary values satisfy  $u_* - g \in \pi\mathbb{Z}$  with possible jumps at the  $a_i$  follows from  $\int_{\partial\Omega} \sin^2(u_\varepsilon - g) \rightarrow 0$  and  $(u_\varepsilon - g)$  being close to  $\pi\mathbb{Z}$  outside the approximate vortex set  $S_\varepsilon$ .

We still have to prove that  $N = 2D$  and that there is a jump by  $-\pi$  at each of the points  $a_j$ . To see this, we note that we can localize parts of the proof of Proposition 4.1 around  $a_j^\varepsilon$  to obtain

$$E_\varepsilon(u_\varepsilon; \Omega \cap B_\eta(a_j^\varepsilon)) \geq \frac{\pi}{2} \sum_m \varkappa_{m,j}^2 \log \frac{1}{\varepsilon} - C(\eta)$$

The jump of  $u_*$  at  $a_j^0$  is  $-\pi d_j$ , where  $d_j = \sum_m \varkappa_{m,j}$  so  $\sum_m \varkappa_{m,j}^2 \geq |d_j|$ . The upper bound on the energy now implies

$$\sum_j |d_j| \leq 2D + \frac{C(\eta)}{\log \frac{1}{\varepsilon}}.$$

Letting  $\varepsilon \rightarrow 0$  we obtain  $\sum_j |d_j| \leq 2D = \sum_j d_j$ , which proves  $d_j \geq 0$ . Since by the lower bound argument, the energy around those  $a_j^0$  with  $d_j > 0$  already suffices to make up for the singular part of the energy, we can use the  $\eta$ -compactness lemma 3.8 to see that  $d_j = 0$  is impossible.

To finish the proof, we need to show  $d_j = 1$ . To this end, we compare the energy of  $u_\varepsilon$  to that of  $u_*$ . Letting  $\varepsilon \rightarrow 0$  in (4.4) and using the weak lower semicontinuity of the Dirichlet integral, we have

$$\int_{\Omega_\sigma^0} |\nabla u_*|^2 \leq 2\pi D \log \frac{1}{\sigma} + C.$$

On the other hand, Proposition 7.1 shows that for  $\sigma$  sufficiently small,

$$\int_{\Omega_\sigma^0} |\nabla u_*|^2 \geq \pi \sum_j d_j^2 \log \frac{1}{\sigma} - C.$$

Combining these estimates shows  $\sum_j (d_j^2 - d_j) \leq 0$ . Since  $d_j \neq 0$ , it follows that  $d_j = 1$  for all  $j$ .  $\square$

## 5 Convergence results by PDE arguments

The  $W^{1,p}$  convergence results of the previous section also hold for general stationary points where upper and lower energy bounds do not match up to a constant as those for minimizers do. Away from the vortices, there also holds convergence in higher norms.

**Proposition 5.1.** There is a constant  $C > 0$  such that for every sequence of stationary points  $u_\varepsilon$  satisfying the energy bound  $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ , there holds

$$\limsup_{\varepsilon \rightarrow 0} \operatorname{osc}_{\overline{\Omega}} u_\varepsilon \leq C$$

In particular, by adding a suitable sequence  $z_\varepsilon \in 2\pi\mathbb{Z}$ , the  $u_\varepsilon$  themselves can be assumed to be uniformly bounded in  $L^\infty$ .

PROOF. From Proposition 3.9 we know that there exist a bounded number of points  $a_i^\varepsilon \in \partial\Omega$  such that  $|\sin(u_\varepsilon - g)| < \frac{1}{4}$  outside  $\bigcup_i B_\varepsilon(a_i^\varepsilon)$ , so the oscillation there is bounded. Inside  $B_\varepsilon(a_i^\varepsilon) \cap \partial\Omega$ , the oscillation is bounded since there we have  $[u_\varepsilon]_{C^{0,1/2}} \leq \frac{C}{\sqrt{\varepsilon}}$  as follows from the proof of Proposition 3.9. By the maximum principle, the bounds extend to  $\overline{\Omega}$ .  $\square$

**Proposition 5.2.** Let  $u = u_\varepsilon$  be a stationary point of  $E_\varepsilon$  and let  $z_0 \in \partial\Omega$ , w.l.o.g.  $z_0 = 0$ . Let  $R > 0$  be such that  $B_R \cap S_\varepsilon = \emptyset$ , where  $S_\varepsilon$  is the approximate vortex set. Let  $G \in H^1(B_R)$  be a function with  $G|_{\partial\Omega \cap B_R(z)} = g$ , and let  $k \in \mathbb{Z}$  such that  $|u - g - k\pi| \leq \arcsin \frac{1}{2}$ . Then for any  $\vartheta < 1$  there holds

$$\int_{B_{\vartheta R} \cap \Omega} |\nabla u|^2 + \frac{1}{\varepsilon} \int_{\partial\Omega \cap B_{\vartheta R}} (u - g - k\pi)^2 \leq C \quad (5.1)$$

PROOF. We test the equation (2.1) with  $\eta^2(u - G - k\pi)$ . This yields

$$\begin{aligned} 0 &= \int_{\Omega} \eta^2 |\nabla u|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega} \eta^2 \sin 2(u - g - k\pi)(u - g - k\pi) \\ &\quad + 2 \int_{\Omega} \eta(u - G - k\pi) \nabla u \cdot \nabla \eta + \int_{\Omega} \eta^2 \nabla u \cdot \nabla G. \end{aligned}$$

By the monotonicity  $\sin 2(u - g - k\pi)(u - g - k\pi) \geq c(u - g - k\pi)^2$  that holds true by choice of  $k$  since  $S_\varepsilon \cap B_R = \emptyset$  and by aid of Young's inequality, we obtain

$$\int_{\Omega} \eta^2 |\nabla u|^2 + \frac{c}{\varepsilon} \int_{\partial\Omega} |u - g - k\pi|^2 \leq C \int_{\Omega} |\nabla \eta|^2 (u - G - k\pi)^2 + \eta^2 |\nabla G|^2.$$

Choosing a standard cut-off function  $\eta$  satisfying  $\eta = 1$  on  $B_{\vartheta R}$  and  $\eta = 0$  outside  $B_R$  with  $|\nabla \eta| \leq \frac{C}{R(1-\vartheta)}$ , we obtain the result.  $\square$

**Proposition 5.3.** Let  $u_\varepsilon$  be stationary points of  $E_\varepsilon$  satisfying  $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ . Assume (by aid of Proposition 3.9) that the approximate vortex set  $S_\varepsilon$  is covered by  $\bigcup_{j=1}^N B_\varepsilon(a_j^\varepsilon)$ . Then for any  $\sigma > 0$ , the energy of  $u_\varepsilon$  on  $\Omega_\sigma^\varepsilon = \Omega \setminus \bigcup_j B_\sigma(a_j^\varepsilon)$  can be estimated as

$$E_\varepsilon(u_\varepsilon; \Omega_\sigma^\varepsilon) \leq C \log \frac{1}{\sigma}. \quad (5.2)$$

PROOF. This follows from Proposition 5.2 since the part of  $\Omega_\sigma^\varepsilon$  near the boundary can always be covered by a logarithmical number of balls, see Figure 1. In the remaining sector, classical interior gradient bounds for harmonic functions also show logarithmic bounds. If many vortices are close together, we can combine the bounds obtained near each vortex similar to the argument in the proof of Proposition 4.1.  $\square$

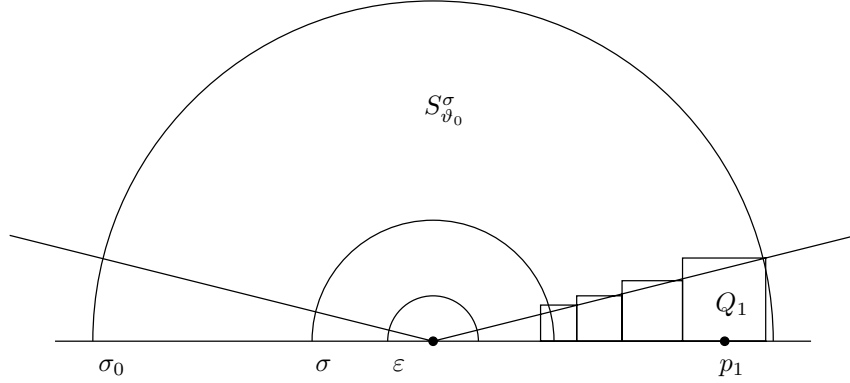


Figure 1: Construction for the proof of Proposition 5.3: The set  $B_{\sigma_0}^+ \setminus B_{\sigma}^+$  is covered by the circular sector  $S_{\vartheta_0}^{\sigma}$  and some squares  $Q_i$  in geometrical progression that can thus be covered by  $C(\vartheta_0) \log \frac{\sigma_0}{\sigma}$  half-balls not touching  $B_{\varepsilon}^+$ .

**Theorem 5.4.** *There is for  $1 \leq p < 2$  a constant  $C = C(g, p, M, \Omega)$  such that for every sequence  $u_{\varepsilon}$  of stationary points of  $E_{\varepsilon}$  satisfying  $E_{\varepsilon}(u_{\varepsilon}) \leq M \log \frac{1}{\varepsilon}$ , there holds*

$$\int_{\Omega} |\nabla u_{\varepsilon}|^p \leq C. \quad (5.3)$$

PROOF. This follows exactly as in the proof of Theorem 4.2 from the estimate (5.2).  $\square$

**Remark 5.5.** Theorem 5.4 can fail without the a priori bound  $E_{\varepsilon}(u_{\varepsilon}) \leq M \log \frac{1}{\varepsilon}$  that provides an a priori bound on the number of vortices. Counterexamples (with  $\Omega = B_1(0)$  and  $g = 0$ ) can be constructed by conformally mapping the periodic half-space solutions given by Toland [11] to the unit disk, see [5, Section 5.5] for some more explicit calculations. This is in contrast to Ginzburg-Landau theory, where the a priori bound always holds in the case of a starshaped domain [1, Chapter X].

To obtain higher bounds, we will flatten the boundary and use a harmonic extension of the forcing function  $g$ . By changing variables we obtain

**Proposition 5.6.** Let  $z_0 \in \partial\Omega$ , w.l.o.g.  $z_0 = 0$  and  $\partial\Omega$  has horizontal tangent at 0. Then for  $\rho > 0$  sufficiently small, the part of  $\Omega$  near  $z_0$  can be written as a graph of a  $C^1$  function  $\gamma$  over its tangent plane, so  $\Psi : \overline{B_{\rho}^+} \rightarrow \overline{\Omega}$ ,  $\Psi(x, y) = (x, y + \gamma(x))$  is a diffeomorphism of  $\overline{B_{\rho}^+}$  onto a (closed) relative neighborhood of  $z_0$  in  $\overline{\Omega}$ .

Let  $u_{\varepsilon}$  be a stationary point of  $E_{\varepsilon}$  and  $G$  a harmonic extension of  $g$  to  $\Psi(B_{\rho}^+)$  with bounded Dirichlet integral. Then the function  $w_{\varepsilon} = (u_{\varepsilon} - G) \circ \Psi$  solves the PDE

$$\int_{B_{\rho}^+} a_{ij} \partial_i w_{\varepsilon} \partial_j \varphi + \int_{\Gamma_{\rho}} \left( \frac{1}{2\varepsilon} \sin 2w_{\varepsilon} + h \right) b \varphi = 0 \quad (5.4)$$

for all  $\varphi \in H^1(B_{\rho}^+)$  that vanish near  $\partial B_{\rho}$ , where  $(a_{ij}) = \begin{pmatrix} 1 & -\gamma' \\ -\gamma' & 1+\gamma'^2 \end{pmatrix}$ ,  $b = \sqrt{1+\gamma'^2}$  and  $h = \frac{\partial G}{\partial \nu} \circ \Psi^{-1}$ .

**Proposition 5.7.** Let  $w = w_{\varepsilon}$  be a solution of (5.4) and  $R > 0$  such that  $|\sin 2w| < \frac{1}{4}$  on  $\Gamma_R$ . Then for any  $\vartheta < 1$

$$\int_{B_{\vartheta R}^+} |\nabla^2 w|^2 + \frac{1}{\varepsilon} \int_{\Gamma_{\vartheta R}} \left| \frac{\partial w}{\partial \tau} \right|^2 \leq C(\vartheta). \quad (5.5)$$

PROOF. We differentiate (5.4) in  $x = x_1$ -direction and test with  $\eta^2 \partial_1 w$  (or equivalently, test (5.4) with  $-\partial_1(\eta^2 \partial_1 w)$ ), where  $\eta$  denotes the usual cut-off function that is 0 outside  $B_R^+$ , 1 inside  $B_{\vartheta R}^+$ , and satisfies  $|\nabla \eta| \leq \frac{C}{R(1-\vartheta)}$ . The claim follows similar to that of Proposition 5.2, using that  $\cos 2w \geq c_0 > 0$ .  $\square$

**Proposition 5.8.** Let  $u_\varepsilon$  be stationary points of  $E_\varepsilon$  satisfying  $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ . Assume the approximate vortex set  $S_\varepsilon$  is covered by  $B_\varepsilon(a_j^\varepsilon)$ , with  $a_j^\varepsilon \rightarrow a_j^0$  as  $\varepsilon \rightarrow 0$ . Then on  $\Omega_\sigma = \Omega \setminus \bigcup B_\sigma(a_j^0)$  there holds the estimate

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega_\sigma} |\nabla^2 u_\varepsilon|^2 \leq C(\sigma) \quad (5.6)$$

as well as

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\partial\Omega \cap \partial\Omega_\sigma} \sin^2(u_\varepsilon - g) = 0. \quad (5.7)$$

PROOF. The first claim follows from (5.5) and a covering argument. For the second, we observe that the  $H^2$  bound implies weak  $H^2$  convergence  $u_\varepsilon \rightarrow u_*$  and thus also  $\frac{\partial u_\varepsilon}{\partial \nu} \rightarrow \frac{\partial u_*}{\partial \nu}$  in  $L^2$ . Now we have (with  $\Gamma = \partial\Omega \cap \partial\Omega_\sigma$ )

$$\frac{1}{\varepsilon} \int_\Gamma \sin^2(u_\varepsilon - g) \leq \frac{C}{\varepsilon} \int_\Gamma \sin^2(u_\varepsilon - g) \cos^2(u_\varepsilon - g) = C\varepsilon \int_\Gamma \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2,$$

which tends to 0 by the convergence of  $\frac{\partial u_\varepsilon}{\partial \nu}$ .  $\square$

## 6 The half-space solutions

Blow-up of the solutions of (2.2)–(2.3) at scale  $\varepsilon$  will lead to a half-space problem. The resulting equation is the Peierls-Nabarro equation known from the theory of crystal dislocations, and its solutions have been classified by Toland [11]. We will use the following essential uniqueness result:

**Theorem 6.1 (Toland [11]).** *Let  $u$  be a bounded solution of*

$$\Delta u = 0 \quad \text{in } \mathbb{R}_+^2 \quad (6.1)$$

$$\frac{\partial u}{\partial \nu} = -\frac{1}{2} \sin 2u \quad \text{on } \mathbb{R}. \quad (6.2)$$

*Then either  $u$  is periodic or constant, or there exist  $n \in \mathbb{Z}$ ,  $a \in \mathbb{R}$ , and a sign such that*

$$u(x, y) = \pm \arctan \frac{x+a}{y+1} + \pi n + \frac{\pi}{2}, \quad (6.3)$$

**Proposition 6.2.** Assume  $u_\varepsilon$  are stationary points of  $E_\varepsilon$  satisfying  $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ , let  $z_0 \in \partial\Omega$  and define  $w_\varepsilon := (u_\varepsilon - G) \circ \Psi$  as above. Then the functions  $V_\varepsilon(z) = w_\varepsilon(\varepsilon z)$  converge weakly in  $H_{\text{loc}}^1(\mathbb{R}_+^2)$  to a nonperiodic solution of (6.1)–(6.2).

PROOF. As in the proof of Proposition 5.3, the energy of  $w_\varepsilon$  satisfies a logarithmic energy bound  $\leq C \log \frac{R}{\varepsilon}$ , giving local bound for Dirichlet energy of  $V_\varepsilon$  in  $B_R^+$ .

It follows that  $V_\varepsilon \rightarrow V$  in  $H^1(B_R)$  for every  $R$ . Since  $V_\varepsilon$  satisfies the PDE

$$\int_{B_{\rho/\varepsilon}^+} a_{ij}^\varepsilon \partial_i V_\varepsilon \partial_j \varphi + \int_{\Gamma_{\rho/\varepsilon}} \left( \frac{1}{2} \sin 2V_\varepsilon + h_\varepsilon \right) b^\varepsilon \varphi = 0 \quad (6.4)$$

for all  $\varphi \in H^1(B_{\rho/\varepsilon}^+)$  vanishing near  $\partial B_{\rho/\varepsilon}$ , where  $a_{ij}^\varepsilon(z) = a_{ij}(\varepsilon z)$ ,  $b^\varepsilon(z) = b^\varepsilon(\varepsilon z)$  and  $h_\varepsilon(z) = \varepsilon \left( \frac{\partial G}{\partial \nu} \circ \Psi \right)(\varepsilon z)$ . Letting  $\varepsilon \rightarrow 0$ , we have  $a_{ij}^\varepsilon \rightarrow \delta_{ij}$ ,  $b^\varepsilon \rightarrow 1$ , and  $h_\varepsilon \rightarrow 0$  uniformly in every  $B_R^+$ . Passing to the limit in (6.4) we thus obtain that  $V$  satisfies the weak form of (6.1)–(6.2). The limit cannot be periodic since this and the strong convergence  $V_\varepsilon \rightarrow V$  in  $L_{\text{loc}}^2(\mathbb{R})$  would otherwise contradict Proposition 3.9 for  $\varepsilon$  sufficiently small.  $\square$

**Corollary 6.3.** If  $(u_\varepsilon)$  are stationary points of  $E_\varepsilon$  with  $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ , then the approximate vortex set  $S_\varepsilon$  can be covered (for a subsequence) by disjoint balls  $B_{\sigma\varepsilon}(a_j^\varepsilon)$  for some  $a_j^\varepsilon \in \partial\Omega$  and some  $\sigma > 0$ . If  $(u_\varepsilon)$  have been minimizers or local minimizers (i.e. with respect to variations of small support) then  $a_j^\varepsilon$  converge to distinct points  $a_j^0$  as  $\varepsilon \rightarrow 0$ .

PROOF. The first part follows from the strong  $L_{\text{loc}}^2(\mathbb{R})$  convergence implied by Proposition 6.2. The second follows as in the proof of Theorem 4.2.  $\square$

## 7 Energy expansion for isolated vortices

In this section we assume that  $u_\varepsilon$  are stationary points of  $E_\varepsilon$  such that  $S_\varepsilon$  can be covered by balls  $B_{\sigma\varepsilon}(a_j^\varepsilon)$  with  $a_j^\varepsilon \rightarrow a_j^0$  that are distinct, and that the jump  $d_j$  near these points is  $\pm 1$ . This holds true for minimizers by Corollary 6.3. We obtain that there is an asymptotic expansion of the energy in terms of a *renormalized energy* depending only on  $(a_j, d_j)$  that can be calculated by solving a linear boundary value problem.

**Proposition 7.1 (Energy expansion for limit functions).** Let  $(a_i)$ , with  $i = 1, \dots, N$  be a collection of distinct points in  $\partial\Omega$ ,  $d_i \in \mathbb{Z}$  with  $\sum_i d_i = 2D$ , and let  $u_*$  be a harmonic function such that  $u_* - g \in \pi\mathbb{Z}$  on  $\partial\Omega$  and  $u_*$  jumps by  $-d_i\pi$  at the points  $a_i$ . Then the Dirichlet energy of  $u_*$  in the domain  $\Omega_\rho = \Omega \setminus \bigcup_{i=1}^N B_\rho(a_i)$  has the following asymptotic expansion:

$$\frac{1}{2} \int_{\Omega_\rho} |\nabla u_*|^2 = \frac{\pi}{2} \sum_{i=1}^N d_i^2 \log \frac{1}{\rho} + W + O(\rho \log \rho), \quad (7.1)$$

where  $W$  is the *renormalized energy* corresponding to  $(a_i, d_i)$  that can be calculated by the expression

$$W = -\pi \sum_{1 \leq i < j \leq N} d_i d_j \log |a_i - a_j| + \int_{\partial\Omega} V g' - \pi \sum_{j=1}^N d_j R(a_j). \quad (7.2)$$

Here  $V$  denotes a solution of the inhomogeneous Neumann problem

$$\Delta V = 0 \quad \text{in } \Omega \quad (7.3)$$

$$\frac{\partial V}{\partial \nu} = g' - \pi \sum_{j=1}^N d_j \delta_{a_j} \quad \text{on } \partial\Omega, \quad (7.4)$$

and  $R$  is a harmonic function, continuous on  $\bar{\Omega}$  and given by

$$R(z) = V(z) - \sum_{j=1}^N d_j \log |z - a_j|. \quad (7.5)$$

PROOF. Similar but simpler calculations than the corresponding proof for interior vortices (where the lifting of  $e^{iu}$  to  $u$  can be done only locally) in [1, Chapter I].  $\square$

**Proposition 7.2.** The configuration  $(a_1, \dots, a_N)$  is a stationary point of the renormalized energy  $W$  if and only if for every  $j$ , the function  $h = u_*(z) - d_j \arg(z - a_j)$  satisfies  $\frac{\partial h}{\partial \nu}(a_j) = 0$ .

PROOF. This follows from essentially the same calculation as the corresponding statement in [1, pp. 87–89].  $\square$

In the following we will assume  $w_\varepsilon$  to be a solution of (5.4) around some point  $z_0$  that satisfies  $|\sin 2w_\varepsilon| < \frac{1}{4}$  on  $\Gamma_{R_0} \setminus \Gamma_{\sigma\varepsilon}$  and  $\sup_{B_R^+} |w_\varepsilon| \leq C$ . Assume in addition that  $|w(\pm x, 0) - k_\pm \pi| < \frac{1}{4}$  for  $x \in (\sigma\varepsilon, R_0)$ , with  $k_\pm \in \mathbb{Z}$  and  $|k_+ - k_-| = 1$ . These assumptions are valid for minimizers by Corollary 6.3.

Let  $\bar{w}_\varepsilon$  be the solution of

$$\begin{aligned} \Delta \bar{w}_\varepsilon &= 0 \quad \text{in } \mathbb{R}_+^2 \\ \frac{\partial \bar{w}_\varepsilon}{\partial \nu} &= -\frac{1}{2\varepsilon} \sin 2\bar{w}_\varepsilon \quad \text{on } \mathbb{R} = \partial\mathbb{R}_+^2 \end{aligned}$$

that satisfies  $\bar{w}_\varepsilon(x, 0) \rightarrow k_\pm$  as  $x \rightarrow \pm\infty$  and  $w_\varepsilon(0, 0) = \frac{k_- + k_+}{2}$ . Without loss of generality, we assume  $k_+ = k_- + 1$ . By Toland's uniqueness result Theorem 6.1,  $\bar{w}_\varepsilon$  is given by

$$\bar{w}_\varepsilon(z) = k_- + W_0\left(\frac{z}{\varepsilon}\right) \quad (7.6)$$

where  $W_0$  is the base solution

$$W_0(z) = \frac{\pi}{2} + \arctan \frac{x}{y+1}. \quad (7.7)$$

**Proposition 7.3.** For  $R < \frac{R_0}{2}$ , there holds

$$\int_{B_R^+} |\nabla w_\varepsilon - \nabla \bar{w}_\varepsilon|^2 + \frac{1}{\varepsilon} \int_{\Gamma_R} |w_\varepsilon - \bar{w}_\varepsilon|^2 \leq C. \quad (7.8)$$

PROOF. The function  $\bar{w}_\varepsilon$  is a solution of

$$\int_{B_R^+} \delta_{ij} \partial_i \bar{w}_\varepsilon \partial_j \varphi + \frac{1}{2\varepsilon} \int_{\Gamma_R} \sin 2\bar{w}_\varepsilon \varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(B_R^+). \quad (7.9)$$

With the notation used already in Proposition 5.6 and setting  $g_j = (a_{ij} - \delta_{ij}) \partial_i \bar{w}_\varepsilon$  and  $H = bh + (b-1) \frac{1}{2\varepsilon} \sin 2\bar{w}_\varepsilon$ , we can rewrite (7.9) as

$$\int_{B_R^+} a_{ij} \partial_i \bar{w}_\varepsilon \partial_j \varphi + \frac{1}{2\varepsilon} \int_{\Gamma_R} \sin 2\bar{w}_\varepsilon b \varphi = \int_{B_R^+} g_j \partial_j \varphi + \int_{\Gamma_R} H \varphi, \quad (7.10)$$

where  $H$  and  $g_j$  satisfy by the definition of  $a_{ij}$  and the explicit form of  $\bar{w}_\varepsilon$  the estimates  $|g_j| \leq Cr \frac{1}{\sqrt{r^2 + \varepsilon^2}} \leq C$  and  $|H| \leq C + C \frac{r}{2\varepsilon} (1 \wedge \frac{\varepsilon}{r}) \leq C$ . Subtracting (7.10) from (5.4) and testing with  $\eta^2(w_\varepsilon - \bar{w}_\varepsilon)$  leads using ellipticity and Young's inequality for any  $\delta > 0$  to the estimate

$$\begin{aligned} & c \int_{B_R^+} \eta^2 |\nabla w_\varepsilon - \nabla \bar{w}_\varepsilon|^2 + \frac{1}{2\varepsilon} \int_{\Gamma_R} \eta^2 (\sin 2w_\varepsilon - \sin 2\bar{w}_\varepsilon)(w_\varepsilon - \bar{w}_\varepsilon) \\ & \leq C \int_{B_R^+} |\nabla \eta|^2 (w_\varepsilon - \bar{w}_\varepsilon)^2 + C \int_{B_R^+} \eta^2 (g_1^2 + g_2^2) + \frac{C\varepsilon}{\delta} \int_{\Gamma_R} \eta^2 H^2 + \frac{\delta}{\varepsilon} \int_{\Gamma_R} \eta^2 (w_\varepsilon - \bar{w}_\varepsilon)^2. \end{aligned}$$

On  $\Gamma_R \setminus \Gamma_{\sigma\varepsilon}$ , we have by assumption that  $(\sin 2w_\varepsilon - \sin 2\bar{w}_\varepsilon)(w_\varepsilon - \bar{w}_\varepsilon) \geq c|w_\varepsilon - \bar{w}_\varepsilon|^2$  and so we can choose  $\delta > 0$  small enough and use  $\eta \leq 1$  and the bounds on  $g_j$  and  $H$  to obtain

$$\begin{aligned} & c \int_{B_R^+} \eta^2 |\nabla w_\varepsilon - \nabla \bar{w}_\varepsilon|^2 + \frac{c}{\varepsilon} \int_{\Gamma_R} \eta^2 |w_\varepsilon - \bar{w}_\varepsilon|^2 \\ & \leq C \int_{B_R^+} |\nabla \eta|^2 (w_\varepsilon - \bar{w}_\varepsilon)^2 + CR^2 + CR\varepsilon + \frac{1}{2\varepsilon} \int_{\Gamma_{\sigma\varepsilon}} |(\sin 2w_\varepsilon - \sin 2\bar{w}_\varepsilon)(w_\varepsilon - \bar{w}_\varepsilon)| + |w_\varepsilon - \bar{w}_\varepsilon|^2. \end{aligned} \quad (7.11)$$

Choosing for  $\eta$  a smooth function that is equal to 1 on  $B_{R/2}^+$ , equal to 0 outside  $B_R^+$  and satisfying  $|\nabla \eta| \leq \frac{C}{R}$ , the right hand side is seen to be bounded by a constant depending on  $R$ .  $\square$

**Proposition 7.4.** For all  $\varepsilon > 0$  there exists  $a_\varepsilon \in \mathbb{R}$  such that for all  $C_1 > 0$  there holds

$$\frac{1}{2\varepsilon} \int_{\Gamma_{C_1\varepsilon}} |w_\varepsilon(x, 0) - \bar{w}_\varepsilon(x - a_\varepsilon\varepsilon, 0)|^2 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (7.12)$$

The shifts  $a_\varepsilon$  are uniformly bounded:  $|a_\varepsilon| \leq C_0$ .

PROOF. Rescaling by  $\varepsilon$ , we obtain the functions  $W_\varepsilon(z) = w_\varepsilon(\varepsilon z)$  and  $\bar{W}_\varepsilon(z) = \bar{w}_\varepsilon(\varepsilon z) = \bar{W}(z)$ . If the assertion were false, then there exists a subsequence  $\varepsilon \rightarrow 0$  and a  $\delta > 0$  such that

$$\frac{1}{2} \int_{\Gamma_{C_1}} |W_\varepsilon(x) - \bar{W}(x - a)|^2 \geq \delta > 0 \quad (7.13)$$

for all  $a$  with  $|a| \leq C_0$ . Repeating up to rescaling the proof of Proposition 6.2, we obtain that  $W_\varepsilon \rightharpoonup W_*$  in  $H^1(B_R^+)$  for all  $R > 0$ , for some  $W_* \in H_{\text{loc}}^1(\mathbb{R}_+^2)$ .  $W_*$  must be a solution of the half-space problem, and by Rellich-Kondrachov embedding on the boundary, we obtain

$$\int_{\Gamma_R} |W_* - \bar{W}|^2 = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_R} |W_\varepsilon - \bar{W}|^2 \leq \int_{\mathbb{R}} |W_\varepsilon - \bar{W}|^2 \leq C$$



by (7.8), in particular the difference  $W_* - \overline{W}$  is in  $L^2(\mathbb{R})$ . From Toland's theorem 6.1 we obtain that  $W_*$  can only be a translation of  $\overline{W}$ , i.e.  $W_*(z) = \overline{W}(z - a)$ . From

$$\int_{\mathbb{R}} |W_* - \overline{W}|^2 \leq C$$

and the explicit form of the solution we deduce that  $|a| \leq C_0$  for some  $C_0$ . The convergence  $W_\varepsilon \rightarrow W_*$  also implies

$$\int_{\Gamma_{C_1}} |W_\varepsilon - W_*|^2 \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ , contradicting (7.13).  $\square$

**Proposition 7.5.** If we redefine  $\overline{w}_\varepsilon$  by choosing the shifts  $a_\varepsilon$  as in Proposition 7.4, then the energies of  $w_\varepsilon$  and  $\overline{w}_\varepsilon$  are asymptotically close:

$$\limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_{B_\rho^+} |\nabla w_\varepsilon - \nabla \overline{w}_\varepsilon|^2 = 0 \quad (7.14)$$

and

$$\limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Gamma_\rho} |w_\varepsilon - \overline{w}_\varepsilon|^2 = 0 \quad (7.15)$$

PROOF. Let  $R \leq \frac{R_0}{4}$ . Then by a suitable Poincaré inequality

$$\int_{B_R^+ \setminus B_{R/2}^+} |w_\varepsilon - \overline{w}_\varepsilon|^2 \leq CR^2 \int_{B_R^+ \setminus B_{R/2}^+} |\nabla w_\varepsilon - \nabla \overline{w}_\varepsilon|^2 + CR \int_{\Gamma_R \setminus \Gamma_{R/2}} (w_\varepsilon - \overline{w}_\varepsilon)^2. \quad (7.16)$$

This and (7.11) show together with Proposition 7.4 that

$$\int_{B_{R/2}^+} |\nabla w_\varepsilon - \nabla \overline{w}_\varepsilon|^2 \leq C \int_{B_R^+ \setminus B_{R/2}^+} |\nabla w_\varepsilon - \nabla \overline{w}_\varepsilon|^2 + C \frac{\varepsilon}{R} + CR^2 + \omega(\varepsilon), \quad (7.17)$$

where  $\omega(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Adding the integral over  $B_{R/2}^+$  to both sides ("filling the hole") leads with  $\vartheta = \frac{C}{C+1} < 1$  to

$$\int_{B_{R/2}^+} |\nabla w_\varepsilon - \nabla \overline{w}_\varepsilon|^2 \leq \vartheta \int_{B_R^+} |\nabla w_\varepsilon - \nabla \overline{w}_\varepsilon|^2 + C \frac{\varepsilon}{R} + CR^2 + \omega(\varepsilon), \quad (7.18)$$

from which we conclude the first claim. The second follows from the first, (7.11) and Proposition 7.4.  $\square$

**Proposition 7.6.** There holds

$$\limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \left| \int_{B_\rho^+} a_{ij} \partial_i w_\varepsilon \partial_j w_\varepsilon - \int_{B_\rho^+} |\nabla \overline{w}_\varepsilon|^2 \right| = 0 \quad (7.19)$$

PROOF. We have

$$\begin{aligned} \int_{B_\rho^+} a_{ij} \partial_i w_\varepsilon \partial_j w_\varepsilon &= \int_{B_\rho^+} a_{ij} \partial_i \overline{w}_\varepsilon \partial_j \overline{w}_\varepsilon \\ &\quad + 2 \int_{B_\rho^+} a_{ij} \partial_i \overline{w}_\varepsilon \partial_j (w_\varepsilon - \overline{w}_\varepsilon) + \int_{B_\rho^+} a_{ij} \partial_i (w_\varepsilon - \overline{w}_\varepsilon) \partial_j (w_\varepsilon - \overline{w}_\varepsilon). \end{aligned} \quad (7.20)$$

The last term goes to 0 by Proposition 7.5. For the other terms, we have

$$\left| \int_{B_\rho^+} a_{ij} \partial_i \overline{w}_\varepsilon \partial_j \overline{w}_\varepsilon - \int_{B_\rho^+} |\nabla \overline{w}_\varepsilon|^2 \right| \leq \int_{B_\rho^+} Cr \frac{1}{r^2} \leq C\rho$$

and

$$\left| \int_{B_\rho^+} a_{ij} \partial_i \bar{w}_\varepsilon \partial_j (w_\varepsilon - \bar{w}_\varepsilon) \right| \leq \int_{B_\rho^+} C r \frac{1}{r} |\nabla w_\varepsilon - \nabla \bar{w}_\varepsilon|,$$

which goes to 0 by Hölder's inequality and (7.14). For the final term, we use the harmonicity of  $\bar{w}_\varepsilon$  and integrate by parts. This shows

$$\int_{B_\rho^+} \nabla \bar{w}_\varepsilon \cdot \nabla (w_\varepsilon - \bar{w}_\varepsilon) = \int_{\Gamma_\rho} \frac{\partial \bar{w}_\varepsilon}{\partial \nu} (w_\varepsilon - \bar{w}_\varepsilon) + \int_{\partial B_\rho \cap \mathbb{R}_+^2} \frac{\partial \bar{w}_\varepsilon}{\partial \nu} (w_\varepsilon - \bar{w}_\varepsilon)$$

The integral over  $\partial B_\rho \cap \mathbb{R}_+^2$  can be estimated since the integrand is bounded by  $\frac{C\varepsilon}{\rho^2}$ , which tends to 0 under the convergence considered. The other is via Hölder's inequality bounded by  $\left(\frac{C}{\varepsilon} \int_{\Gamma_R} |w_\varepsilon - \bar{w}_\varepsilon|^2\right)^{1/2}$ , which tends to 0 by (7.15).  $\square$

**Proposition 7.7.** The energy of  $\bar{w}_\varepsilon$  on  $B_\rho^+$  satisfies

$$\lim_{\varepsilon \rightarrow 0} \left( \frac{1}{2} \int_{B_\rho^+} |\nabla \bar{w}_\varepsilon|^2 + \frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin^2 \bar{w}_\varepsilon - \frac{\pi}{2} \left( \log \frac{\rho}{\varepsilon} + 1 - \log 2 \right) \right) = 0. \quad (7.21)$$

PROOF. This follows from an explicit calculation.  $\square$

**Theorem 7.8.** Assume that  $u_\varepsilon$  are stationary points of  $E_\varepsilon$  with  $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$  and  $u_\varepsilon \rightarrow u_*$  in  $H_{\text{loc}}^1 \cap W^{1,p}(\Omega)$ , where  $u_*$  is the harmonic function corresponding to  $(a_i, d_i)$  as in Proposition 7.1. Assume furthermore that the vortices are isolated, i.e. the centers of the balls covering the approximate vortex set  $S_\varepsilon$  converge to distinct points. (Observe that these conditions are satisfied by subsequences of minimizers by Corollary 6.3). Then as  $\varepsilon \rightarrow 0$ , there holds

$$E_\varepsilon(u_\varepsilon) = \pi D \log \frac{1}{\varepsilon} + W(a_i, d_i) + \pi D(1 - \log 2) + \omega(\varepsilon), \quad (7.22)$$

where  $D = \frac{1}{2} \sum d_i^2$ ,  $W$  is the renormalized energy of Proposition 7.1, and  $\omega(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

The configuration  $(a_i)$  is a stationary point for  $W$  with fixed  $d_i$ , and (locally) minimizing if  $u_\varepsilon$  are (locally, i.e. w.r.t. variations of small support) minimizing.

PROOF. By Proposition 5.8, we have  $u_\varepsilon \rightarrow u_*$  in  $H^1(\Omega_\rho)$ , and in particular for any  $\rho > 0$

$$\lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; \Omega_\rho) = \frac{1}{2} \int_{\Omega_\rho} |\nabla u_*|^2.$$

Inside  $\omega_\rho = B_\rho(a_j) \cap \Omega$ , we use again a harmonic extension  $G$  of  $g$ . With  $v_\varepsilon = u_\varepsilon - G$  there holds

$$\int_{\omega_\rho} |\nabla u_\varepsilon|^2 = \int_{\omega_\rho} |\nabla v_\varepsilon|^2 + 2 \int_{\partial \omega_\rho} v_\varepsilon \frac{\partial G}{\partial \nu} + \int_{\omega_\rho} |\nabla G|^2 = \int_{\omega_\rho} |\nabla u_\varepsilon|^2 + O(\rho).$$

In the limit  $\lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0}$ , we can thus work with  $v_\varepsilon$  instead of  $u_\varepsilon$ . From Proposition 7.6 we already know that the energy of  $v_\varepsilon$  on  $\Psi(B_\rho^+)$  is close to that of  $\bar{w}_\varepsilon$  on  $B_\rho^+$ . The symmetric difference  $\Delta_\rho := \Psi(B_\rho^+) \Delta (B_\rho \cap \Omega)$  does not play a role here since

$$\begin{aligned} & \limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_{\Delta_\rho} |\nabla v_\varepsilon|^2 \\ & \leq C \limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_{\Psi^{-1}(\Delta_\rho)} |\nabla w_\varepsilon|^2 \\ & = C \limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_{\Psi^{-1}(\Delta_\rho)} |\nabla \bar{w}_\varepsilon|^2 = 0 \end{aligned}$$

by (7.19) and the explicit form of  $\bar{w}_\varepsilon$ . Similarly, there also holds (using (7.15))

$$\limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\partial \Delta_\rho \cap \partial \Omega} \sin^2(u_\varepsilon - g) = 0.$$

Since

$$\limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left| \int_{\Gamma_\rho} \sin^2 w_\varepsilon b - \int_{\Gamma_\rho} \sin^2 \bar{w}_\varepsilon \right| = 0$$

as can again be deduced from (7.15) and  $\lim_{\rho \rightarrow 0} \sup_{|t| < \rho} b = 1$ , the energy of  $u_\varepsilon$  in  $B_\rho \cap \Omega$  is thus asymptotically that of  $\bar{w}_\varepsilon$  in  $B_\rho^+$ . We thus obtain the claim from Proposition 7.1 and Proposition 7.7.

To show that  $(a_j)$  is a stationary point of the renormalized energy, we need to show that  $\frac{\partial}{\partial \nu}(u_* - d_j \vartheta)$  is zero at  $a_j$ , where  $\vartheta = \arg(z - a_j)$ . To this end, we calculate using harmonicity of  $u_\varepsilon$  and  $u_*$  and setting  $h = u_* - d_j \vartheta$

$$\begin{aligned} \int_{\partial\Omega \cap B_\rho} \frac{\partial u_\varepsilon}{\partial \nu} &= - \int_{\partial B_\rho \cap \Omega} \frac{\partial u_\varepsilon}{\partial \nu} \xrightarrow{\varepsilon \rightarrow 0} - \int_{\partial B_\rho \cap \Omega} \frac{\partial u_*}{\partial \nu} = \int_{\partial\Omega \cap B_\rho} \frac{\partial u_*}{\partial \nu} \\ &= \int_{\partial\Omega \cap B_\rho} \frac{\partial h}{\partial \nu} \pm \frac{\partial \vartheta}{\partial \nu}. \end{aligned}$$

Using the PDE, we have

$$\int_{\partial\Omega \cap B_\rho} \frac{\partial u_\varepsilon}{\partial \nu} = \frac{1}{2\varepsilon} \int_{\partial\Omega \cap B_\rho} \sin 2(u_\varepsilon - g) = \frac{1}{2\varepsilon} \int_{\Psi^{-1}(\partial\Omega \cap B_\rho)} \sin 2w_\varepsilon b.$$

Using estimates from above, we can again replace  $\Psi^{-1}(\partial\Omega \cap B_\rho)$  by  $\Gamma_\rho$  up to an error that is  $O(\rho^2)$ . Similarly, we can estimate

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \left| \frac{1}{2\varepsilon} \int_{\Gamma_\rho} \sin 2w_\varepsilon b - \int_{\Gamma_\rho} \sin 2\bar{w}_\varepsilon \right| &\leq \limsup_{\varepsilon \rightarrow 0} \left( \int_{\Gamma_\rho} \sin^2 2\bar{w}_\varepsilon \right)^{1/2} \left( \int_{\Gamma_\rho} (b - 1)^2 \right)^{1/2} \\ &\quad + C \limsup_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{\Gamma_\rho} (\sin 2w_\varepsilon - \sin 2\bar{w}_\varepsilon) \\ &= O(\rho^{3/2}) + 0 \end{aligned}$$

by (7.15) and  $|b(s) - 1| \leq Cs$ . Since  $\int_{\partial\Omega \cap B_\rho} \frac{\partial \vartheta}{\partial \nu} = O(\rho^2)$  we obtain that

$$\frac{1}{\rho} \int_{\partial\Omega \cap B_\rho} \frac{\partial h}{\partial \nu} \rightarrow 0$$

as  $\rho \rightarrow 0$ , hence  $\frac{\partial h}{\partial \nu} = 0$  at  $a_j$ .

To show that  $(a_j)$  is (locally) minimizing if we started with (local) minimizers, we can construct for any  $(a'_j)$  a test function  $v_\varepsilon$  similar to that of Proposition 3.1 by interpolating linearly in the radial variable between  $G + \bar{w}_\varepsilon \circ \Psi^{-1}$  inside  $B_\rho$  and  $u_*$  in  $\Omega_{2\rho}$ . It is not hard to show that the resulting function  $v_\varepsilon$  then has an energy whose  $O(1)$  part is given up to a constant by  $W(a'_j, d_j)$ , and by minimality we obtain  $W(a_j, d_j) \leq W(a'_j, d_j)$ .  $\square$

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